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CHAPTER I

EXACT CONTROLLABILITY OF PARABOLIC EQUATIONS

Introduction

Let $(x,t) \in Q = \Omega \times]0,T[$, where $\Omega \subset \mathbb{R}^n$ is a connected bounded domain with boundary $\partial\Omega \subset C^2$, $\nu(x)$ - the external normal to $\partial\Omega$, $T \in (0,+\infty)$ is an arbitrary moment of time. We consider the semilinear parabolic equation

$$G(y) = \frac{\partial y}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial y}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(t,x)\frac{\partial y}{\partial x_i} + c(t,x)y + f(t,x,y) = u + g, \quad u \in \mathcal{U}(\omega), \tag{1}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{\Sigma} = (l_1(t,x)\sum_{i,j=1}^{n} a_{ij}(t,x)\nu_i\frac{\partial y}{\partial x_j} + l_2(t,x)y)\big|_{\Sigma} = 0,$$
$$y(0,x) = v_0(x), \tag{2}$$

where v_0 and g are given, and $u(t,x)$ is a control in the space

$$\mathcal{U}(\omega) = \{u(t,x) \in L^2(Q)\big|\text{supp } u \subset [0,T] \times \overline{\omega}\}.$$

Here ω is an arbitrary fixed subdomain of Ω and $\Sigma =]0,T[\times\partial\Omega$.

By the problem of exact controllability we mean finding a control $u \in \mathcal{U}(\omega)$ such that

$$y(T,x) = v_1(x), \tag{3}$$

where $v_1(x)$ is a given function.

In this paper we also consider the problem of exact boundary controllability, by which we mean finding a control $u(t,x)$ such that

$$G(y) = g \quad \text{in} \quad Q, \quad y(0,x) = v_0(x), \quad y(T,x) = v_1(x), \tag{4}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{]0,T[\times\Gamma_0} = u, \quad (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{]0,T[\times(\partial\Omega\setminus\Gamma_0)} = 0, \tag{5}$$

where Γ_0 is an arbitrary fixed subdomain of $\partial\Omega$, and v_0, v_1, g are given functions.

In the above problems we assume

$$a_{ij} \in C^{1,2}(\overline{Q}), \quad a_{ij} = a_{ji}, \quad b_i \in C^{0,1}(\overline{Q}), \quad c \in L^\infty(Q), \tag{6}$$

where $i, j = 1, \cdots, n$ and the uniform ellipticity: There exists $\beta > 0$ such that

$$a(t,x,\zeta,\zeta) = \sum_{i,j=1}^{n} a_{ij}(t,x)\zeta_i\zeta_j \geq \beta|\zeta|^2 \quad \forall\zeta \in \mathbb{R}^n, \quad (t,x) \in Q, \tag{7}$$

Suppose functions $l_1, l_2 \in C^{1,1}(\overline{\Sigma})$ and

$$\text{either } l_1(t,x) > 0 \ \forall(t,x) \in \Sigma, \text{ or } l_1(t,x) \equiv 0 \text{ and } l_2(t,x) \equiv 1. \tag{8}$$

and compartibility condition of the first order holds

$$\text{if } l_1(t,x) \equiv 0, \text{ then } v_0|_{\partial\Omega} = 0. \tag{9}$$

Firstly exact boundary controllability problem was studied in the work of Yu. V. Egorov [11] for the case of one dimensional equation. For the controllability of the linear heat equation with time independent coefficients there are many developments due to H. Fattorini [15], D. Russel [56] and T. Seidman [58]. Most of the results obtained till 1991 are for parabolic equations in one space dimension and for the heat equation with the control distributed on part of the boundary such that some non-trapping conditions are fulfilled, or in the case of domains of special forms (ball, square,..). In the end of 80's essential progress was made in the theory of exact boundary controllability of hyperbolic equations. Automatically the method introduced by Russel in [57] gives the possibility to prove null controllability for wide class of the linear parabolic equations under non-trapping conditions [33],[34].

For approximate boundary controllability of the semilinear heat equation where the nonlinear term satisfies the sublinear growth condition see C. Fabre, J. P. Puel and E. Zuazua [12]- [14].

The case of exact controllability of semilinear heat equation with Dirichlet boundary conditions was studied in works of O.Yu. Imanuvilov [29]-[32]. The case of Neumann boundary conditions was studied in [9]. For the one dimensional case with analytical nonlinear term there is a result due to W. Littman and Guo [66]. They introduced the method completely different from ours. We should also mention the work of G. Lebeau and L. Robbiano [46] for linear heat equation which used a combination of Russels method, integral transform, and the Carleman inequality for elliptic equations.

Local exact controllabilty results for the Burgers equation were obtained by A.V. Fursikov and O.Yu. Imanuvilov in [16]. In the case when the nonlinear term satisfies the superlinear growth condition there is an estimate for the Burgers equation due to A. V. Fursikov and O. Yu. Imanuvilov [11] which shows that the equation is not approximately controllable with respect to boundary control.

This chapter is organized as follows. In the first section we prove the Carleman estimate for adjoint parabolic equation. In the second section we apply this estimate to solve problems (1)-(3) and (4), (5) for the case of linear parabolic equation. We use a variant of the penalization method.

In section 3 in the case where $f(t,x,\zeta)$ satisfies the global Lipschitz condition in ζ variable with $f(t,x,0)\equiv 0$ we obtain the necessary and sufficient conditions for the global exact controllability, while where $f(t,x,\zeta)$ satisfies the superlinear growth condition in ζ we prove in section 4 the local exact controllability. The exact controllability of the nonlinear problem follows by means of Schauder's fixed point theorem for the global exact controllability and by means of the implicit function theorem for the local exact controllability respectively. Also in section 4 global exact controllability results are proved. In section 5 for some class of parabolic equations we prove an a priori estimates which imply uncontrollability of these equations. Finally in section 6 controllability of Burgers equation is studied.

For more details on the technical assumptions and the results please see the main theorems in following sections.

1. Carleman estimate.

Let us introduce the following spaces:

$$W_p^k(\Omega) = \left\{ w(x) \middle| \, ||w||_{W_p^k(\Omega)} = \Big(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha w|^p dx \Big)^{\frac{1}{p}} < \infty \right\},$$

$\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \partial^{\alpha_1}/\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n}/\partial^{\alpha_n} x_n$,

$$W_p^{1,2}(Q) = \left\{ w(t,x) \middle| \, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j} \in L^p(Q) \; i,j = 1, \dots, n \right\},$$

$$C^{1,2}(\overline{Q}) = \left\{ y(t,x) \middle| \, y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i \partial x_j} \in C(\overline{Q}) \;\; i,j = 1, \cdots, n \right\}.$$

We have

Lemma 1.1. *Let $\omega_0 \Subset \omega$ be an arbitrary fixed subdomain of Ω. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi(x) > 0 \; \forall \, x \in \Omega, \;\; \psi|_{\partial\Omega} = 0, \quad |\nabla \psi(x)| > 0 \quad \forall x \in \Omega \setminus \omega_0. \tag{1.1}$$

The proof of Lemma 1.1 will be given later.

We set

$$\varphi(t,x) = e^{\lambda\psi(x)}/(t(T-t)), \quad \tilde{\varphi}(t,x) = e^{-\lambda\psi(x)}/(t(T-t)), \tag{1.2}$$

$$\begin{aligned} \alpha(t,x) &= (e^{\lambda\psi} - e^{2\lambda||\psi||_{C(\overline{\Omega})}})/(t(T-t)), \\ \tilde{\alpha}(t,x) &= (e^{-\lambda\psi} - e^{2\lambda||\psi||_{C(\overline{\Omega})}})/(t(T-t)), \end{aligned} \tag{1.3}$$

where $\lambda > 0$ and function ψ from Lemma 1.1. Note that

$$\alpha(t,x) \ge \tilde{\alpha}(t,x) \quad \forall (t,x) \in Q.$$

We also set

$$\gamma = \sum_{i,j=1}^n ||a_{ij}||_{C^{1,2}(\overline{Q})} + \sum_{i=1}^n ||b_i||_{C^{0,1}(\overline{Q})} + ||c||_{L^\infty(Q)}, \; \tilde{\gamma} = \sum_{i,j=1}^n ||a_{ij}||_{C^{1,2}(\overline{Q})}.$$

Let us consider the boundary value problem

$$Lz = \frac{\partial z}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t,x) \frac{\partial z}{\partial x_j} \right) + \sum_{i=1}^n b_i(t,x) \frac{\partial z}{\partial x_i} + c(t,x) z = g \quad \text{in} \quad Q, \tag{1.4}$$

$$\left(l_1(t,x) \frac{\partial z}{\partial \nu_A} + l_2(t,x) z\right)\Big|_\Sigma = 0, \quad z(0,\cdot) = z_0. \tag{1.5}$$

We have the following:

Lemma 1.2. *Let (6)- (9) be fulfilled and functions φ, α, $\tilde\varphi$ and $\tilde\alpha$ be defined as in (1.2) and (1.3). Then there exists a number $\hat\lambda > 0$ such that for an arbitrary $\lambda \ge \hat\lambda$ there exists $s_0(\lambda)$ such that for each $s \ge s_0(\lambda)$ the solutions of problem (1.4) - (1.5) satisfy the following inequality:*

$$\int\limits_Q \left(\frac{1}{s\varphi}\left(\left|\frac{\partial z}{\partial t}\right|^2 + |\Delta z|^2\right) + s\varphi|\nabla z|^2 + s^3\varphi^3 z^2\right)(e^{2s\alpha}+e^{2s\tilde\alpha})dx\,dt$$
$$\le c_1\Big(\int\limits_Q |g|^2(e^{2s\alpha}+e^{2s\tilde\alpha})dxdt + \int_{[0,T]\times\omega} s^3\varphi^3 z^2(e^{2s\alpha}+e^{2s\tilde\alpha})dxdt\Big), \tag{1.6}$$

where constant c_1 depends continuously on γ, λ and constant $\hat\lambda$ depends continuously on $\tilde\gamma$.

Since the proof of Lemma 1.2 technically looks very awkward firstly we demonstrate it's main ideas considering the more simple case of the heat equation:

$$\partial_t z + \Delta z = f(t,x) \quad \text{in } Q, \tag{1.7}$$

$$z|_\Sigma = 0, \quad \left.\frac{\partial z}{\partial \nu}\right|_\Sigma = 0. \tag{1.8}$$

We have

Lemma 1.3. *There exists such $s_0 > 0$ that for any $s > s_0$ the solution $z(t,x)$ of (1.7), (1.8) satisfies the Carleman estimate:*

$$\int_Q \left((s\varphi)^{-1}\left(\left|\frac{\partial z}{\partial t}\right|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 z(t,x)}{\partial x_i \partial x_j}\right|^2\right)\right.$$
$$\left. + s\varphi\sum_{j=1}^n \left|\frac{\partial z}{\partial x_i}\right|^2 + s^3\varphi^3 z^2\right) e^{s\alpha(t,x)}\,dx\,dt \le c_3 \int_Q f^2(t,x)e^{s\alpha}\,dx\,dt, \tag{1.9}$$

where the functions $\varphi(t,x)$, $\alpha(t,x)$ are defined in (1.2), (1.3), $\lambda = 1$, $\psi(x) = x_1$ and $c_3 > 0$ does not depend on s.

Proof. We make the change of variables

$$w(t,x) = e^{s\alpha} z(t,x) \tag{1.10}$$

in (1.7), (1.8). As a result in virtue of (1.10) we get

$$L_1w(t,x) + L_2w(t,x) = f_s(t,x) \qquad (t,x) \in Q, \tag{1.11}$$

$$w|_\Sigma = \left.\frac{\partial w}{\partial \nu}\right|_\Sigma = 0, \tag{1.12}$$

where

$$L_1w = \Delta w + s^2\varphi^2 w - s(\partial_t\alpha)w, \tag{1.13}$$

$$L_2w = \partial_t w - 2s\varphi\partial_{x_1}w, \tag{1.14}$$

$$f_s = e^{s\alpha}f + s\varphi w. \tag{1.15}$$

Besides, by virtue of (1.3) and properties of α we have

$$w|_{t=0} = w|_{t=T} = 0. \tag{1.16}$$

Equation (1.11) implies

$$\|L_1w\|^2_{L_2(Q)} + \|L_2w\|^2_{L_2(Q)} + 2(L_1w, L_2w)_{L_2(Q)} = \|f_s\|^2_{L_2(Q)}. \tag{1.17}$$

In virtue of (1.13), (1.14) we get

$$(L_1w, L_2w)_{L_2(Q)} = I_1 + I_2 + I_3, \tag{1.18}$$

where

$$I_1 = \int_Q (\Delta w + s^2\varphi^2 w - s(\partial_t\alpha)w)\partial_t w\,dx\,dt, \tag{1.19}$$

$$I_2 = -\int_Q \Delta w(2s\varphi\partial_{x_1}w)\,dx\,dt, \tag{1.20}$$

$$I_3 = -\int_Q (s^2\varphi^2 - s(\partial_t\alpha))(2s\varphi w\partial_{x_1}w)\,dx\,dt. \tag{1.21}$$

Let us transform I_1, I_2, I_3. Integration by parts in (1.19) with help of (1.12), (1.16) yields

$$I_1 = \int_Q (-\frac{1}{2}\partial_t|\nabla w|^2 + \frac{1}{2}(s^2\varphi^2 - s(\partial_t\alpha))\partial_t|w|^2)\,dx\,dt =$$

$$-\int_Q (s^2\varphi\partial_t\varphi - \frac{s}{2}\partial^2_{tt}\alpha)|w|^2\,dx\,dt. \tag{1.22}$$

Analogously, integration by parts with respect to x in (1.21) with help of (1.12) yields

$$I_3 = -\int_Q (s^2\varphi^2 - s\partial_t\alpha)s\varphi\partial_{x_1}w^2\,dx\,dt = \int_Q (3s^3\varphi^3w^2 - s^2(\partial_t\varphi)\varphi w^2 - s^2(\partial_t\alpha)\varphi w^2)\,dx\,dt. \quad (1.23)$$

Finally, let us estimate term (1.20). Integrating by parts and (1.10) imply

$$I_2 = (\nabla w, \nabla(2s\varphi\partial_{x_1}w))_{(L_2(Q))^n} = \int_Q (2s\varphi(\partial_{x_1}w)^2 + s\varphi\partial_{x_1}|\nabla w|^2)\,dx\,dt = \int_Q (2s^2\varphi(\partial_{x_1}w)^2 - s\varphi|\nabla w|^2)\,dx\,dt. \quad (1.24)$$

We substitute (1.22), (1.23), (1.24) into (1.18) and after that substitute the obtained equality into (1.17). As a result we have

$$\|L_1w\|^2_{L_2(Q)} + \|L_2w\|^2_{L_2(Q)} + 2\int_Q (3s^3\varphi^3|w|^2 - s\varphi|\nabla w|^2 + (\partial_{x_1}w)^2 2s\varphi)\,dx\,dt = \|f_s\|^2_{L_2(Q)} + X_1, \quad (1.25)$$

where

$$X_1 = 2\int_Q (s^2\varphi\partial_t\varphi - \frac{s}{2}\partial^2_{tt}\alpha + s^2\varphi(\partial_t\varphi) + s^2\varphi(\partial_t\alpha))|w|^2\,dx\,dt. \quad (1.26)$$

We get with help of simple estimation of (1.15)

$$\|f_s\|^2_{L_2(\Omega)} \le 2\int_Q (e^{2s\alpha}|f|^2 + s^2\varphi^2|w|^2)\,dx, \quad (1.27)$$

where $c_0 > 0$ does not depend on s, t, x.

Definition (1.2), (1.3) of φ and α imply the inequalities

$$|\partial_t\varphi| \le c_1\varphi^2, \qquad |\partial_t\alpha| \le c_2\varphi^2, \qquad |\partial^2_{tt}\alpha| \le c_3\varphi^3, \quad (1.28)$$

where c_1, $c_2\,c_3$ does not depend on s, t, x. The estimation of (1.26) with help of (1.28) yields

$$|X_1| \le c_4\int_Q (1+s^2)\varphi^3|w|^2\,dx\,dt. \quad (1.29)$$

Scaling (1.11) by $s\varphi w$ in $L_2(Q)$ and taking into account (1.13) we get after integration by parts

$$\int_Q f_s s\varphi w\,dx\,dt = \int_Q (L_2 w)s\varphi w\,dx\,dt + \int_Q (s^3\varphi^3|w|^2 -$$
$$s\varphi(\partial_t\alpha)|w|^2 - s\varphi|\nabla w|^2 + \frac{1}{2}s\Delta\varphi|w|^2)\,dx\,dt.$$

We can rewrite this equality by the form

$$\int_Q s\varphi|\nabla w|^2\,dx\,dt = \int_Q s^3\varphi^3|w|^2\,dx\,dt - X_2, \tag{1.30}$$

where

$$X_2 = \int_Q (f_s^2 s\varphi w - (L_2 w)s\varphi w + s\varphi(\partial_t\alpha)|w|^2 - \frac{1}{2}s\varphi|w|^2)\,dx\,dt. \tag{1.31}$$

We estimate X_2 taking into account (1.27), (1.28):

$$|X_2| \le \frac{1}{4}\|L_2 w\|^2_{L_2(Q)} + c_5 \int_Q (e^{2s\alpha}|f|^2 + (s^2\varphi^2 + s^2\varphi^3 + s\varphi)|w|^2)\,dx\,dt. \tag{1.32}$$

The estimation of (1.25) by means of (1.29), (1.30) yields:

$$\|L_1 w\|^2_{L_2(Q)} + \|L_2 w\|^2_{L_2(Q)} + 2\int_Q (3s^3\varphi^3|w|^2 - s\varphi|\nabla w|^2)\,dx\,dt$$
$$\le \int_Q e^{2s\alpha}|f|^2 dxdt + c_6 \int_Q ((1+s^2)\varphi^3 + s^2\varphi^2)|w|^2\,dx\,dt. \tag{1.33}$$

We express the terms $\int_Q s\varphi|\nabla w|^2\,dx\,dt$ in (1.33) by means of (1.30) and after that use estimation (1.32). As a result we get the upper bound

$$\|L_1 w\|^2_{L_2(Q)} + \|L_2 w\|^2_{L_2(Q)} + 2\int_Q 2s^3\varphi^3|w|^2\,dx\,dt$$
$$\le \frac{1}{2}\|L_2 w\|^2_{L_2(Q)} + c_9 \int_Q (e^{2s\alpha}|f|^2 + s^2\varphi^2 w^2)\,dx\,dt. \tag{1.34}$$

By (1.34) there exists a parameter s_0 such that the following inequality holds:

$$\|L_1 w\|^2_{L_2(Q)} + \|L_2 w\|^2_{L_2(Q)} + \int_Q s^3\varphi^3|w|^2\,dx\,dt$$
$$\le c_{10}\int_Q e^{2s\alpha}|f|^2\,dx\,dt \ \ \forall\ s \ge s_0, \tag{1.35}$$

where c_{10} does not depend on s. After the estimation of right side of (1.30) with help of (1.32), (1.35) we get

$$\int_Q s\varphi|\nabla w|^2\,dx\,dt \le c_{11}\int_Q e^{2s\alpha}|f|^2\,dx\,dt \ \ \forall\ s\ge s_0. \tag{1.36}$$

Multiplying (1.13) on $(s\varphi)^{-\frac{1}{2}}$ and doing estimate with help of (1.35), (1.28) we get

$$\int_Q (s\varphi)^{-1}|\Delta w|^2\,dx\,dt \le c_{12}\int_Q ((s\varphi)^{-1}|L_1 w|^2 + s^3\varphi^3 w^2$$
$$+ (s\varphi)^{-1}|\partial_t\alpha|^2 w^2)\,dx\,dt \le c_{13}\int_Q e^{2s\alpha}|f|^2\,dx\,dt \quad \forall\ s\ge s_0. \tag{1.37}$$

Analogously, multiplying (1.14) on $(s\varphi)^{-\frac{1}{2}}$ we obtain the following inequality by means of (1.35), (1.36):

$$\int_Q (s\varphi)^{-1}|\partial_t w|^2\,dx\,dt \le c_{14}\int_Q e^{2s\alpha}|f|^2\,dx\,dt \quad \forall\ s > s_0. \tag{1.38}$$

After substitution $w = e^{s\alpha}z$ into (1.35) - (1.38) we obtain (1.9).

■

Proof of Lemma 1.2. We give the proof of our lemma for the case $l_1(t,x) > 0$ for all $(t,x)\in\Sigma$. The proof in the case of Dirichlet boundary conditions is more simpler (see [32]). Set $l_3(t,x) = l_2(t,x)/l_1(t,x)$. Then we can rewrite the boundary condition (1.5) as follows

$$(\frac{\partial z}{\partial\nu_A} + l_3(t,x)z)|_\Sigma = 0. \tag{1.39}$$

We can assume without loosing of generality that $l_3(t,x) > 0$ for all $(t,x)\in\Sigma$ otherwise we made the change $z(t,x)\to e^{-\kappa\psi(x)}z(t,x)$ where parameter κ sufficiently large.

Let us consider the operator

$$\hat{L}z = \frac{\partial z}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 z}{\partial x_i\partial x_j}. \tag{1.40}$$

We set

$$\tilde{g}(t,x) = g(t,x) - \sum_{i=1}^{n} b_i(t,x)\frac{\partial z}{\partial x_i} - c(t,x)z + \sum_{i,j=1}^{n} \frac{\partial a_{ij}(t,x)}{\partial x_i}\frac{\partial z}{\partial x_j}. \qquad (1.41)$$

We denote $w(t,x) = e^{s\alpha}z(t,x)$, $\tilde{w}(t,x) = e^{s\tilde{\alpha}}z(t,x)$.
By (1.3) we have

$$w(T,\cdot) = \tilde{w}(T,\cdot) = w(0,\cdot) = \tilde{w}(0,\cdot) = 0 \quad \text{in} \quad \Omega. \qquad (1.42)$$

We define operators $P, \tilde{P}$ as the following:

$$Pw = e^{s\alpha}\hat{L}e^{-s\alpha}w, \quad \tilde{P}w = e^{s\tilde{\alpha}}\hat{L}e^{-s\tilde{\alpha}}w. \qquad (1.43)$$

It follows from (1.4) and (1.40), (1.41) that

$$Pw = e^{s\alpha}\hat{L}e^{-s\alpha}w = e^{s\alpha}\tilde{g} \quad \text{in} \quad Q, \qquad (1.44)$$

$$\tilde{P}\tilde{w} = e^{s\tilde{\alpha}}\hat{L}e^{-s\tilde{\alpha}}\tilde{w} = e^{s\tilde{\alpha}}\tilde{g} \quad \text{in} \quad Q. \qquad (1.45)$$

Operator P can be written explicitly as follows

$$\begin{aligned} Pw =& \frac{\partial w}{\partial t} - \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}\psi_{x_i}\frac{\partial w}{\partial x_j} + s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w \\ & - s^2\lambda^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi)w + s\lambda\varphi w\sum_{i,j=1}^{n} a_{ij}\psi_{x_i x_j} - s\alpha_t w. \qquad (1.46) \end{aligned}$$

We recall that quadratic form $a(t,x,\xi,\eta)$ was defined in (7). We introduce the operators L_1, L_2, $\tilde{L_1}$ and $\tilde{L_2}$ as follows

$$L_1 w = -\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} - \lambda^2 s^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi)w - s\alpha_t w, \qquad (1.47)$$

$$L_2 w = \frac{\partial w}{\partial t} + 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\psi_{x_i}\frac{\partial w}{\partial x_j} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w, \qquad (1.48)$$

$$\tilde{L_1} w = -\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} - \lambda^2 s^2\tilde{\varphi}^2 a(t,x,\nabla\psi,\nabla\psi)w - s\tilde{\alpha}_t w, \qquad (1.49)$$

$$\tilde{L_2}w = \frac{\partial w}{\partial t} - 2s\lambda\tilde{\varphi}\sum_{i,j=1}^{n} a_{ij}\psi_{x_i}\frac{\partial w}{\partial x_j} + 2s\lambda^2\tilde{\varphi}a(t,x,\nabla\psi,\nabla\psi)w. \tag{1.50}$$

It follows from (1.41), (1.46), (1.47) and (1.48) that

$$L_1 w + L_2 w = f_s \quad \text{in} \quad Q, \tag{1.51}$$

where

$$f_s(t,x) = \tilde{g}e^{s\alpha} - s\lambda\varphi w\sum_{i,j=1}^{n} a_{ij}\psi_{x_i x_j} + s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w.$$

Taking L_2-norm of both sides of (1.51), we obtain

$$||f_s||^2_{L^2(Q)} = ||L_1 w||^2_{L^2(Q)} + ||L_2 w||^2_{L^2(Q)} + 2(L_1, w, L_2 w)_{L^2(Q)}. \tag{1.52}$$

By (1.47) and (1.48) we have the following equality:

$$\begin{aligned}
(L_1 w, L_2 w)_{L^2(Q)} = \Big(&- \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} - \lambda^2 s^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi)w \\
- s\alpha_t w, \frac{\partial w}{\partial t} &+ 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w\Big)_{L^2(Q)} - \int_Q (2\lambda^3 s^3\varphi^3 a(t,x,\nabla\psi,\nabla\psi)w \\
&+ 2s^2\lambda\varphi\alpha_t w)a(t,x,\nabla\psi,\nabla w)dxdt \\
&- \int_Q \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j}\right) 2s\lambda\varphi a(t,x,\nabla\psi,\nabla w)dxdt.
\end{aligned} \tag{1.53}$$

Integrating by parts in the first term of the right-hand-side of (1.53), we obtain

$$A_0 = \Big(- \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} - \lambda^2 s^2 \varphi^2 a(t,x,\nabla\psi,\nabla\psi) w - s\alpha_t w, \frac{\partial w}{\partial t}$$

$$+ 2s\lambda^2 \varphi a(t,x,\nabla\psi,\nabla\psi) w \Big)_{L^2(Q)} = \int\limits_Q \Bigg(\frac{\partial w}{\partial t} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w_t}{\partial x_j}$$

$$- \frac{\lambda^2 s^2 \varphi^2}{2} a(t,x,\nabla\psi,\nabla\psi) \frac{\partial w^2}{\partial t} - \frac{s\alpha_t}{2} \frac{\partial w^2}{\partial t} - 2s^3 \varphi^3 \lambda^4 a(t,x,\nabla\psi,\nabla\psi)^2 w^2$$

$$- 2s^2 \lambda^2 \alpha_t \varphi a(t,x,\nabla\psi,\nabla\psi) w^2 + 2\lambda^2 s \varphi a(t,x,\nabla\psi,\nabla\psi) w \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i}$$

$$+ 2s\lambda^2 \varphi a(t,x,\nabla\psi,\nabla\psi) a(t,x,\nabla w,\nabla w) \Bigg)$$

$$+ 2s\lambda^2 w \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial}{\partial x_i} (\varphi a(t,x,\nabla\psi,\nabla\psi)) \Big) dxdt$$

$$- \int\limits_\Sigma \left(\frac{\partial w}{\partial t} + 2s\lambda^2 \varphi a(t,x,\nabla\psi,\nabla\psi) w \right) \frac{\partial w}{\partial \nu_A} d\Sigma. \quad (1.54)$$

Integrating by parts in the second term of the right-hand-side of (1.53), we have

$$-\int\limits_Q (2\lambda^3 s^3 w \varphi^3 a(t,x,\nabla\psi,\nabla\psi) a(t,x,\nabla\psi,\nabla w) + 2s^2 \lambda \alpha_t w \varphi a(t,x,\nabla\psi,\nabla w)) dxdt$$

$$= -\int\limits_Q (\lambda^3 s^3 \varphi^3 a(t,x,\nabla\psi,\nabla\psi) a(t,x,\nabla\psi,\nabla w^2) + s^2 \alpha_t \varphi \lambda a(t,x,\nabla\psi,\nabla w^2)) dxdt$$

$$= \int\limits_Q (3\lambda^4 s^3 \varphi^3 a(t,x,\nabla\psi,\nabla\psi)^2 w^2 + w^2 \varphi^3 \lambda^3 s^3 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij} \psi_{x_j} a(t,x,\nabla\psi,\nabla\psi))$$

$$+ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{s^2 \lambda^2 \alpha_t \varphi}{2} a_{ij} \frac{\partial \psi}{\partial x_i} \right) w^2) dxdt$$

$$- \int\limits_\Sigma (\lambda^3 s^3 \varphi^3 a(t,x,\nabla\psi,\nabla\psi) + s^2 \alpha_t \varphi \lambda) a(t,x,\nabla\psi,\nu) w^2 d\Sigma. \quad (1.55)$$

Finally, integrating by parts for the third term of right-hand-side of (1.53), and taking into account (1.1_2) we have

$$
\begin{aligned}
A_1 &= \int_Q -\left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j}\right)\left(2s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial w}{\partial x_\ell}\right) dxdt \\
&= \int_Q \left(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial w}{\partial x_\ell} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2\right. \\
&+ 2s\lambda\varphi \sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial w}{\partial x_i}\sum_{k,\ell=1}^{n}\frac{\partial}{\partial x_j}(a_{k\ell}\psi_{x_k})\frac{\partial w}{\partial x_\ell}\right) \\
&\left.+ 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}\frac{\partial w}{\partial x_i}\sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial^2 w}{\partial x_j \partial x_\ell}\right) dxdt + \int_\Sigma 2s\lambda\varphi|\nabla\psi|\left|\frac{\partial w}{\partial \nu_A}\right|^2 d\Sigma \\
&= \int_Q \Big(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial w}{\partial x_\ell} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2 \\
&+ 2s\lambda\varphi \sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial w}{\partial x_i}\sum_{k,\ell=1}^{n}\frac{\partial}{\partial x_j}(a_{k\ell}\psi_{x_k})\frac{\partial w}{\partial x_\ell}\right) \\
&- s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&\left.+ s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial}{\partial x_\ell}\sum_{i,j=1}^{n} a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\right) dxdt + \int_\Sigma 2s\lambda\varphi|\nabla\psi|\left|\frac{\partial w}{\partial \nu_A}\right|^2 d\Sigma.
\end{aligned}
\tag{1.56}
$$

Integrating by parts once again, we obtain

$$
\begin{aligned}
A_1 = \int_Q &\left(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\frac{\partial w}{\partial x_\ell} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2 + \right. \\
2s\lambda\varphi &\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial w}{\partial x_i}\sum_{k,\ell=1}^{n}\frac{\partial}{\partial x_j}(a_{k\ell}\psi_{x_k})\frac{\partial w}{\partial x_\ell}\right) - s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&- s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla w,\nabla w) - s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}
\end{aligned}
$$

$$-a(t,x,\nabla w,\nabla w)s\lambda\varphi\sum_{k,\ell=1}^{n}\frac{\partial}{\partial x_\ell}(a_{k\ell}\psi_{x_k})\Bigg)\,dxdt$$
$$+\int\limits_{\Sigma}\left(2s\lambda\varphi|\nabla\psi|\left|\frac{\partial w}{\partial\nu_A}\right|^2-s\lambda\varphi|\nabla\psi|a(t,x,\nabla w,\nabla w)a(t,x,\nu,\nu)\right)d\Sigma. \quad (1.57)$$

Now, let us transform integrals on Σ in (1.54) and (1.57). By virtue of (1.39) for the integral on Σ in (1.54) we have

$$\int\limits_{\Sigma}\left(\frac{\partial w}{\partial t}+2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w\right)a(t,x,\nu,\nabla w)d\Sigma$$
$$=\int\limits_{\Sigma}\left(\frac{\partial w}{\partial t}+2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w)(a(t,x,\nu,\nabla z)e^{s\alpha}+s\lambda a(t,x,\nu,\nabla\psi)w\right)d\Sigma$$
$$=\int\limits_{\Sigma}\left(\frac{\partial w}{\partial t}+2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w\right)(s\lambda a(t,x,\nu,\nabla\psi)-l_3(t,x))wd\Sigma; \quad (1.58)$$

On the other hand, for the integrals on Σ in (1.57) we have

$$\int\limits_{\Sigma}\left(2s\lambda\varphi|\nabla\psi|\left|\frac{\partial w}{\partial\nu_A}\right|^2-s\lambda\varphi|\nabla\psi|a(t,x,\nabla w,\nabla w)a(t,x,\nu,\nu)\right)d\Sigma$$
$$=\int\limits_{\Sigma}(2s\lambda\varphi|\nabla\psi|a(t,x,\nu,\nabla w)^2-s\lambda\varphi|\nabla\psi|a(t,x,\nabla w,\nabla w)a(t,x,\nu,\nu))d\Sigma$$
$$=\int\limits_{\Sigma}(2s\lambda\varphi|\nabla\psi|(-l_3(t,x)w+s\lambda\varphi a(t,x,\nu,\nabla\psi)w)^2$$
$$-s\lambda\varphi|\nabla\psi|a(t,x,e^{s\alpha}(\nabla z+s\lambda\varphi\nabla\psi z),e^{s\alpha}(\nabla z+s\lambda\varphi\nabla\psi z))a(t,x,\nu,\nu))d\Sigma$$
$$=\int\limits_{\Sigma}(2s^3\lambda^3\varphi^3|\nabla\psi|a(t,x,\nu,\nabla\psi)^2w^2+2s\lambda\varphi|\nabla\psi|l_3^2w^2$$
$$+4s^2\lambda^2\varphi^2|\nabla\psi|^2l_3(t,x)a(t,x,\nu,\nu)w^2-2s^2\lambda^2\phi^2|\nabla\psi|^2l_3(t,x)a(t,x,\nu,\nu)w^2$$
$$-|\nabla\psi|e^{2s\alpha}(s\lambda\varphi a(t,x,\nabla z,\nabla z)+s^3\lambda^3\varphi^3a(t,x,\nabla\psi,\nabla\psi)z^2)a(t,x,\nu,\nu))d\Sigma. \quad (1.59)$$

By virtue of (1.54), (1.55) and (1.57) - (1.59) one can rewrite (1.53) as follows.

$$
\begin{aligned}
(L_1 w, L_2 w)_{L^2(Q)} = &\int_Q (\lambda^4 s^3 \varphi^3 a(t,x,\nabla\psi,\nabla\psi)^2 w^2 \\
&+ s\lambda^2 \varphi a(t,x,\nabla\psi,\nabla\psi) a(t,x,\nabla w,\nabla w) + L_2 w \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i} \\
&+ 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2) dxdt + \int_\Sigma (2s^3\lambda^3\varphi^3 w^2 |\nabla\psi| a(t,x,\nu,\nabla\psi)^2 \\
&+ 2s\lambda\varphi|\nabla\psi| l_3^2(t,x) w^2 + 2s^2\lambda^2\varphi^2 l_3(t,x)|\nabla\psi|^2 a(t,x,\nu,\nu) w^2 \\
&- s\lambda\varphi|\nabla\psi| e^{2s\alpha}(a(t,x,\nabla z,\nabla z) + s^2\lambda^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi) z^2) a(t,x,\nu,\nu)) d\Sigma \\
&- \int_\Sigma \left(\frac{\partial w}{\partial t} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi) w\right)(s\lambda\varphi a(t,x,\nu,\nabla\psi) - l_3(t,x)) w d\Sigma \\
&- \int_\Sigma (\lambda^3 s^3\varphi^3 a(t,x,\nabla\psi,\nabla\psi) + s^2\alpha_t\varphi\lambda) w^2 a(t,x,\nabla\psi,\nu) d\Sigma + X_1, \qquad (1.60)
\end{aligned}
$$

where we put

$$
\begin{aligned}
X_1 = &\int_Q \Bigg(2s\lambda^2 w \sum_{i,j=1}^n a_{ij}\frac{\partial w}{\partial x_j}\frac{\partial}{\partial x_i}(\varphi a(t,x,\nabla\psi,\nabla\psi)) \\
&+ \frac{1}{2}\frac{\partial}{\partial t}(\lambda^2 s^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi)) w^2 - \frac{s\alpha_{tt} w^2}{2} \\
&+ 2s\lambda\varphi \sum_{i,j=1}^n \left(a_{ij}\frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n \frac{\partial(a_{k\ell}\psi_{x_k})}{\partial x_j}\frac{\partial w}{\partial x_\ell}\right) - s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell}\psi_{x_k} \\
&\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} - a(t,x,\nabla w,\nabla w) s\lambda\varphi \sum_{k,\ell=1}^n \frac{\partial}{\partial x_\ell}(a_{k\ell}\psi_{x_k}) \\
&- \frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + w^2\varphi^3\lambda^3 s^3 \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\psi_{x_j} a(t,x,\nabla\psi,\nabla\psi)) \\
&- \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(\frac{s^2\alpha_t\varphi\lambda^2}{2} a_{ij}\frac{\partial\psi}{\partial x_i}\right) w^2 \Bigg) dxdt.
\end{aligned}
$$

One can easily prove the following estimate:

$$|X_1| \le c_2 \int_Q ((s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^3)w^2 + (s\lambda\varphi + 1)|\nabla w|^2)dxdt$$
$$s \ge 1, \quad \lambda \ge 1, \tag{1.61}$$

where the constant c_2 is independent on s and λ.

Similarly to (1.51) we have

$$\tilde{L}_1\tilde{w} + \tilde{L}_2\tilde{w} = \tilde{f}_s \quad \text{in} \quad Q, \tag{1.62}$$

where

$$\tilde{f}_s(t,x) = \tilde{g}e^{s\tilde{\alpha}} + s\lambda\tilde{\varphi}w \sum_{i,j=1}^n a_{ij}\psi_{x_ix_j} + s\lambda^2\tilde{\varphi}wa(t,x,\nabla\psi,\nabla\psi).$$

Thus,

$$||\tilde{f}_s||^2_{L^2(Q)} = ||\tilde{L}_1\tilde{w}||^2_{L^2(Q)} + ||\tilde{L}_2\tilde{w}||^2_{L^2(Q)} + 2(\tilde{L}_1\tilde{w}, \tilde{L}_2\tilde{w})_{L^2(Q)}. \tag{1.63}$$

Since $\psi(x)|_{\partial\Omega} = 0$ we have

$$\tilde{w}|_\Sigma = w|_\Sigma; \quad \tilde{\varphi}|_\Sigma = \varphi|_\Sigma; \quad \tilde{\alpha}|_\Sigma = \alpha|_\Sigma. \tag{1.64}$$

By similar arguments one can obtain the analog of equality (1.60) for the scalar product $(\tilde{L}_1\tilde{w}, \tilde{L}_2\tilde{w})_{L^2(Q)}$, and transform it using (1.64).

$$\begin{aligned}
(\tilde{L}_1\tilde{w}, \tilde{L}_2\tilde{w})_{L^2(Q)} = &\int_Q (\lambda^4 s^3\tilde{\varphi}^3 a(t,x,\nabla\psi,\nabla\psi)\tilde{w}^2 + L_2\tilde{w}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial\tilde{w}}{\partial x_i} \\
&+ s\lambda^2\tilde{\varphi}a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla\tilde{w},\nabla\tilde{w}) + 2s\lambda^2\tilde{\varphi}a(t,x,\nabla\psi,\nabla\tilde{w})^2)dxdt \\
&- \int_\Sigma (2s^3\lambda^3\varphi^3 w^2|\nabla\psi|a(t,x,\nu,\nabla\psi)^2 + 2s\lambda\varphi l_3^2(t,x)|\nabla\psi|w^2 \\
&- 2s^2\lambda^2\varphi^2|\nabla\psi|^2 l_3(t,x)a(t,x,\nu,\nu)w^2 + s\lambda\varphi|\nabla\psi|e^{2s\alpha}(a(t,x,\nabla z,\nabla z) \\
&+ s^2\lambda^2\varphi^2 a(t,x,\nabla\psi,\nabla\psi)z^2)a(t,x,\nu,\nu))d\Sigma \\
&+ \int_\Sigma \left(\frac{\partial w}{\partial t} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w\right)(s\lambda\varphi a(t,x,\nu,\nabla\psi) - l_3(t,x))wd\Sigma \\
&+ \int_\Sigma (\lambda^3 s^3\varphi^3 a(t,x,\nabla\psi,\nabla\psi) + s^2\lambda\alpha_t\varphi)w^2 a(t,x,\nabla\psi,\nu)d\Sigma + X_2,
\end{aligned} \tag{1.65}$$

where $|X_2|$ satisfies the estimate

$$|X_2| \le c_3 \int_Q [(s^3\lambda^3\tilde{\varphi}^3 + s^2\lambda^4\tilde{\varphi}^3)\tilde{w}^2 + (s\lambda\tilde{\varphi}+1)|\nabla\tilde{w}|^2]dxdt \ \ \forall\, s \ge 1, \lambda \ge 1. \tag{1.66}$$

Constant c_3 is independent of s and λ.

Hence by virtue of (1.52), (1.59), (1.63) and (1.65) we have

$$\begin{aligned}
&||f_s||^2_{L^2(Q)} + ||\tilde{f}_s||^2_{L^2(Q)} = ||\tilde{L}_1\tilde{w}||^2_{L^2(Q)} + ||L_1 w||^2_{L^2(Q)} + ||\tilde{L}_2\tilde{w}||^2_{L^2(Q)} \\
&+ ||L_2 w||^2_{L^2(Q)} + 2\int_Q (\lambda^4 s^3\varphi^3 a(t,x,\nabla\psi,\nabla\psi)^2 w^2 + \lambda^4 s^3\tilde{\varphi}^3 a(t,x,\nabla\psi,\nabla\psi)^2\tilde{w}^2 \\
&+ s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla w,\nabla w) + s\lambda^2\tilde{\varphi} a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla\tilde{w},\nabla\tilde{w}) \\
&+ (\tilde{L}_2\tilde{w})\left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial \tilde{w}}{\partial x_i}\right) + (L_2 w)\left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i}\right) + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2 \\
&+ 2s\lambda^2\tilde{\varphi} a(t,x,\nabla\psi,\nabla\tilde{w})^2)dxdt + \int_\Sigma (8s^2\lambda^2\varphi^2|\nabla\psi| l_3(t,x)a(t,x,\nu,\nu) \\
&+ 2\frac{\partial l_3}{\partial t} + 8s\lambda^2\varphi l_3 a(t,x,\nabla\psi,\nabla\psi))w^2 d\Sigma + X_1 + X_2.
\end{aligned} \tag{1.67}$$

Applying the Cauchy-Bunyakovskii inequality in (1.67), we get

$$\begin{aligned}
&||\tilde{L}_1\tilde{w}||^2_{L^2(Q)} + ||L_1 w||^2_{L^2(Q)} + \frac{1}{2}||\tilde{L}_2\tilde{w}||^2_{L^2(Q)} + \frac{1}{2}||L_2 w||^2_{L^2(Q)} \\
&+ 2\int_Q (\lambda^4 s^3\varphi^3 a(t,x,\nabla\psi,\nabla\psi)^2 w^2 + \lambda^4 s^3\tilde{\varphi}^3 a(t,x,\nabla\psi,\nabla\psi)^2\tilde{w}^2 + \\
&s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla w,\nabla w) + s\lambda^2\tilde{\varphi} a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla\tilde{w},\nabla\tilde{w}) \\
&- 4\left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial \tilde{w}}{\partial x_i}\right)^2 - 4\left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j}\frac{\partial w}{\partial x_i}\right)^2)dxdt + X_1 + X_2 \\
&+ \int_\Sigma 8s^2\lambda^2\varphi^2|\nabla\psi|^2 l_3(t,x)a(t,x,\nu,\nu) + 2\frac{\partial l_3}{\partial t} + 8s\lambda^2\varphi l_3 a(t,x,\nabla\psi,\nabla\psi))w^2 d\Sigma \\
&\le ||f_s||^2_{L^2(Q)} + ||\tilde{f}_s||^2_{L^2(Q)}.
\end{aligned} \tag{1.68}$$

We recall that by Lemma 1.1

$$|\nabla\psi(x)| > \beta > 0 \quad \forall x \in \overline{\Omega \setminus \omega_0}.$$

Hence, taking parameter $\lambda > 0$ sufficiently large in (1.68), by virtue of (1.61) and (1.66) we obtain: There exists $s_0(\lambda) > 0$ such that

$$||\tilde{L}_1\tilde{w}||^2_{L^2(Q)} + ||L_1 w||^2_{L^2(Q)} + \frac{1}{2}||L_2 w||^2_{L^2(Q)} + \frac{1}{2}||\tilde{L}_2\tilde{w}||^2_{L^2(Q)}$$
$$+ \int\limits_Q (\lambda^4 s^3\varphi^3 w^2 + \lambda^4 s^3\tilde{\varphi^3}\tilde{w}^2 + s\lambda^2\varphi|\nabla w|^2 + s\lambda^2\tilde{\varphi}|\nabla\tilde{w}|^2)dxdt$$
$$\le c_4(\int\limits_{[0,T]\times\omega} (\lambda^4 s^3\varphi^3 w^2 + \lambda^4 s^3\tilde{\varphi}^3\tilde{w}^2 + s\lambda^2\varphi|\nabla w|^2 + s\lambda^2\tilde{\varphi}|\nabla\tilde{w}|^2)dxdt$$
$$+ ||\tilde{g}e^{s\alpha}||^2_{L^2(Q)} + ||\tilde{g}e^{s\tilde{\alpha}}||^2_{L^2(Q)}) \quad \forall s \ge s_0. \tag{1.69}$$

Thus, from (1.47) - (1.50), (1.69) we have

$$\int\limits_Q \Bigg\{ \frac{1}{s\varphi}\left(\frac{\partial w}{\partial t}\right)^2 + \frac{1}{s\tilde{\varphi}}\left(\frac{\partial\tilde{w}}{\partial t}\right)^2 + \frac{1}{s\varphi}\sum_{i,j=1}^n\left(\frac{\partial^2 w}{\partial x_i\partial x_j}\right)^2 + \frac{1}{s\tilde{\varphi}}\sum_{i,j=1}^n\left(\frac{\partial^2\tilde{w}}{\partial x_i\partial x_j}\right)^2$$
$$+ s\lambda^2\varphi|\nabla w|^2 + s\lambda^2\tilde{\varphi}|\nabla\tilde{w}|^2 + \lambda^4 s^3\varphi^3 w^2 + \lambda^4 s^3\tilde{\varphi}^3\tilde{w}^2\Bigg\}dxdt$$
$$\le c_5(\int_Q (\lambda^4 s^3\varphi^3 w^2 + \lambda^4 s^3\tilde{\varphi}^3\tilde{w}^2 + s\lambda^2\varphi|\nabla w|^2 + s\lambda^2\tilde{\varphi}|\nabla\tilde{w}|^2)dxdt$$
$$+ ||\tilde{g}e^{s\alpha}||^2_{L^2(Q)} + ||\tilde{g}e^{s\tilde{\alpha}}||^2_{L^2(Q)}) \quad \forall s \ge s_0. \tag{1.70}$$

Replacing w by $e^{s\alpha}z$ and $\tilde{w}$ by $e^{s\tilde{\alpha}}z$ respectively in (1.70), we get

$$\int\limits_Q \Bigg\{ (\frac{1}{s\varphi}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{1}{s\varphi}\sum_{i,j=1}^n\left(\frac{\partial^2 z}{\partial x_i\partial x_j}\right)^2 + s\lambda^2\varphi|\nabla z|^2 + s^3\lambda^4\varphi^3 z^2)e^{2s\alpha}$$
$$+ (\frac{1}{s\tilde{\varphi}}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{1}{s\tilde{\varphi}}\sum_{i,j=1}^n\left(\frac{\partial^2 z}{\partial x_i\partial x_j}\right)^2 + s\lambda^2\tilde{\varphi}|\nabla z|^2 + s^3\lambda^4\tilde{\varphi}^3 z^2)e^{2s\tilde{\alpha}}\Bigg\}dxdt$$
$$\le c_6(\lambda)[\int\limits_{[0,T]\times\omega} (\lambda^4 s^3\varphi^3 z^2 e^{2s\alpha} + \lambda^4 s^3\tilde{\varphi}^3 z^2 e^{2s\tilde{\alpha}} + s\lambda^2\varphi|\nabla z|^2 e^{2s\alpha}$$
$$+ s\lambda^2\tilde{\varphi}|\nabla z|^2 e^{2s\tilde{\alpha}})dxdt + ||ge^{s\alpha}||^2_{L^2(Q)} + ||ge^{s\tilde{\alpha}}||^2_{L^2(Q)}] \quad \forall s \ge s_1. \tag{1.71}$$

Let us consider the function $\rho(x) \in C_0^\infty(w)$, $\rho(x) \equiv 1$ in ω_0. We multiply the equation (1.44) by $s\lambda^2\varphi z e^{2s\alpha}$ scalarly in $L^2(Q)$. Integrating by parts with respect to t and x, applying the Cauchy-Bunyakovskii inequality, we obtain

$$\int\limits_{[0,T]\times\omega_0} s\lambda^2\varphi|\nabla z|^2 e^{2s\alpha}dxdt \le c_7(\int\limits_{[0,T]\times\omega} s^3\lambda^4\varphi^3 z^2 e^{2s\alpha}dxdt + ||\tilde{g}e^{s\alpha}||^2_{L^2(Q)}), \tag{1.72}$$

where a constant c_7 is independent of s.

Similarly

$$\int\limits_{[0,T]\times\omega_0} s\lambda^2\tilde{\varphi}|\nabla z|^2 e^{2s\tilde{\alpha}}dxdt \le c_8(\int\limits_{[0,T]\times\omega} s^3\lambda^4\tilde{\varphi}^3 z^2 e^{2s\tilde{\alpha}}dxdt + ||\tilde{g}e^{s\tilde{\alpha}}||^2_{L^2(Q)}), \tag{1.73}$$

where a constant c_8 is independent of s.

By virtue of (1.41), (1.71), (1.72) and (1.73) we have

$$\begin{aligned}&\int\limits_Q((\frac{1}{s\varphi}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{1}{s\varphi}\left(\sum_{i,j=1}^n \frac{\partial^2 z}{\partial x_i\partial x_j}\right)^2 + s\lambda^2\varphi|\nabla z|^2 + s^3\lambda^4\varphi^3 z^2)e^{2s\alpha}\\&+(\frac{1}{s\tilde{\varphi}}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{1}{s\tilde{\varphi}}\left(\sum_{i,j=1}^n \frac{\partial^2 z}{\partial x_i\partial x_j}\right)^2 + s\lambda^2\tilde{\varphi}|\nabla z|^2 + s^3\lambda^4\tilde{\varphi}^3 z^2)e^{2s\tilde{\alpha}})dxdt\\&\le c_9[\int\limits_{[0,T]\times\omega}(\lambda^4 s^3\varphi^3 z^2 e^{2s\alpha} + \lambda^4 s^3\tilde{\varphi}^3 z^2 e^{2s\tilde{\alpha}})dxdt\\&+||ge^{s\alpha}||^2_{L^2(Q)} + ||ge^{s\tilde{\alpha}}||^2_{L^2(Q)}] \quad \forall s \ge s_0.\end{aligned} \tag{1.74}$$

We observe that for all $\lambda > 0$ there exist constants $c_{10}(\lambda) > 0, c_{11}(\lambda), c_{12}(\lambda) > 0, c_{13}(\lambda)$ such that the following inequalities hold

$$c_{10}(\lambda)|\varphi| \le |\tilde{\varphi}| \le c_{11}(\lambda)|\varphi|,\ c_{12}(\lambda)\frac{1}{|\varphi|} \le \frac{1}{|\tilde{\varphi}|} \le c_{11}(\lambda)\frac{1}{|\varphi|}\ \forall\,(t,x)\in Q. \tag{1.75}$$

By (1.74), (1.75) we finally obtain (1.6).■

Remark 1.1. Careful examination of the proof of Lemma 1.3 shows, that parameter $\hat{\lambda}$ can be defined by formula

$$\hat{\lambda} = \frac{10}{\beta} \sup_{(t,x)\in Q} \left| \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\psi_{x_i} a(t,x,\nabla\psi,\nabla\psi)) \right|$$
$$+ \frac{10}{\beta} \sup_{(t,x)\in Q} \left| \sum_{k,\ell=1}^{n} a_{k\ell}\psi_{x_k} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_\ell} \right| + \sup_{(t,x)\in Q} 10 \sum_{i,j=1}^{n} \left| \frac{\partial}{\partial x_i}(a_{ij}\psi_{x_j}) \right|,$$

where constant β defined in (7).

Remark 1.2. In the case of the Dirichlet boundary conditions $z|_\Sigma = 0$ instead of (1.6) we can prove the more sharp estimate

$$\int_Q \left(\frac{1}{s\varphi} \left(\left| \frac{\partial z}{\partial t} \right|^2 + |\Delta z|^2 \right) + s\varphi|\nabla z|^2 + s^3\varphi^3 z^2 \right) (e^{2s\alpha} + e^{2s\tilde{\alpha}}) dx\, dt$$
$$+ \int_\Sigma s\varphi \left(\frac{\partial z}{\partial \nu} \right)^2 (e^{2s\alpha} + e^{2s\tilde{\alpha}}) d\Sigma$$
$$\leq c_1 \Big(\int_Q |g|^2 (e^{2s\alpha} + e^{2s\tilde{\alpha}}) dxdt + \int_{[0,T]\times\omega} s^3\varphi^3 z^2 (e^{2s\alpha} + e^{2s\tilde{\alpha}}) dxdt \Big).$$

Proof of the Lemma 1.1. Let us consider a function $\theta(x) \in C^2(\mathbb{R}^n)$ such that

$$\Omega = \{x \big| \ \theta(x) < 0\}, \quad |\nabla\theta(x)| \neq 0 \quad \forall \ x \in \partial\Omega. \tag{1.76}$$

By virtue of the Theorem on density of Morse functions (see [3]) there exist a sequence of Morse functions $\{\theta_k(x)\}_{k=1}^\infty$ such that

$$\theta_k \to \theta \quad \text{in} \quad C^2(\overline{\Omega}) \quad \text{as} \quad k \to +\infty. \tag{1.77}$$

Let us construct a Morse function $\mu \in C^2(\overline{\Omega})$ such that

$$\mu(x)|_{\partial\Omega} = 0, \quad |\nabla\mu(x)| > 0 \quad \forall \ x \in \partial\Omega. \tag{1.78}$$

We denote by $\mathcal{B} = \{x \in \mathbb{R}^n | \nabla\theta(x) = 0\}$ the set of critical points of functions θ. Since $|\nabla\theta|\big|_{\partial\Omega} > 0$ there exists an open set $\Theta \subset \mathbb{R}^n$ such that

$$\overline{\Theta} \cap \mathcal{B} = \{\emptyset\}, \quad \partial\Omega \subset \Theta. \tag{1.79}$$

Let $e(x) \in C_0^\infty(\Theta)$, $e|_{\partial\Omega} \equiv 1$. Set $\mu_k(x) = \theta_k + e(\theta - \theta_k)$. It is obvious that

$$\mu_k|_{\partial\Omega} = 0. \tag{1.80}$$

By definition of the function $e(x)$ we have

$$\nabla\mu_k(x) = \nabla\theta_k(x) \quad \forall x \in \overline{\Omega \setminus \Theta}. \tag{1.81}$$

For all x from the set $\Theta \cap \Omega$

$$\nabla\mu_k(x) = \nabla\theta_k + e(\nabla\theta - \nabla\theta_k) + \nabla e(\theta - \theta_k). \tag{1.82}$$

By virtue of (1.77) and (1.82) we have: $\forall \epsilon > 0\ \exists\ k_0(\epsilon)$ such that

$$\begin{aligned} |\nabla\mu_k| \geq |\nabla\theta_k| - ||e||_{C^1(\overline{\Omega})}|\nabla\theta - \nabla\theta_k| \\ - ||e||_{C^1(\overline{\Omega})}|\theta - \theta_k| \geq |\nabla\theta_k| - \epsilon \quad \forall x \in \Theta \cap \Omega, \end{aligned}$$

where $k > k_0$.

It follows from (1.77), (1.79), (1.81) and this inequality that there exists such $\epsilon > 0$ and $\hat{k}$ that

$$|\nabla\mu_{\hat{k}}| > 0 \quad \text{in} \quad \Theta \cap \Omega. \tag{1.83}$$

Set $\mu(x) = \mu_{\hat{k}}(x)$. By (1.80), (1.81) and (1.83) the Morse function $\mu_{\hat{k}}(x)$ satisfies (1.78).

We denote by $\mathfrak{M}$ the set of critical points of function $\mu(x)$:

$$\mathfrak{M} = \{\hat{x}_i \in \mathbb{R}^n \ \ i = 1, \dots r\}.$$

Let us consider the sequence of functions $\{l_i\}_{i=1}^r \subset C^\infty([0,1];\mathbb{R}^n)$ such that

$$l_i(t) \in \Omega \ \forall \ t \in [0,1], \ \ l_i(t_1) \neq l_i(t_2) \ \ \forall t_1, t_2 \in [0,1] \ \& \ t_1 \neq t_2 \ \ i = 1, \cdots, r; \tag{1.84}$$

$$l_i(1) = \hat{x}_i, \ \ l_i(0) \in \omega_0 \ \ i = 1, \cdots, r; \tag{1.85}$$

$$l_i(t_1) \neq l_j(t_2) \quad \forall\ i \neq j \quad \forall\ t_1, t_2 \in [0,1]. \tag{1.86}$$

By (1.84) - (1.86) there exists a sequence of functions $\{\mathrm{w}^{(i)}\}_{i=1}^r \subset C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\{e_i\}_{i=1}^r \subset C_0^\infty(\Omega)$ such that

$$\frac{dl_i(t)}{dt} = \mathrm{w}^{(i)}(l_i(t)) \quad \forall\, t \in [0,1], \quad i = 1, \cdots, r; \tag{1.87}$$

$$\operatorname{supp} e_i \subset \Omega \quad i = 1, \cdots, r; \tag{1.88}$$

$$\operatorname{supp} e_i \cap \operatorname{supp} e_j = \{\emptyset\} \quad \forall\, i \neq j; \tag{1.89}$$

$$e_i(l_i(t)) = 1 \quad \forall\ t \in [0,1], \quad i = 1, \cdots, r. \tag{1.90}$$

We set

$$V^{(i)}(x) = e_i(x)\mathrm{w}^{(i)}(x).$$

Let us consider the system of the ordinary differential equations

$$\frac{dx}{dt} = V^{(i)}(x), \quad x(0) = x_0. \tag{1.91}$$

We denote by $S_t^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ the operator such that $S_t^{(i)}(x_0) = x(t)$, where $x(t)$ is the solution of problem (1.91).

By (1.85), (1.87) and (1.90) we have

$$S_1^{(i)}(l_i(0)) = \hat{x}_i \quad i = 1, \cdots, r.$$

We set

$$\psi(x) = \mu(g_r(x)), \quad g_r(x) = S_1^{(1)} \circ S_1^{(2)} \circ \cdots \circ S_1^{(r)}(x). \tag{1.92}$$

By (1.88) there exists a domain $\mathfrak{S} \subset \mathbb{R}^n$ such that $\partial\Omega \subset \mathfrak{S}$ and

$$S_1^{(i)}(x) = x \quad \forall\ x \in \mathfrak{S}, \quad i = 1, \cdots, r. \tag{1.93}$$

By (1.93) the mappings $S_1^{(i)}(x)$ - are diffeomorfisms on the domain Ω. So $g_r(x)$ is a diffeomorfism on the domain Ω. By (1.93) $\psi(x) = \mu(x) \quad \forall x \in \mathfrak{S}$. Hence

$$\psi(x)|_{\partial\Omega} = 0. \tag{1.94}$$

We denote by Ψ the set of critical points of function ψ. Since the mapping $g_r : \Omega \to \Omega$ is a diffeomorfism we have

$$\Psi = \{x \in \Omega |\ g_r(x) \in \mathfrak{M}\}. \tag{1.95}$$

By (1.89) and (1.93)

$$g_r(l_i(0)) = \hat{x}_i \quad i = 1, \cdots, r. \tag{1.96}$$

It follows from (1.95) and (1.96) that

$$\Psi \subset \omega_0. \blacksquare$$

2. Exact controllability of linear parabolic equations

In this section we will prove an existence theorem for the problem of exact controllability for linear parabolic equations.

Let us introduce function $\eta(t,x,\lambda)$ as follows:

$$\eta(t,x,\lambda) = (e^{2\lambda\|\psi\|_{C(\overline{\Omega})}} - e^{\lambda\psi})/((T-t)l(t)), \tag{2.1}$$

where $\lambda \geq \hat{\lambda}$. The function $\psi(x)$ and the parameter $\hat{\lambda}$ were defined in Lemmas 1.1, 1.2. We assume that $l(t)$ is a fixed function, which satisfies the following conditions

$$l(t) \in C^1[0,T], \quad l(t) = t \quad \forall\, t \in \big(\frac{3T}{4}, T\big], \quad l(t) > 0 \quad \forall\, t \in [0,T].$$

To formulate our results we need to introduce the following function spaces:

$$Y(Q) = \left\{ y(t,x) \big|\ y \in L^2(0,T;W_2^2(\Omega)), \quad \frac{\partial y}{\partial t} \in L^2(Q) \right\},$$

$$X_s^\lambda(Q) = \{y(t,x) \big|\ e^{s\eta} y \in L^2(Q)\},$$

$$Z_s^\lambda(Q) = \left\{ y(t,x) | \frac{e^{s\eta} y}{(T-t)^{3/2}} \in L^2(Q) \right\},$$

$$\Xi_s^\lambda(Q) = \{y(t,x) \in Z_s^\lambda(Q) |\ |\nabla y| e^{s\eta}/\sqrt{(T-t)},$$
$$\sqrt{(T-t)} \left(\left| \frac{\partial y}{\partial t} \right| + \sum_{i,j=1}^n \left| \frac{\partial^2 y}{\partial x_i \partial x_j} \right| \right) e^{s\eta} \in L^2(Q)\}$$

equipped with the norms

$$\|y\|^2_{Y(Q)} = \|y\|^2_{L^2(0,T;W_2^2(\Omega))} + \|\partial y/\partial t\|^2_{L^2(Q)},$$

$$\|y\|_{X_s^\lambda(Q)} = \|e^{s\eta} y\|_{L^2(Q)},$$

$$\|y\|_{Z_s^\lambda(Q)} = \left\| \frac{e^{s\eta}}{(T-t)^{3/2}} y \right\|_{L^2(Q)},$$

$$||y||_{\Xi_s^\lambda(Q)} = (||y||^2_{Z_s^\lambda(Q)} + |||\nabla y|e^{s\eta}/\sqrt{(T-t)}||^2_{L^2(Q)}$$
$$+ ||\sqrt{(T-t)}(|\frac{\partial y}{\partial t}| + \sum_{i,j=1}^n |\frac{\partial^2 y}{\partial x_i \partial x_j}|)e^{s\eta}||^2_{L^2(Q)})^{\frac{1}{2}}.$$

Let us consider the problem of exact controllability for linear parabolic equations

$$Ly = \frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial y}{\partial x_j}\right) + \sum_{i=1}^n b_i(t,x)\frac{\partial y}{\partial x_i}$$
$$+ c(t,x)y = u + g \ \text{ in } \quad Q, \qquad (2.2)$$

$$u \in \mathcal{U}(\omega), \quad (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad y(0,x) = v_0(x), \qquad (2.3)$$

$$y(T,x) = v_1(x). \qquad (2.4)$$

We have

Theorem 2.1. *Let $\lambda \ge \hat{\lambda}$ and $v_0 \in W_2^1(\Omega)$, $v_1 \equiv 0$, and let conditions (6) -(9) be fulfilled. Then there exist a constant $s_0(\lambda)$ such that if $g \in X_s^\lambda(Q)$ with $s \ge s_0(\lambda)$, then problem (2.2) - (2.4) has a solution $(y,u) \in (Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ which satisfies the following estimate:*

$$||(y,u)||_{(Y(Q)\cap Z_s^\lambda(Q))\times(\mathcal{U}(\omega)\cap X_s^\lambda(Q))} \le c_1(\lambda,s)(||v_0||_{W_2^1(\Omega)} + ||g||_{X_s^\lambda(Q)}). \quad (2.5)$$

Proof. We recall that parameter $\hat{\lambda}$ was defined in the Lemma 1.2.
Let us consider the extremal problem

$$\mathcal{J}_k(y,u) = \frac{1}{2}\int_Q \frac{\rho_k y^2}{(T-t)^3}dxdt + \frac{1}{2}\int_Q \rho_k m_k u^2 dxdt \to \inf, \qquad (2.6)$$

$$Ly = u + g \quad \text{in} \quad Q, \ (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0,$$
$$y(0,x) = v_0, \quad y(T,x) = 0, \quad (2.7)$$

where

$$\rho_k(t,x) = e^{\frac{2s\eta(t,x,\lambda)(T-t)}{(T-t+1/k)}},\ m_k(x) = \begin{cases} 1, x \in \overline{w}, \\ k, x \in \Omega \setminus \overline{w}, \end{cases}$$

and parameters $s \geq s_0(\lambda)$, $\lambda \geq \hat{\lambda}$ are fixed. Here $s_0(\lambda)$ is defined in Lemma 1.2 and function $\eta(t,x,\lambda)$ defined in (2.1) .

It is easy to prove (see [47], [51]) that problem (2.6) - (2.7) has a unique solution, which we denote as $(\hat{y}_k, \hat{u}_k) \in Y(Q) \times L^2(Q)$.

Applying the Lagrange principle to the problem (2.6) - (2.7) (see [1], [47] and [22]) we obtain

$$L\hat{y}_k = g + \hat{u}_k \text{ in } Q, \quad (l_1(t,x)\frac{\partial \hat{y}_k}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0,\ \hat{y}_k(T,\cdot) \equiv 0,\ \hat{y}_k(0,\cdot) = v_0, \tag{2.8}$$

$$L^* p_k = \frac{\rho_k}{(T-t)^3}\hat{y}_k \text{ in } Q, \quad (l_1(t,x)\frac{\partial p_k}{\partial \nu_A} + l_2(t,x)p_k)\big|_\Sigma = 0,$$
$$p_k + m_k\rho_k\hat{u}_k = 0 \text{ in } Q, \tag{2.9}$$

where

$$L^* y = -\frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial y}{\partial x_j}\right) - \sum_{i=1}^n \frac{\partial b_i(t,x)y}{\partial x_i} + c(t,x)y$$

is an operator formally conjugate to the operator L.

By (1.6) and a priori estimates for linear parabolic equations we have

$$\int_Q e^{-2s\eta}|p_k|^2 dxdt + \int_\Omega |p_k(0,x)|^2 dx$$
$$\leq c_2(\lambda,s)(\int_Q \frac{\rho_k^2}{(T-t)^3}e^{-2s\eta}\hat{y}_k^2 dxdt + \int_{[0,T]\times\omega} e^{-2s\eta}p_k^2 dxdt).$$

We observe that $|\rho_k(t,x)e^{-2s\eta(t,x)}| \leq 1\ \forall\ (t,x) \in Q$.

Thus, we have

$$\int_\Omega |p_k(0,x)|^2 dx + \int_Q |p_k|^2 e^{-2s\eta} dxdt$$
$$\leq c_2(\lambda,s)(\int_Q \frac{\rho_k\hat{y}_k^2}{(T-t)^3}dxdt + \int_{[0,T]\times\omega} e^{-2s\eta}\rho_k^2\hat{u}_k^2 dxdt). \tag{2.10}$$

Multiplying (2.9_1) by $\hat{y}_k$ scalarly in $L^2(Q)$, and integrating by parts with respect to t and x, we have

$$
\begin{aligned}
0 &= (L^* p_k - \rho_k \hat{y}_k, \hat{y}_k)_{L^2(Q)} \\
&= -\int_Q \rho_k \hat{y}_k^2 dxdt + (p_k, L\hat{y}_k)_{L^2(Q)} + (p_k(0,\cdot), \hat{y}_k(0,\cdot))_{L^2(\Omega)} \\
&= -\int_Q \frac{\rho_k \hat{y}_k^2}{(T-t)^3} dxdt - \int_Q \rho_k m_k \hat{u}_k^2 dxdt + \int_Q g p_k dxdt + (p_k(0,\cdot), v_0)_{L^2(\Omega)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{J}_k(\hat{y}_k, \hat{u}_k) &= \frac{1}{2}\int_Q \left(\frac{\rho_k \hat{y}_k^2}{(T-t)^3} + \rho_k m_k \hat{u}_k^2 \right) dxdt \\
&= \frac{1}{2}\left(\int_Q g p_k dxdt + (p_k(0,\cdot), v_0)_{L^2(Q)} \right). \qquad (2.11)
\end{aligned}
$$

By (2.10) and (2.11) we obtain

$$
\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \le c_3(||g||_{X_s^\lambda(Q)} + ||v_0||_{L^2(\Omega)})\sqrt{\mathcal{J}_k(\hat{y}_k, \hat{u}_k)}.
$$

It follows that

$$
\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \le c_3^2(||g||_{X_s^\lambda(Q)} + ||v_0||_{L^2(\Omega)})^2. \qquad (2.12)
$$

By virtue of (2.12) we have a subsequence $\{(\hat{y}_k, \hat{u}_k)\}_{k=1}^\infty$ such that

$$
\begin{aligned}
&(\hat{y}_k, \hat{u}_k) \to (y, u) \quad \text{weakly in} \quad Y(Q) \times L^2(Q), \\
&\hat{u}_k \to 0 \quad \text{in} \quad L^2((0,T) \times (\Omega \setminus \omega)), \\
&\sqrt{\rho_k}\hat{u}_k \to e^{s\eta} u \quad \text{weakly in} \quad L^2((0,T) \times \omega), \\
&\frac{\sqrt{\rho_k}}{(T-t)^{3/2}}\hat{y}_k \to \frac{e^{s\eta}}{(T-t)^{3/2}} y \quad \text{weakly in} \quad L^2(Q). \qquad (2.13)
\end{aligned}
$$

Using (2.13), we pass to the limit in (2.8) to obtain that pair (y, u) is a solution of problem (2.2) - (2.4). Estimate (2.5) follows from (2.12), (2.13). ■

Now, we will prove that solutions of controllability problem (2.2) - (2.4) from the Theorem 2.1 have further regularity as described in the following theorem.

THEOREM 2.2. *Let $\lambda \geq \hat{\lambda}$, and $v_0 \in W_2^1(\Omega)$, $v_1 \equiv 0$, and let conditions (6) - (9) be fulfilled. Then there exist a constant $s_0(\lambda)$ such that if $g \in X_s^\lambda(Q)$, with $s \geq s_0(\lambda)$ then problem (2.2) - (2.4) has a solution $(y,u) \in \Xi_s^\lambda(Q) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ which satisfies the following estimate*

$$||(y,u)||_{\Xi_s^\lambda(Q)\times(\mathcal{U}(\omega)\cap X_s^\lambda(Q))} \leq c_4(\lambda,s)(||v_0||_{W_2^1(\Omega)} + ||g||_{X_s^\lambda(Q)}). \tag{2.14}$$

PROOF. By Theorem 2.1 for $v_0 \in W_2^1(\Omega)$, $g \in X_s^\lambda(Q)$ we have a solution $(y,u) \in (Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ of problem (2.2) - (2.4) which satisfies the estimate (2.5). Let us prove that this solution satisfies the estimate (2.14).

Multiplying (2.2) by $\frac{e^{2s\eta}}{(T-t)}y$ scalarly in $L^2(Q)$, and integrating by parts with respect to t and x, we have

$$\int_Q \left(\frac{1}{T-t} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} e^{2s\eta} + \frac{1}{T-t} \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} y \frac{\partial e^{2s\eta}}{\partial x_i} \right.$$
$$\left. - \frac{1}{2} y^2 \frac{\partial}{\partial t}\left(\frac{e^{2s\eta}}{T-t}\right) + \frac{1}{T-t}\left(\sum_{i=1}^n b_i \frac{\partial y}{\partial x_i} + cy\right) y e^{2s\eta} \right) dxdt$$
$$- \frac{1}{2T}\int_\Omega e^{2s\eta(0,x,\lambda)} y^2(0,x)dx = \int_Q (u+g)e^{2s\eta} y dxdt. \tag{2.15}$$

By (6), (2.15) the following equality holds.

$$\int_Q \frac{1}{T-t} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} e^{2s\eta} dxdt \leq c_4 \int_Q \Big(\frac{1}{(T-t)^2}|\nabla y||y|$$
$$+ \frac{y^2}{(T-t)^2}\Big)dxdt + \frac{1}{2T}\int_\Omega e^{2s\eta(0,x,\lambda)} y^2(0,x)dx = \int_Q (u+g)e^{2s\eta} y dxdt. \tag{2.16}$$

By (2.16) and (7) we have the inequality:

$$\int_Q \frac{|\nabla y|^2 e^{2s\eta}}{(T-t)} dxdt \leq c_5 \left(\int_Q \frac{y^2 e^{2s\eta}}{(T-t)^3} dxdt + ||u||^2_{X_s^\lambda(Q)} + ||g||^2_{X_s^\lambda(Q)} \right). \tag{2.17}$$

Let us denote $w(t,x) = e^{s\eta} y\sqrt{T-t}$. By (2.3) we have

$$Lw = yL(e^{s\eta}\sqrt{T-t}) - 2\sqrt{T-t}\sum_{i,j=1}^{n} a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial e^{s\eta}}{\partial x_i}$$
$$+ e^{s\eta}\sqrt{(T-t)}(u+g) \quad \text{in} \quad Q, \quad (l_1(t,x)\frac{\partial w}{\partial \nu_A} + l_2(t,x)w)|_\Sigma = 0, \tag{2.18}$$

$$w(0,x) = v_0(x)e^{s\eta(0,x,\lambda)}\sqrt{T}. \tag{2.19}$$

We also denote

$$q = yL(e^{s\eta}\sqrt{T-t}) - 2\sqrt{T-t}\sum_{i,j=1}^{n} a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial e^{s\eta}}{\partial x_i} + e^{s\eta}\sqrt{T-t}(u+g).$$

Hence, by (2.18), (2.19) function w satisfy

$$Lw = q \quad \text{in} \quad Q, \quad (l_1(t,x)\frac{\partial w}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad w(0,x) = v_0 e^{s\eta(0,x,\lambda)}\sqrt{T}. \tag{2.20}$$

By (2.5), (2.17) we have

$$||q||_{L^2(Q)} \le c_6(||y||_{Z_s^\lambda(Q)} + ||u||_{X_s^\lambda(Q)} + ||g||_{X_s^\lambda(Q)}). \tag{2.21}$$

Then, using well-known a priori estimates for linear parabolic equations, we have

$$||w||_{Y(Q)} \le c_7(||q||_{L^2(Q)} + ||w(0,\cdot)||_{W_2^1(\Omega)}). \tag{2.22}$$

By (2.5), (2.17), (2.21) and (2.22) we obtain

$$\left(\int_Q \left((T-t)\left(\left|\frac{\partial y}{\partial t}\right|^2 + \sum_{i,j=1}^{n}\left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right|^2\right) + \frac{|\nabla y|^2}{(T-t)}\right) e^{2s\eta(t,x,\lambda)} dxdt\right)^{\frac{1}{2}}$$
$$+ ||y||_{Z_s^\lambda(Q)} + ||u||_{X_s^\lambda(Q)} \le c_8(||g||_{X_s^\lambda(Q)} + ||v_0||_{W_2^1(\Omega)}). \tag{2.23}$$

The inequality (2.23) prove Theorem 2.2. ■

Let us consider the problem of exact boundary controllability for linear parabolic equation

$$Ly = g \quad \text{in} \quad Q, \quad (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{]0,T[\times(\partial\Omega\setminus\Gamma_0)} = 0,$$
$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{[0,T]\times\Gamma_0} = u, \tag{2.24}$$

$$y(0,x) = v_0(x), \quad y(T,x) = v_1(x). \tag{2.25}$$

The following theorem is a corollary of Theorem 2.2.

THEOREM 2.3. *Let* $v_0 \in W_2^1(\Omega)$, $v_1 \equiv 0$, *and let conditions (6) - (9) be fulfilled. Then there exists a constant* $\hat\lambda > 0$ *such that for* $\lambda \geq \hat\lambda$ *there exists a constant* $s_0(\lambda)$ *such that if* $g \in X_s^\lambda(Q)$ *where* $\lambda \geq \hat\lambda$, $s \geq s_0(\lambda)$, *then problem (2.24) - (2.25) has a solution* $(y,u) \in \Xi_s^\lambda(Q) \times L^2(0,T; W_2^{\frac{1}{2}}(\partial\Omega))$.

PROOF. Let us consider a connected domain $\tilde\Omega$ such that

$$\tilde\Omega = \Omega \cup \omega, \quad \partial\tilde\Omega \in C^2, \quad \overline{\omega} \cap \overline{(\partial\Omega \setminus \Gamma_0)} = \{\emptyset\},$$

where ω is a connected domain in $\mathbb{R}^n$. Denote by $\tilde Q =]0,T[\times\tilde\Omega$. Set $g(t,x) = 0 \ \forall (t,x) \in [0,T] \times \omega$. We extend the function v_0 on $\tilde\Omega$ such that $v_0 \in W_2^1(\tilde\Omega)$. We also extend coefficients of operator L keeping properties (6) - (9). Applying Theorem 2.1, we get a solution $(y,u) \in \Xi_s^\lambda(\tilde Q) \times (\mathcal{U}(\omega) \cap X_s^\lambda(\tilde Q))$ of the following problem of exact controllability

$$\begin{cases} Ly = g + u \quad \text{in} \quad \tilde Q, \quad \operatorname{supp} u \subset [0,T] \times \omega, \\ (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)|_{[0,T]\times\partial\tilde\Omega} = 0, \quad y(0,x) = v_0(x), \quad y(T,x) \equiv 0. \end{cases}$$

It is easily seen that the pair $(y, l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)|_{[0,T]\times\Gamma_0})$ is a solution of problem (2.24), (2.25). ■

To prove local controllability theorem in the case of superlinear growth of nonlinear term we need to prove existence of solution of problem (1) - (3) in the space $L^\infty(Q)$.

We have

THEOREM 2.4. *Let* $p > max\{2, (n+2)/2\}$, $\lambda \geq \hat\lambda$, $v_0 \in W_\infty^1(\Omega)$, $v_1 \equiv 0$, *and conditions (6)- (9) be fulfilled. Then there exists a constant* $s_0(\lambda)$ *such that if* $g \in X_s^\lambda(Q) \cap L^p(Q)$ *with* $s \geq s_0(\lambda)$, *and* $\lambda \geq \hat\lambda$, *then problem (2.2) - (2.4) has a solution* $(y,u) \in (W_p^{1,2}(Q) \cap \Xi_s^\lambda(Q)) \times (\mathcal{U}(w) \cap X_s^\lambda(Q) \cap L^p(Q))$ *which satisfies the following estimate*

$$||(y,u)||_{(W_p^{1,2}(Q)\cap\Xi_s^\lambda(Q))\times(X_s^\lambda(Q)\cap L^p(Q))} \leq c_{10}(\lambda,s)(||v_0||_{W_\infty^1(\Omega)} + ||g||_{X_s^\lambda(Q)\cap L^p(Q)}). \quad (2.26)$$

PROOF. To construct solution (y,u) of the problem (2.2) - (2.4) firstly we consider boundary value problem (2.2) - (2.3) when $u(t,x) \equiv 0$:

$$L\tilde y = g \text{ in } Q, \quad (l_1(t,x)\frac{\partial \tilde y}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad \tilde y(0,x) = v_0(x). \quad (2.27)$$

It is well known (see [51],[38]), that under assumptions of Theorem 2.4 for arbitrary $g \in L^p(Q)$ and $v_0 \in W^1_\infty(\Omega)$ there exists a unique solution of this problem $\tilde{y} \in W^{1,2}_p(Q)$.

Let $l(t) = exp(-t/(T-t)^3)$. We set $\overline{y}(t,x) = l(t)\tilde{y}(t,x)$. By (2.27) the function $\overline{y}$ satisfies the following:

$$L\overline{y} = g\ell(t) + \ell'(t)\tilde{y}, \quad (l_1(t,x)\frac{\partial \overline{y}}{\partial \nu_A} + l_2(t,x)\overline{y})\big|_\Sigma = 0, \quad \overline{y}(0,x) = v_0(x),$$
$$\overline{y}(T,x) = 0. \quad (2.28)$$

Let us consider the problem of exact controllability

$$Lz = -g\ell(t) - \ell'(t)\tilde{y} + g + v, \ \ v \in \mathcal{U}(\omega_0), \tag{2.29}$$

$$(l_1(t,x)\frac{\partial z}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \ \ z(0,x) = 0, \ \ z(T,x) = 0, \tag{2.30}$$

where $\omega_0 \Subset \omega$. By virtue of definition of function $\ell(t)$ we have

$$\ell'(t)\tilde{y} \in Z^\lambda_s(Q) \quad \forall \ \lambda > 0, \quad s > 0.$$

By Theorem 2.2 problem (2.29) - (2.30) has a solution $(z,v) \in \Xi^\lambda_s(Q) \times (\mathcal{U}(\omega_0) \cap X^\lambda_s(Q))$ which satisfies the estimate (2.13). Let $\omega_0 \Subset \omega_1 \Subset \omega$, $\rho(x) \in C^\infty(\overline{\Omega})$, $\rho(x) \equiv 1 \ \forall x \in \Omega \setminus \omega_1$, $\rho(x) \equiv 0 \, \forall \ x \in \omega_1$. We set $z_1(t,x) = \rho(x)z(t,x)$, $u_1(t,x) = -z\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial \rho}{\partial x_j}\right) - a(t,x,\nabla z,\nabla\rho)$ $+ \rho\sum_{i=1}^n b_i(t,x)\frac{\partial z}{\partial x_i} + \rho u + (\rho-1)g + (1-\rho)(g\ell(t) + \ell'(t)\tilde{y})$. By (2.28) the pair $(z_1(t,x), u_1(t,x))$ satisfies the equations

$$Lz_1 = -g\ell(t) - \ell'(t)\tilde{y} + g + u_1, \quad u_1 \in \mathcal{U}(\omega), \tag{2.31}$$

$$(l_1(t,x)\frac{\partial z_1}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad z_1(0,x) = 0, \quad z_1(T,x) = 0. \tag{2.32}$$

Using well known results on the regularity of solutions of parabolic equations, we get that pair

$$(z_1, u_1) \in (W^{1,2}_p(Q) \cap \Xi^\lambda_s(Q)) \times (\mathcal{U}(\omega) \cap X^\lambda_s(Q) \cap L^p(Q))$$

and satisfies the inequality (2.26). Then the pair $(y,u) = (\overline{y} - z_1, v - u_1)$ is a solution of problem (2.2) - (2.4) which also satisfies the estimate (2.26). ∎

3. Exact controllability of semilinear parabolic equation.

Let us assume that

$$f(t,x,0)=0 \quad \forall \ (t,x)\in Q, \tag{3.1}$$

and function $f(t,x,y)$ satisfies the Lipschitz condition

$$|f(t,x,\zeta_1)-f(t,x,\zeta_2)|\le K|\zeta_1-\zeta_2| \quad \forall\,(t,x,\zeta)\in Q\times\mathbb{R}^1, \tag{3.2}$$

Let us consider the boundary value problem for parabolic equation

$$G(y)=g, \quad (l_1(t,x)\frac{\partial y}{\partial \nu_A}+l_2(t,x)y)|_\Sigma=0, \quad y(0,\cdot)=v_0. \tag{3.3}$$

The following theorem proved in [38]

THEOREM 3.1. *Let (6)-(9), (3.2) be fulfilled. Then for every* $(v_0,g)\in W^1_2(\Omega)\times L^2(Q)$ *there exists a unique solution of problem (3.3)* $y\in Y(Q)$ *which satisfy inequality*

$$\|y\|_{Y(Q)}\le c_1(\|v_0\|_{W^1_2(\Omega)}+\|g\|_{L^2(Q)}). \tag{3.4}$$

We have

THEOREM 3.2. *Let* $v_0\in W^1_2(\Omega)$, $v_1\equiv 0$, *and let the conditions (6)- (9), (3.1) and (3.2) be fulfilled. Then there exists* $\hat\lambda>0$ *such that for* $\lambda\ge\hat\lambda$ *there exists such constant* $s_0(\lambda)$ *such that if* $g\in X^\lambda_s(Q)$, *with* $\lambda\ge\hat\lambda$, $s\ge s_0(\lambda)$ *then there exists a solution pair* $(y,u)\in Y(Q)\times\mathcal{U}(\omega)$ *of the problem (1) - (3).*

PROOF. Let us consider the following family of problems of exact controllability

$$G_\varepsilon(y)=Ly+f_\varepsilon(t,x,y)-f_\varepsilon(t,x,0)=u+g \quad \text{in} \quad Q, \quad u\in\mathcal{U}(\omega), \tag{3.5}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A}+l_2(t,x)y)|_\Sigma=0, \quad y(0,x)=v_0(x), \tag{3.6}$$

where $f_\varepsilon(t,x,y)=\frac{1}{\varepsilon}\int\limits_{R^1}\omega\left(\frac{|\tau-y|}{\varepsilon}\right)f(t,x,\tau)d\tau$, $\omega(x)\ge 0\ \forall x\in\mathbb{R}^1$, $\omega(x)=\omega(|x|)$, $\operatorname{supp}\omega\subset\{x||x|\le 1\}$, $\int\limits_{R^1}\omega dx=1$ and operator L was defined in (2.2).

We have

$$(f_\varepsilon(t,x,\zeta)-f_\varepsilon(t,x,0))|_{\zeta=0}=0 \quad (t,x)\in Q. \tag{3.7}$$

Moreover

$$|f_\varepsilon(t,x,\zeta_1)-f_\varepsilon(t,x,\zeta_2)|\le \frac{1}{\varepsilon}|\int\limits_{R^1}\omega\left(\frac{|\tau-\zeta_1|}{\varepsilon}\right)f(t,x,\tau)d\tau$$
$$-\int\limits_{R^1}\omega\left(\frac{|\tau-\zeta_2|}{\epsilon}\right)f(t,x,\tau)d\tau\Bigg|=\frac{1}{\epsilon}\Bigg|\int\limits_{R^1}(\omega\left(\frac{|\tau|}{\varepsilon}\right)f(t,x,\tau-\zeta_1)$$
$$-\omega\left(\frac{|\tau|}{\varepsilon}\right)f(t,x,\tau-\zeta_2))d\tau\Big|\le\frac{K}{\varepsilon}\int\limits_{R^1}\omega\left(\frac{|\tau|}{\varepsilon}\right)d\tau|\zeta_1-\zeta_2|\quad\forall\ (t,x)\in Q. \tag{3.8}$$

By (3.7) and (3.8) we obtain

$$\begin{aligned}&f_\varepsilon(t,x,\zeta)-f_\varepsilon(t,x,0)=\tilde f_\varepsilon(t,x,\zeta)\zeta,\\&|\tilde f_\varepsilon(t,x,\zeta)|\le K\quad\forall\ (t,x,\zeta)\in Q\times\mathbb{R}^1.\end{aligned} \tag{3.9}$$

where the constant K is from (3.2).

It follows from (3.9) that for linear parabolic operator $R_\varepsilon(y)z=Lz+\tilde f_\varepsilon(t,x,y)z$ the parameter $\gamma(y)$ defined

$$\gamma(y)=\sum_{i,j=1}^n||a_{ij}||_{C^{1,2}(\overline{Q})}+\sum_{i=1}^n||b_i||_{C^{0,1}(\overline{Q})}+||c(t,x)+\tilde f_\varepsilon(t,x,y)||_{L^\infty(Q)}$$

for every $y\in L^2(Q)$ satisfies the inequality

$$\gamma(y)\le c_2. \tag{3.10}$$

where c_1 is a constant independent of y and ε.

Let us consider the problem of exact controllability of parabolic equations

$$\begin{aligned}&R_\varepsilon(y)z=u+g\quad\text{in } Q,\quad u\in\mathcal{U}(\omega),\\&(l_1(t,x)\frac{\partial z}{\partial\nu_A}+l_2(t,x)z)\big|_\Sigma=0,\quad z(0,x)=z_0(x),\quad z(T,x)=0.\end{aligned} \tag{3.11}$$

By (3.10) and Theorem 2.2 we obtain that there exists $\hat\lambda>0$ such that for $\lambda\ge\hat\lambda$ there exists $s_0(\lambda)$ that if $\lambda\ge\hat\lambda$ the problem of exact controllability

(3.11) has solutions in the space $\Xi_s^\lambda(Q)\times(\mathcal{U}(\omega)\cap X_s^\lambda(Q))$ for all initial data $(v_0,g)\in W_2^1(\Omega)\times X_s^\lambda(Q)$, moreover, these solutions satisfy (2.14) with $c_1(\lambda)$ independent of $y\in L^2(Q)$ and $\epsilon\in(0,1)$.

Let us introduce the mappings $\Psi^{(\varepsilon)}: y\to\hat z_\varepsilon$ and $\Psi_1^{(\varepsilon)}: y\to(\hat z_\varepsilon,\hat u_\varepsilon)$ as follows: For $y\in L^2(Q)$ a pair $(\hat z_\varepsilon,\hat u_\varepsilon)$ is a solution of the extremal problem:

$$\mathcal{J}(z,u)=\int\limits_Q\frac{e^{2s\eta(t,x,\lambda)}}{(T-t)^3}z^2dxdt+\int\limits_Q e^{2s\eta(t,x,\lambda)}u^2dxdt\to\inf, \tag{3.12}$$

$$R_\varepsilon(y)z=g+u\quad\text{in}\quad Q,\quad u\in\mathcal{U}(\omega),$$
$$(l_1(t,x)\frac{\partial z}{\partial\nu_A}+l_2(t,x)z)\big|_\Sigma=0,\quad z(0,x)=v_0(x),\quad z(T,x)=0. \tag{3.13}$$

By virtue of Theorem 2.2 for $y\in L^2(Q)$ there exists the unique solution $(\hat z_\varepsilon,\hat u_\varepsilon)\in(Y(Q)\cap Z_s^\lambda(Q))\times(\mathcal{U}(\omega)\cap X_s^\lambda(Q))$ of the problem (3.12) - (3.13). So mappings $\Psi^{(\varepsilon)}$ and $\Psi_1^{(\varepsilon)}$ are well defined on the whole space $L^2(Q)$.

Let us prove that $\Psi^{(\varepsilon)}\in C(L_2(Q),L_2(Q))$ is a continuous mapping. Let $\varepsilon>0$ be fixed. Assume the contrary. Then there exists a function $y\in L^2(Q)$ and a sequence $\{(y_i,\hat z_i,\hat u_i)\}$ such that

$$y_i\to y\quad\text{in}\quad L^2(Q),\quad \Psi^{(\varepsilon)}(y_i)=\hat z_i\to z\quad\text{weakly in } Y(Q)\cap Z_s^\lambda(Q),$$
$$\hat u_i\to u\quad\text{weakly in}\quad \mathcal{U}(\omega)\cap X_s^\lambda(Q), \tag{3.14}$$

$$\Psi_1^{(\varepsilon)}(y)=(\hat z,\hat u)\neq(z,u),\quad \hat z\in Z_s^\lambda(Q), \tag{3.15}$$

the triple $(y_i,\hat z_i,\hat u_i)$ satisfies (3.13) and

$$\mathcal{J}(\hat z,\hat u)<\mu_0<\mathcal{J}(\hat z_i,\hat u_i)\quad\forall i\in\mathbb{Z}_+. \tag{3.16}$$

By (3.14) and (3.15)

$$\hat z(\tilde f_\varepsilon(t,x,y_i)-\tilde f_\varepsilon(t,x,y))\to 0\quad\text{in}\quad Z_s^\lambda(Q)\text{ as } i\to+\infty. \tag{3.17}$$

By (3.17) and Theorem 2.1 there exists a subsequence $\{(\delta_i,q_i)\}_{i=1}^\infty\subset(Y(Q)\cap Z_s^\lambda(Q))\times(\mathcal{U}(\omega)\cap X_s^\lambda(Q))$ such that

$$L\delta_i+\tilde f_\varepsilon(t,x,y)\delta_i=\hat z(f(t,x,y_i)-f(t,x,y))+q_i\quad\text{in}\quad Q, \tag{3.18}$$

$$(l_1(t,x)\frac{\partial \delta_i}{\partial \nu_A} + l_2(t,x)\delta_i)|_\Sigma = 0, \quad \delta_i(0,x) = \delta_i(T,x) = 0, \tag{3.19}$$

$$||\delta_i||_{Y(Q)\cap Z_s^\lambda(Q)} + ||q_i||_{X_s^\lambda(Q)} \to 0 \quad \text{as} \quad i \to +\infty. \tag{3.20}$$

We set

$$\tilde{z}_i = \hat{z} - \delta_i, \quad \tilde{u}_i = \hat{u} - q_i. \tag{3.21}$$

By (3.15), (3.18) and (3.19) the following holds:

$$L\tilde{z}_i + \tilde{f}_\varepsilon(t,x,y_i)\tilde{z}_i = g + \tilde{u}_i \quad \text{in} \quad Q, \quad \tilde{u}_i \in \mathcal{U}(\omega), \tag{3.22}$$

$$(l_1(t,x)\frac{\partial \tilde{z}_i}{\partial \nu_A} + l_2\tilde{z}_i)\big|_\Sigma = 0, \quad \tilde{z}_i(0,x) = v_0(x), \quad \tilde{z}_i(T,x) = 0. \tag{3.23}$$

Moreover, by (3.20)

$$\mathcal{J}(\tilde{z}_i, \tilde{u}_i) \to \mathcal{J}(\hat{z}, \hat{u}). \tag{3.24}$$

By (3.16), (3.22) and (3.23) the pair $(\tilde{z}_i, \tilde{u}_i)$ is an admissible element of extremal problem (3.12) - (3.13). So by definition of the mapping Ψ_1^ε the following inequality holds

$$\mathcal{J}(\hat{z}_i, \hat{u}_i) \le \mathcal{J}(\tilde{z}_i, \tilde{u}_i). \tag{3.25}$$

Now (3.24) and (3.25) contradict to (3.15). We reached to contradiction.

Denote by B_r a ball in $L^2(Q)$ with the center at zero, and having as a radius r. By (2.5) and (3.10) for all sufficiently large r we obtain

$$\Psi^{(\varepsilon)}(B_r) \subset B_r.$$

where r is independent on ϵ. Moreover, if $\mathfrak{S}$ is a bounded set in $L^2(Q)$, then by (2.14) the set $\Psi^{(\varepsilon)}(\mathfrak{S})$ is bounded in $Y(Q)$. Since imbedding $Y(Q) \subset L^2(Q)$ is compact, the mapping $\Psi^{(\varepsilon)}$ is a compact mapping.

Applying the Schauder fixed point theorem, we find that there exists a fixed point y_ε of the mapping $\Psi^{(\varepsilon)}$:

$$\Psi^{(\varepsilon)}(y_\varepsilon) = y_\varepsilon$$

and

$$||\Psi_1^{(\varepsilon)}(y_\varepsilon)||_{Y(Q)\times\mathcal{U}(\omega)} \le c_2, \tag{3.26}$$

where c_2 is a constant independent of ε.

Obviously a pair $\Psi_1^{(\varepsilon)}(y_\varepsilon) = (y_\varepsilon, u_\varepsilon)$ is a solution of the exact controllability problem:

$$Ly_\varepsilon + f_\varepsilon(t,x,y_\varepsilon) - f_\varepsilon(t,x,0) = u_\varepsilon + g \quad \text{in} \quad Q, \quad u_\epsilon \in \mathcal{U}(\omega), \tag{3.27}$$

$$(l_1(t,x)\frac{\partial y_\varepsilon}{\partial \nu_A} + l_2(t,x)y_\varepsilon)|_\Sigma = 0, \quad y_\varepsilon(0,x) = v_0(x), \quad y_\varepsilon(T,x) = 0. \tag{3.28}$$

By (3.26) taking, if necessary, a subsequence, we can pass to the limit in (3.27) and (3.28). This limit is a solution of the problem (1) - (3). ■

By similar procedure leading to Theorem 2.3 we obtain the following theorem from Theorem 3.1.

THEOREM 3.3. *Let $v_0 \in W_2^1(\Omega)$ and $v_1 \equiv 0$ and conditions (6)- (9), (3.1) and (3.2) be fulfilled. Then there exists $\hat\lambda > 0$ such that for all $\lambda \geq \hat\lambda$ there exists a constant $s_0(\lambda)$ such that if $g \in X_s^\lambda(Q)$ where $\lambda \geq \hat\lambda$, and $s \geq s_0(\lambda)$ then there exists a solution pair $(y,u) \in \Xi_s^\lambda(Q) \times L_2(0,T; W_2^{\frac{1}{2}}(\partial\Omega))$ of the problem (4) - (5).*

Now, let us consider the case, $v_1 \not\equiv 0$. Let us assume

Condition 3.1. There exists a constant $\tau > 0$ and function $\tilde u \in \mathcal{U}(\omega)$ such that the boundary value problem

$$L\tilde y + f(t,x,\tilde y) = \tilde u + g \text{ in } [T-\tau, T]\times\Omega,$$
$$(l_1(t,x)\frac{\partial \tilde y}{\partial \nu_A} + l_2(t,x)\tilde y)\big|_{[T-\tau,T]\times\partial\Omega} = 0, \ \tilde y(T,\cdot) = v_1$$

has a solution $\tilde y \in Y(Q)$.

We have

THEOREM 3.4. *Let $v_0 \in W_2^1(\Omega)$ and $g \in L^2(Q)$ and let (6)- (9), (3.1) and (3.2) be fulfilled, and the finally let functions v_1 and g satisfy the condition 3.1. Then there exists a solution $(y,u) \in Y(Q) \times \mathcal{U}(\omega)$ of problem (1) - (3).*

PROOF. We denote pair $(y,u) \in Y(Q) \times \mathcal{U}(\omega)$ which is a solution of the problem (1.1) - (1.3). We set $u(t,x) = 0 \ \forall \ (t,x) \in [0, T-\tau]\times\Omega$. For $(t,x) \in [0, T-\tau]\times\Omega$ we define the function $y(t,x)$ as a solution of boundary value problem

$$Ly + f(t,x,y) = g \text{ in } \quad [0,T-\tau]\times\Omega, \ (l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)|_{[0,T-\tau]\times\partial\Omega} = 0,$$
$$y(0,x) = v_0(x).$$

In the cylinder $[T-\tau, T] \times \Omega$ we are looking for solution of the problem (1) - (3) in the following form

$$(y, u) = (\hat{y}, \hat{u}) + (\tilde{y}, \tilde{u}), \tag{3.30}$$

where $(\tilde{y}, \tilde{u})$ is the pair from condition 3.1, and a pair $(\hat{y}, \hat{u})$ satisfies the equations

$$L\hat{y} + f(t, x, \hat{y} + \tilde{y}) - f(t, x, \tilde{y}) = \hat{u} - \tilde{u} \quad \text{in} \quad [T-\tau, T] \times \Omega, \ \hat{u} \in \mathcal{U}(\omega), \tag{3.31}$$

$$(l_1(t,x)\frac{\partial \hat{y}}{\partial \nu_A} + l_2(t,x)\hat{y})|_{[T-\tau,T]\times\partial\Omega} = 0, \quad \hat{y}(T-\tau, x) = y(T-\tau, x). \tag{3.32}$$

By virtue of (3.6), (3.7) and condition 3.1 the function $f_0(t, x, \zeta) = f(t, x, \zeta + \tilde{y}) - f(t, x, \tilde{y})$ satisfies (3.1) and (3.2). So applying Theorem 3.1, we obtain a solution of (3.31) and (3.32). Hence, the solution of (1) - (3) defined by formula (3.30) is obtained for $t \in [T-\tau, T]$. ■

Condition 3.2. There exists a constant $\tau > 0$ and a function $\tilde{u} \in L^2(T-\tau, T; W_2^{\frac{1}{2}}(\Omega))$ such that the following boundary value problem

$$L\tilde{y} + f(t, x, \tilde{y}) = g \quad \text{in} \quad [T-\tau, T] \times \Omega, \quad \tilde{y}(T, x) = v_1(x),$$
$$(l_1(t,x)\frac{\partial \tilde{y}}{\partial \nu_A} + l_2(t,x)\tilde{y})|_{[T-\tau,T]\times\Gamma_0} = \tilde{u},$$
$$(l_1(t,x)\frac{\partial \tilde{y}}{\partial \nu_A} + l_2(t,x)\tilde{y})|_{[T-\tau,T]\times(\partial\Omega\setminus\Gamma_0)} = 0 \tag{3.33}$$

has a solution $\tilde{y} \in Y(Q)$.

From Theorem 3.3, using the similar methods used in the proof of Theorem 2.3, we can obtain

THEOREM 3.5. *Let $v_0 \in W_2^1(\Omega)$ and $g \in L^2(Q)$ and let (6)- (9), (3.1) and (3.2) be fulfilled, and finally let function v_1 and g satisfy the condition 3.2. Then there exists a solution pair $(y, u) \in Y(Q) \times L^2(0, T; W_2^{\frac{1}{2}}(\Omega))$ of problem (4) - (5).*

Note that conditions **3.1** and **3.2** are not only sufficient, but also necessary. Indeed if the problems (1)-(3) or (4)-(5) has a solution (y, u) then functions v_1, g satisfy condition **3.1** or **3.2** with $(\tilde{y}, \tilde{u}) = (y, u)$ and $\tau = T$.

Example. Let us assume that there exist $\tau_0 > 0$ such that $g(t, x) = 0$ for all $(t, x) \in [T-\tau, T] \times \Omega$. Then by (3.1) functions $v_1 \equiv 0$ and $g \equiv 0$ satisfy condition **3.1** or **3.2** with $(\tilde{y}, \tilde{u}) = (0, 0)$.

If the coefficients of operator G and function g are independent of t, and v_1 is a steady-state solution of equation (1):$G(v_1(x)) = g(x)$ in Ω, $v_1|_{\partial\Omega} = 0$ then the pair (v_1, g) satisfy condition **3.1** or **3.2**.

Numerous results on solvability of (3.33) for the linear parabolic operators with analytic coefficients were obtained in [10].

Let us consider parabolic equation of the form

$$\frac{\partial y}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) + c(x)y = 0 \text{ in } Q, \tag{3.34}$$

where a_{ij}, c satisfy (6),(7).

The following theorem due to E. Landis and O. Oleinik [40].

THEOREM 3.6. *Let $y(t,x)$ be a solution of equation (3.34) in $(0,T] \times B_r(0)$. Suppose $|x_0| < R$ and that as $x \to x_0$ the function $y(T,x)$ decays faster than any polynomial, that is, for each k there exists c_k such that $|y(T,x)| \le c_k|x - x_0|^k$. Then $y(T,x) \equiv 0$.*

To find a function $v_1(x)$ which satisfy condition 3.1 one should solve the problem (3.33) for parabolic equation. It is well known (see [45]) that this problem is ill posed. If operator G is a linear operator there is the following result due to J.L. Lions.

THEOREM 3.7. *Let $f \equiv 0$ and conditions (6)-(9) be fulfilled. Then set of initial dates (v_1, g) for which exists solution $y \in Y(Q)$ dense in the space $W_2^1(\Omega) \times L^2(Q)$.*

Now we prove approximate boundary controllability of the parabolic equation (4_1).

THEOREM 3.8. *Let $v_1, v_0 \in W_2^1(\Omega)$, $g \in L^2(Q)$ and let (6)- (9), (3.1) and (3.2) be fulfilled. Then for every $\varepsilon > 0$ there exists a control $u_\varepsilon \in \mathcal{U}(\omega)$ such that solution of problem (1),(2) $y_\varepsilon \in Y(Q)$ satisfy the inequality*

$$\|y_\varepsilon(T,\cdot) - v_1\|_{W_2^1(\Omega)} \le \varepsilon. \tag{3.35}$$

Proof. By Theorem 3.1 for every $\varepsilon > 0$ one can find $\delta > 0$ such that the solution of boundary value problem

$$G(z_\varepsilon) = g \quad (t,x) \in (T-\delta, T) \times \Omega,\ (l_1(t,x)\frac{\partial z_\varepsilon}{\partial \nu_A} + l_2(t,x)z_\epsilon)|_{\partial\Omega} = 0,$$
$$z_\varepsilon(T-\delta,\cdot) = v_1(x) \tag{3.36}$$

satisfy the inequality

$$\|z_\varepsilon(T,\cdot)-v_1\|_{W_2^1(\Omega)} \le \varepsilon. \tag{3.37}$$

By (3.36) functions z_ε, g satisfy Condition 3.1. So the initial datum (v_0, $z_\varepsilon(T,\cdot), g$) satisfy to all assumptions of Theorem 3.5. Thus applying this theorem we get a solution of the problem (1)-(3) which satisfy (3.35) by virtue of (3.37). ■

Remark 3.1. We can consider analog of the problem (1)-(3)

$$\frac{\partial y}{\partial t} + Ay + \sum_{k=1}^{n} B_k \frac{\partial y}{\partial x_k} + Cy + f(t,x,y) = u + g, \quad u \in (\mathcal{U}(\omega))^n, \tag{3.38}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad y(0,x) = v_0(x), \tag{3.39}$$

$$y(T,x) = v_1(x), \tag{3.40}$$

and problem (4)-(5)

$$\frac{\partial y}{\partial t} + Ay + \sum_{k=1}^{n} B_k \frac{\partial y}{\partial x_k} + Cy + f(t,x,y) = g \quad \text{in} \quad Q, \tag{3.41}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{]0,T[\times\Gamma_0} = u,$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_{]0,T[\times(\partial\Omega\setminus\Gamma_0)} = 0, \tag{3.42}$$

$$y(0,\cdot) = v_0, \quad y(T,\cdot) = v_1 \tag{3.43}$$

for the system of parabolic equations. Here $y(t,x) = (y_1(t,x),...,y_n(t,x))$, $y_0(x) = (y_1^0(x),...,y_n^0(x)), y_1(x) = (y_1^1(x),...,y_n^1(x)), u(t,x) = (u_1(t,x),...,$ $u_n(t,x)), g(t,x) = (g_1(t,x),...,g_n(t,x)), f(t,x,y) = (f_1(t,x,y),...,f_n(t,x,y))$,

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}\right), B_k = \left\{b_{ij}^{(k)}(t,x)\right\}_{i,j=1}^n, C = \left\{c_{ij}(t,x)\right\}_{i,j=1}^n.$$

We assume that

$$a_{ij} \in C^{0,1}(\overline{Q}),\ a_{ij} = a_{ji},\ b_{ij}^{(k)} \in C^{0,1}(\overline{Q}),\ c_{ij} \in L^\infty(Q)\ k,i,j = 1,\dots,n \tag{3.44}$$

and initial datum (v_1, g) satisfy to the

Condition 3.3. There exists a constant $\tau > 0$ and function $\tilde{u} \in (\mathcal{U}(\omega))^n$ such that the boundary value problem

$$\frac{\partial \tilde{y}}{\partial t} + A\tilde{y} + \sum_{k=1}^{n} B_k \frac{\partial \tilde{y}}{\partial x_k} + C\tilde{y} + f(t,x,\tilde{y}) = \tilde{u} + g \text{ in } [T-\tau, T] \times \Omega,$$

$$(l_1(t,x)\frac{\partial \tilde{y}}{\partial \nu_A} + l_2(t,x)\tilde{y})\big|_{[T-\tau,T]\times\partial\Omega} = 0,\ \tilde{y}(T,\cdot) = v_1$$

has a solution $\tilde{y} \in (Y(Q))^n$.

or to the

Condition 3.4. There exists a constant $\tau > 0$ and a function $\tilde{u} \in L^2(T-\tau, T; (W_2^{\frac{1}{2}}(\Omega))^n)$ such that the following boundary value problem

$$\frac{\partial \tilde{y}}{\partial t} + A\tilde{y} + \sum_{k=1}^{n} B_k \frac{\partial \tilde{y}}{\partial x_k} + C\tilde{y} + f(t,x,\tilde{y}) = g \quad \text{in} \quad [T-\tau, T] \times \Omega,$$

$$\tilde{y}(T,x) = v_1(x), \quad (l_1(t,x)\frac{\partial \tilde{y}}{\partial \nu_A} + l_2(t,x)\tilde{y})|_{[T-\tau,T]\times\Gamma_0} = \tilde{u},$$

$$(l_1(t,x)\frac{\partial \tilde{y}}{\partial \nu_A} + l_2(t,x)\tilde{y})|_{[T-\tau,T]\times(\partial\Omega\setminus\Gamma_0)} = 0$$

has a solution $\tilde{y} \in (Y(Q))^n$.

We have

THEOREM 3.9. *Let $v_0 \in (W_2^1(\Omega))^n$ and $g \in (L^2(Q))^n$ and let (3.44), (7)-(9), (3.1) and (3.2) be fulfilled, and the finally let functions v_1 and g satisfy the condition 3.3. Then there exists a solution $(y,u) \in (Y(Q))^n \times (\mathcal{U}(\omega))^n$ of problem (3.38) - (3.40).*

THEOREM 3.10. *Let $v_0 \in (W_2^1(\Omega))^n$ and $g \in (L^2(Q))^n$ and let (3.44), (7)-(9), (3.1) and (3.2) be fulfilled, and finally let function v_1 and g satisfy the condition 3.4. Then there exists a solution pair $(y,u) \in (Y(Q))^n \times L^2(0,T; (W_2^{\frac{1}{2}}(\Omega))^n)$ of problem (3.41) - (3.43).*

4. Local exact controllability of semilinear parabolic equations

In §3 we proved the global existence theorem for the problem of exact boundary controllability (1) - (3) in the case of sublinear growth of nonlinear term. When nonlinear term has a superlinear growth (for example $f(t,x,y) =$

$|y|^p y$, where $p > 0$) in general case the statement of the Theorem 3.3 and 3.4 isn't true. We will discuss this situation in details in the next section. Below in the case of superlinear growth we can prove only a theorem on local exact controllability of the parabolic equation.

Let $(\hat{y}, \hat{u}) \in W_p^{1,2}(Q) \times (\mathcal{U}(\omega) \cap L^p(Q))$ satisfy (1) - (2) :

$$G(\hat{y}) = \hat{u} + g \quad \text{in} \quad Q, \quad (l_1(t,x)\frac{\partial \hat{y}}{\partial \nu_A} + l_2(t,x)\hat{y})\big|_\Sigma = 0. \tag{4.1}$$

Definition 4.1. Let X, Z are the Banach spaces. A linear continous operator $A : X \to Z$ is called epimorphism if it maps the space X onto the whole space Z.

We recall theorem on right inverse operator

THEOREM 4.1. *Let X, Z are the Banach spaces and*

$$A : X \to Z \tag{4.2}$$

is a continuously differentiated mapping. Let us assume that for some $x_0 \in X$, and $z_0 \in Z$ equality holds

$$A(x_0) = z_0, \tag{4.3}$$

and derivative

$$A'(x_0) : X \to Z \tag{4.4}$$

of the operator A at x_0 is a epimorphism. Then for sufficiently small $\varepsilon > 0$ exists mapping $M(z) : B_\varepsilon(z_0) \to X$, defined on the ball

$$B_\varepsilon(z_0) = \{z \in Z : \|z - z_0\|_Z < \varepsilon\},$$

which satisfy conditions

$$A(M(z)) = z, \qquad z \in B_\varepsilon(z_0), \tag{4.5}$$

$$\|M(z) - x_0\|_X \le k \, \|A(x_0) - z\|_Z \text{ for all } \qquad z \in B_\varepsilon(z_0), \tag{4.6}$$

where $k > 0$ some number.

This theorem is a simple corollary of the generalized implicit function theorem which has been proved in V. M. Aleksev, V. M. Tikhomirov and S. V. Fomin [1].

We have

THEOREM 4.2. *Let* $p > \max\{2, (n+2)/2\}$*, and* $f(t,x,y) \in C^1(\overline{Q} \times \mathbb{R}^1)$*, conditions (6) - (9) be fulfilled, and* $v_0 \in W^1_\infty(\Omega)$ *and* $v_1(x) = \hat{y}(T,x)$*, and let* $(\hat{y}, \hat{u}) \in W^{1,2}_p(Q) \times (\mathcal{U}(\omega) \cap L^p(Q))$ *be a solution of (4.1). Then there exist* $\varepsilon > 0$ *such that if* $||v_0 - \hat{y}(0,\cdot)||_{W^1_\infty(\Omega)} \leq \varepsilon$ *the problem (1) - (3) has a solution* $(y(t,x), u(t,x)) \in W^{1,2}_p(Q) \times (\mathcal{U}(\omega) \cap L^p(Q))$*.*

PROOF. We are looking for solution in the form

$$y(t,x) = \hat{y}(t,x) + w(t,x), \quad u(t,x) = \hat{u}(t,x) + q(t,x). \tag{4.7}$$

The substitution of (4.2) into equation (1) and (2) and subtraction from them of the same equation as (4.1) for $(\hat{y}, \hat{u})$ yields

$$N(w,q) = Lw + f(t,x,\hat{y}+w) - f(t,x,\hat{y}) - q = 0 \text{ in } Q,\ q \in \mathcal{U}(\omega), \tag{4.8}$$

$$(l_1(t,x)\frac{\partial w}{\partial \nu_A} + l_2(t,x)w)\big|_\Sigma = 0, \quad w(0,x) = v_0(x) - \hat{y}(0,x), \tag{4.9}$$

$$w(T,x) = 0. \tag{4.10}$$

We introduce the mapping $A(w,q)$ by:

$$A(w,q) = (N(w,q), w(0,\cdot)).$$

Let us consider the space

$$V^\lambda_s(Q) = \{(w(t,x), q(t,x)) \in \Xi^\lambda_s(Q) \times (\mathcal{U}(\omega) \cap X^\lambda_s(Q) \cap L^p(\Omega)), |\hat{L}w \in$$
$$L^p(Q) \cap X^\lambda_s(Q),\ y(0,x) \in W^1_\infty(\Omega),\ l_1\partial w/\partial\nu_A + l_2 w = 0\ \forall\,(t,x) \in \Sigma\ \},$$

where operator $\hat{L}$ was defined in (1.40). We note (see [38], [51]) $V^\lambda_s(Q) \subset L^\infty(Q)$. It is obvious that for all $\lambda > 0$ and $s > 0$ we have

$$A \in C^1(V^\lambda_s(Q), (X^\lambda_s(Q) \cap L_p(Q)) \times W^1_\infty(\Omega)). \tag{4.11}$$

By Theorem 2.4 there exist $\hat{s} > 0$ and $\hat{\lambda} > 0$ such that

$$A'(0,0)V^{\hat{\lambda}}_{\hat{s}}(Q) = (X^{\hat{\lambda}}_{\hat{s}}(Q) \cap L_p(Q)) \times W^1_\infty(\Omega). \tag{4.12}$$

We set $X = V^{\hat{\lambda}}_{\hat{s}}(Q)$, $Z = (X^{\hat{\lambda}}_{\hat{s}}(Q) \cap L_p(Q)) \times W^1_\infty(\Omega)$, $x_0 = (0,0)$ and $z_0 = (0,0)$.

By (4.11) and (4.12) all assumptions of theorem on a right inverse operator are fulfilled. So, applying this theorem, we complete the proof of Theorem 4.1. ■

Let $\hat{y} \in W^{1,2}_p(Q)$ satisfy equation

$$G(\hat{y}) = g \quad \text{in} \quad Q, \quad (l_1(t,x)\frac{\partial \hat{y}}{\partial \nu_A} + l_2(t,x)\hat{y})\big|_{[0,T]\times(\partial\Omega\setminus\Gamma_0)} = 0. \tag{4.13}$$

We have

THEOREM 4.3. *Let $p > \max\{2, (n+2)/2\}$, $f(t,x,y) \in C^1(\overline{Q} \times \mathbb{R}^1)$, and let conditions (6) - (9) be fulfilled, $v_0 \in W^1_\infty(\Omega)$ and $v_1(x) = \hat{y}(T,x)$ where $\hat{y}(t,x) \in W^{1,2}_p(Q)$ is a solution of (4.13). Then there exist $\varepsilon > 0$ such that if $||v_0 - \hat{y}(0,\cdot)||_{W^1_\infty(\Omega)} \le \varepsilon$ the problem (4), (5) has a solution $(y(t,x), u(t,x)) \in W^{1,2}_p(Q) \times L^2(0,T; W^{\frac{1}{2}}_2(\partial\Omega))$.*

PROOF. Let us consider a connected domain $\tilde{\Omega}$ such that

$$\tilde{\Omega} = \Omega \cup \omega, \quad \partial\tilde{\Omega} \in C^2, \quad \overline{\omega} \cap \overline{(\partial\Omega \setminus \Gamma_0)} = \{\emptyset\},$$

where ω is a nonempty set in $\mathbb{R}^n$. Set $Q_0 = (0,T) \times \tilde{\Omega}$. We extend function $\hat{y}$ from $W^{1,2}_p(Q)$ up to $W^{1,2}_p(Q_0)$, function $f(t,x,y)$ from $C^1(\overline{Q} \times \mathbb{R}^1)$ up to $C^1(\overline{Q}_0 \times \mathbb{R}^1)$ and coefficients of the operator L on Q_0 keeping the properties (6)-(9).

Let us consider the problem of exact controllability

$$G(y + \hat{y}) = u + g \quad \text{in} \quad Q_0, \quad u \in \mathcal{U}(\tilde{\omega}), \tag{4.14}$$

$$(l_1(t,x)\frac{\partial y}{\partial \nu_A} + l_2(t,x)y)\big|_\Sigma = 0, \quad y(0,x) = v_0(x), \quad y(T,x) = 0. \tag{4.15}$$

By Theorem 4.2 there exist $\varepsilon > 0$ such that for all

$$||v_0||_{W^1_\infty(\Omega)} \le \varepsilon$$

the problem (4.14) - (4.15) has a solution

$$(y(t,x), u(t,x)) \in W^{1,2}_p(Q_0) \times (\mathcal{U}(\tilde{\omega}) \cap L^p(\tilde{Q})).$$

Restricting function $y(t,x)$ on Q we find that the pair $(y + \hat{y}, (l_1\partial y/\partial\nu_A + l_2 y + l_1\partial\hat{y}/\partial\nu_A + l_2\hat{y})|_\Sigma)$ is a solution of problem (4) - (5). ■

One of the possible applications of this theorem is as follows. Let $\hat{y}(t,\cdot)$ is a smooth periodical solution of problem (4)

$$G(\hat{y}) = g \quad \text{in} \quad \mathbb{R}^1 \times \Omega, \quad \hat{y}(t + l, x) = \hat{y}(t,x).$$

By Theorem 4.2 there exists a neighborhood $\mathfrak{S}$ of this curve $\hat{y}(t,\cdot)$ in the space $W^1_\infty(\Omega)$ such that an arbitrary point v_0 from $\mathfrak{S}$ can be transferred on this curve by means of boundary control.

Let us consider the problem (4) - (5) for one-dimensional semilinear parabolic equation

$$G_1(y) = \frac{\partial y}{\partial t} - a(x)\frac{\partial^2 y}{\partial x^2} + b(x)\frac{\partial y}{\partial x} + c(x)y + f(x,y) = g(x) \quad \text{in} \quad [0,1], \tag{4.16}$$

$$y(t,0) = u_1(t), \quad y(t,1) = u_2(t), \tag{4.17}$$

$$y(0,x) = v_0(x), \quad y(T,x) = v_1(x), \tag{4.18}$$

where

$$a(x), b(x), c(x) \in C^1(\overline{\Omega}), \quad a(x) > 0. \tag{4.19}$$

Let us assume that function $f(x,y) \in C^2([0,1] \times \mathbb{R}^1)$ satisfy the following inequality

$$-f(x,y)y \geq c_1|y|^p - c_2 \quad \forall (x,y) \in [0,1] \times \mathbb{R}^1, \tag{4.20}$$

where

$$c_1 > 0, \quad c_2 > 0, \quad p > 1.$$

We have

THEOREM 4.4. *Let conditions (4.19) and (4.20) be fulfilled and, let $v_0, v_1 \in C^1(\overline{\Omega})$ be the steady-state solutions of the equation (4.16). Then there exists $\hat{T} > 0$ such that for arbitrary $T \geq \hat{T}$ the problem (4.16) - (4.18) has a solution.*

PROOF. Let $\ell(t) \in C^2([0,1]; \mathbb{R}^2)$ such that

$$\ell(0) = \left(v_0(0), \frac{\partial v_0}{\partial x}(0)\right), \quad \ell(1) = \left(v_1(1), \frac{\partial v_1}{\partial x}(1)\right).$$

By (4.19) and (4.20) there exist a function $v(t,x) \in C^{1,2}([0,1] \times [0,1])$ such that

$$a(x)\frac{\partial^2 v(t,x)}{\partial x^2} + b(x)\frac{\partial v(t,x)}{\partial x} + c(x)v(t,x) + f(x, v(t,x)) = g(x),$$
$$\left(v(t,0), \frac{\partial v}{\partial x}(t,0)\right) = \ell(t).$$

By Theorem 4.3 there exist finite number of points

$$t_0 = 0 < t_1 \cdots < t_{k-1} < t_k = 1$$

such that there exist a solution of the following problem of exact controllability

$$G_1(y_i) = g(x) \quad \text{in} \quad [0,1] \times [0,1],\ y_i(0,x) = v(t_{i-1},x), y_i(1,x) = v(t_i,x).$$

Set $\hat{T} = k$. We define the solution of problem (4.16) - (4.18) by the formula :

$$t \in [i-1,i] \quad \text{implies} \quad y(t,x) = y_i(t+i-1,x).$$

The control functions $u_1(t)$ and $u_2(t)$ are well defined by the formulas (4.17). ■

From now we assume that nonlinear term of parabolic equation function f is independent on t, x and satisfy the following growth condition: There exists a constant $p > 1$ such that

$$c_4|y|^{p+1} - c_5 \leq f(y)y \leq c_6(|y|^{p+1}+1); \quad f'(y) \geq c_7 \quad \forall y \in R^1, \qquad (4.21)$$

where $c_4 > 0, c_5, c_6, c_7$ are independent constants.

Let $\Omega = [0,L]$. We consider the dynamical system

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} + f(y) = 0, \quad x \in \Omega, \ \frac{\partial y(t,0)}{\partial x} = \frac{\partial y(t,L)}{\partial x} = 0,\ y(0,\cdot) = v_0. \ (4.22)$$

The evolution dynamical system (4.22) described by a family of operators $S(t)$, $t \geq 0$, that map $L^2(0,L)\,(W_2^1(0,L))$ into itself and enjoy the usual semigroup properties

$$S_{t+s} = S_t S_s \quad \forall s,t \geq 0,$$
$$S_0 = I \quad (I - \text{identity in } L^2(0,L) \ \ (W_2^1(0,L))),$$
$$y(t,\cdot) = S_t v_0.$$

We need to remind some facts of the theory of infinite-dimensional dynamical systems (see [5],[64]). Let E be a Banach space. Set

$$dist_E(X,Y) = \sup_{x \in X} \inf_{y \in Y} \|x-y\|_E.$$

Definition 4.1. Let $S_t : E \to E$ be a semigroup in a Banach space E. The set X is called point-attracting in E if $dist_E(S_t u, X) \to 0$ as $t \to +\infty$ for any point $u \in E$.

Definition 4.2. A functional $\Phi : E \to R$ is called a Lyapunov function of semigroup $\{S_t\}$ on E if first for any $u \in E$ the function $\Phi(S_t(u))$ of the variable t is a monotonously decreasing in t and, the second, if the equality $\Phi(u) = \Phi(S_t u)$ for some $t > 0$ implies that $u = S_t u = z$ is an equilibrium point of the semigroup $\{S_t\}$, that is $S_t z = z \ \forall t \geq 0$.

The following theorem proved in [5].

THEOREM 4.5. *Let E be a Banach space and let a semigroup $\{S_t\}$, $S_t : E \to E$ have a point-attracting set X compact in E. Let $\{S_t\}$ be continuous on E and have the Lyapunov function defined, continuous and bounded from below in a neighborhood of X in E. Let $\mathcal{M}$ be the set of all equilibrium points of $\{S_t\}$. Then*

$$dist(S_t u, \mathcal{M}) \to 0 \quad as \ \to +\infty \ \forall u \in E.$$

Denote by $\mathfrak{B}$ the collection of all bounded subset in $L^2(0, L)$.

Definition 4.3. A set $X \subset W_2^1(0, L)$ is called $(L^2(0, L), W_2^1(0, L))$-attracting if for any $B \subset \mathfrak{B}$ $\exists T > 0$ such that $S_t \subset W_2^1(0, L)$ for $t > T$ and $S_t B \to X$ in $W_2^1(0, L)$ as $t \to +\infty$.

Definition 4.4. A set $\mathfrak{U} \subset W_2^1(0, L)$ is called a maximal $(L^2(0, L), W_2^1(0, L))$ attractor of the semigroup S_t if it has a following properties:

1. $\mathfrak{U}$ is compact in $W_2^1(0, L)$.
2. $\mathfrak{U}$ is an $(L^2(0, L), W_2^1(0, L))$ - attracting set.
3. $\mathfrak{U}$ is strictly invariant, i.e. $S_t \mathfrak{U} = \mathfrak{U}$ $\forall t \geq 0$.

Now let us return again to the dynamical system (4.22). It is well known that under assumptions (4.21) there exists an attractor of dynamical system (4.22). The following theorem is a special case of the general theorem proved in [5, pp.127].

THEOREM 4.6. *Let (4.21) be fulfilled. The semigroup $S(t) : L^2(0, L) \to L^2(0, L)$ possesses an $(L^2(0, L), W_2^1(0, L))$- maximal attractor $\mathfrak{U}$ which is bounded in $W_2^1(0, L)$, compact and connected in $L^2(0, L)$.*

We introduce the Lyapunov function of dynamical system (4.22) by the formula

$$\Phi(u) = \int_0^L \left(\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + F(u) \right) dx, \quad F(u) = \int_0^u f(v) dv.$$

Let us check the properties of the Lyapunov function. Firstly we show that function $\Phi(y(t, \cdot))$ decrease in t on the trajectories of dynamical system. Differentiation of $\Phi(y(t, \cdot))$ respect to variable t gives

$$\frac{\partial}{\partial t} \Phi(y(t, \cdot)) = \int_0^L \left(\frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} + f(y) \frac{\partial y}{\partial t} \right) dx =$$
$$\int_0^L \left(-\frac{\partial^2 y}{\partial x^2} + f(y) \right) \frac{\partial y}{\partial t} dx = - \int_0^L \left(-\frac{\partial^2 y}{\partial x^2} + f(y) \right)^2 dx \leq 0.$$

Now we assume that for some $t_0 > t_1$ $\Phi(y(t_0,\cdot)) = \Phi(y(t_1,\cdot))$. Multiplying equation (4.22) on y scalarly in $L^2(0,L)$ and integrating it on the segment $[t_0,t_1]$ we obtain

$$0 = \int_{t_0}^{t_1}\int_0^L y_t^2 dxdt + \Phi(y(t,\cdot))|_{t_0}^{t_1} = \int_{t_0}^{t_1}\int_0^L y_t^2 dxdt.$$

This equality in turn imply

$$y(t,x) = y(x).$$

So the functional Φ posses all properties necessary to be Lyapunov function of dynamical system (4.22). Note that by (4.21) there exists a constant c_7 such that

$$\Phi(v) > c_7 \quad \forall v \in W_2^1(0,L).$$

Set $E = W_2^1(0,L)$. As a point-attracting set $X \subset E$ we consider the maximal $(L^2(0,L), W_2^1(0,L))$ attractor, which existence was establish in the Theorem 4.6. Thus all conditions necessary to apply the Theorem 4.5 are fulfilled.

Let $\mathcal{M}$ be the set of equilibrium points of dynamical system (4.22) i.e $\mathcal{M}$ is a collection of all functions $z(x) \in W_2^2(0,L)$ such that

$$-\frac{d^2z}{dx^2} + f(z) = 0 \ x \in (0,L), \ \frac{\partial z(0)}{\partial x} = \frac{\partial z(L)}{\partial x} = 0. \tag{4.23}$$

By Theorem 4.5 we have

THEOREM 4.7. *Let (4.21) be fulfilled and $y(t,x)$ be solution of the problem (4.22) with initial datum $v_0 \in W_2^1(0,L)$. Then*

$$dist_{W_2^1(0,L)}(y(t,\cdot),\mathcal{M}) \to 0 \ as \ t \to +\infty. \tag{4.24}$$

Now we consider the problem of exact controllability for the equation (4.22_1) when control concentrated on the part of the boundary

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} - f(y) = 0, \ x \in [0,L], \tag{4.25}$$

$$\frac{\partial y}{\partial x}(t,0) = 0, \quad \frac{\partial y}{\partial x}(t,L) = u(t), \tag{4.26}$$

$$y(0,x) = v_0(x), \quad y(T,x) = v_1(x), \tag{4.27}$$

where (we remind) $u(t)$ is a control, function v_0 is a initial state, and v_1 is a target function.

We introduce numbers m and M by formulas

$$m = \inf_{a \in \{f(a)=0\}} a, \quad M = \sup_{a \in \{f(a)=0\}} a.$$

Denote by $\mathfrak{R}(v_0)$ the set of reachability of the function v_0. i.e.

$$\mathfrak{R}(v_0) = \{v(x) | \text{there exists a pair } (y(t,x), u(t)) \text{ which satisfy } (4.25), (4.26) \\ \text{such that } y(0,\cdot) = v_0, y(T,\cdot) = v.\}$$

We would like to investigate the following problem: Is the set $\mathcal{M}$ belong to $\mathfrak{R}(v_0)$ for an arbitrary v_0 or not?

Let us consider the Cauchy problem for the second order differential equation

$$-\frac{d^2z}{dx^2} + f(z) = 0, \ z(0) = z_0, \ z_x(0) = 0. \tag{4.28}$$

We have

THEOREM 4.8. *Let (4.21) be fulfilled. Then there exist* $0 < \hat{L} \le \infty$ *that for* $L < \hat{L}$ *for an arbitrary* $v_0 \in W_2^1(0,L)$ $\mathcal{M} \subset \mathfrak{R}(v_0)$ *and for* $L > \hat{L}$ *there exists a open set* $\mathfrak{O} \subset W_2^1(0,L)$ *such that for every* $v_0 \in \mathfrak{O}$ $\mathcal{M} \not\subset \mathfrak{R}(v_0)$. *Moreover* $\hat{L} \ne +\infty$ *if and only if for some* $L_1 > 0$ *there exists* $z_0 \in (m, M)$ *such that there is not solution of the problem (4.28) on segment* $[0, L_1]$.

Proof. Multiplying the (4.23) by z scalarly in $L^2(0,L)$ and integrating by parts we have

$$\int_0^L (|\nabla z|^2 + |z|^{p+1})dx \le c_8(L), \tag{4.29}$$

where constant c_8 dependes continuously on L. The estimate (4.29) and the Sobolev imbedding theorem imply

$$\|z\|_{C[0,L]} \le c_9(L), \tag{4.30}$$

Let us show that constant c_9 dependes continuously on L. Our proof by contradiction. Let us assume

$$\|z\|_{C[0,L]} \to +\infty \text{ as } L \to +0. \tag{4.31}$$

By (4.29), (4.31)

$$\inf_{x\in(0,L^*)} |z(x)| \to +\infty \quad \text{as } L \to +0, \tag{4.32}$$

where $L^* \in (0, L)$ the first point, such $\frac{\partial z(L^*)}{\partial x} = 0$.

Integrating (4.23_1) on $[0, L^*]$, bearing in mind boundary conditions, we obtain

$$\int_0^{L^*} f(y)dx = 0. \tag{4.33}$$

From (4.21), (4.33) we get the contradiction to (4.32). But (4.30) in turn imply

$$\|z\|_{C^2[0,L]} \le c_{10}, \tag{4.34}$$

where constant c_{10} also dependes continuously on L.

Now let $v_0 \in W_2^1(0, L)$ be an arbitrary function. By Theorem 4.7 we get for any $\varepsilon > 0$ there exists a function $z_\varepsilon \in \mathcal{M}$ such that for some $t_\epsilon \in R_+^1$ we have inequality

$$\|z_\varepsilon - S(t_\varepsilon)v_0\|_{W_2^1(0,L)} \le \varepsilon.$$

By (4.34) the set $\mathcal{M}$ bounded in $C^2[0, L]$. So applying the Theorem 4.2 for suitable ε, we can reach some target function $\hat{z} \in \mathcal{M}$ at some moment $\hat{t}$. Thus we prove that for any $v_0 \in W_2^1(0, L)$ there exists $\hat{z} \in \mathcal{M} \cap \mathfrak{R}(v_0)$.

Let $\tilde{z}$ be an arbitrary function from $\mathcal{M}$. Now we assume that for any $z_0 \in (m(L), M(L))$ there exist a solution of the problem (4.28) $z(x) \in C^2[0, L]$. Thus one can find a function $z(\tau, x)$ for any fixed $\tau \in [0, 1]$, $z(\tau, \cdot)$ is a solution of the following problem:

$$-\frac{d^2 z(\tau,\cdot)}{dx^2} + f(z(\tau,\cdot)) = 0,\ z(\tau, 0) = (1-\tau)\hat{z}(0) + \tau\tilde{z}(0),\ \frac{\partial z(\tau, 0)}{\partial x} = 0.$$

Thanks to local existence and uniqueness theorem for O.D.E. (see [2]) the mapping $\tau \to z(\tau, \cdot)$ is continuous in the space $W_2^1(0, L)$. By uniqueness theorem for ordinary differential equations $z(0, \cdot) = \hat{z}$, $z(1, \cdot) = \tilde{z}$. By arguments similar to the proof of Theorem 4.3 we obtain that the function $\tilde{z}$ belong to the set of reachability of the function $\hat{z}$.

On the other hand if there exists $z_0 \in (m(L), M(L))$ such that the solution of the problem (4.28) $\overline{z}(x)$ blow up at the moment $x_0 < L$. Then by (4.21) there are only two possibilities

$$\text{either } \lim_{x\to x_0-0} \overline{z}(x) = +\infty, \text{ or } \lim_{x\to x_0-0} \overline{z}(x) = -\infty. \tag{4.35}$$

Assume that first one holds. Set $\mathfrak{O} = \{v(x) \in W_2^1(0,L) | v(x) < \overline{z}(x)\ x \in [0, x_0]\}$. Obviously that interiority of this set in $W_2^1(0,L)$ is not empty. Let $y(t,x)$ is a solution of the problem (4.25)-(4.27) with some control $u(t)$ and initial condition $v_0 \in \mathfrak{O}$. Let us continue the functions $y(t,x), v_0(x)$ on the segment $[-L,0]$ by formula $y(t,x) = y(t,-x), v_0(x) = v_0(-x)$. Evidently function $y(t,x)$ satisfy the equations

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} + f(y) = 0,\ x \in [-L, L], \tag{4.36}$$

$$y(t,-L) = u(t), \quad y(t,L) = u(t), \tag{4.37}$$

$$y(0,x) = v_0(x). \tag{4.38}$$

Applying to (4.36)-(4.38) the maximum principle ,bearing in mind (4.35), we have

$$y(t,x) \le \overline{z}(x). \tag{4.39}$$

By definition of $m(L), M(L)$ there exist a function $\tilde{z} \in \mathcal{M}$ such that $\tilde{z}(0) > \overline{z}(0)$. Thus if we consider the problem (4.25)-(4.27) with initial datum $v_0 \in \mathfrak{O}$ and $v_1 = \tilde{z}$ the inequality (4.39) imply that there is no solution of this problem. ■

§5. Some results on uncontrollability of semilinear parabolic equations

Let us consider the problem of exact controllability (1)-(3) under the following assumptions on nonlinear term of parabolic equation (1): There exist constants $c_1 > 0, c_2, p > 1$ such that

$$f(t,x,y)y > c_1|y|^{p+1} - c_2 \quad \forall (t,x) \in Q_\infty, y \in R^1, \tag{5.1}$$

where $Q_\infty = R^1_+ \times \Omega$.

We also assume that functions a_{ij}, b_i, c satisfy conditions (6), (7) where $\overline{Q}$ replaced by $\overline{Q}_\infty$. Let $\omega' \subset \Omega$ be subdomain of Ω such that $\omega \subset \omega'$, $\partial\omega' \in C^\infty$. Denote by $\rho \in C^\infty(\overline{\Omega})$ a function such that

$$\rho|_{\partial\Omega} = 0, \quad \rho|_{\omega'} = 0, \quad \rho(x) > 0\ \forall x \in \Omega \setminus \omega'. \tag{5.2}$$

Firstly we prove the a priori estimate for solutions of problem (1)-(3). We have

THEOREM 5.1. *Let $k > 2(p+1)/(p-1)$, $l_1 \equiv 0$,(6)- (9), (5.1) be fulfilled, function ρ satisfy (5.2) and $(y,u) \in Y(Q) \times \mathcal{U}(\omega)$ be a solution of problem (1)-(3). Then the estimate holds*

$$\frac{d}{dt}\int_\Omega \rho^k y^2(t,x)dx + \frac{c_1}{4}\int_\Omega \rho^k |y|^{p+1}dx \le c_3(\|g(t,\cdot)\|^2_{L^2(\Omega)} + 1). \tag{5.3}$$

Proof. Multiplying equation (1) by $\rho^k y$ scalarly in $L^2(\Omega)$ and integrating by parts respect to variable x we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^k y^2\,dx + \int_\Omega \sum_{i,j=1}^n \left(a_{ij}\rho^k \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} + a_{ij}\frac{\partial \rho^k}{\partial x_i} y \frac{\partial y}{\partial x_j}\right)dx$$
$$+\int_\Omega \{\sum_{i=1}^n b_i \rho^k y \frac{\partial y}{\partial x_i} + \rho^k c y^2 + \rho^k f(t,x,y)y\}dx = \int_\Omega (u+g)\rho^k y dx = \int_\Omega g\rho^k y dx. \tag{5.4}$$

Integrating by parts in the left hand side of (5.4) again and carry out some terms from the left part of this equality to right part we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^k y^2\,dx + \int_\Omega \left(\rho^k \sum_{i,j=1}^n a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} + \rho^k f(t,x,y)y\right)dx =$$
$$\int_\Omega \left(\frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\rho^k}{\partial x_i}\right)y^2 + \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i\rho^k)y^2 - \rho^k c y^2\right)dx + \int_\Omega \rho^k g y dx. \tag{5.5}$$

Note that

$$\left|\frac{\partial^2 \rho^k(x)}{\partial x_i \partial x_j}\right| \le c_4(k)\rho^{k-2}(x), \qquad \left|\frac{\partial \rho^k(x)}{\partial x_i}\right| \le c_5(k)\rho^{k-1}(x)\ \forall x \in \Omega. \tag{5.6}$$

Hence by (5.1), (5.5) we get from (5.4)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^k y^2 dx + \int_\Omega c_1\rho^k |y|^{p+1}dx \le c_7\|g(t,\cdot)\|^2_{L^2(\Omega)} + c_6\int_\Omega \rho^{k-2}y^2 dx, \tag{5.7}$$

where c_6, c_7 are independent constants.

By Hölder inequality one can estimate the last integral in the right hand side of (5.7) as follows

$$\left|\int_\Omega \rho^{k-2}y^2dx\right| \leq \left(\int_\Omega |y|^{p+1}\rho^{(p+1)(k-2)/2}dx\right)^{\frac{2}{p+1}}\left(\int_\Omega 1dx\right)^{\frac{p-1}{p+1}}$$
$$\leq c_8\left(\int_\Omega |y|^{p+1}\rho^{(p+1)(k-2)/2}dx\right)^{\frac{2}{p+1}} \leq c_9\left(\int_\Omega |y|^{p+1}\rho^k dx\right)^{\frac{2}{p+1}}. \quad (5.8)$$

Applying the inequality $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ to the right side of (5.8) we have

$$\left|\int_\Omega \rho^{k-2}y^2dx\right| \leq \frac{c_1}{2}\int_\Omega |y|^{p+1}\rho^k dx + c_{10}. \quad (5.9)$$

Replacing in (5.6) the last term by right part of (5.8) we get (5.2). ■

We have

THEOREM 5.2. *Let (5.1), (6)-(9) be fulfilled, $l_1 \equiv 0$, $v_0 \in L^2(\Omega)$, $g \in L^2(\Omega)$. Then there exists a function $v_1 \in C^\infty(\overline{\Omega})$ such that for any $T > 0$ the problem (1)-(3) has not solution $y \in W_2^{1,2}(Q)$.*

Proof. Let us introduce function $m(t)$ by formula $m(t) = \int_\Omega \rho^k y^2(t,x)dx$ where $k > (p+1)/(p-1)$ and function ρ defined in (5.2). By (5.3) we have

$$\frac{1}{2}\frac{d}{dt}m + \frac{c_1}{4}\int_\Omega \rho^k|y|^{p+1}dx \leq c_4(\|g\|^2_{L^2(\Omega)} + 1) \quad \forall t \in (0,+\infty). \quad (5.10)$$

By Hölder inequality there exists some $\mu > 0$ such that

$$\mu m^{p+1} \leq \frac{c_1}{4}\int_\Omega \rho^k|y|^{p+1}dx.$$

Thus (5.10) imply the inequality

$$\frac{1}{2}\frac{d}{dt}m + \mu m^{p+1} \leq c_{11}(\|g\|^2_{L^2(\Omega)} + 1) \quad \forall t \in (0,+\infty). \quad (5.11)$$

It follows from this inequality that

$$m(t) \leq A = max\left\{\int_\Omega \rho^k v_0^2dx, \left(\frac{c_{12}(\|g\|^2_{L^2(\Omega)} + 1)}{\mu} + 1\right)^{\frac{1}{p+1}}\right\} \quad \forall t \in [0,T]. \quad (5.12)$$

So, if for given $(v_0(x), g(x))$ one choose a function $v_1(x)$ such that

$$\int_\Omega \rho^k v_1^2 dx > A$$

the inequality (5.12) imply that there is no solution of problem (1)- (3).■

Note that nonexistence results of such type were proved in [25] for particular case of semilinear parabolic equations.

We assume that nonlinear term of equation (1) satisfies the relation

$$c_{12}|y|^{p+1} - c_{13} \leq f(t,x,y), \tag{5.13}$$

where $p > 0, c_{12} > 0$.

It is known (see [48]) that in this case for some initial datum v_0, g the problem does not have a global solution in t. For similar equations J.L. Lions posed in [51] the following "stabilization" problem: by choice of the control u arrange that the solution of the problem (1), (2) exists on a given time interval $[0, T]$. We have

THEOREM 5.3. *Suppose, that (6)-(9),(5.13) be fulfilled and $g(t,x) \equiv 0$, $l_1(t,x) \equiv 0$. Then there exists a constant $T_0 > 0$ such that for $T > T_0$ the boundary value problem has no solution in the space $L^2(0,T; L^{p+1}(\Omega))$ for any control $u \in \mathcal{U}(\omega)$.*

Proof. Multiply (1) by ρ^k, where

$$k > 2(1 + \frac{1}{p}). \tag{5.14}$$

By integrating the resulting equation over Ω we get

$$\frac{d}{dt}\int_\Omega \rho^k y dx + \int_\Omega \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \rho^k}{\partial x_i}\right) y \right.$$
$$\left. - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\rho^k b_i) y + \rho^k c y + f(t,x,y)\rho^k \right) dx = 0. \tag{5.15}$$

It follows from (5.15) by (5.13) and by the inequality

$$\left|\frac{\partial^2 \rho^k(x)}{\partial x_i \partial x_j}\right| + \left|\frac{\partial \rho^k(x)}{\partial x_i}\right| \leq c_{14}\rho^{k-2}(x) \quad \forall x \in \overline{\Omega},$$

that

$$-\frac{d}{dt}\int_\Omega \rho^k y dx \geq c_{12}\int_\Omega \rho^k y^{p+1}dx - c_{15}\int_\Omega \rho^{k-2}|y|dx - c_{16}. \tag{5.16}$$

Application of Hölder's inequality to the last integral in right part of (5.16) yield

$$-\frac{d}{dt}\int_\Omega \rho^k y dx \geq c_{12}\int_\Omega \rho^k |y|^{p+1}dx - c_{17}\left(\int_\Omega \rho^{(k-2)(p+1)}|y|^{p+1}dx\right)^{\frac{1}{p+1}} - c_{16}.$$

It follows from (5.13) and this inequality that

$$-\frac{d}{dt}\int_\Omega \rho^k y dx \geq \frac{c_{12}}{2}\int_\Omega \rho^k |y|^{p+1}dx - c_{17} \geq c_{18}\left|\int_\Omega \rho^k y dx\right|^{p+1} - c_{19}, \tag{5.17}$$

where constants $c_{18} > 0, c$ are independent on v_0. Setting $\beta(t) = -\int_\Omega \rho^k y(t,x)dx$ we deduce that by (5.17) the function $\beta(t)$ satisfies the differential inequality

$$\frac{d\beta}{dt} \geq c_{18}\beta^{p+1} - c_{19}.$$

It is known that this differential inequality does not have a global solution in t if $\beta(0) = -\int_\Omega \rho^k y_0 dx$ is sufficiently large.$\blacksquare$

§6. Exact controllability of Burgers equation

From now we start studying the problems of exact controllability of evolution equations which describe the fluid flow. The simplest of them is the Burgers equation. In this section we show that steady state solutions of Burgers equation with zero right hand side belongs to the set of reachability of any initial condition v_0. On the other hand we prove that the Burgers equation is not approximately controllable on the arbitrary bounded time intervals. Let us consider the Burgers equation

$$R(y) = \frac{\partial y(t,x)}{\partial t} - \frac{\partial^2 y(t,x)}{\partial x^2} + 2y(t,x)\frac{\partial y(t,x)}{\partial x} = u(t,x) \quad (t,x) \in \ [0,T]\times[0,L], \tag{6.1}$$

where $\infty > L > 0$ and $T > 0$ are arbitrary fixed numbers. We suppose that $y(t,x)$ satisfies zero boundary and initial conditions

$$y(t,0) = y(t,L) = 0, \quad y(0,\cdot) = v_0, \quad y(T,\cdot) = v_1, \tag{6.2}$$

where v_0 is a given initial data, v_1 is a target function. Assume that control $u(t,x) \in L_2([0,T]\times[0,L])$ and that for any $t \in [0,T]$

$$\text{supp}\; u(t,x) \subset [b,e], \quad 0 < b < e < L. \tag{6.3}$$

We also consider the problem of exact boundary controllability for Burgers equation

$$R(y) = 0 \; x \in [0,L], \; y(t,0) = u_1(t), \; y(t,L) = u_2(t), \; y(0,\cdot) = v_0, \; y(T,\cdot) = v_1. \tag{6.4}$$

It is well-known that for an arbitrary $u(t,x) \in L_2([0,T]\times[0,L])$ there exists a unique solution $y(t,x) \in L_2(0,T; W_2^2(0,L))$ of problem (6.1)-(6.2). It is possible to see, that $\partial y/\partial t \in L_2([0,T]\times[0,L])$. The following Lemma describes the set of steady state solutions of Burgers equation with zero right hand side i.e. the set of functions $z(x)$ such that

$$-\frac{\partial^2 z}{\partial x^2} + \frac{\partial z^2}{\partial x} = 0, \quad x \in (0,L), \tag{6.5}$$

$$z(0) = \alpha_1, \quad z(L) = \alpha_2. \tag{6.6}$$

LEMMA 6.1. *For an arbitrary finite $\alpha_1 \le \alpha_2$ there exists the unique solution of the problem (6.4),(6.5). Moreover*

$$\begin{cases} \text{if } \alpha_2 - \alpha_1 > L\alpha_1\alpha_2 \text{ then } z(x) = \sqrt{c}\, tg(\sqrt{c}(x+d)); \\ \text{if } \alpha_2 - \alpha_1 = L\alpha_1\alpha_2 \text{ then } z(x) = -1/(x+d); \\ \text{if } \alpha_2 - \alpha_1 < L\alpha_1\alpha_2 \text{ then } z(x) = \sqrt{c}\, cth(\sqrt{c}(x+d)). \end{cases} \tag{6.7}$$

For $\alpha_1 \ge \alpha_2$ problem (6.5), (6.6) has a solution

$$z(x) \equiv \alpha_1, \quad \text{if } \alpha_1 = \alpha_2; \tag{6.8}$$

$$z(x) = -\sqrt{c}\, cth(\sqrt{c}(x+d)), \; \text{if } \alpha_1 > \alpha_2. \tag{6.9}$$

The constants c, d are determinatead uniquely by α_1, α_2.

Proof. Integrating (6.5) in x we obtain

$$\frac{\partial y}{\partial x} = y^2 + c. \tag{6.10}$$

If $c > 0$ then integrating (6.10) we obtain the equality

$$\frac{1}{\sqrt{c}} arctg \frac{y}{\sqrt{c}} = x + d \tag{6.11}$$

which implies (6.7). Let us show that the constants $c > 0, d$ in this inequality is determinatead by α_1, α_2. It follows from (6.11), (6.6) that

$$L\sqrt{c} = artg \frac{\alpha_2}{\sqrt{c}} - artg \frac{\alpha_1}{\sqrt{c}}.$$

Applying to the both parts of this equality the operator tg we obtain that

$$tg(L\sqrt{c}) = \sqrt{c}(\alpha_2 - \alpha_1)/(c + \alpha_1\alpha_2).$$

Solving this equation by the graphics method we obtain that if α_1, α_2 satisfy condition (6.7_1) then the unique positive solution c of this equation exists.

If $c = 0$ then we obtain (6.7_2) after integrating (6.10) . Equation (6.10) with $c < 0$ implies the equality

$$\frac{z - \sqrt{c_1}}{z + \sqrt{c_1}} = e^{2\sqrt{c_1}(x+d)}, \tag{6.12}$$

where $c_1 = c$. It follows from (6.12) (6.6) that

$$e^{2\gamma L} = \left| \frac{(\alpha_2 - \gamma)(\alpha_1 + \gamma)}{(\alpha_2 + \gamma)(\alpha_1 - \gamma)} \right|,$$

where $\gamma = \sqrt{c_1}$. Solving this equation by graphics method, one can easily to show that this equation has a unique positive solution if α_1, α_2 satisfy condition (6.7_3), (6.9). The case (6.8) is evident.The proof of the theorem is complete.■

LEMMA 6.2. *Let $\alpha_1, \alpha_2 \in R$ satisfy condition $\alpha_2 \geq \alpha_1$ and $z(x)$ is a solution of problem (6.5), (6.6) and $y(t,x)$ is a solution of the problem (6.4) with $u(t) = \alpha_1$ and $u(t) = \alpha_2$. Then there exists $\lambda > 0$ such that*

$$\|z - y(t,\cdot)\|^2_{L^2(0,L)} \leq e^{-\lambda t}\|v_0 - z\|^2_{L^2(0,L)}, \tag{6.13}$$

$$\int_0^{+\infty} \left\| \frac{\partial(z - y(t,\cdot))}{\partial x} \right\|^2_{L^2(0,L)} dt \leq \|z - v_0\|^2_{L^2(0,L)}. \tag{6.14}$$

Proof. Set $w(t,x) = y(t,x) - z(x)$. By virtue of (6.4)-(6.6) $w(t,x)$ is a solution of the problem

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial (wz)}{\partial x} + \frac{\partial w^2}{\partial x} = 0 \quad x \in [0,L], \tag{6.15}$$

$$w(t,0) = w(t,L) = 0, \quad w(0,x) = z(x) - v_0(x). \tag{6.16}$$

Scaling in $L^2(0,L)$ both parts of (6.15) by the $w(t,x)$ and taking into account (6.16) we obtain after simple transformations, that

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(0,L)} + \left\|\frac{\partial w}{\partial x}\right\|^2_{L^2(0,L)} + \int_0^L \left(\frac{\partial z}{\partial x}\right) w^2(t,x)dx = 0. \tag{6.17}$$

Let λ_1 be the minimal eigenvalue of the spectral problem

$$-\frac{\partial^2 v(x)}{\partial x^2} + \frac{\partial v(x)}{\partial x} = \lambda v(x) \; x \in [0,L], \quad v(0) = v(L) = 0.$$

Since by Lemma 6.1 the inequality $\frac{\partial z(x)}{\partial x} \geq 0$ holds, then $\lambda_1 > 0$. It follows from (6.17) that

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(0,L)} + \lambda_1 \|w(t,\cdot)\|^2_{L^2(0,L)} \leq 0.$$

This inequality imply (6.13). Integrating (6.17) on segment $(0,\infty)$ bearing in mind (6.13) we get (6.14).■

THEOREM 6.1. *Let $\hat{y}(t,x) \in W_2^{1,2}(Q)$ be a solution of problem (6.1). Then there exists $\varepsilon > 0$ such that for every*

$$\|v_0 - \hat{y}(0,\cdot)\|_{W_2^1(0,L)} \leq \varepsilon$$

there exists a solution of the problem (6.4) with initial datum $(v_0, \hat{y}(T,\cdot))$. The constant ϵ dependes on T, $\|\hat{y}\|_{W_2^{1,2}(Q)}$ continuously and monotonicaly.

The proof of this theorem similar to the proof of Theorem 4.2. We leave it to readers as exercise.

THEOREM 6.2. *Let function v_1 satisfy (6.5). Then for an arbitrary $v_0 \in W_2^1(0,L)$ one can find $T_0(v_0, v_1)$ such that for $T > T_0$ there exist a solution of the problem (6.4) $(y,u) \in L^2(0,T; W_2^1(0,L)) \times L^2(0,T)$.*

Proof. Let $\tilde{z}(x)$ be an arbitrary function which satisfy (6.5), (6.6) and let $v_0(x)$ be an arbitrary function from the space $W_2^1(0,L)$. Denote by $y(t,x)$ the solution of $(6.4_1) - (6.4_4)$ with the zero boundary conditions $u_1(t) = u_2(t) = 0$. Thanks to Lemma 6.2 $y(t,\cdot) \to 0$ in $L^2(0,L)$ as $t \to +\infty$ and inequality holds

$$\int_0^{+\infty} \left\| \frac{\partial y(t,\cdot)}{\partial x} \right\|^2_{L^2(0,L)} dt < \infty.$$

This inequality imply that for every $\varepsilon > 0$ there exists t_ε such that $\|z_0 - y(T_\varepsilon, \cdot)\|_{W_2^1(0,L)} \le \varepsilon$. Then thanks Theorem 6.1 there exists $\varepsilon > 0$ and t_ε such that the following problem has a solution

$$\frac{\partial y(t,x)}{\partial t} - \frac{\partial^2 y(t,x)}{\partial x^2} + 2y(t,x)\frac{\partial y(t,x)}{\partial x} = 0 \quad (t,x) \text{ in } [t_\epsilon, t_\epsilon + \tau] \times [0,L], \tag{6.18}$$

$$y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \tag{6.19}$$

$$y(t_\varepsilon, \cdot) = \lim_{t \to t_\varepsilon - 0} y(t,\cdot), \quad y(t_\varepsilon + 1, \cdot) = 0. \tag{6.20}$$

Let t_ε be fixed. Now we construct the solution of problem 6.4 in following manner. For $t \in [0, t_\varepsilon]$ we set $u_1(t) = u_2(t) = 0$. For $t \in [t_\varepsilon, t_\varepsilon + 1]$ we choose (y, u_1, u_2) as a solution of problem (6.18)-(6.20).

Note that the problem (6.5), (6.6) has a unique solution. Really existence of solution was proved in Lemma 6.1. Let us assume that this problem has two solutions $z(x), u(x)$. Set $\delta = z - u$. This function should satisfy equations

$$-\frac{\partial^2 \delta}{\partial x^2} + u\frac{\partial \delta}{\partial x} + \delta \frac{\partial u}{\partial x} = 0, \ x \in [0,L], \quad \delta(0) = \delta(L) = 0. \tag{6.21}$$

The integration of (6.21) gives $-\frac{\partial \delta}{\partial x} + u\delta = \text{const}$. By the second integration, bearing in mind the boundary conditions (6.21_1) we get that $\delta \equiv 0$. We introduce function $z(\tau, x)$ $(\tau, x) \in [0,1] \times [0,L]$ as follows: for every $\tau \in [0,1]$ function $z(\tau, \cdot)$ is a solution of (6.5) with boundary conditions $z(\tau, 0) = (1-\tau)\tilde{z}(0), z(\tau, L) = (1-\tau)\tilde{z}(L)$. Since we proved the uniqueness of solution of problem (6.5), (6.6) this function correctly defined.

Now let us prove that the mapping $t \to z(t,\cdot)$ is continuous in the space $W_2^1(0,L)$. If this mapping is discontinuous than there exists a point $\hat{\tau} \in [0,1]$ and sequence $\{\tau_i\}$ such that

$$z(\tau_i,\cdot) \to z(\hat{\tau},\cdot) \text{ in } W_2^1(0,L) \text{ as } \tau \nrightarrow \hat{\tau}. \tag{6.22}$$

Note that (6.5), (6.6) imply inequality

$$\|z(\tau_i,\cdot)\|_{W_2^1(0,L)} \le c_2,$$

where c is independent on i. This inequality in turn imply

$$\|z(\tau_i,\cdot)\|_{W_2^2(0,L)} \le c_3.$$

By (6.22) and this inequality and Rellix-Kondrashov theorem one can find a subsequence τ_{i_k} such that

$$z(\tau_{i_k},\cdot) \to u \ne z(\hat{\tau},\cdot) \text{ in } W_2^1(0,L) \text{ as } \tau_{i_k} \to \hat{\tau}.$$

Evidently u is a solution of (6.5) with boundary conditions $u(0) = (1 - \hat{\tau})\tilde{z}(0), u(L) = (1-\hat{\tau})\tilde{z}(L)$. Since above uniqueness of solution of problem (6.5),(6.5) was proved we have contradiction.

By Theorem 6.1 there exist finite number of points

$$0 < \tau_1 < \dots < \tau_s \dots < \tau_k = 1$$

such that the following problems of exact boundary controllability have a solution

$$R(y_i) = 0 \ (t,x) \in [0,1]\times[0,L], \quad y_i(0,x) = z(t_{i-1},x),\ y(1,x) = z(t_i,x). \tag{6.23}$$

No we finish construction of control $u_1(t), u_2(t)$. Set $T_0 = t_\varepsilon + k + 1$,

$$u_1(t) = y_i(t+\varepsilon+i,0) \text{ for } t \in [t_\epsilon + i, t_\epsilon + i + 1],$$

$$u_2(t) = y_i(t+t_\varepsilon+i,L) \text{ for } t \in [t_\epsilon + i, t_\epsilon + i + 1],$$

where $i \in \{1,\dots,k\}$ and $y_i(t,x)$ is a solution of problem (6.23). Since $z(t_k,\cdot) = z(1,\cdot) = \tilde{z}$ the theorem is proved. ■

We proof one estimate for solution $y(t,x)$ of problem (6.1), (6.2) which simply implies the uncontrollability of this problem.

LEMMA 6.2. *Let $u(t,x) \in L_2([0,T] \times [0,L])$ satisfy (6.3) and $y(t,x)$ be the solution of the problem (6.1), (6.2). Denote $y_+(t,x) = max(y(t,x),0)$. Then for arbitrary $N > 5$ the estimate*

$$\frac{d}{dt}\int_0^b (b-x)^N y_+^4(t,x)dx < \alpha(N)b^{N-5} \tag{6.24}$$

holds where b is the constant from (6.3) and $\alpha(N) > 0$ is a constant, depending on N only.

Proof. We multiply both sides of (6.1) by $(b-x)^N y_+^3(t,x)$ and integrate them with respect to x from 0 to b. Integrating by parts in the second term of the left hand side of the obtained identity we shall have

$$\int_0^b (b-x)^N(\partial_t y)y_+^3(t,x)dx + \int_0^b (b-x)^N 3y_+^2(\partial_x y_+)(\partial_x y)dx \\ -\int_0^b N(b-x)^{N-1}y_+^3(\partial_x y)dx + \int_0^b 2(b-x)^N y_+^4(\partial_x y)dx = 0. \tag{6.25}$$

It follows from the theorem on the smoothness of a solution of the Burgers equation that $y(t,x) \in C^\infty((0,T)\times(0,L))$. Denote $y_- = min(y,0)$. Then

$$y_+^3\frac{\partial y}{\partial x} = y_+^3\left(\frac{\partial y_+}{\partial x} + \frac{\partial y_-}{\partial x}\right) = y_+^3\frac{\partial y_+}{\partial x} = \frac{1}{4}\frac{\partial y_+^4}{\partial x}.$$

The following identities are proved in an analogous way

$$y_+^2\frac{\partial y_+}{\partial x}\frac{\partial y_+}{\partial x} = y_+^2\left(\frac{\partial y_+}{\partial x}\right)^2, \quad y_+^k = \frac{1}{k+1}\frac{\partial y_+^{k+1}}{\partial x}.$$

Using these equalities and integrating by parts in last two terms of equation (6.25), we obtain

$$\int_0^b (b-x)^N\frac{1}{4}\partial_t y_+^4(t,x)dx + \int_0^b (b-x)^N 3y_+^2(\partial_x y_+)^2 dx \\ -\int_0^b \frac{N}{4}(N-1)(b-x)^{N-2}y_+^4 dx + \int_0^b \frac{2N}{5}(b-x)^{N-1}y_+^5 dx = 0. \tag{6.26}$$

By the Hölder inequality

$$\int_0^b (b-x)^{N-2} y_+^4(t,x)dx \leq \left(\int_0^b (b-x)^{N-6}dx\right)^{1/5} \left(\int_0^b (b-x)^{N-1} y_+^5(t,x)dx\right)^{4/5} = \frac{b^{(N-5)/5}}{(N-5)^{1/5}} \left(\int_0^b (b-x)^{N-1} y_+^5 dx\right)^{4/5}. \quad (6.27)$$

Using the Young inequality, we shall have

$$\frac{N}{5}\int_0^b (b-x)^{N-1} y_+^5(t,x)dx - \frac{N(N-1)}{4(N-5)^{1/5}} b^{(N-5)/5} \left(\int_0^b (b-x)^{N-1} y_+^5 dx\right)^{4/5} \geq - \alpha(N) b^{N-5}, \quad (6.28)$$

where $\alpha(N)$ is a positive constant, depending on $N > 5$ only. Substituting (6.27)-(6.28) into (6.26) we obtain (6.24).■

We have

THEOREM 6.3. *Let $T > 0$ be an arbitrary finite number. Then problem (6.1)-(6.2) is not $L_2(0,L)$ - approximately controllable with respect to set of controls $u \in L_2((0,T)\times(0,L))$ satisfying (6.3).*

Proof. Let $\hat{y}(x) \in L_2(0,a)$, $\hat{y}(x) \geq 0$, y be a solution of problem (6.1)-(6.2). Then

$$\left(\int_0^b |\hat{y}(x) - y(T,x)|^2 dx\right)^{1/2} \geq \left(\int_0^{b/2} |\hat{y}(x) - y_+(T,x)|^2 dx\right)^{1/2} \geq \|\hat{y}\|_{L_2(0,b/2)} - \|y_+(T,\cdot)\|_{L_2(0,b/2)}. \quad (6.29)$$

By the Cauchy-Bunyakovskii inequality, we have:

$$\|y_+(T,\cdot)\|_{L_2(0,b/2)} \leq \left(\int_0^{\frac{b}{2}} (b-x)^{-N} dx\right)^{1/2} \left(\int_0^b (b-x)^N |y_+|^4 dx\right)^{1/2} \leq \left(\frac{b^{1-N}(2^{N-1}-1)}{N-1}\right)^{1/2} \left(\int_0^b (b-x)^N |y_+|^4 dx\right)^{1/2}. \quad (6.30)$$

In virtue of (6.24) for any $T > 0$ the inequality

$$\int_0^b (b-x)^N |y_+|^4 dx \leq T\alpha(N) b^{N-5}$$

holds. Let $T > 0$ be fixed and $\hat{y}(x) \in L_2(0,L)$ satisfies condition

$$\|\hat{y}\|_{L_2(0,b/2)} > \left(\frac{b^{1-N}(2^{N-1}-1)}{N-1} T\alpha(N) N^{N-5} \right)^{1/2} + 1. \tag{6.31}$$

Then it follows from (6.30) - (6.31) that for any control $u \in L_2((0,T)\times(0,L))$ satisfying (6.3), the solution y of problem (6.1)-(6.2) satisfies inequality

$$\|\hat{y} - y(T,\cdot)\|_{L_2(0,L)} > 1.$$

The inequality implies the approximate uncontrollability of problem (6.1)-(6.2). ■

Now we consider the Burgers equation with boundary control u:

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} + 2y\frac{\partial y}{\partial x} = 0 \quad (t,x) \in \ [0,T] \times [0,L], \tag{6.32}$$

$$y(t,0) = 0, \quad y(t,b) = u(t), \quad y(0,x) = 0, \quad u \in L_2(0,T). \tag{6.33}$$

We have

THEOREM 6.4. *Problem (6.32)-(6.33) is not $L_2(0,L)$-approximately controllable with respect to the control space $L_2(0,T)$ for an arbitrary $T > 0$.*

Proof. Estimate (6.24) holds for solution y of problem (6.32)-(6.33) and its proof does not differ from the proof of Lemma 6.2. We obtain the assertion of the theorem by means of this estimate after repeating the proof of Theorem 6.3 word by word.■

CHAPTER II

EXACT CONTROLLABILITY OF BOUSSINESQ SYSTEM

Introduction

We study the local controllability problem for the Boussinesq equation that describe the incompressible fluid flow coupled to thermal dynamics. The control function is the Dirichlet boundary condition of the velocity and temperature vector field of the fluid flow. More precisely, the investigated problem is as follows: Suppose that

$$\partial_t y(t,x) + A(y) = f(t,x), \qquad t \in (0,T), \qquad x \in \Omega \tag{1}$$

is a symbolic writing of the Boussinesq equations defined in a bounded domain $\Omega \subset \mathbf{R}^n$ n=2,3, where $y(t,x)$ is a velocity and temperature vector field and $f(t,x)$ is an external forces vector field, $t \in (0,T)$ is a time. We assume that a solution $\hat{y}(t,x)$ of (1)

$$\partial_t \hat{y}(t,x) + A(\hat{y}) = f(t,x)$$

as well as an initial condition $y_0(x)$ are given and they satisfy the proximity condition

$$\|\hat{y}(0,\cdot) - y_0(\cdot)\| \leq \epsilon, \tag{2}$$

where $\|\cdot\|$ is the norm of corresponding initial conditions space and $\epsilon > 0$ is sufficiently small magnitude. One has to find such control u defined on the lateral surface $\Sigma = (0,T) \times \partial\Omega$ of the cylinder $(0,T) \times \Omega$:

$$y|_\Sigma = u \tag{3}$$

that the solution $y(t,x)$ of (1), (3) supplied by the initial condition

$$y|_{t=0} = y_0 \tag{4}$$

coincides with the given solution $\hat{y}(t,x)$ at instant $t=T$

$$y|_{t=T} = \hat{y}|_{t=T}. \tag{5}$$

One useful application of local exact controllability problem is as follows. Let $f(t,x) \equiv f(x)$ be independent on t and $\hat{y}(x)$ be a steady-state solution of (1) with zero boundary condition which, by definition, is an unstable point in the phase space of the dynamical system generated by equation (1) supplied by zero boundary conditions. Then the problem (1), (3), (5) solvability implies that one can transfer an arbitrary point y_0 belonging to a small neighborhood of $\hat{y}$ to $\hat{y}$ a solution of (1) by means of boundary control.

We are intrested in the Boussinesq equations because the investigation of a fluid flow stability in the free convection problem is one of the basic area in the theory of hydrodynamical stability (See D.Joseph [37]). Besides as we understand, the local exact controllability problem connected with area outlined by J.L. Lions [49], [50] which contains in particular, certain problems about climate. It, in particular, explains our interest to the Boussinesq equations. The first step of the controllability property proof is the reduction of nonlinear problem (1),(2)-(5) to the solvability of the analogous problem for the linearization of (1). We do it with help of one variant of implicit function theorem. To establish the solvability of controllability problem we prove the density of data set for which the linear controllabilty problem is solvable (§3) and closure of this set (§5). The main difficulties of proof connected with the pressure term in the Boussinesq equations. To overcome this difficulty we introduce some nonstandard functional spaces for investigation of our problem and construct in these spaces a decomposition of a vector field on solenoidal and potential component (§4). This is based on the Carleman estimate for Laplace operator (L. Hörmander [27], [28]) and for heat equation (chapter I §6).

1. Statement of problem and formulation of the main result.

In a bounded domain $\Omega \subset \mathbf{R}^n$ (n=2 or 3) with C^∞-boundary $\partial\Omega$ we consider the Boussinesq system

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \theta(t,x)e_0 + \nabla p(t,x) = f(t,x), \tag{1.1}$$

$$\mathbf{div}\, v \equiv \sum_{j=1}^{n} \partial_{x_j} v_j = 0, \tag{1.2}$$

$$\partial_t\theta(t,x) - \Delta\theta + (v,\nabla\theta) + (v,e_0) = h(t,x), \tag{1.3}$$

$$v(t,x)|_{t=0} = v_0(x), \qquad \theta(t,x)|_{t=0} = \theta_0(x), \tag{1.4}$$

$$v|_\Sigma = u_v, \qquad \theta|_\Sigma = u_\theta. \tag{1.5}$$

Here $(t,x) \in Q = (0,T) \times \Omega$, $v(t,x) = (v_1(t,x),\ldots,v_n(t,x))$ is a fluid velocity at point x at instant t, $\theta(t,x)$ is a fluid temperature, p is a pressure gradient, $f(t,x)$ is the density of external forces, $h(t,x)$ is the density of external heat sources, $e_0 \in \mathbf{R}^n$ is the vector of the gravity force direction, u_v, u_θ are Dirichlet boundary conditions (in our case they are control functions), v_0, θ_0 are initial conditions. Besides, $\Sigma = (0,T) \times \partial\Omega$, $\partial_t = \partial/\partial t$, $\partial_{x_j} = \partial/\partial_{x_j}$, Δ is the Laplace operator, $(v,\nabla)v = \sum v_j\partial_{x_j}v$, $(v,\nabla\theta) = \sum_{j=1}^n v_j\partial_{x_j}\theta$. We investigate the local exact controllability problem for Boussinesq equations which is as follows. Let $\hat v(t,x)$, $\nabla\hat p(t,x)$, $\hat\theta(t,x)$ be sufficiently smooth* solution of Boussinesq equations (1.1)-(1.3):

$$\partial_t\hat v(t,x) - \Delta\hat v + (\hat v,\nabla)\hat v + \hat\theta(t,x)e_0 + \nabla\hat p(t,x) = f(t,x),$$

$$\mathbf{div}\,\hat v = 0,$$

$$\partial_t\hat\theta(t,x) - \Delta\hat\theta(t,x) + (\hat v,\nabla\hat\theta) + (\hat v,e_0) = h(t,x),$$

and initial conditions $v_0(x)$, $\theta_0(x)$ are sufficiently closed to $\hat v(0,x)$, $\hat\theta(0,x)$ with respect to an appropriate norm. One has to find such boundary control (u_v,u_θ) defined on the lateral surface Σ of the cylinder Q, that the components $(v(t,x),\theta(t,x))$ of solution of boundary value problem (1.1)-(1.5) coincide at instant $t=T$ with the given solutions components $(\hat v,\hat\theta)$:

$$v(T,x) \equiv \hat v(T,x), \qquad \theta(T,x) \equiv \hat\theta(T,x). \tag{1.6}$$

Let us introduce the functional spaces to set precisely the controllability problem and to formulate the main result. Besides the Sobolev spaces $W_p^k(\Omega)$, $1 \le p < \infty$ introduced in chapter I we define the functional space $V^k(\Omega)$ of solenoidal vector fields

$$V^k(\Omega) = \{v(x) \in (W_2^k(\Omega))^n : \mathbf{div}\,v(x) = 0\}. \tag{1.7}$$

We need the following spaces of functions defined in the cylinder Q:

$$W^{1,2(k)}(Q) = \{\theta(t,x) \in L_2(0,T;W_2^{k+2}(\Omega)) : \partial_t\theta \in L_2(0,T;W_2^k(\Omega))\}, \tag{1.8}$$

$$V^{1,2(k)}(Q) = \{v(t,x) \in (W^{1,2(k)}(Q))^n : \mathbf{div}\,v = 0\}. \tag{1.9}$$

The main result of this chapter is as follows

*The precise smoothness conditions are formulated below

THEOREM 1.1. *Suppose that* $f(t,x) \in (W^{1,2(2)}(Q))^n$, $h(t,x) \in W^{1,2(2)}(Q)$ *are given data and* $(\hat{v}(t,x), \hat{p}(t,x), \hat{\theta}(t,x)) \in V^{1,2(2)}(Q) \times L_2(0,T; W_2^3(\Omega)) \times W^{1,2(2)}(Q)$ *are a solution of equations (1.1)-(1.3), satisfying the property*

$$\int_{\Gamma_j} (\hat{v}(t,x), \nu(x))\, d\sigma = 0, \qquad j = 1, \dots, r, \qquad t \in [0,T) \tag{1.10}$$

where Γ_j *are components of* $\partial\Omega$*:* $\partial\Omega = \cup_{j=0}^{r} \Gamma_j$ *,*$\Gamma_j \cap \Gamma_k = \{\emptyset\}$*, if* $j \neq k$*,* $\nu(x)$ *is the vector field of outside normals to* $\partial\Omega$ *. Suppose that* $(v_0(x), \theta_0(x)) \in V^1(\Omega) \times W_2^1(\Omega)$ *is a given initial datum satisfying conditions*

$$\int_{\Gamma_j} (v_0(x), \nu(x))\, d\sigma = 0, \qquad j = 1, \dots, r \tag{1.11}$$

which is closed to $(\hat{v}(0,x), \hat{\theta}(0,x))$*:*

$$\|v_0 - \hat{v}(0,\cdot)\|^2_{V^1(\Omega)} + \|\theta_0 - \hat{\theta}(0,\cdot)\|^2_{W_2^1(\Omega)} < \epsilon \tag{1.12}$$

where $0 < \epsilon \leq \epsilon_0$ *and* ϵ_0 *is sufficiently small magnitude depending on* $(\hat{v}, \hat{\theta})$*. Then there exists such boundary control* $(u_v, u_\theta) \in (L_2(\Sigma))^n \times L_2(\Sigma)$ *that there exists the solution* $(v, p, \theta) \in V^{1,2(0)}(Q) \times L_2(0,T; W_2^1(\Omega)) \times W^{1,2(0)}(Q)$ *of problem (1.1)-(1.5) and this solution satisfies condition (1.6). Moreover, there exist constants* $\kappa > 0$*,* $c_1 > 0$ *that*

$$\|v(t,\cdot) - \hat{v}(t,\cdot)\|^2_{V^1(\Omega)} + \|\theta(t,\cdot) - \hat{\theta}(t,\cdot)\|^2_{W_2^1(\Omega)} \leq c_1 e^{-\frac{\kappa}{(T-t)}} \qquad \text{as } t \to T. \tag{1.13}$$

The remaining part of this chapter is devoted to prove this theorem.

Remark 1.1. The condition $(\hat{v}, \hat{p}, \hat{\theta}) \in V^{1,2(2)}(Q) \times L_2(0,T; W_2^3(\Omega)) \times W^{1,2(2)}(Q)$ of Theorem 1.1 can be weakened. Namely, the assertion of Theorem 1.1 remains true when instead of above assumption we suppose that

$$\hat{v}(t,x) \in V^{1,2(1/2)}(Q) \cap (L_\infty(Q))^n, \qquad \hat{\theta} \in W^{1,2(1/2)}(Q). \tag{1.14}$$

In the case of assumption (1.14) we would have to add the Theorem 1.1 proof in several points by some complicated applications of the Sobolev imbedding theorem and also by one technical method mentioned below in Remark 5.1.

2. Reduction to a linear controllability problem.

2.1. We begin with certain simply but useful remarks about the investigated problem. First of all, note that we will not construct specially the boundary control (v_u, θ_u) but study the solvability of problem (1.1)-(1.4),(1.6) without boundary conditions (1.5). We will find a boundary control $(v_u,\ \theta_u)$ at the very end of proof with help of restriction of constructed solution (v, θ) at the boundary Σ. Besides, we show that it is possible to reduce the controllability problem mentioned above to the case of simply connected bounded domain Ω. Indeed , let Γ_0 be the external component of the boundary $\partial\Omega$. By G we denote the bounded domain with the boundary Γ_0. Evidently

$$G = \Omega \cup \cup_{j=1}^{r}(\Omega_j \cup \Gamma_j),$$

where Ω_j is the bounded domain with the boundary Γ_j. To reduce the proof of Theorem 1.1 to the case of simply connected domain G we have to extend continuously functions $(\hat{u}, \hat{p}, \hat{\theta}) \in V^{1,2(2)}(Q) \times L_2(0,T; W_2^3(\Omega)) \times W^{1,2(2)}(Q)$ up to $(\tilde{u}, \tilde{p}, \tilde{\theta}) \in V^{1,2(2)}(\hat{Q}) \times L_2(0,T; W_2^3(G)) \times W^{1,2(2)}(\hat{Q})$ where $\hat{Q} = (0,T) \times G$ and initial conditions $(v_0, \theta_0) \in V^1(\Omega) \times W_2^1(\Omega)$ up to $(\tilde{v}, \tilde{\theta}) \in V^1(G) \times W_2^1(G)$. After this extension we substitute $(\tilde{v}, \tilde{p}, \tilde{\theta})$ into (1.1), (1.3) and calculate the right side $(\tilde{f}, \tilde{h})$ of these equations. Naturally, $(\tilde{f}, \tilde{h})$ will be an extension of (f, h). When we will prove Theorem 1.1 in the case of simply connected domain G, we will restrict the solution of controllability problem at $\partial\Omega = \cup_{j=0}^{r}\Gamma_j$. Then the constructed function (v_u, θ_u) will be the control which solves the controllability problem in the case of multiconnected domain Ω.

Proposition 2.1. *For an arbitrary natural number l there exists the extension operator $L : L\theta(x)|_\Omega \equiv \theta(x)$ such that the maps*

$$L : W_2^k(\Omega) \to W_2^k(G)$$

are bounded for $k = 0, .., l$.

Although the proof of this proposition is well-known we remind briefly the extension construction, taking into account our future goals. After application of a partition unity and restrictifying the boundary we obtain the problem of extension of a function $u(x)$ defined in $\mathbf{R}_+^n = \{x = (x_1, \ldots, x_n), x_n >$

$0\}$ up to a function on $\mathbf{R}^n$. The extension operator L is now defined by the formula

$$Lu(x',x_n) = \begin{cases} u(x',x_n), & \text{when } x_n \geq 0, \\ \sum_{j=1}^{l} \lambda_j u(x',-x_n/k), & \text{when } x_n < 0, \end{cases}$$

where $\lambda_1, \dots, \lambda_n$ are the solution of system

$$\sum_{k=1}^{l} \left(-\frac{1}{k}\right)^j \lambda_k = 1 \qquad (j = 0, 1, \dots, l-1).$$

This construction allows to prove estimates declared in Proposition 2.2 (see [4], [59]) This construction and Proposition 2.1 imply

PROPOSITION 2.2. *For an arbitrary natural k there exists a bounded extension operator*

$$L : W^{1,2(k)}(Q) \to W^{1,2(k)}(\hat{Q}), \qquad \hat{Q} = (0,T) \times G,$$

$$L : L_2(0,T; W_2^k(\Omega)) \to L_2(0,T; W_2^k(G)).$$

Let us consider functional spaces of solenoidal vector fields. We define the space

$$\hat{V}^k(\Omega) = \{v_0 \in V^k(\Omega) : v_0 \text{ satisfies } (1.11)\}.$$

Remark 2.1. In the case of $dim\,\Omega = 2$ we define the operator **rot** by formula

$$\mathbf{rot}\, u = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

We have

PROPOSITION 2.3. *i) For an arbitrary natural number k there exists the extension operator $\hat{L}$ such that the maps*

$$\hat{L} : \hat{V}^k(\Omega) \to V^k(G)$$

are bounded for $k = 0, 1, \dots, l$.

ii) For an arbitrary natural number k there exist bounded extension operators

$$\hat{L} : V^{1,2(k)}(Q) \to V^{1,2(k)}(\hat{Q}), \qquad \hat{Q} = (0,T) \times G$$

$$\hat{L} : L_2(0,T; V^k(\Omega)) \to L_2(0,T; V^k(G)).$$

Proof. Denote $H_\sigma = \{v \in V^0(\Omega) : (v,\nu)|_{\partial\Omega} = 0\}$, where (v,ν) understands in $W^{-1/2}(\Omega)$ (See details in R. Temam [63]). For $u \in V^k(\Omega)$ we consider the boundary value problem

$$\mathbf{rot}\, v = u \qquad x \in \Omega,$$

$$\mathbf{div}\, v = 0 \qquad x \in \Omega,$$

$$(v,\nu)|_{\partial\Omega} = 0.$$

In R. Temam [63] it was shown that there exists a solution $v \in V^k(\Omega) \cap H_\sigma$ of this problem, which satisfies the estimate

$$\|v\|_{V^{k+1}(\Omega)} \leq c_1(\|u\|_{V^k(\Omega)} + \|v\|_{(L_2(\Omega))^n}).$$

Moreover if we will take v from orthogonal complement to $Ker\, \mathbf{rot}\, V^1(\Omega)$ in the space H_σ then (see R. Temam [63])

$$\|v\|_{(L_2(\Omega))^n} \leq c_2\|u\|_{(L_2(\Omega))^n}.$$

Hence, for such v we have the estimate

$$\|v\|_{V^{k+1}(\Omega)} \leq c_3\|u\|_{V^k(\Omega)}.$$

Now, for $u \in V^k(\Omega)$ we define the restriction operator $\hat{L}$ by formula

$$\hat{L}u = \mathbf{rot}\, Lv,$$

where L is a extension operator from Proposition 2.1 and v is the solenoidal vector field constructed above by u. Evidently, estimate for v written above and Propositions 2.1, 2.2 imply assertions i) and ii) of Proposition 2.3.

■

2.2. Now we reduce the proof of Theorem 1.1 to the case of a linear controllability problem. Applying the well-known formula of vectorial analysis

$$(v,\nabla)v = -v \times \mathbf{rot}\, v + \nabla(|v|^2/2),$$

where $\times$ is the operation of vectorial multiplication we can rewrite equation (1.1) in the form

$$\partial_t v(t,x) - \Delta v - v \times \mathbf{rot}\, v + \theta(t,x)e_0 + \nabla p'(t,x) = f(t,x) \tag{2.1}$$

if we denote $\nabla p' = \nabla(p + |v|^2/2)$. We write the solution (v, θ) which we are looking in the form

$$v(t,x) = \hat{v}(t,x) + w(t,x), \qquad \theta(t,x) = \hat{\theta}(t,x) + \tau(t,x). \tag{2.2}$$

The substitution of (2.2) into equation (2.1), (1.2), (1.3) and subtraction from them of the same equations for $(\hat{v}, \hat{p}, \hat{\theta})$ yields the equations

$$\mathcal{N}(w,q,\tau) = \partial_t w(t,x) - \Delta w - \hat{v} \times \mathbf{rot}\, w - w \times \mathbf{rot}\, \hat{v} - w \times \mathbf{rot}\, w + \nabla q + \tau e_0 = 0, \tag{2.3}$$

$$\mathbf{div}\, w = 0, \tag{2.4}$$

$$\mathcal{H}(w,\tau) = \partial_t \tau(t,x) - \Delta\tau + (\hat{v}, \nabla\tau) + (w, \nabla\hat{\theta}) + (w, \nabla\tau) + (w, e_0) = 0, \tag{2.5}$$

where $\nabla q = \nabla p' - \nabla \hat{p}$. The functions w, τ satisfy the initial conditions:

$$w(0,x) = w_0(x), \qquad \tau(0,x) = \tau_0(x), \tag{2.6}$$

where $w_0(x) = v_0(x) - \hat{v}(0,x)$, $\tau_0(x) = \theta_0(x) - \hat{\theta}(0,x)$. Evidently we have reduced our problem to construction of solution $(w(t,x), \tau(t,x))$ of problem (2.3)-(2.6) which satisfies the equalities

$$w(T,x) = 0, \qquad \theta(T,x) = 0. \tag{2.7}$$

Remark 2.2. In the two dimensional case we will rewrite the nonlinear term $(v, \nabla)v$ as follows

$$(v, \nabla)v = (-v_2\, \mathbf{rot}\, v, v_1\, \mathbf{rot}\, v) + \nabla(|v|^2/2).$$

Despite of the system (2.3)-(2.7) is different the proof of Theorem 1.1 is same.

We will solve problem (2.3)-(2.7) with help of the variant of the implicit function theorem formulated in the section 4 of chapter I.

In our case X will be a space of triplets $x = (w, q, \tau)$

$$\mathcal{A}(x) = (\mathcal{N}(w,q,\tau), \mathcal{H}(w,\tau), w|_{t=0}, \tau|_{t=0}) \tag{2.8}$$

and the space Z be defined by collection of components in (2.8). We mark that we will guarantee of (2.7) by introduction of special weights in the norm of X. We take as x_0 and z_0 the zero elements: $x_0 = (0,0,0)$, $z_0 = (0,0,0)$. Then equation (1.4.3) for operator (2.8), (2.3), (2.5) is fulfilled. Thus, the main condition which we have to verify applying the right inverse operator theorem, is the assertion on solvability of equation $\mathcal{A}'(0)x = z$ for any $z \in Z$. This equation in our case has the following form:

$$\mathcal{N}'(0)(v,p,\theta) = \partial_t v(t,x) - \Delta v - \hat{v} \times \mathbf{rot}\, v - v \times \mathbf{rot}\, \hat{v} + \theta e_0 + \nabla p = f, \tag{2.9}$$

$$\mathbf{div}\, v = 0, \tag{2.10}$$

$$\mathcal{H}'(0)(v,\theta) = \partial_t \theta(t,x) - \Delta\theta + (\hat{v}, \nabla\theta) + (v, \hat{\theta}) + (v, e_0) = h, \tag{2.11}$$

$$v|_{t=0} = v_0, \qquad \theta|_{t=0} = \theta_0, \tag{2.12}$$

$$v|_{t=T} = 0, \qquad \theta|_{t=T} = 0. \tag{2.13}$$

2.3. We define now the functional spaces X, Z corresponding to the problem (2.3)-(2.7). Let

$$\eta(t,x) \equiv \eta^s(t,x) = s(e^{2\hat{x}_1} - e^{x_1})/((T-t)l(t)) \tag{2.14}$$

be the weight function where $s > 0$ is a parameter which will be chosen below, $\hat{x}_1 = \max_{x=(x_1\dots x_n)\in\Omega} |x_1|$ and $l(t)$ is a fixed function which satisfies the following conditions

$$l(t) \in C^1[0,T], \quad l(t) = t \quad \forall\, t \in \big(\frac{3T}{4}, T\big], \quad l(t) > 0 \quad \forall\, t \in [0,T].$$

Denote

$$L_2(Q,\eta) \equiv L_2(Q,\eta^s) = \{y(t,x) : \|y\|^2_{L_2(Q,\eta)} = \int_Q e^{2\eta^s} |y|^2\, dx\, dt < \infty\}. \tag{2.15}$$

Below we will use also the space $L_2(Q,\beta)$ with weights of different form. We define the space $\Theta(Q,\eta)$ of components $\theta(t,x)$ in (2.9) - (2.13):

$$\Theta(Q,\eta) \equiv \Theta(Q,\eta^s) = \{\theta(t,x), (t,x) \in Q :\ \|\theta\|^2_{\Theta(Q,\eta^s)} \equiv \|\partial_t\theta - \Delta\theta\|^2_{L_2(Q,\eta^s)}$$
$$+ \|(T-t)^{-3/2}\theta\|^2_{L_2(Q,\eta^s)} + \|(T-t)^{-1/2}|\nabla\theta|\|^2_{L_2(Q,\eta^s)}$$
$$+ \|(T-t)^{1/2}\partial_t\theta\|^2_{L_2(Q,\eta^s)} + \sum_{i,j=1}^n \|(T-t)^{1/2}\partial^2_{x_i x_j}\theta\|^2_{L_2(Q,\eta^s)} < \infty\}. \tag{2.16}$$

The space of right components f in (2.9)-(2.13) is as follows

$$F(Q,\eta) \equiv F(Q,\eta^s) = \{f \in (L_2(Q))^n :\ \exists\ f_1 \in (L_2(Q,\eta))^n,$$
$$\exists\ f_2 \in L_2(0,T; W_2^1(\Omega)) \text{ such that } f = f_1 + \nabla f_2\}. \tag{2.17}$$

The norm of the space $F(Q,\eta)$ is defined by the relation

$$\|f\|_{F(Q,\eta^s)} = \inf_{\substack{f_1,\nabla f_2 \\ f=f_1+\nabla f_2}} (\|f_1\|^2_{(L_2(Q,\eta^s))^n} + \|\nabla f_2\|^2_{(L_2(Q))^n})^{1/2}. \tag{2.18}$$

Remark 2.3. Note that $F(Q,\eta^s)$ is a Hilbert space. Really, since the functional $J(f_1,f_2) = (\|f_1\|^2_{(L_2(Q,\eta^s))^n} + \|\nabla f_2\|^2_{(L_2(Q))^n})^{1/2}$ is the strictly convex for any $f \in F(Q,\eta^s)$ there exists only one pair $(\hat{f}_1, \hat{\nabla} f_2) \in (L_2(Q,\eta^s))^n \times (L_2(Q))^n$ such that $\|f\|_{F(Q,\eta^s)} = J(\hat{f}_1, \hat{f}_2)$.

We define operator B by formula $Bf = (\hat{f}_1, \nabla \hat{f}_2)$. Obviously $B \in C(F(Q,\eta^s), (L_2(Q,\eta^s))^n \times (L_2(Q))^n)$. One can easily check that B is the linear operator. Thus we can introduce scalar product in $F(Q,\eta^s)$ by formula

$$(f,\tilde{f})_{F(Q,\eta^s)} = (Bf, B\tilde{f})_{(L_2(Q,\eta^s))^n \times (L_2(Q))^n}.$$

The space $V(Q,\eta)$ of components v in (2.9)-(2.13) we define with the help of inequality

$$\begin{aligned} V(Q,\eta) \equiv V(Q,\eta^s) = \{v(t,x) : \ \mathbf{div}\, v = 0, \ \ \|v\|^2_{V(Q,\eta^s)} \equiv \\ \|\partial_t v - \Delta v\|^2_{F(Q,\eta^s)} + \|(T-t)^{-1} v\|^2_{(L_2(Q,\eta^s))^n} + \|\nabla v\|^2_{(L_2(Q,\eta^s))^n} + \\ \|(T-t)\partial_t v\|^2_{(L_2(Q,\eta^s))^n} + \sum_{i,j=1}^{n} \|(T-t)\partial^2_{x_i x_j} v\|^2_{(L_2(Q,\eta^s))^n} < \infty\}. \end{aligned} \tag{2.19}$$

Now we can define the spaces X and Z in the case of problems (2.3)-(2.7) or (2.9)-(2.13):

$$X = X^s(Q) = V(Q,\eta^s) \times L_2(0,T; W_1^2(\Omega)) \times \Theta(Q,\eta), \tag{2.20}$$

$$Z = Z^s(Q) = F(Q,\eta^s) \times L_2(Q,\eta^s) \times V^1(\Omega) \times W_2^1(\Omega). \tag{2.21}$$

Since the weight $\eta^s(t,x)$ increases exponentially as $t \to T$, the functions $v \in V(Q,\eta^s)$, $\theta \in \Theta(Q,\eta^s)$ decrease exponentially as $t \to T$ and therefore equalities (2.13) are true.

2.4. Let us show that for an arbitrary parameter $s > 0$ the operator (I.4.2) and its derivative

$$\mathcal{A}'(0) : X^s(Q) \to Z^s(Q), \tag{2.22}$$

are continuous, where $\mathcal{A}(x)$ is defined in (2.8), (2.3), (2.5).

LEMMA 2.1. *Suppose that* $\hat{v} \in V^{1,2(2)}(Q)$, $\hat{\theta} \in W^{1,2(2)}(Q)$

$$\mathcal{A}'(0)(v,p,\theta) = (\mathcal{N}'(0)(v,p,\theta), \mathcal{H}'(0)(v,\theta), v|_{t=0}, \theta|_{t=0}) \tag{2.23}$$

where $\mathcal{N}'(0)$, $\mathcal{H}'(0)$ *are defined by (2.9), (2.11). Then for* $s > 0$ *the operator (2.22) is continuous.*

Proof. Evidently the embeddings $V(Q,\eta) \subset V^{1,2(0)}(Q)$, $\Theta(Q,\eta) \subset W^{1,2(0)}(Q)$ are continuous. Since the restriction operator $\gamma_0 y = y|_{t=0}$ acts continuously from $W^{1,2(0)}(Q)$ to $W_2^1(\Omega)$ and from $V^{1,2(0)}(Q)$ to $V^1(\Omega)$ (see [52]), the inequalities

$$\|\gamma_0 v\|_{V^1(\Omega)} \leq c_4 \|v\|_{V(Q,\eta)}, \qquad \|\gamma_0 \theta\|_{W_2^1(\Omega)} \leq c_5 \|\theta\|_{\Theta(Q,\eta)} \tag{2.24}$$

holds. Let us prove the continuity of the operator

$$\mathcal{H}'(0) : V(Q,\eta) \times \Theta(Q,\eta) \to L_2(Q,\eta) \tag{2.25}$$

defined in (2.11). Since the embeddings

$$V^{1,2(2)}(Q) \subset (C(0,T;C^1(\bar{\Omega}))^n, \qquad W^{1,2(2)}(Q) \subset C(0,T;C^1(\bar{\Omega})) \tag{2.26}$$

are continuous, for $n \leq 3$ we obtain taking into account (2.11), (2.15), (2.16)

$$\|\mathcal{H}'(0)(v,\theta)\|_{L_2(Q,\eta)} \leq \|\partial_t \theta - \Delta\theta\|_{L_2(Q,\eta)} + \|\hat{v}\|_{(C(\bar{Q}))^n} \|\nabla\theta\|_{(L_2(Q,\eta))^n}$$
$$+ \|\nabla\hat{\theta}\|_{(C(\bar{Q}))^n} \|v\|_{(L_2(Q,\eta))^n} \leq (1 + \|\hat{v}\|_{V^{1,2(2)}(Q)}) \|\theta\|_{\Theta(Q,\eta)} + \|\nabla\hat{\theta}\|_{(C(\bar{Q}))^n} \|v\|_{V(Q,\eta)}. \tag{2.27}$$

The relations (2.9),(2.15)-(2.19) yields

$$\begin{aligned}
&\|\mathcal{N}'(0)(v,p,\theta)\|_{F(Q,\eta)} \leq \|\partial_t v - \Delta v - \hat{v} \times \mathbf{rot}\, v - v \times \mathbf{rot}\, \hat{v} + \theta e_0\|_{(L_2(Q,\eta))^n} \\
&+ \|\nabla p\|_{(L_2(Q))^n} \leq \|\partial_t v + \Delta v\|_{(L_2(Q,\eta))^n} + \|\hat{v}\|_{C(0,T;(C^1(\bar{\Omega}))^n)} (\|v\|_{(L_2(Q,\eta))^n} \\
&\qquad + \||\nabla v|\|_{(L_2(Q,\eta))^n}) + c_6 \|\theta\|_{L_2(Q,\eta)} + \|\nabla p\|_{(L_2(Q))^n} \\
&\leq (1 + \|\hat{v}\|_{V^{1,2(2)}(Q)}) \|v\|_{V(Q,\eta)} + c_7 \|\theta\|_{\Theta(Q,\eta)} + \|p\|_{L_2(0,T;W_2^1(\Omega))}.
\end{aligned} \tag{2.28}$$

The inequalities (2.24), (2.27), (2.28) imply the desired assertion. ■

LEMMA 2.2. *Suppose that* $\hat{v} \in V^{1,2(2)}(Q)$, $\hat{\theta} \in W^{1,2(2)}(Q)$, *and* $\mathcal{A}$ *is operator (2.8). Then for arbitrary* $s > 0$ *the operator*

$$\mathcal{A} : X^s(Q) \to Z^s(Q)$$

is continuous.

Proof. To prove this Lemma we need only to complete the proof of Lemma 2.1 by the estimate of the terms $w \times \mathbf{rot}\, w$ and $(w, \nabla\tau)$. The Cauchy-Bouniakovskii inequality and the Sobolev embedding theorem yield

$$\begin{aligned}\|e^{\eta}(w, \nabla\tau)\|_{L_2(Q)} &\le \int_0^T \|e^{\frac{\eta}{2}} w(t, \cdot)\|_{(L_4(\Omega))^n} \|e^{\frac{\eta}{2}} |\nabla\tau|\|_{L_4(\Omega)}\, dt \\ &\le c_8 \int_0^T \|e^{\frac{\eta}{2}} w(t, \cdot)\|_{V^1(\Omega)} \|e^{\frac{\eta}{2}} \tau\|_{W_2^2(\Omega)}\, dt \\ &\le c_9 \|e^{\frac{\eta}{2}} w\|_{C(0,T;V^1(\Omega))} \|e^{\frac{\eta}{2}} \tau\|_{L_2(0,T;W_2^2(\Omega))} \\ &\qquad \le c_{10} \|e^{\frac{\eta}{2}} w\|_{V^{1,2(0)}(Q)} \|e^{\frac{\eta}{2}} \tau\|_{L_2(0,T;W_2^2(\Omega))}. \end{aligned} \tag{2.29}$$

By definition of the norms of spaces $V^{1,2(0)}(Q)$, $L_2(0, T; W_2^2(\Omega))$ in the right side of (2.28), taking into account (2.14) and evident inequality

$$(T - t)^{-k} \le c(k) e^{\eta/2}$$

we get the estimate

$$\begin{aligned}&\|e^{\frac{\eta}{2}} w\|_{V^{1,2(0)}(Q)} \|e^{\frac{\eta}{2}} \tau\|_{L_2(0,T;W_2^2(\Omega))} \le c_{11}(\|e^{\frac{\eta}{2}} (T - 2)^{-2} w\|_{(L_2(Q))^n} \\ &+ \|e^{\frac{\eta}{2}} (T - t)^{-1} |\nabla w|\|_{(L_2(Q))^n} + \sum_{i,j=1}^n \|e^{\frac{\eta}{2}} \partial^2_{x_i x_j} w\|_{(L_2(Q))^n} (\|e^{\frac{\eta}{2}} (T - t)^{-2} \tau\|_{L_2(Q)} \\ &+ \|e^{\frac{\eta}{2}} (T - t)^{-1} |\nabla\tau|\|_{L_2(Q)} + \sum_{i,j=1}^n \|e^{\frac{\eta}{2}} \partial^2_{x_i x_j} \tau\|_{L_2(Q)}) \le c_{12} \|w\|_{V(Q,\eta)} \|v\|_{\Theta(Q,\eta)}. \end{aligned} \tag{2.30}$$

One can estimate the term $(w, \nabla)w$ analogously. ■

Thus, to have the possibility to apply the theorem on right inverse operator we must prove that the operator $\mathcal{A}'(0) : X^s(Q) \to Z^s(Q)$ is epimorphism. To prove this assertion we will show that the image of this operator is dense in $Z^s(Q)$ and besides, it is a closed subset of $Z^s(Q)$ for s sufficiently large.

These assertions imply that the image of $\mathcal{A}'(0)$ coincides with the whole space $Z^s(Q)$.

3. The solvability of the linear controllability problem for dense set of data.

3.1. To prove the controllabilty problem for a dense set of data we need the Carleman estimate for elliptic and inverse parabolic equations.

We consider the Cauchy problem for the Laplace operator

$$\Delta z(x) = f(x), \qquad x \in \Omega, \qquad z|_{\partial\Omega} = \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{3.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with C^∞ boundary, $\partial/\partial\nu$ is the derivative along outside normal ν to $\partial\Omega$.

LEMMA 3.1. *Let $f(x) \in L_2(\Omega)$. There exists such $s_0 > 0$ that for any $s > s_0$ the solution $z(x) \in W_2^2(\Omega)$ of (3.1) satisfies the Carleman estimate:*

$$\int_\Omega \left(\frac{1}{s} \sum_{i,j=1}^n \left| \frac{\partial^2 z(x)}{\partial x_i \partial x_j} \right|^2 + s|\nabla z|^2 + s^3 z^2 \right) exp(se^{x_1})\, dx \\ \leq c_1 \int_\Omega f^2(x) exp(se^{x_1})\, dx, \tag{3.2}$$

where x_1 is the first component of $x = (x_1, ..., x_n) \in \Omega$ and $c_1 > 0$ does not depend on s.

For the proof of Lemma 3.1 refer to L. Hörmander [27], [28].

We introduce function

$$\gamma(t)$$

by formula

$$\gamma(t) = (T - t)t. \tag{3.3}$$

We define $\varphi(t, x)$, $\alpha(t, x)$ by relations:

$$\varphi(t, x) = \frac{e^{x_1}}{\gamma(t)}, \qquad \alpha(t, x) = (e^{x_1} - e^{2\hat{x}_1})/\gamma(t), \tag{3.4}$$

$$\text{where } \hat{x}_1 = \max_{x=(x_1,\dots,x_n)\in\Omega} |x_1|.$$

Corollary 3.1. *Let $f(x) \in L_2(\Omega)$ and s is just the same as in Lemma 3.1. Then for any $t \in (0,T)$ the following estimate is true*

$$\int_\Omega \left(\frac{\gamma(t)}{s} \sum_{i,j=1}^{n} \left| \frac{\partial^2 z(x)}{\partial x_i \partial x_j} \right|^2 + \frac{s}{\gamma(t)} |\nabla z|^2 + \frac{s^3}{\gamma(t)^3} |z|^2 \right) e^{s\varphi(t,x)}\, dx \leq c_2 \int_\Omega f^2(x) e^{s\varphi(t,x)}\, dx. \quad (3.5)$$

Proof. We substitute $s = (s_1\gamma(t))^{-1}$ into (3.2) and obtain (3.5) where instead of s_1 we write s. In virtue of Lemma 3.1 the estimate (3.5) is true when $s > s_0\gamma(t)$. Since $0 < \gamma(t) \leq 1$ for $t \in (0,T)$ this inequality is also true when $s > s_0$. ■

3.2. Firstly instead of problem (2.9)-(2.13) we consider an auxiliary problem. Let $\Omega_0 \subset \mathbf{R}^n$ be a bounded domain with C^∞ -boundary $\partial\Omega_0$ which contains the closure $\bar{\Omega}$ of Ω : $\bar{\Omega} \subset \Omega_0$ and satisfies the condition $\sup_{x\in\Omega_0} |x_1| < 2 \sup_{x\in\Omega} |x_1|$. Therefore function the η from (2.14) is positive and the function α from (3.4) is negative. We denote

$$Q_0 = (0,T) \times \Omega_0, \quad \Sigma_0 = (0,T) \times \partial\Omega_0, \quad \omega = \Omega_0 \setminus \bar{\Omega}.$$

In Q_0 we consider the linearized Boussinesq equation with the distributed control concentrated in $(0,T) \times \omega$:

$$\hat{\mathcal{N}}'(w,p,\tau,u) = \partial_t w(t,x) - \Delta w - \hat{v} \times \mathbf{rot} w + w \times \mathbf{rot} \hat{v} + \nabla p + \tau(t,x) e_0 + u'(t,x) = f(t,x), \quad (3.6)$$

$$\mathbf{div}\, w = 0, \quad (3.7)$$

$$\hat{\mathcal{H}}'(w,\tau,u) = \partial_t \tau(t,x) - \Delta\tau + (w, \nabla\hat{\theta}) + (\hat{v}, \nabla\tau) + (w, e_0) + u_{n+1}(t,x) = h(t,x), \quad (3.8)$$

$$w(0,x) = w_0(x), \qquad \tau(0,x) = \tau_0(x), \quad (3.9)$$

$$w(T,x) = 0, \qquad \tau(T,x) = 0, \quad (3.10)$$

where $u(t,x) = (u'(t,x), u_{n+1}(t,x)) = (u_1, .., u_n, u_{n+1})$ is the distributed control concentrated in $Q_\omega = (0,T) \times \omega$: $\operatorname{supp} u \subset Q_\omega$. The functional space for data (f, h, w_0, τ_0) of problem (3.6)-(3.10) are as follows:

$$(f, h, w_0, \tau_0) \in \Phi^s(Q_0) = (L_2(Q_0, \eta^s))^n \times L_2(Q_0, \eta^s) \times V^1(\Omega_0) \times W_2^1(\Omega_0), \quad (3.11)$$

where $s > 0$ is arbitrary fixed number. We define the functional space of problem (3.6)-(3.10) by formula

$$(w, \nabla p, \tau, u) \in U^s(Q_0) \equiv V(Q_0, \eta^s) \times L_2(Q_0, \eta^s) \times \Theta(Q_0, \eta^s) \\ \times (\hat{L}_2(Q_\omega, \eta^s))^{n+1}, \quad (3.12)$$

where $\hat{L}_2(Q_\omega, \eta^s)$ is the set of functions which belong to $L_2(Q_0, \eta^s)$ and equal zero on the set $Q_0 \setminus Q_\omega$; the constant s, in (3.12) is just the same as in (3.11). We suppose that the functions $\hat{v}, \hat{\theta}$ in (3.6), (3.8) satisfy the condition

$$\hat{v} \in V^{1,2(1/2)}(Q_0), \qquad \hat{\theta} \in W^{1,2(1/2)}(Q_0). \quad (3.13)$$

As in Lemma 2.1 one can prove easily that the operator

$$\hat{\mathcal{A}}' : U^s(Q_0) \to \Phi^s(Q_0) \quad (3.14)$$

is continuous, where $\Phi^s(Q)$, $U^s(Q)$ are defined in (3.11), (3.12) and

$$\hat{\mathcal{A}}'(w, \nabla p, \tau, u) = (\hat{\mathcal{N}}'(w, \nabla p, \tau, u), \hat{\mathcal{H}}'(w, \tau, u), \gamma_0 w, \gamma_0 \tau) \quad (3.15)$$

with $\hat{\mathcal{N}}', \hat{\mathcal{H}}'$ defined in (3.6), (3.8).

LEMMA 3.2. *The image of the operator (3.14), (3.15) is dense in the space $\Phi^s(Q_0)$.*

Proof. Suppose that the assertion of Lemma 3.2 is not true. Then there exists not zero collection $\phi \equiv (m(t,x), \zeta(t,x), z_0(x), \psi_0(x)) \in \Phi^s(Q_0)$, that

$$(\mathcal{A}'(w, \nabla p, \tau, u), \phi)_{\Phi^s(Q_0)} = 0, \qquad \forall\ (w, \nabla p, \tau, u) \in U^s(Q_0). \quad (3.16)$$

One can rewrite equality (3.16) in the form

$$\int_{Q_0} (\partial_t w(t,x) - \Delta w - \hat{v} \times \mathbf{rot}\, w - w \times \mathbf{rot}\, \hat{v} + \nabla p(t,x) + \tau(t,x) e_0 \\ + u'(t,x), m(t,x)) e^{2\eta^s(t,x)}\, dx\, dt + \int_{Q_0} (\partial_t \tau(t,x) - \Delta \tau + (w, \nabla \hat{\theta}) + (\hat{v}, \nabla \tau) \\ + (w, e_0) + u_{n+1}) \zeta(t,x) e^{2\eta^s}\, dx\, dt + (w(0,\cdot), z_0)_{V^1(\Omega_0)} + (\theta(0,\cdot), \psi_0)_{W_2^1(\Omega_0)} = 0. \quad (3.17)$$

We set in (3.17)

$$z(t,x) = m(t,x)e^{2\eta^s(t,x)}, \qquad \psi(t,x) = \zeta(t,x)e^{2\eta^s(t,x)}, \tag{3.18}$$

$\nabla p(t,x) \equiv 0$, $u(t,x) \equiv 0$, $w \in V(Q_0,\eta) \cap (C_0^\infty(Q_0))^n$, $\tau \in \Theta(Q_0,\eta) \cap C_0^\infty(Q_0)$. Then the integrating by parts in (3.17) yields the equations

$$\partial_t z + \Delta z = \mathbf{rot}\,(\hat{v} \times z) + z \times \mathbf{rot}\,\hat{v} + \psi(\nabla\hat{\theta} + e_0) + \nabla p \text{ in } Q_0, \tag{3.19}$$

$$\partial_t \psi + \Delta\psi = -\nabla(\psi\hat{v}) + (e_0, z) \qquad \text{in}\, Q_0. \tag{3.20}$$

If one set in (3.17) $u \in (\hat{L}_2(Q_\omega,\eta))^{n+1}$, $\nabla p = 0$, $w = 0$, $\tau = 0$, he will obtain the equalities

$$z(t,x) \equiv 0, \qquad \psi(t,x) = 0, \qquad (t,x) \in Q_\omega = Q_0 \setminus Q. \tag{3.21}$$

In particularly (3.21) means that z and ψ equal zero in a neighborhood of $\Sigma_0 = (0,T) \times \partial\Omega_0$. After setting in (3.17) $\nabla p \in (L_2(Q_0,\eta))^n$, $w = 0$, $\tau = 0$, $u = 0$ and taking into account (3.21) we get

$$\mathbf{div}\, z = 0 \text{ in}\, Q_0. \tag{3.22}$$

Equalities (3.19), (3.21) yields that

$$\nabla\tilde{p}(t,x) \equiv 0 \qquad (t,x) \in Q_\omega. \tag{3.23}$$

Applying to both parts of (3.19) the operator **div** and taking into account (3.22) and the formula $\mathbf{div}\,\mathbf{rot}\, y = 0$ we obtain

$$-\Delta\tilde{p} = \mathbf{div}(z \times \mathbf{rot}\,\hat{v}) + \mathbf{div}((\nabla\hat{\theta} + e_0)\psi). \tag{3.24}$$

Our main goal now is to deduce from relations (3.19)-(3.24) that $z \equiv 0$, $\psi \equiv 0$. We will make it with help of Carleman estimates (3.5), (I.1.9). We can suppose that s_0 in Lemma 3.1 and in Lemma I.1.3 are equal. Otherwise, we can interchange them in both Lemmas on their maximum.

Let

$$\sigma \geq \max(s, s_0), \tag{3.25}$$

where s is the constant from $Z^s(Q_0)$ in Lemma 3.2 formulation. We take magnitude σ instead of s in (I.1.9) and apply estimate (I.1.9) to the equations

(3.19), (3.20). Note that boundary conditions (I.1.8) are fulfilled in our case in virtue of (3.21). We have

$$\int_{Q_0}(\sigma\varphi|\nabla z|^2+(\sigma\varphi)^3|z|^2)e^{\sigma\alpha(t,x)}\,dx\,dt + \int_{Q_0}(\sigma\varphi|\nabla\psi|^2+(\sigma\varphi)^3|\psi|^2)e^{\sigma\alpha}\,dx\,dt \le c_1\int_{Q_0}e^{\sigma\alpha}(|\hat v|^2|\nabla z|^2+|\nabla\hat v|^2|z|^2+|\psi|^2(1+|\nabla\hat\theta|^2)+|\nabla\tilde p|^2 + |\psi|^2|\nabla\hat v|^2+|\nabla\psi|^2|\hat v|^2+|z|^2)\,dx\,dt. \quad (3.26)$$

In the right side (3.26) we need to estimate $\nabla\tilde p$. We do it by means of (3.24),(3.23). Note that $\tilde p$ is defined to within an arbitrary constant. We fix it by the condition

$$\tilde p(t,x)\equiv 0, \qquad (t,x)\in Q_\omega. \quad (3.27)$$

Taking into account (3.23), (3.27) we apply to (3.24) estimate (3.5). After multiplication of inequality (3.5) on $(\gamma(t)/\sigma)exp(-e^{2\hat x_1}/\gamma(t))$ scalarly in $L_2(\Omega)$ and integration respect to t we get

$$\int_{Q_0}|\nabla\tilde p|^2e^{\sigma\alpha}\,dx\,dt\le c_5\int_{Q_0}\frac{\gamma(t)}{\sigma}(|\nabla z|^2|\nabla\hat v|^2+|z|^2|\nabla\mathbf{rot}\hat v|^2 + |\nabla\psi|^2(1+|\hat\theta|^2)+|\psi|^2|\Delta\hat\theta|^2)e^{\sigma\alpha}\,dx\,dt$$
$$\le c_6(\|\hat v\|^2_{C(0,T;(C^1(\bar\Omega))^n)}+\|\hat\theta\|^2_{C(0,T;C^1(\bar\Omega))}+1)\int_{Q_0}\frac{\gamma(t)}{\sigma}(|\nabla z|^2+|\nabla\psi|^2)e^{\sigma\alpha}\,dx\,dt+$$
$$c_7\Bigg(\|\mathbf{rot}\,\hat v\|^2_{L_\infty(0,T;(W_4^1(\Omega_0))^n)}\int_0^T\left(\int_\Omega\left(e^{2\sigma\alpha}\left(\frac{\gamma(t)}{\sigma}\right)^2|z|^4\right)dx\right)^{\frac12}dt + \|\Delta\hat\theta\|^2_{L_\infty(0,T;L_4(\Omega_0))}\int_0^T\left(\left(\int_\Omega\left(\frac{\gamma(t)}{\sigma}\right)^2\psi^4e^{2\sigma\alpha}\,dx\right)^{\frac12}dt\right)\Bigg). \quad (3.28)$$

We estimate the right side of (3.28) using the continuity of embedding $W^{1,2(2)}(Q_0)\subset C(0,T;C^1(\bar\Omega_0))$, $W^{1,2(2)}(Q_0)\subset L_\infty(0,T;W_4^2(\Omega_0))$, $W_2^1(\Omega_0)\subset L_4(\Omega_0)$ when $dim\,\Omega_0\le 3$ and taking into account that in virtue of (3.4)

$$|\partial_{x_j}(e^{\frac{\sigma\alpha}{2}}z)|^2\le c_8(\sigma^2\varphi^2|z|^2+|\nabla z|^2)e^{\sigma\alpha}.$$

As a result we obtain the inequality

$$\int_{Q_0} |\nabla\tilde{p}|^2 e^{\sigma\alpha}\,dx\,dt \le c_9(\|\hat{v}\|^2_{V^{1,2(2)}(Q_0)}$$

$$+\|\hat{\theta}\|^2_{W^{1,2(2)}(Q_0)}+1)\int_{Q_0}\left(\frac{\gamma(t)}{\sigma}(|\nabla z|^2+|\nabla\psi|^2)+\frac{\sigma e^{2x_1}}{\gamma(t)}(|z|^2+|\psi|^2)\right)e^{\sigma\alpha}\,dx\,dt. \tag{3.29}$$

The substitution of (3.29) into (3.26) and simple transformations give us the upper bound:

$$\int_{Q_0}\left(\frac{\sigma e^{x_1}}{\gamma(t)}(|\nabla z|^2+|\nabla\psi|^2)+\frac{\sigma^3 e^{3x_1}}{\gamma(t)^3}(|z|^2+|\psi|^2)\right)e^{\sigma\alpha}\,dx\,dt$$

$$\le c_{10}(\|\hat{v}\|^2_{V^{1,2(2)}(Q_0)}+\|\hat{\theta}\|^2_{W^{1,2(2)}(Q_0)}+1)\int_{Q_0}\left(\left(\frac{\gamma(t)}{\sigma}+1\right)(|\nabla z|^2+|\nabla\psi|^2)\right.$$

$$\left.+\left(\frac{\sigma\ e^{2x_1}}{\gamma(t)}+1\right)(|z|^2+|\psi|^2)\right)e^{\sigma\alpha}\,dx\,dt. \tag{3.30}$$

Note that (3.30) is true for arbitrary σ satisfying (3.25). We choose σ so large that estimates

$$\frac{\sigma e^{x_1}}{\gamma(t)} > c_{10}(\|\hat{v}\|^2_{V^{1,2(2)}(Q_0)}+\|\hat{\theta}\|^2_{W^{1,2(2)}(Q_0)}+1)\left(\frac{\gamma(t)}{\sigma}+1\right),$$

$$\frac{\sigma^3 e^{3x_1}}{\gamma(t)^3} > c_{10}(\|\hat{v}\|^2_{V^{1,2(2)}(Q_0)}+\|\hat{\theta}\|^2_{W^{1,2(2)}(Q_0)}+1)\left(\frac{\sigma e^{2x_1}}{\gamma(t)}+1\right),$$

hold for all $(t,x)\in Q_0$. Then (3.30) yields that

$$z(t,x)\equiv 0, \qquad \psi(t,x)\equiv 0. \tag{3.31}$$

In virtue of (3.18)-(3.21), (3.31) integrating by parts in (3.17) when $\nabla p\equiv 0$, $u\equiv 0$, $w\in V(Q_0,\eta)$, $\tau\equiv 0$ and $\nabla p\equiv 0$, $u\equiv 0$, $w\equiv 0$, $\tau\in\Theta(Q_0,\eta)$ gives us the equalities

$$(w(0,\cdot),z_0)_{V^1(\Omega_0)} = (w(0,\cdot),z(0,\cdot))_{L_2(\Omega_0)} = 0,$$

$$(w(0,\cdot),\psi_0)_{W_2^1(\Omega_0)} = (\tau(0,\cdot),\psi(0,\cdot))_{L_2(\Omega)} = 0.$$

Therefore

$$z_0 = 0, \qquad \psi_0 = 0. \tag{3.32}$$

Hence, by (3.18), (3.31), (3.32) $\phi\equiv(m(t,x),\zeta(t,x),z_0(x),\psi_0(x))\equiv 0$. ■

3.3. Now we can prove the main result of this section.

THEOREM 3.1. *Suppose that $\hat{v} \in V^{1,2(2)}(Q), \hat{\theta} \in W^{1,2(2)}(Q)$, the operator $\mathcal{A}'(0)$ is defined in (2.23), (2.9), (2.11) and the spaces $X^s(Q), Z^s(Q)$ are defined in (2.14)-(2.21) and the parameter s of these spaces is arbitrary positive number. Then the image of operator*

$$\mathcal{A}'(0) : X^s(Q) \to Z^s(Q)$$

is the dense in the space $Z^s(Q)$.

Proof. Let Ω_0, Q_0 be the sets introduced in the beginning of section 3.2. By Propositions 2.2, 2.3 we extend the functions $\hat{v}(t,x), \hat{\theta}(t,x)$ continuously from
$V^{1,2(2)}(Q)$ up to $V^{1,2(2)}(Q_0)$ and from $W^{1,2(2)}(Q)$ up to $W^{1,2(2)}(Q_0)$ correspondingly and denote these new functions also by $\hat{v}(t,x), \hat{\theta}(t,x)$. Comparing (3.6)-(3.9) and (2.9)-(2.12) we see that the restriction of operator (3.14), (3.15) on the cylinder Q, coincides with the operator

$$\mathcal{A}'(0) : U^s(Q) \to \Phi^s(Q) \tag{3.33}$$

where $\mathcal{A}'(0)$ is the operator (2.23) and in contrast to (3.12)

$$U^s(Q) = V(Q,\eta^s) \times L_2(Q,\eta^s) \times \Theta(Q,\eta^s) \tag{3.34}$$

because the restrictions of arbitrary function from $\hat{L}_2(Q_\omega,\eta^s)$ to Q is identical zero. Therefore in virtue of Lemma 3.2 the image of operator (3.33), (2.23) is dense in $\Phi^s(Q)$. Let $(f,h,v_0,\theta_0) \in Z^s(Q)$ (see (2.21)) be an arbitrary element. Since $f \in F^s(Q)$ (see (2.17)) then $f = f_1 + \nabla f_2$ where $f_1 \in L_2(Q,\eta^s)$, $f_2 \in L_2(0,T;W_2^1(\Omega))$ and therefore $(f_1,h,v_0,\theta_0) \in \Phi^s(Q)$. By the density of the image of operator (3.36), for every $\epsilon > 0$ there exists $(f_1^\epsilon, h^\epsilon, v^\epsilon, \theta^\epsilon) \in \Phi^s(Q)$ possessing preimage $(v^\epsilon, p^\epsilon, \theta^\epsilon) \in U^s(Q)$:

$$\mathcal{A}'(0)(v^\epsilon, p^\epsilon, \theta^\epsilon) = (f_1^\epsilon, h^\epsilon, v_0^\epsilon, \theta_0^\epsilon) \tag{3.35}$$

and satisfying the inequality:

$$\|(f_1 - f_1^\epsilon, h - h^\epsilon, v_0 - v_0^\epsilon, \theta_0 - \theta_0^\epsilon)\|_{\Phi^s(Q)} \le \epsilon. \tag{3.36}$$

In virtue of (2.9) and (3.35)

$$\mathcal{A}'(0)(v^\epsilon, p^\epsilon + f_2, \theta^\epsilon) = (f_1^\epsilon + \nabla f_2, h^\epsilon, v_0^\epsilon, \theta_0^\epsilon). \tag{3.37}$$

Since $f-(f_1^\epsilon+\nabla f_2)=f_1-f_1^\epsilon$ then by (2.21), (2.18), (3.11), (3.33) we have:

$$\|(f-(f_1^\epsilon+\nabla f_2),h-h^\epsilon,v_0-v_0^\epsilon,\theta_0-\theta^\epsilon)\|_{Z^s(Q)} \\ \leq \|(f-f_1^\epsilon,h-h^\epsilon,v_0-v_0^\epsilon,\theta_0-\theta_0^\epsilon)\|_{\Phi^s(Q)}<\epsilon. \quad (3.38)$$

By (3.12), (2.20) the inclusion $(v^\epsilon,p^\epsilon,\theta^\epsilon)\in U^s(Q)$ involve the inclusion $(v^\epsilon,p^\epsilon+f_2,\theta^\epsilon)\in X^s(Q)$. Hence, by (3.37) $(v^\epsilon,p^\epsilon+f_2,\theta^\epsilon)$ is preimage of $(f_1^\epsilon+\nabla f_2,h^\epsilon,v_0^\epsilon,\theta_0^\epsilon)$. This proves theorem. ■

Note that for the case of the Stokes system the results similar to Theorem 3.1 were proved in [23], [24].

4.On a decomposition of Weyl type.

In this section we investigate the decomposition of the Weyl type

$$y(t,x)=v(t,x)+\nabla q \qquad (t,x)\in Q_0, \quad (4.1)$$

where $\mathbf{div}\, v=0$ and $\nabla q=(\partial_{x_1}q,\dots,\partial_{x_n}q)$ is the gradient of a function. We do not impose any boundary conditions on v or ∇q but look for v belonging to the space $V(Q_0,\eta)$ when $y\in(\Theta(Q,\eta))^n$. We do not look for natural uniqueness conditions for the decomposition (4.1) but need that the following assumption would be fulfilled:

$$\text{if } \mathbf{div}\, y(0,x)\equiv 0 \text{ then } y(0,x)\equiv v(0,x). \quad (4.2)$$

To find decomposition (4.1) we consider the extermal problem:

$$J(u)=\int_{Q_0}\frac{|u(t,x)|^2e^{2\eta}}{(T-t)^4}\,dx\,dt\to \inf, \quad (4.3)$$

$$\Delta u(t,x)=\mathbf{div}\, y(t,x), \qquad (t,x)\in Q_0, \quad (4.4)$$

where $y(t,x)\in(\Theta(Q_0,\eta))^n$ is a given function. If a solution $m(t,x)$ of problem (4.3), (4.4) would exist then we denote $v=y-\nabla m$ and by (4.4) the equality $\mathbf{div}\, v=0$ and therefore decomposition (4.1) would be true.

LEMMA 4.1. *There exists such s_0 that if $y(t,x) \in (\Theta(Q,\eta^s))^n$ where $s \geq s_0$, the problem (4.3), (4.4) has the unique solution $m(t,x) \in L_2(Q_0, \eta - 2\ln(T-t))$. This solution satisfies the estimates:*

$$\int_{Q_0} \frac{|m(t,x)|^2}{(T-t)^4} e^{2\eta^s}\, dx\, dt \leq c_1 \int_{Q_0} \frac{|\mathbf{div}\, y|^2}{(T-t)} e^{2\eta^s}\, dx\, dt, \tag{4.5}$$

$$\int_{Q_0} |\partial_t m(t,x)|^2 e^{2\eta^s}\, dx\, dt \leq c_2 \|y\|^2_{\Theta(Q_0,\eta^s)}. \tag{4.6}$$

Proof. Let s_0 defined in Lemma 3.1. We denote $Q_\epsilon = (0, T-\epsilon) \times \Omega_0$ and instead of (4.3), (4.4) consider the extremal problem

$$J_\epsilon(u) = \int_{Q_\epsilon} \frac{|u(t,x)|^2}{(T-t)^4} e^{2\eta}\, dx\, dt \to \inf, \tag{4.7}$$

$$\Delta u(t,x) = \mathbf{div}\, y(t,x), \qquad (t,x) \in Q_\epsilon. \tag{4.8}$$

The weight $e^{2\eta}(T-t)^{-4}$ is bounded above and below on Q_ϵ. Hence the space $U_\epsilon = \{u \in L_2(Q_\epsilon) : \Delta u \in L_2(Q_\epsilon)\}$ is natural for the problem (4.7), (4.8) and the set of its admissible elements is as follows:

$$A_\epsilon = \{u \in U_\epsilon : \Delta u = \mathbf{div}\, y\}.$$

As well-known, the limit $m_\epsilon \in A_\epsilon$ of weakly converging subsequence of minimizing sequence u_k: $J_\epsilon(u_k) \to \inf_{v \in A_\epsilon} J_\epsilon(v)$ is the solution of problem (4.7), (4.8). The uniqueness of m_ϵ follows from the functional J_ϵ strict-convexity. For $\epsilon_1 > \epsilon_2$, $m_{\epsilon_1}(t,x)$ coincides almost everywhere with restriction of $m_{\epsilon_2}(t,x)$ on Q_{ϵ_2}. Indeed, if it is not so then $J_{\epsilon_1}(m_{\epsilon_1}) < J_{\epsilon_1}(m_{\epsilon_2})$. But in this occasion m_{ϵ_2} is not solution because the function

$$\hat{m}(t,x) = \begin{cases} m_{\epsilon_1}(t,x), & (t,x) \in Q_{\epsilon_1}, \\ m_{\epsilon_2}(t,x), & (t,x) \in Q_{\epsilon_2}. \end{cases}$$

satisfies (4.8) and inequality $J_{\epsilon_2}(\hat{m}) < J_{\epsilon_2}(m_{\epsilon_2})$. That is why below we use the notation: $m_\epsilon = m$. Since operator $\Delta : U_\epsilon \to L_2(Q_\epsilon)$ is epimorphism we can apply to problem (4.7), (4.8) the Lagrange principle (see [1]). This principle asserts that there exists $p_\epsilon \in (L_2(Q_\epsilon))^n$ such that the Lagrange function

$$\mathcal{L}(u, p_\epsilon) \equiv \int_{Q_\epsilon} \left(\frac{1}{2} \frac{|u(t,x)|^2}{(T-t)^4} e^{2\eta} + (\Delta u - \mathbf{div}\, y) p_\epsilon(t,x) \right) dx\, dt$$

satisfies the equality $\partial_u \mathcal{L}(u, p_\epsilon)|_{u=m} = 0$, i.e. for any $h \in U_\epsilon$

$$\int_{Q_\epsilon} \left(\frac{m(t,x)h(t,x)}{(T-t)^4} e^{2\eta} + \Delta h p_\epsilon(t,x) \right) dx\, dt = 0. \tag{4.9}$$

It follows from (4.9) that

$$\Delta p_\epsilon(t,x) + \frac{m(t,x)}{(T-t)^4} e^{2\eta} = 0, \text{ in } Q_0, \qquad p_\epsilon|_{\partial\Omega_0} = \left.\frac{\partial p_\epsilon}{\partial \nu}\right|_{\partial\Omega_0} = 0. \tag{4.10}$$

Relations (4.10) imply that p_ϵ does not depend on ϵ and therefore, below, we use the notation: $p_\epsilon = p$. We apply to (4.10) Carleman estimate (3.2), substitute in this estimate $s = s_1(T-t)^{-1}$, multiply it on $(T-t)^4$ and integrate with respect to t. As a result we have an estimate:

$$\int_{Q_\epsilon} (T-t) p^2 e^{-2\eta}\, dx\, dt \le c_3 \int_{Q_\epsilon} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt, \tag{4.11}$$

where $c_3 > 0$ does not depend on ϵ . After scaling equation (4.4) for m by p in $L_2(Q_\epsilon)$,, integrating by parts and applying (4.10) we get

$$0 = \int_{Q_\epsilon} (\Delta m - \mathbf{div}\, y) p\, dx\, dt = \int_{Q_\epsilon} (m \Delta p - p\, \mathbf{div}\, y)\, dx\, dt =$$
$$- \int_{Q_\epsilon} \left(\frac{m^2}{(T-t)^4} e^{2\eta} + p\, \mathbf{div}\, y \right) dx\, dt.$$

This equality and (4.11) yields

$$\int_{Q_\epsilon} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt \le c_4 \left(\int_{Q_\epsilon} \frac{|\mathbf{div}\, y|^2}{(T-t)} e^{2\eta} dx dt \right)^{\frac{1}{2}} \left(\int_{Q} (T-t)|p|^2 e^{-2\eta}\, dx\, dt \right)^{\frac{1}{2}}$$
$$\le c_5 \int_{Q_\epsilon} \frac{|\mathbf{div}\, y|^2}{(T-t)} e^{2\eta}\, dx\, dt + \frac{1}{2} \int_{Q_\epsilon} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt.$$

that gives us upper bound

$$\int_{Q_\epsilon} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt \le c_6 \int_{Q_\epsilon} \frac{|\mathbf{div}\, y|^2}{(T-t)} e^{2\eta}\, dx\, dt, \tag{4.12}$$

where c_6 does not depend on ϵ. Hence we can pass to limit in (4.12) as $\epsilon \to 0$ and obtain (4.5). Let $\hat{m}$ be the solution of problem (4.3), (4.4). Since m is the solution of (4.7), (4.8) we have

$$\int_{Q_\epsilon} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt \le \int_{Q_\epsilon} \frac{\hat{m}^2}{(T-t)^4} e^{2\eta}\, dx\, dt \qquad \forall \epsilon > 0$$

and therefore

$$\int_{Q_0} \frac{m^2}{(T-t)^4} e^{2\eta}\, dx\, dt = \int_{Q_0} \frac{\hat{m}^2}{(T-t)^4} e^{2\eta}\, dx\, dt.$$

This equation implies the equality $m = \hat{m}$ because of the uniqueness of solution of problem (4.3), (4.4). After differentiation of the equations in (4.3), (4.10) with respect to t we get

$$\Delta \partial_t m = \mathbf{div}\, \partial_t y, \tag{4.13}$$

$$\Delta \partial_t p + (\partial_t m) \frac{e^{2\eta}}{(T-t)^4} + m \partial_t \left(\frac{e^{2\eta}}{(T-t)^4} \right) = 0. \tag{4.14}$$

Applying to (4.14) the Carleman estimate (3.2) by the same way as in (4.11) we obtain

$$\int_{Q_0} |\nabla \partial_t p|^2 (T-t)^7 e^{-2\eta}\, dx\, dt \le c_7 \int_{Q_0} (|\partial_t m|^2 + (T-t)^{-4} |m|^2) e^{2\eta}\, dx\, dt. \tag{4.15}$$

Scaling equation (4.13) by $\partial_t p$ in $L_2(Q_0)$, integration by parts, and application (4.14) yield

$$\begin{aligned} 0 = \int_{Q_0} (T-t)^4 (\Delta \partial_t m - \mathbf{div} \partial_t y) \partial_t p\, dx\, dt = \int_{Q_0} (T-t)^4 (\partial_t m \Delta \partial_t p \\ - (\partial_t y, \nabla \partial_t p))\, dx\, dt = \int_{Q_0} (-|\partial_t m|^2 e^{2\eta} - ((\partial_t m) m \partial_t \frac{e^{2\eta}}{(T-t)^4})(T-t)^4 \\ - (T-t)^4 (\partial_t y, \nabla \partial_t p))\, dx\, dt. \end{aligned}$$

From this equality we get taking into account (4.15):

$$\begin{aligned} \int_{Q_0} |\partial_t m|^2 e^{2\eta}\, dx\, dt \le c_8 \int_{Q_0} (|\partial_t m| |m| \frac{e^{2\eta}}{(T-t)^2} \\ + e^{\eta} (T-t)^{1/2} |\partial_t y| e^{-\eta} (T-t)^{\frac{7}{2}} |\nabla \partial_t p|)\, dx\, dt \le \frac{1}{4} \int_{Q_0} |\partial_t m|^2 e^{2\eta}\, dx\, dt \\ + c_9 \int_{Q_0} \left(\frac{|m|^2}{(T-t)^4} e^{2\eta} + (T-t) |\partial_t y|^2 e^{2\eta} \right) dx\, dt. \end{aligned}$$

This inequality and (4.5) imply (4.6). ∎

Let

$$\rho(x) \in C^\infty(\bar{\Omega}_0), \qquad \rho|_{\partial\Omega_0} = 0, \qquad \rho(x) > 0, \ \forall\, x \in \Omega_0.$$

Below, we use the following space

$$M(Q_0,\eta) = \{f = (f_1,\dots,f_n) : \|f\|^2_{M(Q_0,\eta)} = \|(T-t)^{-1}f\|^2_{(L_2(Q_0,\eta^s))^n} + \||\nabla f|\|^2_{(L_2(Q_0,\eta^s))^n} + \|(T-t)\partial_t f\|^2_{(L_2(Q_0,\eta^s))^n} + \sum_{i,j=1}^n \|(T-t)\partial^2_{x_ix_j} f\|^2_{(L_2(Q_0,\eta^s))^n} < \infty\}. \tag{4.16}$$

Lemma 4.2. *Let $m(t,x)$ be the solution of problem (4.3), (4.4) constructed in Lemma 4.1. Then*

$$\|\rho^3\nabla m\|^2_{M(Q_0,\eta)} \le c_{10}\|y\|^2_{(\Theta(Q_0,\eta))^n}. \tag{4.17}$$

Proof. Set $\tilde{m} = m\rho$. Then by (4.4) for m

$$\Delta\tilde{m} = m\Delta\rho + 2(\nabla\rho,\nabla m) + \rho\,\mathbf{div}\, y. \tag{4.18}$$

We multiply this equation by $-e^{2\eta}\tilde{m}(T-t)^{-2}$ scalarly in $L_2(Q_0)$, integrate by parts and have as a result

$$\int_{Q_0} |\nabla\tilde{m}|^2 e^{2\eta}(T-t)^{-2}\,dx\,dt = \int_{Q_0}(T-t)^{-2}(\frac{1}{2}|\tilde{m}|^2\Delta e^{2\eta} - m^2\rho\Delta\rho e^{2\eta} + \frac{1}{2}m^2(\Delta\rho^2 e^{2\eta} + (\nabla\rho^2,\nabla e^{2\eta})) - \rho^2 m e^{2\eta}\mathbf{div}\, y)\,dx\,dt$$
$$\le c_{11}\int_{Q_0}\left(\frac{m^2}{(T-t)^4} + |\mathbf{div}\, y|^2\right)e^{2\eta}\,dx\,dt.$$

This inequality, (4.5) and the definition (2.16) of space $\Theta(Q_0,\eta)$ yield:

$$\int_{Q_0}\frac{|\rho\nabla\rho m|^2}{(T-t)^2}e^{2\eta}\,dx\,dt \le c_{12}\int_{Q_0}\frac{|\nabla(\rho m)|^2}{(T-t)^2}e^{2\eta}\,dx\,dt + \int_{Q_0}\frac{m^2|\nabla\rho|^2}{(T-t)^2}e^{2\eta}\,dx\,dt \le c_{13}\|y\|^2_{(\Theta(Q_0,\eta))^n}. \tag{4.19}$$

Denote $m_0 = m\rho^2 e^\eta$. Then we have analogously to (4.18)

$$\Delta m_0 = g, \qquad m_0|_{\partial\Omega_0} = 0, \tag{4.20}$$

where $g = m\Delta(\rho^2 e^\eta) + 2(\nabla(\rho^2 e^\eta), \nabla m) + \rho^2 e^\eta \mathbf{div}\, y$.

By (4.5), (4.19) we get

$$\|g\|_{L_2(Q)} \le c_{14}\|y\|_{(\Theta(Q_0,\eta))^n}. \tag{4.21}$$

Applying to elliptic boundary value problem (4.20) well-known estimate of its solution and taking into account (4.21) we obtain

$$\begin{aligned}\|m_0\|^2_{L_2(0,T;W_2^2(\Omega_0))} &= \|m\rho^2 e^\eta\|^2_{L_2(0,T;W_2^2(\Omega_0))} \\ &\le c_{15}\|g\|^2_{L_2(Q_0)} \le c_{16}\|y\|^2_{\Theta(Q_0,\eta)^n}.\end{aligned} \tag{4.22}$$

Since

$$\begin{aligned}|\partial^2_{x_ix_j}(\rho^2 m e^\eta)|^2 &\ge \frac{1}{2}|\partial_{x_i}(\rho^2\partial_{x_j} m)e^\eta|^2 \\ &\quad - c_{17}(|\rho^2(\partial_{x_j} m)\partial_{x_j} e^\eta|^2 + |(\partial_{x_i} m)\partial_{x_j}(\rho^2 e^\eta)|^2 + |m\partial^2_{x_ix_j}(\rho^2 e^\eta)|^2);\end{aligned}$$

then inequalities (4.22), (4.19), (4.5) imply the estimate

$$\begin{aligned}\int_{Q_0} e^{2\eta}\sum_{j=1}^n |\partial_{x_j}(\rho^2\nabla m)|^2\,dx\,dt &\le c_{16}\int_{Q_0}\sum_{i,j=1}^n(|\partial^2_{x_ix_j}(\rho^2 m e^\eta)|^2 \\ &\quad + |\rho^2(\partial_{x_j} m)\partial_{x_i} e^\eta|^2 + |(\partial_{x_i} m)\partial_{x_j}(\rho^2 e^\eta)|^2 + |m\partial^2_{x_ix_j}(\rho^2 e^\eta)|^2)\,dx\,dt \\ &\le c_{19}\|y\|^2_{(\Theta(Q_0,\eta))^n}.\end{aligned} \tag{4.23}$$

Denote $m_i = \rho^3(\partial_{x_i} m)e^\eta(T-t)$. Then by virtue of (4.4) with $u = m$

$$\Delta m_i = g_i, \qquad m_i|_{\partial\Omega_0} = 0, \tag{4.24}$$

where

$$g_i = (\partial_{x_i} m)\Delta(\rho^3 e^\eta(T-t)) + 2(\nabla(\rho^3 e^\eta(T-t), \partial_{x_i}\nabla m) + \rho^3 e^\eta(T-t)\partial_{x_i}\mathbf{div}\, y.$$

Applying to the solution m_i of problem (4.24) estimate of solution of Laplace equation we get as in (4.22) taking into account (4.5), (4.19), (4.23):

$$\begin{aligned}\|\rho^3(\partial_{x_i} m)e^\eta(T-t)\|^2_{L_2(0,T;W_2^2(\Omega_0))} &\le c_{20}(\|(\partial_{x_i} m)\Delta(\rho^3 e^\eta(T-t))\|^2_{L_2(Q_0)} \\ &\quad + \|(\frac{1}{\rho^2}\nabla(\rho^3 e^\eta(T-t)), (\partial_{x_i}(\rho^2\nabla m) - 2(\partial_{x_i}\rho)\rho\nabla m))\|^2_{L_2(Q_0)} + \\ &\quad \|\rho^3 e^\eta(T-t)\partial_{x_i}\mathbf{div}\, y\|^2_{L_2(Q_0)}) \le c_{21}\|y\|^2_{(\Theta(Q_0,\eta))^n}.\end{aligned} \tag{4.25}$$

As in (4.22), inequalities (4.25) with $i = 1, \ldots, n$, (4.23), (4.19), (4.5) yield:

$$\int_{Q_0} e^{2\eta} \sum_{k,l=1}^{n} |\partial^2_{x_k x_l}(\rho^3 \nabla m)|^2 (T-t)^2 \, dx \, dt \le c_{22} \|y\|^2_{(\Theta(Q_0,\eta))^n}. \tag{4.26}$$

In virtue of (4.13)

$$\Delta(\rho \partial_t m) = \partial_t m \Delta\rho + 2(\nabla\rho, \nabla\partial_t m) + \rho \,\mathbf{div} \partial_t y. \tag{4.27}$$

Scaling (4.27) by $-(\rho\partial_t m)e^{2\eta}(T-t)^2$ in $L_2(Q_0)$ and integrating by parts we have

$$\int_{Q_0} |\nabla(\rho\partial_t m)|^2 (T-t)^2 \, dx \, dt = \int_{Q_0} (\frac{1}{2}\rho(\partial_t m)^2 \Delta e^{2\eta}(T-t)^2 - \rho(\partial_t m)^2 \Delta\rho e^{2\eta}(T-t)^2$$
$$+ \frac{1}{2}(T-t)^2(\partial_t m)^2 \mathbf{div}(e^{\eta}\nabla\rho^2) - \rho^2 \partial_t m(\mathbf{div}\, \partial_t y) e^{2\eta}(T-t)^2) \, dx \, dt.$$

This equality implies

$$\int_{Q_0} |\rho\nabla\partial_t m|^2 (T-t)^2 e^{2\eta} \, dx \, dt \le c_{23} \int_{Q_0} |\partial_t m|^2 (c_{24}|\nabla\rho|^2 (T-t)^2 + \rho^2$$
$$+ |\rho\Delta\rho|(T-t)^2 + c_{25}(T-t)(|\nabla\rho^2| + (T-t)|\Delta\rho^2|) e^{2\eta} \, dx \, dt$$
$$+ \int_{Q_0} [(\rho\nabla\partial_t m, \partial_t y)\rho e^{2\eta}(T-t)^2 + \partial_t m(\nabla(\rho^2 e^{2\eta}, \partial_t y)(T-t)^2] \, dx \, dt$$
$$\le c_{26} \int_{Q_0} |\partial_t m|^2 e^{2\eta} \, dx \, dt + \frac{1}{2} \int_{Q_0} |\rho\nabla\partial_t m|^2 (T-t)^2 e^{2\eta} \, dx \, dt$$
$$+ c_{27} \int_{Q_0} e^{2\eta} |\partial_t y|^2 (T-t)^2 \, dx \, dt \le c_{28} \|y\|^2_{(\Theta(Q_0,\eta))^n}. \tag{4.28}$$

After transfering the term with $\rho\nabla\partial_t m$ from the right side of (4.28) to the left side we get with help of (4.6) and (2.16)

$$\int_{Q_0} |\rho\nabla\partial_t m|^2 (T-t)^2 e^{2\eta} \, dx \, dt \le c_{28} \|y\|^2_{(\Theta(Q_0,\eta))^n}.$$

Upper bounds (4.19), (4.23), (4.26) imply (4.17). ■

We prove now the main result of this section.

THEOREM 4.1. *Let s satisfies to the condition of Lemma 4.1. An arbitrary vector field $y \in (\Theta(Q_0, \eta^s))^n$ admits decomposition (4.1), where $v(t,x) \equiv 0$ and $\rho^3 \nabla q \in M(Q_0, \eta^s)$ and if $y(t,x)$ satisfies equality $\mathbf{div}\, y(0,x) \equiv 0$, then $y(0,x) \equiv z(0,x)$.*

Proof. We define the function $\varphi(t) \in C^\infty(0,T)$, such that $\varphi(t) \equiv 0$ when $t \in (0, T/4)$, $\varphi(t) \equiv 1$ when $t \in [\frac{3}{4}T, T]$. Let $m(t,x)$ be solution of problem (4.3), (4.4) constructed in Lemma 4.1. Since $y \in (\Theta(Q_0, \eta))^n$ then for almost all $t \in (0,T)$, $\Delta m(t,\cdot) \in L_2(\Omega_0)$ and in virtue of (4.5) $m(t,\cdot) \in L_2(\Omega_0)$. Hence, (see J.L. Lions, E. Magenes [53]) the restriction $m(t,\cdot)|_{\partial\Omega}$ is defined and belongs to $W_2^{1/2}(\partial\Omega_0)$. We introduce the function $\zeta(t,x)$, defined on $(0,T) \times \partial\Omega_0$ by formula

$$\zeta(t,x) = \varphi(t) m(t,x) \qquad t \in (0,T),\ x \in \partial\Omega_0$$

and consider the following Dirichlet problem

$$\Delta q(t,x) = \mathbf{div}\, y(t,x), \qquad (t,x) \in Q_0, \tag{4.29}$$

$$q|_{(0,T)\times\partial\Omega_0} = \zeta. \tag{4.30}$$

The unique solution $q(t,x)$ of (4.29), (4.30) exists (see J.L. Lions, E. Magenes [53]) and in virtue of properties of $\zeta(t,x)$

$$q(x,t) \equiv m(t,x) \qquad \forall\ (t,x) \in [3/4T, T] \times \Omega_0 \tag{4.31}$$

and

$$\forall\, (t,x) \in [0, T/4] \qquad \mathbf{div}\, y(t,x) = 0 \qquad \text{imply } q(t,x) \equiv 0. \tag{4.32}$$

In virtue of (4.31) and (4.17) we have: $\rho^3 \nabla q \in M(Q_0, \eta)$. Besides (4.2) follows from (4.32). ■

5. The proof of main results.

5.1. First of all we want to prove the exact controllability problem for linearized Boussinesq equations (2.9)-(2.13). To do it we apply the analogous controllability result for parabolic equation and parabolic system which is formulated below. We consider the controllability problem for heat equation

$$\partial_t \theta(t,x) - \Delta\theta(t,x) = h(t,x) \qquad (t,x) \in Q_0, \tag{5.1}$$

$$\theta|_{t=0} = \theta_0(x), \qquad \theta|_{t=T} = 0, \qquad x \in \Omega_0, \tag{5.2}$$

where the functions $h \in L_2(Q_0, \eta)$, $\theta_0 \in W_2^1(\Omega)$ are given .

THEOREM 5.1. *There exists a number s_1 that for any $s > s_1$ and for arbitrary given $\theta_0 \in W_2^1(\Omega_0)$, $h \in L_2(Q_0, \eta^s)$ there exists the solution $\theta \in \Theta(Q_0, \eta^s)$ of problem (5.1), (5.2).*

We consider also the controllability problem for the following parabolic system

$$\partial_t y(t,x) - \Delta y - \hat{v} \times \mathbf{rot}\, y = f, \qquad (t,x) \in Q_0, \tag{5.3}$$

$$y|_{t=0} = y_0, \qquad y|_{t=T} = 0, \qquad x \in \Omega_0. \tag{5.4}$$

THEOREM 5.2. *Let $\hat{v}(t,x) \in V^{1,2(2)}(Q_0)$ be given. Then there exists a number s_2 such that for any $s > s_2$ and for arbitrary given data $y_0 \in (W_2^1(\Omega_0))^n$, $f \in (L_2(Q_0, \eta^s))^n$ there exists the solution $y \in (\Theta(Q_0, \eta^s))^n$ of problem (5.3), (5.4).*

One can prove the Theorems 5.1, 5.2 in absolutely same way as Theorems I.2.1 , I.3.10 using the carleman estimate (1.9) instead of (1.6). Let us prove one abstract lemma.

LEMMA 5.1. *Suppose that X, Y are Hilbert spaces, a bounded linear operator $B : X \to Y$ is epimorfism and $K : X \to Y$ is a linear compact operator. Then the image of operator $B + K$ is closed in Y.*

Proof. For an arbitrary $\epsilon > 0$ there exists the operator K_ϵ that has finite dimension image and

$$\|K - K_\epsilon\| < \epsilon. \tag{5.5}$$

The equality

$$B + K = B_\epsilon + K_\epsilon \qquad \text{where } B_\epsilon = B + (K - K_\epsilon)$$

is true. If in (5.5) ϵ is small enough then the image of operator B_ϵ coincides with whole Y. Thus, we reduce the Lemma 5.1 to the case when operator $K : X \to Y$ has a finite dimensional image. We can suppose also that $Ker\, B \cap Ker\, K = 0$. Indeed, if it is not so we introduce the factor space $X_1 = X/(Ker\, B \cap Ker\, K)$, define operators B_1 and K_1 by formulas

$$B_1 \tilde{x} = Bx, \qquad K_1 \tilde{x} = Kx, \qquad \text{where} \qquad \tilde{x} = x + Ker\, B \cap Ker\, K$$

and consider the problem on closure of operators $B_1 + K_1 : X_1 \to Y$ image. Since operator K has a finite dimension image then there exists a finite linear independent system of vectors $e_1, \dots, e_k \in Y$ and linear independent system of bounded functionals $f_1, \dots, f_n$ on X such that

$$Kx = \sum_{j=1}^{k} f_j(x) e_j.$$

The linear independentness of $f_1, \dots, f_n$ means that there exist such linear independent vectors $g_1, \dots, g_k \in X$ that $f_j(g_i) = \delta_{i,j}$, where is Kronecher symbol. Hence the space X admits the decomposition

$$X = [g_1, \dots, g_k] + Ker\, K,$$

where $[g_1, \dots, g_k]$ is a linear span of $g_1, \dots, g_k$. Since $Ker\, B \cap Ker\, K = 0$ then $dim Ker\, B \le k$ and X admits the decomposition

$$X = S + Ker\, B + Ker\, K,$$

where S is a certain finite dimension space. Let B_2, K_2 be the restrictions at the space $S + Ker\, K$ of the operators B and K respectively. Since the operator

$$B : S + Ker\, K \to Y$$

is isomorphism then by Fredholm theorem the image $B_2 + K_2$ is closed and has a finite codimension in Y. The coincidence $(B_2 + K_2)(S + Ker\, K) = (B + K)(S + Ker\, K)$ implies the including

$$(B + K)(S + Ker\, K) \subset (B + K)X.$$

Hence $(B + K)X = (B + K)(S + Ker\, K) + S_1$, where S_1 is a certain finite dimensional subspace of Y. Being a finite dimensional space the subspace S_1 is closed. Hence $(B + K)$ is closed. ■

5.2. Now we prove the assertion on closure of set of data for which the controllability problem for the Boussinesq equations has a solution.

THEOREM 5.3. *Let $\hat{v}(t,x) \in V^{1,2(2)}(Q), \hat{\theta}(t,x) \in W^{1,2(2)}(Q)$. Then the set of data (f, h, v_0, θ_0) for which there exists a solution $(v, p, \theta) \in X^s(Q)$ of problem (2.9)-(2.13) is closed in the space $Z^s(Q)$ when magnitude of parameter s is sufficiently large (spaces $X^s(Q), Z^s(Q)$ are defined in (2.20), (2.21)).*

Proof. To proof this theorem we intend to apply Lemma 5.1. We decompose the operator generated by the problem (2.9)-(2.13) by the sum $B+K$, where B is the operator generated by the problem

$$\partial_t v(t,x) - \Delta v - \hat{v}\times \mathbf{rot}\, v + \nabla p = f(t,x), \qquad \mathbf{div}\, v = 0, \qquad v(0,x) = v_0(x), \tag{5.6}$$

$$\partial_t \theta(t,x) - \Delta\theta = h(t,x), \qquad \theta(0,x) = \theta_0(x), \tag{5.7}$$

$$v(T,x) \equiv 0, \qquad \theta(T,x) \equiv 0. \tag{5.8}$$

The operator K is defined by the formula

$$K(v,p,\theta) = (-v\times \mathbf{rot}\,\hat{v} + \theta e_0, (\hat{v},\nabla\theta) + (v,\nabla\hat{\theta}) + (v,e_0), 0, 0). \tag{5.9}$$

The boundness of the operator

$$B : X^s(Q) \to Z^s(Q) \tag{5.10}$$

is proved in Lemma 2.1. To prove that operator (5.10) is epimorphism we firstly, change (5.6) for the more simple equations:

$$\partial_t y(t,x) - \Delta y - \hat{v}\times \mathbf{rot}\, y = f_1(t,x), \qquad y(0,x) = y_0(x). \tag{5.11}$$

Let Q_0, Ω_0 be the set introduced in the beginning of the section 3.2. We extend continuously $\hat{v}(t,x)$ from $V^{1,2(2)}(Q)$ to $V^{1,2(2)}(Q_0)$ as well as $\hat{\theta}(t,x)$ from $W^{1,2(2)}(Q)$ to $W^{1,2(2)}(Q_0)$ using Proposition 2.3 and consider the problem (5.11), (5.7) on Q_0. Note that $y_0(x) \in V^1(\Omega)$ is an extension of $v_0 \in V^1(\Omega_0)$.

We choose parameter s satisfying conditions of Theorems 4.1, 5.1 and 5.2 simultaneously. Then by virtue of these theorems for an arbitrary $(f_1, h, y_0, \theta_0) \in (L_2(Q_0, \eta^{s,}))^n \times L_2(Q_0,\eta^s) \times V^1(\Omega_0) \times W_2^1(\Omega_0)$ there exists a solution $(y,\theta) \in ((\Theta(Q_0,\eta^s))^n \times \Theta(Q_0,\eta^s)$ of problem (5.11), (5.7) on Q_0. With help of Theorem 4.1 we decompose the component y of this section as follows:

$$y(t,x) = v(t,x) + \nabla q, \tag{5.12}$$

where $\mathbf{div}\, v = 0$, $\rho^3\nabla q \in M(Q_0,\eta^s)$ where $M(Q_0,\eta)$ is space (4.16) and $y(0,x) = v(0,x) = y_0(x)$. We substitute (5.12) into (5.11) and verify that $v(t,x)$ satisfies the equation

$$\partial_t v(t,x) - \Delta v - \hat{v}\times \mathbf{rot}\, v + \nabla m = f_1(t,x),\ \mathbf{div}\, v = 0,\ v(0,x) = y_0(x), \tag{5.13}$$

$$m = (\partial_t q - \Delta q). \tag{5.14}$$

Now we can prove that (5.10) is epimorphism. Indeed let $(f, h, v_0, \theta_0) \in Z^s(Q) = F(Q, \eta^s) \times L_2(Q, \eta^s) \times V^1(\Omega) \times W_2^1(\Omega)$. By the definition of the space $F(Q, \eta)$ the decomposition

$$f = f_1 + \nabla f_2, \qquad f_1 \in (L_2(Q, \eta^s))^n, \qquad f_2 \in L_2(0, T; W_2^1(\Omega))$$

holds. After extension of f_1, f_2, h from Q to Q_0 and v_0, θ_0 from Ω to Ω_0 we get as was shown above the function (v, m, θ) which satisfy (5.13), (5.7), (5.8).Evidently, if we define

$$p = m + f_2 \tag{5.15}$$

then (v, p, θ) satisfy (5.6)-(5.8). After the restriction of (v, p, θ) at Q this triplet satisfies (5.6)-(5.8) which considered as defined on Q. We made the extension from Q to Q_0 and after that restriction from Q_0 to Q to have the equality (5.12) on Q with $\nabla q \in M(Q, \eta^s)$ (the restriction to Q allows us to take off the multiplier ρ^3 including $\rho^3 \nabla q \in M(Q_0, \eta^s)$. Since $\nabla q \in M(Q, \eta^s)$ then in virtue of (5.14), (5.15) $p \in L_2(0, T; W_2^1(\Omega))$.

Equality (5.12) and inclusions $\nabla q \in M(Q, \eta), y \in (\Theta(Q, \eta))^n$ give us that all terms in definition (2.19) of $\| \cdot \|_{V(Q,\eta)}$ for v are finite expect, may be $\|\partial_t v - \Delta v\|_{F(Q,\eta)}$. Let us show that this term is also finite. In virtue of (5.6), (5.15), (5.14)

$$\begin{aligned}\|\partial_t v - \Delta v\|_{F(Q,\eta)} = \|f_1 + \hat{v} \times \mathbf{rot}\, v + \nabla f_2 - \nabla p\|_{F(Q,\eta)} \le \|f_1 + \hat{v} \times \mathbf{rot}\, v\|_{(L_2(Q,\eta))^n} \\ + \|\nabla f_2 - \nabla p\|_{(L_2(Q,\eta))^n} \le c_1(\|f_1\|_{(L_2(Q,\eta))^n} + \\ \|\hat{v}\|_{(C(\bar{Q}))^n} \||\nabla v|\|_{(L_2(Q,\eta))^n} + \||\nabla(\partial_t q - \Delta q)|\|_{(L_2(Q,\eta))^n}) < \infty.\end{aligned}$$

Hence $v \in V(Q, \eta)$ and therefore we have proved that the operator (5.10) is epimorphism. We prove now that the operator

$$K : X^s(Q) \to Z^s(Q) \tag{5.16}$$

is compact, where K is define in (5.9). This assertion is reduced to prove compactness of the operator

$$K_1 : X^s(Q) \to (L_2(Q, \eta))^n \times L_2(Q, \eta), \tag{5.17}$$

where

$$K_1(v, p, \theta) = (-v \times \mathbf{rot}\, \hat{v} + \theta e_0, (\hat{v}, \nabla\theta) + (v, \nabla\hat{\theta}) + (v, e_0)). \tag{5.18}$$

We have

$$\int_{T-\delta}^{T}\int_{\Omega} e^{2\eta}(|v\times \mathbf{rot}\,\hat v|^2+|(\hat v,\nabla\theta)+(v,\nabla\hat\theta)+(v,e_0)|^2)\,dx\,dt$$
$$\le c_2(\|\hat v\|^2_{C(0,T;(C^1(\bar\Omega))^n)}+\|\hat\theta\|^2_{C(0,T;C^1(\bar\Omega))}+1)\int_{T-\delta}^{T}\int_{\Omega} e^{2\eta}(|v|^2+|\theta|^2+|\nabla\theta|^2)\,dx\,dt$$
$$\le c_3(\|\hat v\|^2_{V^{1,2(2)}(Q)}+\|\hat\theta\|^2_{W^{1,2(2)}(Q)}+1)\delta\int_{T-\delta}^{T}\int_{\Omega} e^{2\eta}((T-t)^{-2}(|v|^2+|\theta|^2)$$
$$+(T-t)^{-1}|\nabla\theta|^2)\,dx\,dt\le c_4c_3\delta(\|\hat v\|^2_{V^{1,2(2)}(Q)}+\|\hat\theta\|^2_{W^{1,2(2)}(Q)}) \quad (5.19)$$

uniformly with respect to

$$(v,\theta)\in\Phi\equiv\{(v,\theta):\|v\|^2_{V(Q,\eta)}+\|\theta\|^2_{\Theta(Q,\eta)}\le c_4\}.$$

Evidently, at $Q^\delta=(0,T-\delta)\times\Omega$ we have

$$V(Q^\delta,\eta)=V^{1,2(0)}(Q^\delta),\qquad \Theta(Q^\delta,\eta)=W^{1,2(0)}(Q^\delta),\qquad L_2(Q^\delta,\eta)=L_2(Q^\delta)$$

and by the Sobolev embedding theorem the operator

$$K: V^{1,2(0)}(Q^\delta)\times L_2(0,T;W_2^1(\Omega))\times W^{1,2(0)}(Q^\delta)\to (L_2(Q^\delta))^{n+1}\times V^1(\Omega)\times W_2^1(\Omega)$$

is compact. This property of operator K and (5.19) prove the compactness of operator (5.17), (5.18). Hence, all assumptions of Lemma 5.1 are true and by this lemma we get assertion of Theorem 5.3. ■

Now we can prove immediately

THEOREM 5.4. *Let $\hat v\in V^{1,2(2)}(Q),\hat\theta\in W^{1,2(2)}(Q)$ and a magnitude of parameter s is sufficiently large*. Then for an arbitrary data $(f,h,v_0,\theta_0)\in Z^s(Q)$ there exists a solution $(v,p,\theta)\in X^s(Q)$ of problem (2.9)-(2.13).*

Proof. By Theorem 3.1 for a dense set of data $(f,h,v_0,\theta_0)\in Z^s(Q)$ there exist the solution $(v,p,\theta)\in X^s(Q)$ of problem (2.9)-(2.11). By Theorem 5.3 the set of data (f,h,v_0,θ_0) for which there exists a solution is closed in $Z^s(Q)$. Hence, the set of data for which there exists a solution of problem (2.9)-(2.12) coincides with $Z^s(Q)$. ■

The proof of Theorem 1.1. Firstly we apply the right inverse operator theorem to problem (2.3)-(2.7). Let $\mathcal{A}$ be operator (2.8), (2.3), (2.5) and

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*More precisely, s simultaneously satisfy the conditions of Theorems 4.1, 5.1, 5.2

the spaces $X = X^s(Q)$, $Z = Z^s(Q)$ are defined in (2.20), (2.21), (2.15)-(2.19).Taking into account that $\mathcal{A}$ is a sum of linear and quadratic operators we can assert that continuous differentiability of operator (I.4.2) follows from Lemmas 2.1 and 2.2. Equality (I.4.3) is evident for $x_0 = 0$, $z_0 = 0$. At last, the assertion that operator

$$\mathcal{A}'(0) : X^s(Q) \to Z^s(Q)$$

is epimorphizm was proved in Theorem 5.4. So, all assumptions of the right inverse operator theorem are fulfilled and therefore there exists such $\epsilon > 0$, that for any initial data (w_0, τ_0) satisfying inequality

$$\|w_0\|^2_{V^1(\Omega)} + \|\tau_0\|^2_{W^1_2(\Omega)} \leq \epsilon$$

and for zero right sides of equation (2.3), (2.5) the problem (2.3)- (2.7) posses the solution $(v, q, \theta) \in V(Q, \eta^s) \times L_2(0, T; W^1_2(\Omega)) \times \Theta(Q, \eta^s)$. After returning from problem (2.3)-(2.7) to problem (1.1)-(1.4), (1.6) by change of variables (2.2) we get the assertion of Theorem 1.1. ■

Remark 5.1. As we pointed out in Remark 1.1 the smoothness condition on the given solution $(\hat{v}, \hat{p}, \hat{\theta})$ in Theorem 1.1 can be changed on more weak condition (1.14). This changement of condition would lead to the complication of Theorem 5.3 proof which we show below. That is why we approximate functions $\hat{v}, \hat{\theta}$ by a functions $\hat{v}_\epsilon \in V^{1,2(2)}(Q)$, $\hat{\theta}_\epsilon \in W^{1,2(2)}(Q)$:

$$\|\hat{v} - \hat{v}_\epsilon\|_{V^{1,2(1/2)}(Q) \cap (L_\infty(Q))^n} \leq \epsilon, \qquad \|\hat{\theta} - \hat{\theta}_\epsilon\|_{W^{1,2(1/2)}(Q) \cap L_\infty(Q)} < \epsilon, \tag{5.20}$$

where ϵ is sufficiently small. We can write:

$$B + K = B + R_\epsilon + K_\epsilon,$$

where

$$K_\epsilon(v, \theta) = (-v \times \mathbf{rot}\, \hat{v}_\epsilon + \theta e_0, (\hat{v}_\epsilon, \nabla\theta) + (v, \nabla\hat{\theta}_\epsilon) + (v, e_0), 0, 0),$$

$$R_\epsilon(v, \theta) = (-v \times \mathbf{rot}\, (\hat{v} - \hat{v}_\epsilon), (\hat{v} - \hat{v}_\epsilon, \nabla\theta) + (v, \nabla(\hat{\theta} - \hat{\theta}_\epsilon)), 0, 0).$$

In virtue of (5.20) the operator $R_\epsilon : X^s(Q) \to Z^s(Q)$ has a small norm and therefore the operator $B + R_\epsilon : X^s(Q) \to Z^s(Q)$ is epimorphism. The compactness of operator $K_\epsilon : X^s(Q) \to Z^s(Q)$ has been proved in Theorem 5.3. Hence by Lemma 5.1 the image of operator $B + R_\epsilon + K_\epsilon$ coincides with $Z^s(Q)$. ■

CHAPTER IV

EXACT CONTROLLABILITY OF HYPERBOLIC EQUATIONS

Introduction

In this chapter we study problems of exact boundary controllability of second order hyperbolic equations. In the first section we concern on the case of the linear hyperbolic equation. As in chapters I-III to solve controllability problem firstly we prove some a priori inequalities of Carleman inequality for the adjoint hyperbolic equation. To convert this inequality into an existence theorem we use duality arguments. Finally existence result is obtained under an assumption of existence of psevdoconvex function (see condition 1.1). The section 2 is devoted to the study of exact controllability problem for the one dimensional second order hyperbolic equations. In this case, the situation is more complicated compared to the linear case, and solvability of the problem depends on a behavior of nonlinear term at infinity.

Firstly the problem of exact controllability of linear hyperbolic equations was studied in the works of D.L. Russel and H. O. Fattorini. They introduced the following methods (see the excellent survey paper [56]).

1.Reduction of controllability problem to the moment problem.

2.Extension method to the whole space.

3 Use of harmonic analysis in control theory.

4.Introduction of stabilization operators.

5.Multiplier method.(see also [26], [39], [52] and references there in)

For controllability of hyperbolic equations with constant coefficients and control distributed on the whole boundary there is a method based on the Fourier and Radon transforms introduced by W. Littman in [54]. During past few years there has been a marked progress in controllability theory of linear hyperbolic equations. Two powerful methods were introduced. The first one based on the theory of pseudo-differential operators and microlocal analysis (see [6], [33], [34]). Existence theorems proved by this method under non-trapping condition are sharp. Unfortunately the use of pseudo-differential operators requirs that the coefficients of main part of hyperbolic equation and boundary of domain belongs to C^∞. The second method based on Carleman inequalities can be applied to wide class of evolution equations (see [61]). This method does not demand high smoothness of coefficients of hyperbolic equations. Despite of the results obtained in [60] and [61] are general they are not sharp, since the control is distributed on the whole boundary. Below the sharp Carleman's inequality is proved for particular case of second order hyperbolic equations. Existence theorems proved by second method demand the existance of a pseudoconvex function. There is a very interesting (and still open to the author's knowledge) question : Does the fulfillment of non-trapping condition imply the existence of pseudoconvex function?

Unfortunately, to the author's knowledge there are not so many results on controllability of semilinear hyperbolic equations. First there is the local existence theorems, similar to what we proved for the Navier-Stokes system. For the case of nonlinear term with sublinear growth there is an existence theorem due to I. Lasieska and R. Triggiani. The results, presented in section 2, are from [36], while results of section 1 are from [35] and [65].

1. Controllability of linear hyperbolic equations

Let $\Omega \subset R^n$ be a bounded domain with the boundary $\Gamma = \partial\Omega \in C^2$, Γ_0 be an arbitrary subdomain of Γ and $\Gamma_1 = \Gamma \backslash \Gamma_0$. Denote $Q_T =]0,T[\times\Omega, \Sigma_T =]0,T[\times\Gamma, \Sigma_T^0 =]0,T[\times\Gamma_0, \Sigma_T^1 =]0,T[\times\Gamma_1$. Denote $x = (x_0, x') = (x_0, x_1, ..., x_n)$, $\zeta = (\zeta_0, \zeta') = (\zeta_0, \zeta_1, ..., \zeta_n)$. Here we use notation $t = x_0$ for the time variable and x' for the space variable. Benefits of such notation will be clear below.

Let function $y(x_0, x')$ satisfy the boundary value problem

$$Py = \frac{\partial^2 y}{\partial x_0^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a(x')\frac{\partial y}{\partial x_j}\right) + \sum_{i=0}^{n} b_i(x')\frac{\partial y}{\partial x_i} + c(x')y = g \text{ in } Q_T, \quad (1.1)$$

$$y|_{\Sigma_T^1} = 0, \quad y|_{\Sigma_T^0} = u, \quad (1.2)$$

$$y(0,x') = v_0(x'), \quad \frac{\partial y}{\partial x_0}(0,x') = v_1(x'), \quad (1.3)$$

where functions v_0, v_1, g are given, and u is a control function. Let we have target functions v_2, v_3. To solve exact controllability problem one should find control u such that function y at moment T satisfy equations

$$y(T, x') = v_2(x'), \quad \frac{\partial y}{\partial x_0}(T, x') = v_3(x'). \tag{1.4}$$

We assume that coefficients of the linear operator P satisfy conditions

$$a_{ij} \in C^2(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad b_i \in C^1(\overline{\Omega}), \quad c \in L^\infty(\Omega), \tag{1.5}$$

where $i, j = 1, \cdots, n$ and the uniform ellipticity: There exists $\beta > 0$ such that

$$a(x', \zeta, \zeta) = \sum_{i,j=1}^{n} a_{ij}(x')\zeta_i\zeta_j \geq \beta|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^{n+1}, \quad x' \in \Omega. \tag{1.6}$$

For two an arbitrary smooth functions $\phi(x, \zeta), \psi(x, \zeta)$ we define Poisson bracket by the formula

$$\{\phi, \psi\} = \sum_{i=0}^{n} \left(\frac{\partial \phi}{\partial \zeta_i}\frac{\partial \psi}{\partial x_i} - \frac{\partial \phi}{\partial x_i}\frac{\partial \psi}{\partial \zeta_i} \right).$$

Denote by $p(x, \zeta)$ the main symbol of operator P:

$$p(x, \zeta) = \zeta_0^2 - \sum_{i,j=1}^{n} a_{ij}(x')\zeta_i\zeta_j.$$

To formulate our results we introduce the functional spaces

$$X_T = \{y(x_0, x') | y \in L^\infty(0, T; W_2^1(\Omega)), \ \frac{\partial y}{\partial x_0} \in L^\infty(0, T; L^2(\Omega))\},$$

$$Y_T = \{y(x_0, x') | y \in L^\infty(0, T; L^2(\Omega)), \ \frac{\partial y}{\partial x_0} \in L^\infty(0, T; W_2^{-1}(\Omega))\}$$

equipped with norms

$$\|y\|_{X_T} = \|y\|_{L^\infty(0,T;W_2^1(\Omega))} + \left\| \frac{\partial y}{\partial x_0} \right\|_{L^\infty(0,T;L^2(\Omega))},$$

$$\|y\|_{Y_T} = \|y\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial y}{\partial x_0}\right\|_{L^\infty(0,T;W_2^{-1}(\Omega))}.$$

Let us consider the boundary value problem

$$P^*z = \frac{\partial^2 z}{\partial x_0^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a(x')\frac{\partial z}{\partial x_j}\right) - \sum_{i=0}^n \frac{\partial}{\partial x_i}(b_i(x')z) + c(x')z = 0 \text{ in } Q_T, \tag{1.7}$$

$$z|_{\Sigma_T} = 0, \;\; z(0,x') = z_0(x'), \;\; \frac{\partial z}{\partial x_0}(0,x') = z_1(x'). \tag{1.8}$$

The following Theorem proved in [52],[44].

THEOREM 1.1. *Let (1.5), (1.6) be fulfilled. Then for any initial date* $z_0 \in W_0^1(\Omega)$, $z_1 \in L^2(\Omega)$ *there exist a unique solution of the problem (1.7), (1.8)* $z \in X_T$ *which satisfy inequality*

$$\|z\|_{X_T} + \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma_T)} \le c_1(\|z_0\|_{W_2^1(\Omega)} + \|z_1\|_{L^2(\Omega)}). \tag{1.9}$$

We assume that the following condition holds

CONDITION 1.1. *There exists a function* $\phi_0(x') \in C^2(\overline{\Omega})$ *such that*

$$\{a(x',\zeta',\zeta'),\{a(x',\zeta',\zeta'),\phi_0(x')\}\} < 0$$

$$\forall x' \in \overline{\Omega}, \zeta' \in R^n \setminus 0, \;\; \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} a(x',\zeta)\frac{\partial \phi_0}{\partial x_i} = 0$$

and inclusion holds

$$\Gamma_0 \supset \{x' \in \Gamma | \;\; a(x',\nu,\nabla\phi_0(x')) < 0\}.$$

We have

THEOREM 1.2. *Let* $z_0 \in W_2^1(\Omega)$, $z_1 \in L^2(\Omega)$, *(1.5),(1.6) and condition 1.1 be fulfilled. Then there exists a constant* T_0 *such that for any* $T \ge T_0$ *solutions of the problem (1.7),(1.8) satisfy the estimate*

$$\|z\|_{X_T} \le c_2(T)\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma_T^0)}. \tag{1.10}$$

Proof. Set $\phi(x) = \frac{\varepsilon}{T}(x_0 - T/2)^2 + \phi_0(x')$, where $\phi_0(x')$ is a function introduced in the Condition 1.1. The magnitude of parameters $\varepsilon \in (0,1)$ and $T > 0$ will be defined below.

Follow to [27] we introduce the notations:

$$P^{(j)}(x,\zeta) = \frac{\partial}{\partial\zeta_j} p(x,\zeta), \quad P^{(j,k)}(x,\zeta) = \frac{\partial^2}{\partial\zeta_j\partial\zeta_k} p(x,\zeta),$$
$$P_j(x,\zeta) = \frac{\partial}{\partial x_j} p(x,\zeta).$$

Set

$$q(x) = -c(x')z + \sum_{i=0}^{n} \frac{\partial}{\partial x_i}(b_i(x')z) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x') \frac{\partial z}{\partial x_j}.$$

By (1.7) equality holds

$$Lz = \frac{\partial^2 z}{\partial x_0^2} - \sum_{i,j=1}^{n} a_{ij}(x') \frac{\partial^2 z}{\partial x_i \partial x_j} = q \text{ in } Q_T. \tag{1.11}$$

Denote $u(x) = z(x)e^{-s\phi}, q_s(x) = qe^{-s\phi}$. It follows from (1.11) that

$$\Psi u = e^{-s\phi} L e^{s\phi} u = e^{-s\phi} Lz = q_s \text{ in } Q_T. \tag{1.12}$$

The short calculations gives equation

$$Lu + L_1 u = g_s \text{ in } Q_T, \quad u|_{\Sigma_T} = 0, \tag{1.13}$$

where

$$L_1 u = \sum_{i=0}^{n} s\phi_{x_i} P^{(i)}(x, \nabla u),$$
$$g_s(x) = q_s + \left(\sum_{i,j=1}^{n} a_{ij}(s^2 \phi_{x_i}\phi_{x_j} + s\phi_{x_i x_j}) - s\phi_{x_0 x_0} - s^2 \phi_{x_0}^2 \right) u.$$

Taking L_2- norm of both parts of (1.13_1) we obtain

$$\|g_s\|^2_{L^2(Q_T)} = \|Lu\|^2_{L^2(Q_T)} + \|L_1 u\|^2_{L^2(Q_T)} + 2(L_1 u, Lu)_{L^2(Q_T)}. \tag{1.14}$$

Let us transform the last term from right side of (1.14). We have

LEMMA 1.1. *The following equality holds*

$$(L_1u, Lu)_{L^2(Q_T)} = \int_\Omega \Bigg(\frac{\partial u}{\partial x_0} \sum_{i=0}^n s\phi_{x_i} P^{(i)}(x, \nabla u)$$

$$- s\phi_{x_0} p(x, \nabla u) + \frac{s}{2}\theta \frac{\partial u}{\partial x_0} u \Bigg) dx' \Big|_0^T$$

$$+ s \int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 a(x', \nu, \nu) a(x, \nu, \nabla\phi) d\Sigma - \frac{s}{2} \int_{Q_T} \Bigg(\{p, \{p, \phi\}\}(x, \nabla u) +$$

$$\sum_{k,i=0}^n P_k^{(k)}(x, \nabla u) \phi_{x_i} P^{(i)}(x, \nabla u) - \sum_{i=0}^n s\phi_{x_i} P^{(i)}(x, \nabla u)\theta u + g_s\theta u -$$

$$\sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} \theta u + a_{ij} \frac{\partial u}{\partial x_j} u \frac{\partial \theta}{\partial x_i} \right) - \frac{\partial u}{\partial x_0} \frac{\partial \theta}{\partial x_0} u \Bigg) dx, \quad (1.15)$$

where *

$$\theta(x) = \sum_{l,m=0}^n (\phi_{x_l x_m} P^{(l,m)}(x, \nabla u) + \phi_{x_l} P_m^{(l,m)}(x, \nabla u)).$$

Proof. Note, that since $u|_{\Sigma_T} = 0$, then

$$\frac{\partial u}{\partial x_i}\bigg|_{\Sigma_T} = \nu_i \frac{\partial u}{\partial \nu} \quad \forall\ i = 1, ..., n. \quad (1.16)$$

Bearing in mind (1.16) and integrating by parts the last term in right side of (1.14) we get

$$(L_1u, Lu)_{L^2(Q_T)} = \int_\Omega \frac{\partial u}{\partial x_0} \sum_{i=0}^n s\phi_{x_i} P^{(i)}(x, \nabla u) dx' \Big|_0^T$$

$$+ 2s \int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 a(x', \nu, \nu) a(x', \nu, \nabla\phi) d\Sigma$$

$$- \frac{s}{2} \int_{Q_T} \sum_{k,i=0}^n (P^{(k)}(x, \nabla u) \{\phi_{x_i x_k} P^{(i)}(x, \nabla u)$$

$$+ \phi_{x_k} (P_k^{(i)}(x, \nabla u) + P^{(i)}(x, \nabla \frac{\partial u}{\partial x_k}))\} +$$

$$\phi_{x_i} P_k^{(k)}(x, \nabla u) P^{(i)}(x, \nabla u)) dx. \quad (1.17)$$

*note that function θ is independent of u.

We claim, that the following identity holds

$$\sum_{k,i=0}^{n} P^{(k)}(x,\nabla u)\phi_{x_i}P^{(i)}(x,\nabla\frac{\partial u}{\partial x_k}) =$$
$$\sum_{k,i=0}^{n} \phi_{x_k}P^{(k,i)}(x,\nabla u)\left\{\frac{\partial}{\partial x_i}p(x,\nabla u) - P_i(x,\nabla u)\right\}. \quad (1.18)$$

Really, the short calculation gives

$$\sum_{k,i=0}^{n} P^{(k)}(x,\nabla u)\phi_{x_i}P^{(i)}(x,\nabla\frac{\partial u}{\partial x_k}) = \sum_{k,i=0}^{n} 4\phi_{x_i}\sum_{l,j=0}^{n} a_{jk}a_{il}\frac{\partial}{\partial x_l}\left(\frac{\partial u}{\partial x_k}\right)\frac{\partial u}{\partial x_j} =$$

$$\sum_{l,i=0}^{n} 4\phi_{x_i}a_{il}\sum_{k,j=0}^{n} a_{jk}\frac{\partial}{\partial x_l}\left(\frac{\partial u}{\partial x_k}\right)\frac{\partial u}{\partial x_j} = \sum_{i,l=0}^{n} 2\phi_{x_i}a_{il}\sum_{k,j=0}^{n} a_{jk}\frac{\partial}{\partial x_l}\left(\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j}\right)$$

$$= \sum_{l,i=0}^{n} \phi_{x_i}P^{(i,l)}(x,\nabla u)(\frac{\partial}{\partial x_l}p(x,\nabla u) - P_l(x,\nabla u)) =$$
$$\sum_{k,i=0}^{n} \phi_{x_i}P^{(i,k)}(x,\nabla u)(\frac{\partial}{\partial x_k}p(x,\nabla u) - P_k(x,\nabla u)).$$

Let us transform (1.17) using identity (1.18). As a result we have

$$(L_1u, Lu)_{L^2(Q_T)} = \int_\Omega \frac{\partial u}{\partial x_0}\sum_{i=0}^{n} s\phi_{x_i}P^{(i)}(x,\nabla u)dx'\Big|_0^T +$$
$$2s\int_{\Sigma_T}\left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$-\frac{s}{2}\int_{Q_T}\left\{\sum_{k,i=0}^{n} P^{(k)}(x,\nabla u)[\phi_{x_ix_k}P^{(i)}(x,\nabla u)\right.$$
$$+\phi_{x_i}P_k^{(i)}(x,\nabla u)] + \sum_{k,i=0}^{n}\phi_{x_i}P^{(i,k)}(x,\nabla u)\left(\frac{\partial}{\partial x_k}p(x,\nabla u) - P_k(x,\nabla u)\right)$$
$$\left. + \sum_{k,i=0}^{n}\phi_{x_i}P_k^{(k)}(x,\nabla u)P^{(i)}(x,\nabla u)\right\}dx. \quad (1.19)$$

Integrating by parts in (1.19) we get the equality

$$
\begin{aligned}
(L_1u, Lu)_{L^2(Q_T)} = &\int_\Omega \frac{\partial u}{\partial x_0}\sum_{i=0}^n s\phi_{x_i}P^{(i)}(x,\nabla u)dx'\Big|_0^T \\
&- s\int_\Omega \phi_{x_0}p(x,\nabla u)dx'\Big|_0^T + s\int_{\Sigma_T}\left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma \\
&- \frac{s}{2}\int_{Q_T}\sum_{k,i=0}^n (P^{(k)}(x,\nabla u)[\phi_{x_ix_k}P^{(i)}(x,\nabla u)+ \\
&\phi_{x_i}P^{(i)}_k(x,\nabla u)] - \phi_{x_i}P^{(i,k)}(x,\nabla u)P_k(x,\nabla u) \\
&- (\phi_{x_ix_k}P^{(i,k)}(x,\nabla u) + \phi_{x_i}P^{(i,k)}_k(x,\nabla u))p(x,\nabla u) \\
&+ \phi_{x_i}P^{(k)}_k(x,\nabla u)P^{(i)}(x,\nabla u))dx. \quad (1.20)
\end{aligned}
$$

Short calculation give the identity

$$
\begin{aligned}
\{p,\{p,\phi\}\}(x,\nabla u) = &\sum_{i,k=0}^n (P^{(k)}(x,\nabla u)[P^{(i)}(x,\nabla u)\phi_{x_ix_k} + P^{(i)}_k(x,\nabla u)\phi_{x_i}] \\
&- \phi_{x_k}P^{(k,i)}(x,\nabla u)P_i(x,\nabla u)). \quad (1.21)
\end{aligned}
$$

Using identity (1.21) one can rewrite (1.20) as follows

$$
\begin{aligned}
(L_1u, Lu)_{L^2(Q_T)} = &\int_\Omega \frac{\partial u}{\partial x_0}\sum_{i=0}^n s\phi_{x_i}P^{(i)}(x,\nabla u)dx'\Big|_0^T \\
&- s\int_\Omega \phi_{x_0}p(x,\nabla u)dx'\Big|_0^T + \\
&s\int_{\Sigma_T}\left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma - \frac{s}{2}\int_{Q_T}\Big(\{p,\{p,\phi\}\}(x,\nabla u) \\
&- \sum_{k,i=0}^n (\phi_{x_ix_k}P^{(i,k)}(x,\nabla u) + \phi_{x_i}P^{(i,k)}_k(x,\nabla u))p(x,\nabla u) + \\
&\sum_{k,i=0}^n P^{(k)}_k(x,\nabla u)\phi_{x_i}P^{(i)}(x,\nabla u)\Big)dx. \quad (1.22)
\end{aligned}
$$

Let us multiply equation (1.13) by θu scalarly in $L^2(Q_T)$ and integrate by parts. As result we obtain equality

$$\int_{Q_T} \theta p(x,\nabla u)dx = \int_\Omega \theta \frac{\partial u}{\partial x_0} u dx'\Big|_0^T - \int_{Q_T} g_s \theta dx + \int_{Q_T} \theta u \sum_{i=0}^n s\phi_{x_i} P^{(i)}(x,\nabla u)dx$$
$$+ \int_{Q_T} \left(\sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} \theta u + a_{ij} \frac{\partial u}{\partial x_j} u \frac{\partial \theta}{\partial x_i} \right) - \frac{\partial u}{\partial x_0} \frac{\partial \theta}{\partial x_0} u \right) dx. \quad (1.23)$$

Equalities (1.22), (1.23) imply (1.15).■

Let us continue the proof of Theorem 1.2. Denote

$$A_1 = \int_\Omega \frac{\partial u}{\partial x_0} \sum_{i=0}^n s\phi_{x_i} P^{(i)}(x,\nabla u)dx'\Big|_0^T - s\int_\Omega \phi_{x_0} p(x,\nabla u)dx'\Big|_0^T$$
$$+ \frac{s}{2}\int_\Omega \sum_{k,i=0}^n (\phi_{x_i x_k} P^{(i,k)}(x,\nabla u) + \phi_{x_i} P_k^{(i,k)}(x,\nabla u)) \frac{\partial u}{\partial x_0} u dx'\Big|_0^T,$$

$$A_2 = \frac{s}{2}\int_{Q_T} \left(\sum_{i=0}^n s\phi_{x_i} P^{(i)}(x,\nabla u)\theta u + \sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_i} \theta u + a_{ij} \frac{\partial u}{\partial x_j} u \frac{\partial \theta}{\partial x_i} \right) \right.$$
$$\left. - \frac{\partial u}{\partial x_0} \frac{\partial \theta}{\partial x_0} u - g_s \theta u \right) dx.$$

Short calculations give the estimate

$$|A_2| \le c_3(\|g_s\|^2_{L^2(Q_T)} + \sqrt{s}\|\nabla u\|^2_{L^2(Q_T)} + s^4\|u\|^2_{L^2(Q_T)}). \quad (1.24)$$

By virtue (1.14), (1.15) for every $s \ge 2$ we have

$$\|g_s\|^2_{L^2(Q_T)} \ge \|L_1 u\|^2_{L^2(Q_T)} + 2A_1 + 2A_2 + 2s\int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$

$$- s\int_{Q_T} \{p,\{p,\phi\}\}(x,\nabla u)dx - \|L_1 u\|_{L^2(Q_T)} \|\sum_{k=0}^n P_k^{(k)}(x,\nabla u)\|_{L^2(Q_T)} \ge$$
$$\frac{1}{2}\|L_1 u\|^2_{L^2(Q_T)} + 2A_1 + 2A_2 + 2s\int_{\Sigma_T} \left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$- \int_{Q_T} s\{p,\{p,\phi\}\}(x,\nabla u)dx - \frac{1}{2}\int_{Q_T} (\sum_{k=0}^n P_k^{(k)}(x,\nabla u))^2 dx. \quad (1.25)$$

By definition of the function ϕ the Poisson bracket $\{p,\{p,\phi\}\}$ can be written as follows

$$\{p,\{p,\phi\}\}(x,\nabla u) = \frac{8\varepsilon}{T}\left(\frac{\partial u}{\partial x_0}\right)^2 + \{p,\{p,\phi_0\}\}(x,\nabla u) =$$
$$\frac{8\varepsilon}{T}\left(\frac{\partial u}{\partial x_0}\right)^2 + \{a(x',\nabla u,\nabla u),\{a(x',\nabla u,\nabla u),\phi_0\}\}. \quad (1.26)$$

Note that condition 1.1 imply the existence of constants $\mu > 0$ and $c_4 > 0$ such that

$$c_4\left(\sum_{i=1}^n \phi_{0x_i}P^{(i)}(x,\zeta)\right)^2 - \{p,\{p,\phi_0\}\}(x,\zeta) \geq \mu a(x',\zeta,\zeta)\ \forall\zeta\in R^{n+1}. \quad (1.27)$$

By (1.25)-(1.27) for any $s > 2$ we have

$$\|g\|^2_{L^2(Q_T)} \geq \frac{1}{2}\|L_1u\|^2_{L^2(Q_T)} + 2A_1 + 2A_2 + 2s\int_{\Sigma_T}\left(\frac{\partial u}{\partial\nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$+\int_{Q_T}\left(s\mu a(x',\nabla u,\nabla u) - \frac{8\varepsilon s}{T}\left(\frac{\partial u}{\partial x_0}\right)^2\right.$$
$$\left. -\frac{1}{2}(\sum_{k=0}^n P_k^{(k)}(x,\nabla u))^2 - c_4 s(\sum_{i=1}^n \phi_{0x_i}P^{(i)}(x,\nabla u))^2\right)dx. \quad (1.28)$$

Let us multiply (1.13) by u scalarly in $L^2(Q_T)$. Integrating by parts we obtain

$$\int_{Q_T}\left(\frac{\partial u}{\partial x_0}\right)^2 dx = \int_{Q_T} a(x',\nabla u,\nabla u)dx + A_3, \quad (1.29)$$

where

$$A_3 = \int_\Omega u\frac{\partial u}{\partial x_0}dx'\Big|_0^T + \int_{Q_T}\left(uL_1u + \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}u - g_s u\right)dx. \quad (1.30)$$

By (1.29) the estimate holds,

$$4c_4\int_{Q_T}\frac{\varepsilon}{T}\left|(x_0-\frac{T}{2})\frac{\partial u}{\partial x_0}\sum_{k=1}^n\phi_{0x_k}P^{(k)}(x,\nabla u)\right|dx$$
$$\leq \varepsilon c_5\left\|\frac{\partial u}{\partial x_0}\right\|_{L^2(Q_T)}(\int_{Q_T}a(x',\nabla u,\nabla u)dx)^{\frac{1}{2}} \leq c_6\int_{Q_T}a(x',\nabla u,\nabla u)dx + c_7|A_3|, \quad (1.31)$$

there constants c_6, c_7 are independent on ε, T.

From (1.28) we deduce that

$$\|g\|^2_{L^2(Q_T)} \geq 2A_1 + 2A_2 + 2s\int_{\Sigma_T}\left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$+\int_{Q_T}\left\{\frac{7s\mu}{8}a(x',\nabla u,\nabla u) - \frac{8\epsilon s}{T}\left(\frac{\partial u}{\partial x_0}\right)^2 + c_4 s((\sum_{k=0}^{n}\phi_{x_k}P^{(k)}(x,\nabla u))^2\right.$$
$$\left. -(\sum_{k=1}^{n}\phi_{x_k}P^{(k)}(x,\nabla u))^2)\right\}dx = 2A_1 + 2A_2$$
$$+2s\int_{\Sigma_T}\left(\frac{\partial u}{\partial \nu}\right)^2 a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$+\int_{Q_T}\left\{\frac{7s\mu}{8}a(x',\nabla u,\nabla u) - \frac{8\epsilon s}{T}\left(\frac{\partial u}{\partial x_0}\right)^2\right.$$
$$+4s\left(\frac{\varepsilon}{T}\left(x_0 - \frac{T}{2}\right)\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2$$
$$\left.+\frac{4s\varepsilon}{T}(x_0 - T/2)\frac{\partial u}{\partial x_0}\sum_{k=1}^{n}\phi_{0x_k}P^{(k)}(x,\nabla u)\right\}dx. \quad \forall s > s_0. \tag{1.32}$$

Let us choose ε from interval $(0, \min\{1, \frac{T\mu}{64}, \mu/(c_6 8)\})$. The (1.28) imply the inequality. Using the estimate (1.31) in inequality (1.32) for any $s > s_0$ we obtain

$$\int_{Q_T} s\left(-\frac{8\varepsilon}{T}\left(\frac{\partial u}{\partial x_0}\right)^2 + \frac{3}{4}\mu a(x',\nabla u,\nabla u)\right)dx + 2s\int_{\Sigma_T} a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$\leq c_8(|A_1| + |A_2| + |A_3| + s^4\int_{Q_T} u^2 dx). \tag{1.33}$$

By virtue of (1.24), (1.30), (1.33) there exists a constant s_1 such that

$$\int_{Q_T}\frac{1}{4}s\mu\left(\left(\frac{\partial u}{\partial x_0}\right)^2 + a(x',\nabla u,\nabla u)\right)dx + 2s\int_{\Sigma_T} a(x',\nu,\nu)a(x',\nu,\nabla\phi)d\Sigma$$
$$\leq c_9(s(T+1)\int_{\Omega}(|\nabla u(T,x')|^2 + |\nabla u(0,x')|^2 + |u(T,x')|^2 + |u(0,x')|^2)dx'$$
$$+ s^4\int_{Q_T} u^2 dx) \quad \forall s \geq s_1. \tag{1.34}$$

Now we return in (1.34) from variable u to z. We obtain

$$\int_{Q_T} \frac{1}{4} s\mu \left(\left(\frac{\partial z}{\partial x_0} \right)^2 + a(x', \nabla z, \nabla z) \right) e^{-2s\phi} dx$$
$$+ 2s \int_{\Sigma_T} \left(\frac{\partial z}{\partial \nu} \right)^2 a(x', \nu, \nu) a(x', \nu, \nabla\phi) e^{-2s\phi} d\Sigma \leq c_9 (s(T+1) \int_\Omega (|\nabla z(T, x')|^2$$
$$+ |\nabla z(0, x')|^2 + |z(T, x')|^2 + |z(0, x')|^2) e^{-2s\phi(T,x')} dx'$$
$$+ s^4 \int_{Q_T} z^2 e^{-2s\phi} dx) \quad \forall s \geq s_2. \quad (1.35)$$

Now we take parameter $T_0 > 1$ such, that

$$\gamma = \min_{x' \in \overline{\Omega}} \phi(T_0, x') > \beta = \max_{x \in [T_0/4, 3T_0/4] \times \overline{\Omega}} \phi(x).$$

Then there exists a constant s_3 such, that for any $s \geq s_3$ the inequality holds

$$\int_{[T_0/4, 3T_0/4] \times \Omega} \frac{1}{8} s\mu \left(\left(\frac{\partial z}{\partial x_0} \right)^2 + a(x', \nabla z, \nabla z) \right) e^{-2s\phi} dx \geq$$
$$\int_{[T_0/4, 3T_0/4] \times \Omega} \frac{1}{8} s\mu \left(\left(\frac{\partial z}{\partial x_0} \right)^2 + a(x', \nabla z, \nabla z) \right) e^{-2s\beta} dx \geq$$
$$c_9 s(T_0+1) \int_\Omega (|\nabla z(T, x')|^2 + |\nabla z(0, x')|^2 + |z(T, x')|^2 + |z(0, x')|^2) e^{-2s\gamma} dx' \geq$$
$$c_9 s(T_0+1) \int_\Omega (|\nabla z(T, x')|^2 + |\nabla z(0, x')|^2 + |z(T, x')|^2 + |z(0, x')|^2) e^{-2s\phi(0,x')} dx'. \quad (1.36)$$

The (1.35), (1.36) imply the inequality

$$\int_{Q_{T_0}} \frac{1}{8} s\mu \left(\left(\frac{\partial z}{\partial x_0} \right)^2 + a(x', \nabla z, \nabla z) \right) e^{-2s\phi} dx +$$
$$2s \int_{\Sigma_{T_0}} \left(\frac{\partial z}{\partial \nu} \right)^2 a(x', \nu, \nu) a(x', \nu, \nabla\phi) e^{-2s\phi} d\Sigma \leq c_{11} s^4 \int_{Q_T} z^2 e^{-2s\phi} dx$$
$$\forall s \geq \max(s_2, s_3). \quad (1.37)$$

From this moment we set

$$s = \max(s_2, s_3).$$

Note that

$$a(x', \nu, \nabla\phi) = a(x', \nu, \nabla\phi_0) \ \forall x \in R^1_+ \times \partial\Omega.$$

Hence (1.37) and Condition 1.1 imply the estimate

$$\int_{Q_T} |\nabla z|^2 dx \le c_{12}(T) \left(\int_{\Sigma^0_{T_0}} \left(\frac{\partial z}{\partial \nu}\right)^2 a(x', \nu, \nu) a(x', \nu, \nabla\phi_0) e^{-2s\phi} d\Sigma \right.$$
$$\left. + \int_{Q_T} z^2 dx \right) \ \forall\, T \ge T_0. \quad (1.38)$$

Denote

$$E_T = \{(v_0, v_1) \in W^1_2(\Omega) \times L^2(\Omega) |\ P^* z = 0 \text{ in } Q_T, z|_{\Sigma_T} = 0,$$
$$\left.\frac{\partial z}{\partial \nu}\right|_{\Sigma^0_T} = 0,\ z(0, x') = v_0, \frac{\partial z}{\partial x_0}(0, x') = v_1\}.$$

Evidently, that

$$E_{T_1} \subset E_{T_2} \quad \forall\, T_1 > T_2 \ge T_0. \quad (1.39)$$

By virtue of (1.38), (1.39) on segment $[T_0, \infty)$ function $\ell(t) = dim E_t$ finite and decrease monotonically. But values of the function $\ell(t)$ belong to $\mathbf{Z}_+$. So, for any $\delta > 0$ there exist $T_1, T_2 \in [T_0, T_0 + \delta]$ such that

$$dim\, E_{T_1} = dim\, E_{T_2}. \quad (1.40)$$

By virtue of (1.39) the equality (1.40) imply

$$E_{T_1} = E_{T_2}.$$

But since the coefficients of operator P^* are independent on x_0, we have

$$E_{T_1} = E_\infty \quad \forall\, T_1 > T_0. \quad (1.41)$$

Let pair (v_0, v_1) is an arbitrary element of the space E_∞. Let us consider the boundary problem

$$P^* z = 0 \text{ in } R^1 \times \Omega,\ z|_{R^1 \times \partial\Omega} = 0,\ z(0, x') = v_0,\ \frac{\partial z}{\partial x_0}(0, x') = v_1. \quad (1.42)$$

Let us show, that

$$\frac{\partial z}{\partial \nu}\bigg|_{R^1\times\Gamma_0} = 0. \tag{1.43}$$

Let $\tau > 0$ be an arbitrary number. By (1.42) there exist a pair $(\tilde v_0, \tilde v_1) \in E_\infty$ such that equalities holds

$$P^*\tilde z = 0 \text{ in } Q_\infty,\ \tilde z|_{R^1\times\partial\Omega} = 0,\ \tilde z(0,x') = v_0,\ \frac{\partial \tilde z}{\partial x_0}(0,x') = v_1.$$

and

$$(\tilde z(\tau,x'), \frac{\partial \tilde z}{\partial x_0}(\tau,x')) = (v_0, v_1).$$

By Theorem 1.1 we have

$$z(x) = \tilde z(x_0 - \tau, x').$$

But since $(\tilde v_0, \tilde v_1) \in E_\infty$, then

$$\frac{\partial z}{\partial \nu}\bigg|_{[-\tau,0]\times\Gamma_0} = 0.$$

This equality proves (1.43).

In [62] proved, that any function z which satisfy (1.42), (1.43) equal zero in R^1. Hence

$$dim\, E_T = 0 \quad \forall\, T > T_0. \tag{1.44}$$

Let us assume that there exists a sequence of functions $z_k \in X$ which are the solution of problem (1.7), (1.8) such that

$$\|z_k\|_{W_2^1(Q_T)} = 1,\ \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma_T^0)} \to 0 \quad \text{as } k \to +\infty. \tag{1.45}$$

and

$$z_k \to z \text{ weakly in } W_2^1(Q_T),\ z_k \to z \text{ in } L^2(Q_T).$$

Passing to the limit as $k \to +\infty$ we obtain that function z satisfy (1.7), (1.8). Moreover by (1.9), (1.45) function z satisfy (1.43). Hence $z \equiv 0$. But this is impossible by virtue of (1.38), (1.45). This contradiction completes the proof of the theorem.■

We have

THEOREM 1.3. *Let (1.5), (1.6) be fulfilled. Then for any initial date* $v_0 \in L^2(\Omega)$, $v_1 \in W_2^{-1}(\Omega)$, $g \in L^1(0,T;W_2^{-1}(\Omega))$ *there exist a unique solution of the problem (1.1)- (1.3)* $y \in Y_T$ *and inequality holds*

$$\|y\|_{Y_T} \le c_1(\|v_0\|_{L^2(\Omega)} + \|v_1\|_{W_2^{-1}(\Omega)} + \|g\|_{L^1(0,T;W_2^{-1}(\Omega))}). \tag{1.46}$$

Proof. We define the linear functional $l(q)$ on the space $L^1(0,T;L^2(\Omega))$ by formula

$$l(q) = (g,z)_{L^2(Q_T)} + (v_0, z_{x_0}(0,\cdot))_{L^2(\Omega)} - (v_1, z(0,\cdot))_{L^2(\Omega)} \\ + (b_0 v_0, z(0,\cdot))_{L^2(\Omega)} - (\frac{\partial z}{\partial \nu_A}, u)_{L^2(\Sigma_T^0)}, \tag{1.47}$$

where functions q and z are connected by relations

$$P^* z = q \text{ in } Q_T,\ z|_{\Sigma_T} = 0,\ z(T,\cdot) = z_{x_0}(T,\cdot) = 0.$$

By Theorem 1.1 the functional l is bounded and the following estimate holds

$$\|l\| \le C(\|v_0\|_{L^2(\Omega)} + \|v_1\|_{W_2^{-1}(\Omega)} + \|v_2\|_{L^2(\Omega)} \\ + \|v_3\|_{W_2^{-1}(\Omega)} + \|g\|_{L^1(0,T;W_2^{-1}(\Omega))}). \tag{1.48}$$

Thus the functional l is continuous. It is known that any linear continuous functional on the space $L^1(0,T;L^2(\Omega))$ can be written as follows

$$l(q) = (y,q)_{L^2(Q_T)}, \tag{1.49}$$

where y is some function from the space $L^\infty(0,T;L^2(\Omega))$.

Using (1.49) we can rewrite (1.47):

$$(y,q)_{L^2(Q_T)} = (g,z)_{L^2(Q_T)} - (v_0, z_{x_0}(0,\cdot))_{L^2(\Omega)} + (v_1, z(0,\cdot))_{L^2(\Omega)} \\ (b_0 v_0, z(0,\cdot))_{L^2(\Omega)} - \left(\frac{\partial z}{\partial \nu_A}, u\right)_{L^2(\Sigma_T^0)}. \tag{1.50}$$

So function y satisfy (1.1_1) in the sense of theory of distributions. By (1.48), (1.49) and (1.50) we obtain

$$\|y\|_{L^\infty(0,T;L^2(\Omega))} \le C(\|v_0\|_{L^2(\Omega)} + \|v_1\|_{W_2^{-1}(\Omega)} + \|g\|_{L^1(0,T;W_2^{-1}(\Omega))}). \tag{1.51}$$

Since $y \in L^{\infty}(0,T;L^2(\Omega))$ it follows from (1.50) that $\frac{\partial^2 y}{\partial x_0^2} \in L^1(0,T;W_2^{-2}(\Omega))$. Moreover inequality holds

$$\left\|\frac{\partial^2 y}{\partial x_0^2}\right\|_{L^1(0,T;W_2^{-2}(\Omega))} \leq C(\|v_0\|_{L^2(\Omega)} + \|v_1\|_{W_2^{-1}(\Omega)} + \|g\|_{L^1(0,T;W_2^{-1}(\Omega))}). \tag{1.52}$$

Note that

$$\left\|\frac{\partial y}{\partial x_0}\right\|^2_{L^{\infty}(0,T;W_2^{-1}(\Omega))} \leq C(\left\|\frac{\partial^2 y}{\partial x_0^2}\right\|^2_{L^1(0,T;W_2^{-2}(\Omega))} + \|y\|^2_{L^{\infty}(0,T;L^2(\Omega))}).$$

This inequality together with (1.51), (1.52) gives (1.46).■

The following theorem, proved in [52] is a corollary of Hilbert Uniquenesses Method . But here we gives other version of its proof based on Reiz representation theorem for Hilbert spaces and Hanh-Banach extension theorem.

THEOREM 1.4. *Let (1.5), (1.6) be fulfilled and constant $T > 0$ such that for any solution of (1.7), (1.8) inequality (1.10) holds. Then for any initial date $v_0, v_2 \in L^2(\Omega)$, $v_1, v_3 \in W_2^{-1}(\Omega)$, $g \in L^1(0,T;W_2^{-1}(\Omega))$ there exist a solution of the problem (1.1)-(1.4) a pair $(y,v) \in Y_T \times L^2(\Sigma_T^0)$.*

Proof. Let us introduce the space F by formula

$$F = \{m(t,x)\ (t,x) \in \Sigma_T^0 | \text{there exists } z \in X_T,\ P^*z = 0 \text{ in } Q_T,$$
$$z|_{\Sigma_T} = 0, \frac{\partial z}{\partial \nu_A}|_{\Sigma_T^0} = m(t,x)\}.$$

And equipped it with norm $\|m\|_F = \|m\|_{L^2(\Sigma_T^0)}$.Note that all assumptions of Theorem 1.3 are fulfilled. This imply that there exists a constant C that

$$\|z\|_{Y_T} \leq C\|m\|_{L^2(\Sigma_T^0)}, \tag{1.53}$$

where functions z and m connected by relations

$$P^*z = 0 \text{ in } Q_T, \quad z|_{\Sigma_T} = 0, \quad \left.\frac{\partial z}{\partial \nu_A}\right|_{\Sigma_T^0} = m(t,x). \tag{1.54}$$

Thus (1.53) imply that F is a Banach space. Let us consider the linear functional $l(m)$ defined on the space F by formula

$$l(m) = (v_0, z_{x_0}(0,\cdot))_{L^2(\Omega)} - (v_1, z(0,\cdot))_{L^2(\Omega)} + (b_0v_0, z(0,\cdot))_{L^2(\Omega)}$$
$$-(b_0v_2, z(T,\cdot))_{L^2(\Omega)} - (v_2, z_{x_0}(T,\cdot))_{L^2(\Omega)} + (v_3, z(T,\cdot))_{L^2(\Omega)} + (g,z)_{L^2(Q_T)},$$

where functions $(v_0, v_1, v_2, v_3, g) \in L^2(\Omega) \times W_2^{-1}(\Omega) \times L^2(\Omega) \times W_2^{-1}(\Omega) \times L^1(0,T; W_2^{-1}(\Omega))$ are given, and function z and m connected by relation (1.54). By (1.53) this functional is correctly defined on F. The short calculations gives

$$\begin{aligned}
&|l(m)| \le \|v_0\|_{L^2(\Omega)}\|z_t(0,\cdot)\|_{L^2(\Omega)} + \|v_1\|_{W_2^{-1}(\Omega)}\|z(0,\cdot)\|_{W_2^1(\Omega)} \\
&+ \|v_0\|_{L^2(\Omega)}\|z(0,\cdot)\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}\|z(T,\cdot)\|_{L^2(\Omega)} + \|v_2\|_{L^2(\Omega)}\|z_t(T,\cdot)\|_{L^2(\Omega)} \\
&+\|v_3\|_{W_2^{-1}(\Omega)}\|z(T,\cdot)\|_{W_2^1(\Omega)}+\|g\|_{L^1(0,T;W_2^{-1}(\Omega))}\|z\|_{L^\infty(0,T;W_2^1(\Omega))} \le C(\|v_0\|_{L^2(\Omega)} \\
&+ \|v_1\|_{W_2^{-1}(\Omega)} + \|v_2\|_{L^2(\Omega)} + \|v_3\|_{W_2^{-1}(\Omega)} + \|g\|_{L^1(0,T;W_2^{-1}(\Omega))})\|m\|_{L^2(\Sigma_T^0)}.
\end{aligned}$$

Thus by Hanh-Banach extension theorem the functional l can be extended onto the hole space $L^2(\Sigma_T^0)$, keeping its norm. Applying the Reiz theorem on representation of a linear functional in Hilbert space we obtain that there exists a function $u(t,x) \in L^2(\Sigma_T^0)$ such that

$$l(m) = -(u,m)_{L^2(\Sigma_T^0)} \ \forall\, m \in F.$$

Set $u(t,x) = 0$ $(t,x) \in \Sigma_T^1$. For any $(v_0, v_1) \in L^2(\Omega) \times W_2^{-1}(\Omega)$ denote by $y(t,x) \in Y_T$ the unique solution of the following boundary value problem

$$Py = g \text{ in } Q_T,\ y|_{\Sigma_T^1} = 0,\ y|_{\Sigma_T^0} = u,\ y(0,x') = v_0(x'),\ \frac{\partial y}{\partial x_0}(0,x') = v_1(x') \tag{1.55}$$

which exists by Theorem 1.3. Let us prove that

$$y(T,\cdot) = v_2, \quad \frac{\partial y(T,\cdot)}{\partial x_0} = v_3.$$

Let function $z \in X_T$ be a solution of boundary value problem

$$P^*z = 0 \text{ in } Q_T, \quad z|_{\Sigma_T} = 0,\ z(T,\cdot) = z_0,\ z_{x_0}(T,\cdot) = z_1. \tag{1.56}$$

Multiplying (1.55_1) by z scalarly in $L^2(Q_T)$ and integrating by parts we have

$$\begin{aligned}
&(z_{x_0}(T,\cdot), y(T,\cdot))_{L^2(\Omega)} - (z(T,\cdot), y_{x_0}(T,\cdot))_{L^2(\Omega)} - (\frac{\partial z}{\partial \nu_A}, u)_{L^2(\Sigma_T^0)} \\
&\qquad + (b_0 z(T,\cdot), y(T,\cdot))_{L^2(\Omega)} - (b_0 y(0,\cdot), z(0,\cdot)_{L^2(\Omega)} \\
&\qquad\qquad - (z_{x_0}(0,\cdot), v_0)_{L^2(\Omega)} + (z(0,\cdot), v_1)_{L^2(\Omega)} = (g,z)_{L^2(Q_T)}.
\end{aligned}$$

By definition of the functional l we have

$$(z_{x_0}(T,\cdot), y(T,\cdot))_{L^2(\Omega)} - (z(T,\cdot), y_{x_0}(T,\cdot))_{L^2(\Omega)} + (b_0 y(T,\cdot), z(T,\cdot))_{L^2(\Omega)} \\ - (b_0 v_2, z(T,\cdot))_{L^2(\Omega)} - (z_{x_0}(T,\cdot), v_2)_{L^2(\Omega)} - (z(T,\cdot), v_3)_{L^2(\Omega)} = 0. \quad (1.57)$$

Since (z_0, z_1) are an arbitrary functions from the space $W_2^1(\Omega)\times L^2(\Omega)$ equality (1.57) imply

$$y(T,\cdot) = v_2, \quad y_{x_0}(T,\cdot) = v_3.$$

This proves our theorem. ■

The Theorem 1.3 and Theorem 1.4 imply

THEOREM 1.5. *Let (1.5),(1.6) and condition 1.1 be fulfilled. Then there exists a constant T_0 such that for $T > T_0$ and for any initial date $v_0, v_2 \in L^2(\Omega)$, $v_1, v_3 \in W_2^{-1}(\Omega)$, $g \in L^2(0,T;W_2^{-1}(\Omega))$ there exist a solution of the problem (1.1)-(1.4) a pair $(y,v) \in Y_T \times L^2(\Sigma_T^0)$.*

As a example of application of the Theorem 1.5 we consider the problem of exact boundary controllability of hyperbolic operator which in principal part be the same as the wave operator $\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Set

$$\Gamma_0 = \{x' \in \Gamma | \sum_{i=1}^n \nu_i(x_i - \overline{x}_i) > 0\},$$

where $\overline{x} \in R^n$ is an arbitrary point.

Let function $y(x)$ satisfy equations

$$\frac{\partial^2 y}{\partial x_0^2} - \Delta y + \sum_{i=0}^n b_i(x')\frac{\partial y}{\partial x_i} + c(x')y = g \text{ in } Q_T, \quad (1.58)$$

$$y|_{\Sigma_T^1} = 0, \quad y|_{\Sigma_T^0} = u, \quad (1.59)$$

$$y(0,x') = v_0(x'), \quad \frac{\partial y}{\partial x_0}(0,x') = v_1(x'), \quad (1.60)$$

$$y(T,x') = v_2(x'), \quad \frac{\partial y}{\partial x_0}(T,x') = v_3(x'). \quad (1.61)$$

We have

THEOREM 1.6. *Let (1.5) be fulfilled. Then there exists a constant T_0 such that for $T > T_0$ and for any initial date $v_0, v_2 \in L^2(\Omega)$, $v_1, v_3 \in W_2^{-1}(\Omega)$, $g \in L^2(0,T;W_2^{-1}(\Omega))$ there exist a solution of the problem (1.58)-(1.61) a pair $(y,v) \in Y_T \times L^2(\Sigma_T^0)$.*

Proof. We set $\phi_0(x') = -\sum_{i=1}^n (x_i - \overline{x}_i)^2$. The short calculations shows, that function $\phi_0(x')$ satisfy to Condition 1.1. Application of the Theorem 1.5 gives the statement of the Theorem 1.6.■

2. Boundary control by semilinear hyperbolic equations.

We consider the following problem

$$G(y) = \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + b_1(t,x)\frac{\partial y}{\partial x} + b_2(t,x)\frac{\partial y}{\partial t} - f(t,x,y) = 0 \quad \text{in} \quad Q, \quad (2.1)$$

$$y(t,0) = v_1(t), \quad y(t,L) = v_2(t), \quad (2.2)$$

$$y(0,x) = y_0(x), \quad \frac{\partial y(0,x)}{\partial t} = y_1(x), \quad (2.3)$$

where $y_0 \in W_2^1(0,L)$ and $y_1 \in L^2(0,L)$ are given functions. Suppose that we have the functions $y_2 \in W_2^1(0,L)$ and $y_3 \in L^2(0,L)$. It is required to find $v_1(t), v_2(t) \in W_2^1(0,T)$ such that at time T the following inequality hold:

$$y(T,x) = y_2(x), \quad \frac{\partial y(T,x)}{\partial t} = y_3(x). \quad (2.4)$$

Thus the solution of the problem (2.1)-(2.4) is a triple of functions $(y(t,x), v_1(t), v_2(t)) \in W_2^1(Q) \times W_2^1(0,T) \times W_2^1(0,T)$.

We set $K_1 = \{(t,x) \in Q | (L/2 - t) \geq |x - L/2|\}$, $K_2 = \{(t,x) \in Q | (t - T + L/2) \geq |x - L/2|\}$.

We shall assume the following condition:

Condition 2.1. In cone K_1 there exists a solution $y(t,x) \in W_2^1(K_1)$ of the Cauchy problem (2.1), (2.3). In cone K_2 there exists a solution $y(t,x) \in W_2^1(K_1)$ of the Cauchy problem (2.1), (2.4).

We have the following theorem

THEOREM 2.1. *Suppose that Condition 2.1 holds, $b_1, b_2 \in L^\infty(\overline{Q}), f \in C^1(\overline{Q} \times R^1)$ and that there is a number $p \geq 0$ such that*

$$|f(t,x,y)| + \Big|\frac{\partial f(t,x,y)}{\partial x}\Big| + \Big|\frac{\partial f(t,x,y)}{\partial t}\Big| \leq C(|y|^p + 1),$$

$$C_1|y|^{p+1} \leq \int_0^y f(t,x,\zeta)\,d\zeta + C_2 \quad \forall\ (t,x,y) \in Q \times R^1, C_1 > 0.$$

Then:

a) *if $T > L$, then there exist infinitely many solutions of the problem (2.1)-(2.4).*

b) *if $T \leq L$, then there exist $y_0, y_1, y_2, y_3 \in W_2^1(0,L)$ such that problem (2.1)-(2.4) has no solutions.*

Proof. We set $A = (0,0), B = (0,T), C = (L,T), D = (L,0)$. We denote by $E = (L/2, L/2), F = (L/2, T - L/2)$ the vertices of the cones K_1 and K_2, by K_3, K_4 the trapeziums $AEFB$, $DEFC$ and by S_1, S_2 the polygonal line $AEFB$ and $DEFC$. We claim that for any $u \in L^2(K_3)$, $z_0 \in W_2^1(S)$, $z_1 \in L^2(E,F)$ there exists a solution $z \in W_2^1(K_3) \cap L^\infty(Q)$ of the following problem:

$$G(z) = u \quad \text{in } K_3, \quad z|_{S_1} = z_0, \quad \frac{\partial z}{\partial x}|_{[E,F]} = z_1 \tag{2.5}$$

We scalar multiply (2.5_1) by $\frac{\partial z}{\partial x}e^{Nx}$ in $L^2(K_3)$. For sufficiently large N we obtain upon integrating by parts with respect to x and t the a estimate

$$\|z\|_{W_2^1(K_3)\cap L^\infty(K_3)} \leq c(\|u\|_{L^2(K_3)} + \|z_0\|^{p+2}_{W_2^1(S_1)} + \|z_1\|_{L^2(E,F)} + 1).$$

Thus the image of operator

$$P(z) = (G(z),\ z|_{S_1},\ \frac{\partial z}{\partial x}|_{[E,F]}).$$

is closed in the space $Y = L^2(K_3) \times W_2^1(S) \times L^2(E,F)$.

Let us introduce the operator G_1 by formula

$$G_1(z) = \frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} + b_1(t,x)\frac{\partial z}{\partial x} + b_2(t,x)\frac{\partial z}{\partial t} + c(t,x)z,$$

where $c \in L^\infty(K_3)$. Let us consider the following boundary problem

$$G_1 z = u \quad \text{in } K_3, \quad z|_{S_1} = z_0, \quad \frac{\partial z}{\partial x}|_{[E,F]} = z_1. \tag{2.6}$$

For all $(u, z_0, z_1) \in Y$ there exist the unique solution of problem (2.6) which satisfy the estimate

$$\|z\|_{W_2^1(K_3)\cup L^\infty(K_3)} \leq c\|(u, z_0, z_1)\|_Y.$$

Applying the implicit function theorem we find that $\operatorname{Im} P$ is open in the space Y. Thus

$$\operatorname{Im} P = Y.$$

We construct a solution of the problem (2.1)-(2.4) in the following manner. In the cones K_1, K_2 it coincides with the solutions of the problems (2.1), (2.3) and (2.1), (2.4) which exist by virtue of Condition **2.1**.

Let φ_0, φ_1 are an arbitrary functions which satisfy the following properties

$$\varphi_0 \in W_2^1(E, F), \quad \varphi_0(L/2, L/2) = y(L/2, L/2), \quad \varphi_1 \in L^2(E, F).$$

In the trapezium $AEFB$ we set $y(t, x)$ equal to the solution of problem (2.5) with $u = 0, \quad z_0 = \varphi_0$ on $[E, F]$, $z_0 = y$ on $[A, E] \cup [F, B]$, $z_1 = \varphi_1$. To find y in the trapezium $DEFC$, we solve the following problem in it that is analogous to (2.5):

$$G(y) = 0 \text{ in } K_4, \ y|_{S_2} = \phi_2, \ \frac{\partial z}{\partial x}|_{[E,F]} = \phi_1,$$

where $\phi_2 = \phi_0(t, x) \in [E, F]$ and $\phi_2 = y(t, x) \in [D, E] \cup [E, C]$.■

Condition 2.2 Let f does not depend on t, x and ether

$$\lim_{y\to+\infty} f(y) = -\infty, \quad \overline{\lim_{y\to-\infty}}|f(y)| < \infty$$

or

$$\overline{\lim_{y\to+\infty}}|f(y)| \leq \infty, \quad \lim_{y\to-\infty} f(y) < +\infty.$$

We have

THEOREM 2.2. *Let $T > 3L$, $b_1 = b_2 = 0$, $f(y) \in C^1(R^1)$ and suppose that condition 2.2 holds. Then there exists a solution of the problem (2.1) - (2.4).*

Proof. Let $\varphi(\zeta) \in W_2^1(0, l)$, $\varphi(0) = 0, \psi_n(\tau) = -n\tau, \tau \in [0, \tau_0]$, $\psi_n(\tau) = n\tau_0$, $\tau \in [\tau_0, l]$, where $0 < \tau_0 < l$. In the region $Q_1 = (0, l) \times (0, l)$ we consider the Goursat problem

$$\frac{\partial^2 z}{\partial\tau\partial\zeta} + f(z) = 0, \quad z(0, \zeta) = \varphi, \quad z(\tau, 0) = \psi_n.$$

Let $\lim_{y\to+\infty} f(y) = -\infty$. We claim that under condition 2 there exists $n_0(\tau_0,\varphi)$ such that problem (2.7) has a solution for all $n > n_0$. We set $f_k(z) = f(z),\ z \in (-\infty, k),\ f_k(z) = f(k) + f'(k)(z-k),\ z \in (k, +\infty)$. We denote by z_k the solution of problem (2.7) in which the function f in the equation is replaced by f_k:

$$\frac{\partial^2 z_k}{\partial\tau\partial\zeta} + f(z_k) = 0, \quad z_k(0,\zeta) = \varphi, \quad z_k(\tau, 0) = \psi_n. \tag{2.7}$$

Using Condition 2.2 it can be shown that there are numbers C and N_0 such that for all $n > n_0,\ z_k(t,x) \le C$ for all $(t,x) \in Q_1$. Scalar multiplying equation (2.7) by $\partial z_k/\partial\zeta$ and $\partial z_k/\partial\tau$ in $L^2(0,l)$ and using the upper estimate, we have

$$\|z_k\|_{W_2^1(Q_1)\cap L^\infty(Q_1)} \le C_1(\|\varphi\|_{W_2^1(0,l)} + \|\psi_n\|_{W_2^1(0,l)} + 1)\ \forall\ k \ge 0.$$

Consequently there is a number k_0 such that z_k is a solution of problem (2.7) for all $k > k_0$. Let $\epsilon \in (0, T-3L)$. We set $A = (0,0), B = (0,L), C = (0, L+\epsilon), D = (0,T), E = (L,T), F = (L, T-L), I = (L, 2L+\epsilon), J = (L,0)$. We denote by M the point of intersection of the characteristics BJ and CI, and by P the point of intersection of the characteristics CI and DF. The solution of problem (2.1) - (2.4) is constructed as follows. In the triangles ABJ and DEF it coincides with the solutions of problems (2.1), (2.3) and (2.1), (2.4) the existence of which is proved in [55]. To find y in rectangle $BDFI$ we solve the Goursat problem for equation (2.1) in the triangles JMI and CPD. The initial data for these problems are already defined on the intervals $[B,J]$ and $[D,F]$, while on the interval $[C,I]$ we set $y(t,x) = -N$. According to what has been proved above, we can choose the initial data on the polygonal lines BMC and IPF such that for some N both Goursat problems can be solved simultaneously.■æ

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CHAPTER III

EXACT CONTROLLABILITY FOR 2-D NAVIER-STOKES SYSTEM

Introduction

In this chapter we are concerned on the local exact controllability of the 2-D Navier-Stokes system, defined in a bounded domain $\Omega \subset R^2$ for the control distributed on the whole boundary $\partial\Omega$, or on it's part. The case of local distributed control is also studied.

The case of the local exact controllability with control distributed on the part of the boundary $\Gamma_1 \subset \partial\Omega$ is very interesting from theoretical point of view, and is important in practice. In this chapter we made only first step to solve this problem. Namely, the local exact controllability was proved for control distributed on part of the boundary Γ_1 if for complement $\Gamma_0 = \partial\Omega \backslash \Gamma_1$ the following boundary conditions holds

$$(y(x), \nu(x))\,|_{\Gamma_0} = 0, \qquad \mathbf{rot}\, y(x)|_{\Gamma_0} = 0,$$

where ν- outward normal to $\partial\Omega$, $\mathbf{rot}\, y = \partial_{x_2} y_1 - \partial_{x_1} y_2$.

We also consider the local exact controllability when control is a function $u(t, x)$ in the right hand side of the Navier-Stokes system with support in the given subdomain $\omega \subset \Omega$:

$$\mathbf{supp}\, u \subset (0, T) \times \omega.$$

The case of the locally distributed control is a basic case of this work. The results on local exact boundary controllability are deduced from the results on local exact distributed controllability.

This chapter is organized as follows. In section 1 we state exact controllability problems and formulate main results. In section 2 we introduce the stream function $\psi(t, x)$ and equation for it. Then using this equation and implicit function theorem, we reduce our original problem to the case of linear exact controllability problem. Sections 3-5 is devoted to prove solvability of this problem. In section 6 we prove main theorems. Note that in §3 we use the Carleman estimate for parabolic equation

$$-\frac{\partial\Delta\psi}{\partial t}+\Delta^2\psi=f$$

which is proved in section 7. This Carleman's estimate is slightly different from one proved in Chapter I. We close the section by mentioning some previous works on this subject. The cases of the 2-D and 3-D Navier-Stokes system with control on the whole boundary were studied in [17], [21] and [20]. The ϵ-controllability of the Stokes system was proved in [23],[24]. There is a very interesting nonlocal result on 2-D Euler equation due to Coron. In [7] and [8] for the Euler equation

$$\frac{\partial y}{\partial t}-(y,\nabla)y=f+u, \quad \text{div } y=0, \ y(0,\cdot)=v_0$$

the global ϵ-controllability and for some cases global exact controllability were proved. Thus additional argument was supplied for J.L. Lions conjecture on global ϵ-controllability of the Navier-Stokes system. The Coron's techniques of proof is qwite different from ours and relies on special structure of nonlinear term of Euler equation and it's invertibility respect to time.

1. The statement of the problem and formulation of main results.

1.1. In a bounded domain $\Omega\subset R^2$ with boundary $\partial\Omega\in C^\infty$ we consider the Navier-Stokes system

$$\partial_t y(t,x)-\Delta y(t,x)+(y,\nabla)\,y+\nabla p(t,x)=f(t,x), \tag{1.1}$$

$$\mathbf{div}\,y=\partial_{x_1}y_1+\partial_{x_2}y_2=0, \tag{1.2}$$

$(t,x)\in Q\equiv(0,T)\times\Omega,$ where $y(t,x)=(y_1(t,x),y_2(t,x))$ - velocity of fluid, $\nabla p(t,x)$ - pressure gradient, $\partial_t=\frac{\partial}{\partial t},\ \partial_{x_j}=\frac{\partial}{\partial x_j},\ (y,\nabla)y=\sum_{j=1}^2 v_j\partial_{x_j}y,\quad \Delta-$ Laplace operator, $f=(f_1,f_2)-$ density of external forces. We assume that

$$y(t,x)|_{t=0}=y_0(x), \tag{1.3}$$

where $y_0(x)=(y_{01},y_{02})$ is a given initial condition.

Let Γ_0 be an open subset of $\partial\Omega$,

$$\Gamma_0 \subset \partial\Omega, \quad \Gamma_1 = \partial\Omega \backslash \Gamma_0, \quad \Sigma = (0,T) \times \partial\Omega, \quad \Sigma_i = (0,T) \times \Gamma_i, \tag{1.4}$$

$i = 0, 1$. We set on Σ_0 the boundary conditions

$$(\mathbf{rot}\, y)\,|_{\Sigma_0} = 0, \quad (y, \nu)\,|_{\Sigma_0} = 0, \tag{1.5}$$

where $\nu = (\nu_1, \nu_2)$ is a vector field of outward normal to $\partial\Omega$, $(y, \nu) = y_1\nu_1 + y_2\nu_2$, $\mathbf{rot}\, y = \partial_{x_1} y_2 - \partial_{x_2} y_1$.

On the part of the lateral surface Σ_1 Dirichlet boundary conditions

$$y\,|_{\Sigma_1} = u, \tag{1.6}$$

are posed, where u is a boundary value of the vector field y, which in the our case is a control.

Since ∇p easily can be determinate from (1.1) by f, y below, if we say about solutions of system (1.1), instead of pair $(y, \nabla p)$ we are writing y.

Now we can set the problem of exact controllability. Let we have a solution $\hat{y} \in V^{1,2(1)}(Q)$ of equation (1.1), (1.2) and initial condition $y_0 \in V^2(\Omega)$ satisfying the inequality

$$\|\hat{y}(0,\cdot) - y_0\|^2_{V^2(\Omega)} < \varepsilon, \tag{1.7}$$

where $\varepsilon > 0$ is sufficiently small. Assume that for any connected component $\partial\Omega_j$ of the boundary $\partial\Omega$ the following equalities hold:

$$\int_{\partial\Omega_j} (y_0, \nu) d\sigma = 0, \quad \int_{\partial\Omega} (\hat{y}, \nu) d\sigma = 0. \tag{1.8}$$

Moreover the initial datum y_0 satisfy the compatibility conditions

$$\mathbf{rot}\, y_0|_{\Gamma_0} = 0, \quad (y_0, \nu)|_{\Gamma_0} = 0. \tag{1.9}$$

The local exact controllability problem is to find control $u \in W^{1,2(1/2)}(\Sigma_1)$, such that the solution $y \in V^{1,2(1)}(Q)$ of (1.1)-(1.3), (1.5), (1.6) satisfies for $t = T$ equation

$$y(t,x)|_{t=T} = \hat{y}(T,x). \tag{1.10}$$

Below we will prove

THEOREM 1.1. *Let Γ_0 be connected in $\partial\Omega$, $\Gamma_1 \equiv \partial\Omega\backslash\Gamma_0 \neq \emptyset$ ($\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$), $\hat{y} \in V^{1,2(1)}(\Omega)$ is a given solution (1.1), (1.2), $y_0 \in V^2(\Omega)$ and conditions (1.8),(1.9),(1.7) are fulfilled with sufficiently small $\varepsilon > 0$. Then one can find a control $u \in W^{1,2(1/2)}(\Sigma_1)$ such that there exists a solution of the problem (1.1)-(1.3), (1.5), (1.6) in the space $V^{1,2(1)}(Q)$ and for $t = T$ satisfy (1.10). Moreover inequality holds*

$$\|y(t,\cdot) - \hat{y}(t,\cdot)\|^2_{V^2(\Omega)} \leq c\, exp\left\{\frac{-k}{T-t}\right\} \qquad as \quad t \to T, \tag{1.11}$$

where $c > 0, k > 0$ some constants.

Remark 1.1. In particular, the set $\Gamma_0 \subset \partial\Omega$ may be empty i.e. the control u from (1.6) can be distributed on the whole lateral boundary Σ.

1.2. Now let us consider the Navier-Stokes equation, governed by distributed control, concentrated in some fixed subdomain $\omega \subset \Omega$ i.e. the case of local distributed control. Let $\Gamma_0 = \partial\Omega$, thus $\Sigma_0 = \Sigma$, $\Gamma_1 = \emptyset$, $\Sigma_1 = \emptyset$. We replace (1.1) by the equation

$$\partial_t y(t,x) - \Delta y(t,x) + (y,\nabla)\, y + \nabla p(t,x) = f(t,x) + u(t,x), \tag{1.12}$$

where $u(t,x) = (u_1,u_2)$ is a control, concentrated in the subdomain $\omega \subset \Omega$:

$$u(t,x) \equiv \chi_\omega(x) u(t,x), \quad \text{where} \quad \chi_\omega(x) = \begin{cases} 1, & x \in \omega, \\ 0, & x \notin \omega. \end{cases} \tag{1.13}$$

Let $\hat{y}(t,x) \in V^{1,2(1)}(Q)$ be a given solution of equations (1.1), (1.2) and $y_0(x) \in V^2(\Omega)$ be initial condition connected with $\hat{y}$ by inequality (1.7).

To solve exact controllability problem with locally distributed control we have to construct a control $u(t,x)$ such that solution of the problem (1.12), (1.2), (1.3), (1.5) for $t = T$ satisfies to equation (1.10).

For $\omega \subset \Omega$ set $Q^\omega = (0,T) \times \omega$.

THEOREM 1.2. *$\Gamma_0 = \partial\Omega$ be connected, $\hat{y}(t,x) \in V^{1,2(1)}(Q)$ be a given solution of (1.1), (1.2), (1.5) and $y_0(x) \in V^2(\Omega)$ satisfy (1.9), (1.7) with sufficiently small $\varepsilon > 0$. Then there exists a local distributed control $u(t,x) \in L_2(Q)$, $supp\, u \subset Q^\omega$, such that corresponding solution $y(t,x) \in V^{1,2(1)}(Q)$ of the problem (1.12), (1.2), (1.3), (1.5) exists and satisfy (1.11), (1.10).*

2. Reduction to a linear control problem.

2.1. To get rid of pressure we transform the Navier-Stokes system to the equation for stream function ψ which is connected with velocity field $y(t,x) = (y_1, y_2)$ by equations

$$\partial_{x_1}\psi = -y_2, \qquad \partial_{x_2}\psi = y_1. \tag{2.1}$$

Application the operator ∂_{x_2} to the first of equations (1.1) and operator $-\partial_{x_1}$ to the second one, adding of this two new equations yields the equation for the stream function:

$$\partial_t(-\Delta\psi(t,x)) + \Delta^2\psi + \partial_{x_2}((\partial_{x_1}\psi)\Delta\psi) - (\partial_{x_1}((\partial_{x_2}\psi)\Delta\psi) = u + g. \tag{2.2}$$

In the right-hand-side of (2.2) instead of $\mathbf{rot}\, f$ we substitute $u(t,x)+g(t,x)$, where $g = \mathbf{rot}\, f$ and u is a control. Just this form of the right-hand-side we need below. First of boundary condition (1.5) by virtue of (2.1) can be rewritten as follows

$$(-\Delta\psi)|_\Sigma = 0, \qquad \Sigma = (0,T)\times\partial\Omega. \tag{2.3}$$

The second one is transformed to the equation

$$\partial_\tau\psi|_\Sigma = 0, \tag{2.4}$$

where $\tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$ is the vector tangential to the $\partial\Omega$. By this equality

$$\psi|_{\partial\Omega} = \text{const},$$

and since $\partial\Omega$ is a connected set* function ψ can be determined by (2.1) up to constant arbitraryness. Without the loosing of generality we can assume that

$$\psi|_\Sigma = 0. \tag{2.5}$$

By virtue of (2.1), (1.5) instead of the initial condition (1.3) we have

$$\psi(t,x)|_{t=0} = \psi_0(x), \tag{2.6}$$

*Only here,deducing condition (2.5) we used connectedness of $\partial\Omega$. Therefore, below controllability problem for current function studied without assumption of connectedness of $\partial\Omega$.

where ψ_0 can be determined by the equalities

$$\partial_{x_1}\psi_0 = -y_{02}, \qquad \partial_{x_2}\psi_0 = y_{01}.$$

According to (1.9), (2.5) following compatibility conditions should be fulfilled

$$\psi_0|_{\partial\Omega} = 0, \qquad \Delta\psi_0|_{\partial\Omega} = 0. \tag{2.7}$$

Let us assume similarly to the section §1 that a solution $\hat{\psi}(t,x) \in W^{1,2(2)}(Q)$ of (2.2) with $u(t,x) \equiv 0$ and right-hand-side $g \in L_2(Q)$ are given. Moreover the function $\hat{\psi}(t,x)$ satisfies to the boundary conditions (2.4), (2.5) and the inequality

$$\left\|\hat{\psi}(0,\cdot) - \psi_0(\cdot)\right\|^2_{W_2^3(\Omega)} < \varepsilon, \tag{2.8}$$

holds, where parameter $\varepsilon > 0$ is sufficiently small. The local exact controllability problem consists in the constructing of such control $u(t,x) \in L_2(Q)$, $\operatorname{supp} u \subset Q^\omega$, such that the solution of boundary value problem (2.2)-(2.6) function $\psi(t,x)$ satisfy the condition

$$\psi(t,x)|_{t=T} = \hat{\psi}(t,x)|_{t=T}. \tag{2.9}$$

We are looking for solution $\psi(t,x)$ in the following form

$$\psi(t,x) = w(t,x) + \hat{\psi}(t,x), \tag{2.10}$$

where w is a new unknown function. Substitution of (2.10) in (2.2) - (2.6) yields the equation for the function w :

$$\partial_t(-\Delta w(t,x)) + \Delta^2 w + B(\hat{\psi} + w, w) + B(w, \hat{\psi}) = u(t,x), \tag{2.11}$$

where

$$B(\psi,\varphi) = \partial_{x_2}((\partial_{x_1}\psi)\Delta\varphi) - \partial_{x_1}((\partial_{x_2}\psi)\Delta\varphi). \tag{2.12}$$

This also gives boundary and initial conditions

$$(-\Delta w)|_\Sigma = 0, \quad w|_\Sigma = 0, \tag{2.13}$$

$$w(t,x)|_{t=0} = w_0. \tag{2.14}$$

Here $w_0(x) = \psi_0(x) - \hat{\psi}(0,x)$. By virtue of (2.10), (2.7), (2.8) we have

$$w_0|_{\partial\Omega} = \Delta w_0|_{\partial\Omega} = 0, \qquad \|w_0\|^2_{W_2^3(\Omega)} < \varepsilon. \tag{2.15}$$

In sections 2-7 will be proved

THEOREM 2.1. *Let* $\hat{\psi} \in W^{1,2(2)}(Q)$ *satisfies (2.2) with* $u \equiv 0$, *(2.3), (2.5), and initial condition* $w_0 \in W_2^3(\Omega)$ *satisfies (2.15) with sufficiently small* $\varepsilon > 0$. *Then one can find such control* $u \in L_2(Q)$, $\operatorname{supp} u \subset (0,T) \times \omega$, *that the corresponding solution* $w \in W^{1,2(2)}(Q)$ *of the problem (2.11) -(2.14) exists and satisfies equality*

$$w(t,x)|_{t=T} = 0. \tag{2.16}$$

2.2. To prove Theorem 2.1 we use the theorem on right inverse operator which was formulated in §4 of the chapter I.

In our case the space X consists of pairs $x = (w,u)$, and operator $A(x)$ defined by formula (2.11):

$$A(x) = (-\partial_t \Delta w + \Delta^2 w + B(\hat{\psi} + w, w) + B(w, \hat{\psi}) - u, \, w|_{t=0}) \tag{2.17}$$

(the condition $w|_{t=T} = 0$ and boundary conditions for w are included to the space X definition.) The space Z will be determined by set of pairs (2.17). Set $x_0 = (0,0), z_0 = (0,0)$. Evidently equality (I.4.3) is fulfilled.

To the check of the epimorphism condition of the operator (I.4.4) we write out equation

$$A'(x_0)x = z.$$

In our case this equation is as follows:

$$Lw - u \equiv \partial_t(-\Delta w) + \Delta^2 w + B(\psi, w) + B(w, \psi) - u = f, \tag{2.18}$$

where $u = \chi_\omega u$, the function χ_ω be determined in (1.13),

$$w|_\Sigma = \Delta w|_\Sigma = 0, \tag{2.19}$$

$$w|_{t=0} = w_0, \qquad w|_{t=T} = 0. \tag{2.20}$$

Note that if $x_0 = (0,0)$, $z_0 = (0,0)$ then function ψ from (2.18) coincides with $\hat{\psi}$. However we will prove solvability of problem (2.18)-(2.20) for an arbitrary function $\psi \in W^{1,2(1)}(Q)$. This result below give us possibility to strengthen the statement of the Theorem 2.1. (see Remark 6.1 below.)

Now, let us define the spaces X, Z which corresponding to problems (2.11)-(2.14) and (2.18)-(2.20). Set

$$\eta(t,x) \equiv \eta^\lambda(t,x) = (e^{\frac{4\lambda}{3}\|\beta\|_{C(\bar{\Omega})}} - e^{\lambda\beta(x)})/(T-t), \tag{2.21}$$

where parameter $\lambda > 0$ (magnitude of λ will be fixed below), function $\beta(x) \in C^2(\bar{\Omega})$ satisfies conditions

$$\nabla\beta(x) \neq 0,\ \forall x \in \Omega \backslash w', \qquad (\nabla\beta(x), \nu(x)) \leq 0,\ \forall x \in \partial\Omega, \tag{2.22}$$

$$\beta(x) \geq \ln 3,\ \forall x \in \bar{\Omega}, \qquad \min_{x\in\bar{\Omega}} \beta(x) > \frac{3}{4} \max_{x\in\bar{\Omega}} \beta(x). \tag{2.23}$$

Here $\omega' \subset\subset \omega \subset\subset \Omega$ are subdomains of Ω , $\nu(x)$ is outward normal to $\partial\Omega$. Existence of function $\beta \in C^2(\bar{\Omega})$ which satisfies (2.23) proved in Lemma I.1.1. For validity of (2.23) one has to increase β on sufficiently large constant. Let $\kappa(t,x) > 0,\ (t,x) \in Q$. Set

$$L_2(Q,\kappa) = \left\{ u(t,x), (t,x) \in Q : \|u\|^2_{L_2(Q,\kappa)} \equiv \int_Q \kappa^2(t,x)u^2(t,x)dxdt < \infty \right\} \tag{2.24}$$

Weight functions used below are constructed by means of the function (2.21). One of such weight functions defined by the formula θe^η, where

$$\theta(t,x) = \chi_\omega(x)(T-t)^{\frac{1}{2}} + (1-\chi_\omega(x))(T-t), \tag{2.25}$$

and χ_ω is a characteristic function of the set ω (see (1.13)). We introduce the space

$$\begin{aligned} Y(Q) \equiv \Big\{ & y(t,x) \in W^{1,2(2)}(Q) : y|_\Sigma = \Delta y|_\Sigma = 0, \\ & \|y\|^2_{Y(Q)} \equiv \left\|\partial_t(-\Delta y) + \Delta^2 y\right\|^2_{L_2(Q,\theta e^\eta)} + \|y\|^2_{W^{1,2(2)}(Q)} + \\ & + \int_Q \left(\sum_{|\alpha|\leq 2} (T-t)^{2(|\alpha|-1)} |D^\alpha_x \Delta y|^2 + \sum_{|\alpha|\leq 3} (T-t)^{2|\alpha|-6} |D^\alpha_x y|^2 \right) e^{2\eta} dxdt \Big\}, \end{aligned} \tag{2.26}$$

where functions θ, η are defined in (2.25), (2.21). Define also

$$\begin{aligned} U_\omega(Q) = \{ & u(t,x) \in L_2(Q) : \operatorname{supp} u \subset Q^\omega, \\ & \|u\|^2_{U_\omega(Q)} \equiv \int_{Q^\omega} (T-t)e^{2\eta} |u|^2 dxdt < \infty \Big\}, \end{aligned} \tag{2.27}$$

where remind $Q^\omega = (0,T)\times\omega$.

To apply the Theorem I.4.1 in order to establish solvability of (2.11), (2.13), (2.14), (2.16) we define spaces X , Z as follows

$$X = Y(Q)\times U_\omega(Q), \qquad Z = L_2(Q,\theta e^\eta)\times \hat{W}_2^3(\Omega), \tag{2.28}$$

where

$$\hat{W}_2^3(\Omega) = \left\{v(x)\in W_2^3(\Omega) : v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0\right\}. \tag{2.29}$$

We have

PROPOSITION 2.1. *Let the spaces X, Y defined in (2.28), operator $A(x)$ defined by formula (2.17). Then the mapping (2.17) continuously differentiated for any point $x_0\in X$.*

Proof. Definition (2.26)-(2.28) of the spaces $X,\ Z$ implies directly continuity of the operator

$$(w,u)\ \to\ (\partial_t(-\Delta w)+\Delta^2 w - u,\ w|_{t=0})\ :\ X\ \to\ Z$$

Being linear this operator belongs to $C^1(X,Z)$. The operator B from (2.17) defined by (2.12) is bilinear one. Thus to prove proposition 2.1 one has to establish continuity of bilinear operator

$$B : Y(Q)\times Y(Q)\to L_2(Q,\theta e^\eta). \tag{2.30}$$

Taking into account (2.12), (2.25)-(2.27), we get simple calculations

$$\|B(\varphi,\psi)\|^2_{L_2(Q,\theta e^\eta)} \le c\int\limits_Q \theta^2 e^{2\eta}(|\partial_{x_1}\psi|^2\,|\nabla\Delta\varphi|^2 + |\nabla\Delta\psi|^2\,|\partial_{x_1}\varphi|^2)dxdt \le$$

$$\le c(\|\nabla\psi\|^2_{C(\bar{Q})}\,\|\nabla\Delta\varphi\|^2_{L_2(Q,e^\eta)} + \|\nabla\varphi\|^2_{C(\bar{Q})}\|\nabla\Delta\psi\|^2_{L_2(Q,e^\eta)}) \le c\,\|\psi\|^2_{Y(Q)}\,\|\varphi\|^2_{Y(Q)}.$$

This estimate proves continuity of the operator (2.30).■

Evidently, equality (I.4.3) holds for mapping (2.17) when $x_0 = (w^0,u^0) = 0,\ z_0 = 0$. So, to apply Theorem I.4.1 now we have to establish only that image of operator (I.4.4) coincides with Z. This reduced to the proof of solvability of problem (2.18)-(2.20) for any $(f,w_0)\in Z$. Sections 3-5 are devoted to achievement of this aim.

3. Auxiliary extremal problem and solvability of it's optimal system

We start to prove the solvability of problem (2.18)-(2.20). Obviously, this is an ill-posed problem, because the number of boundary conditions on Σ is unsufficient and there are too much conditions on the time axis. That is why first of all we reduce it's solution to solvability of some coercive boundary problem. To write out it we consider the following extremal problem:

$$J(w,u) = \frac{1}{2}\int\limits_{Q} \frac{e^{2\eta}}{(T-t)^6} w^2(t,x)dxdt + \frac{1}{2}\int\limits_{Q^\omega} (T-t)e^{2\eta}u^2(t,x)dxdt \to \inf, \tag{3.1}$$

where a pair (w,u) satisfy (2.18)-(2.20). The optimality system of problem (3.1), (2.18)-(2.20) is as follows

$$L^*p \equiv \partial_t(\Delta p(t,x)) + \Delta^2 p + B_2^*(\psi,p) + B_1^*(p,\psi) = -\frac{e^{2\eta}}{(T-t)^6}w, \tag{3.2}$$

$$p|_\Sigma = \Delta p|_\Sigma = 0, \tag{3.3}$$

$$\chi_\omega(x)p(t,x) \equiv (T-t)e^{2\eta}u(t,x), \tag{3.4}$$

where $B_1^*(\cdot,\psi)$, $B_2^*(\psi,\cdot)$ are operators adjoint formaly to linear operators $B(\cdot,\psi)$, $B(\psi,\cdot)$ respectively. By definition (2.12) of operator $B(\psi,\varphi)$ we have

$$B_1^*(h,\psi) = \partial_{x_1}(\Delta\psi\partial_{x_2}h) - \partial_{x_2}(\Delta\psi\partial_{x_1}h), \tag{3.5}$$

$$B_2^*(\psi,h) = \Delta(\partial_{x_1}h\partial_{x_2}\psi - \partial_{x_2}h\partial_{x_1}\psi). \tag{3.6}$$

To deduce (formally) system (3.2)-(3.4) one can, for example, apply Lagrange principle (see [1]). Since, the fact that (3.2)-(3.4) is the optimality system of extremal problem (3.1), (2.18)-(2.20) never used below, here we do not prove this. Note, that can be obtain as in [21].

Now instead of problem (2.18) - (2.20) we are investigating the problem (2.18)- (2.20), (3.2)-(3.4). First of all let us get over from (2.18)-(2.20), (3.2)-(3.4) to the boundary problem with one unknown function $p(t,x)$. For this we express function w from (3.2) and function u from (3.4) and substitute these formulas into (2.18). As a result we have the equation for the function $p(t,x)$

$$-L((T-t)^6 e^{-2\eta}L^*p) - (T-t)^{-1}e^{-2\eta}\chi_w(x)p = f. \tag{3.7}$$

The boundary conditions (2.20) we rewrite using (3.2):

$$-(T-t)^6 e^{-2\eta} L^* p\Big|_{t=0} = w_0, \qquad -(T-t)^6 e^{-2\eta} L^* p\Big|_{t=T} = 0. \tag{3.8}$$

Now, let us prove that if parameter λ in the function $\eta \equiv \eta^\lambda$ sufficiently large the problem (3.7), (3.8), (3.3) has a unique solution. For this we need the Carleman's inequality of the following type.

THEOREM 3.1. *Let function $\eta = \eta^\lambda$ defined in (2.21), function β satisfy conditions (2.22), (2.23) and functions p, w satisfy (3.2), (3.3), where coefficient ψ from (3.2) belongs to the space $W^{1,2(2)}(Q)$. Then there exists $\hat{\lambda} > 0$ such that for $\lambda > \hat{\lambda}$ inequality holds*

$$I_\lambda(p) \equiv \int_Q ((T-t)^7 |\partial_t \Delta p|^2 + \sum_{|\alpha|\le 2} (T-t)^{3+2|\alpha|} |D_x^\alpha \Delta p|^2 +$$

$$+ \sum_{|\alpha|\le 4} (T-t)^{2|\alpha|} |D_x^\alpha \Delta p|^2) e^{-2\eta^\lambda} dxdt \le c(\int_Q (T-t)^{-6} |w|^2 e^{2\eta^\lambda} dxdt +$$

$$+ \int_{Q^w} (T-t)^{-1} |p|^2 e^{2\eta^\lambda} dxdt), \tag{3.9}$$

where the constant c depends on λ and $\|\psi\|_{W^{1,2(2)}(Q)}$. Moreover dependness of constant c on the second argument is continuous and monotonic.

The proof of this theorem, because of it's technically awkward will be given in the end of the paper in the section 7.

To define the generalized solution of problem (3.7), (3.8), (3.3) we introduce the space Φ_λ by formula

$$\Phi_\lambda = \left\{ p(t,x) : \|p\|^2_{\Phi_\lambda} \equiv I_\lambda(p) + \int_{Q^w} (T-t)^{-1} |p|^2 e^{2\eta^\lambda} dxdt + \right.$$

$$\left. + \int_Q (T-t)^6 e^{-2\eta^\lambda} |L^* p|^2 dxdt < \infty, \qquad p|_\Sigma = \Delta p|_\Sigma = 0 \right\}, \tag{3.10}$$

where functional $I_\lambda(p)$ defined in (3.9). Note that traces $p|_\Sigma$, $\Delta p|_\Sigma$ are correctly defined by virtue of the inequality $\|p\|_{\Phi_\lambda} < \infty$.

Definition 3.1. *Let* $f \in L_2(Q, e^{\eta^\lambda})$, $w_0 \in W_2^2(\Omega)$. *Function* $p(t,x) \in \Phi_\lambda$, *is called the generalized solution of problem (3.7), (3.8), (3.3) if for any* $q \in \Phi_\lambda$ *the inequality*

$$\int\limits_Q (T-t)^6 e^{-2\eta^\lambda} L^* p \cdot L^* q dx dt + \int\limits_{Q^w} (T-t)^{-1} e^{2\eta^\lambda} pq dx dt =$$

$$= -\int\limits_Q f q dx dt + \int\limits_\Omega w_0(x) \Delta q(0,x) dx, \tag{3.11}$$

holds where operator L^* *defined by formula (3.2).*

We have.

Theorem 3.2. *Let* $w_0 \in W_2^2(\Omega)$, $f \in L_2(Q, e^{\eta^\lambda})$, *where* $\lambda > \hat{\lambda}$ *and* $\hat{\lambda}$ *is defined in Theorem 3.1. Then there exists an unique generalized solution* p *of the problem (3.7), (3.8), (3.3). Function* p *satisfies (3.7) in the sense of distributions theory.*

Proof. Let us consider the bilinear form, defined on the space Φ_λ,

$$a(p,q) = \int\limits_Q (T-t)^6 e^{-2\eta^\lambda} L^* p \cdot L^* q dx dt + \int\limits_{Q^\omega} (T-t)^{-1} e^{-2\eta^\lambda} pq dx dt.$$

By virtue of Theorem 3.1 this form is continuous and coercive on Φ_λ :

$$a(p,q) \geq c \|q\|^2_{\Phi_\lambda}.$$

Obviously, the functional

$$F(q) = -\int\limits_Q f q dx dt + \int\limits_\Omega w_0(x) \Delta q(0,x) dx,$$

is continuous on Φ_λ. So by the Riez theorem on representation of linear functional there exists an unique solution $p \in \Phi_\lambda$ of the equation (3.2). Setting in (3.11) $q \in C_0^\infty(Q)$, we get the equality (3.7) in the distributions theory sense. ■

Now we are fix parameter λ, chosen in Theorem 3.2 till the end of section 6.

Let p be a generalized solution constructed in Theorem 3.2. Using the function p we can define function w by equation (3.2). Our aim is to prove that w is a solution of linear controllability problem (2.18) – (2.20_1). In the next section we will show, that w is a solution of boundary problem (2.18)-(2.20_1).

4. Properties of the function w.

We start from the following Lemma.

LEMMA 4.1. *Let $p(t,x)$ be a generalized solution of problem (3.7), (3.8), (3.3), constructed in Theorem 3.2, and functions p and u defined by (3.2), and (3.4) respectively. Then*

$$w \in L_2(Q,(T-t)^{-3}e^{\eta}), \quad u \in L_2(Q,(T-t))^{-3}e^{\eta}), \quad supp\, u \subset Q^{\omega},$$

and estimate

$$\int\limits_{Q} (T-t)^{-6}e^{2\eta}w^2(t,x)dxdt + \int\limits_{Q^{\omega}} (T-t)e^{2\eta}u^2(t,x)dxdt \leq$$

$$\leq c(\int\limits_{Q} e^{2\eta}f^2(t,x)dxdt + \int\limits_{\Omega} (w_0(x))^2dx), \tag{4.1}$$

holds where c dependes continuously and monotonicaly on $\|\psi\|_{W^{1,2(2)}(Q)}$ only. Moreover, functions w and u satisfy equation (2.18) in the distribution theory sense.

Proof. Let us substitute $p = q$ into (3.11), than in virtue (3.2), (3.4) express $(L^*p)^2$ by w^2, and p^2 by u^2. Applying to the right-hand-side of the obtained equality the Cauchy-Bynakovskii estimate, and doing simple transformations we get:

$$\left\|(T-t)^{-3}e^{\eta}w\right\|^2_{L_2(Q)} + \left\|(T-t)^{\frac{1}{2}}e^{\eta}u\right\|^2_{L_2(Q_w)} \leq \varepsilon(\left\|e^{-\eta}p\right\|^2_{L_2(Q)} +$$

$$+\int\limits_{\Omega}(\Delta p(0,x))^2dx) + \frac{c}{\varepsilon}(\|e^{\eta}f\|^2_{L_2(Q)} + \|w_0\|^2_{L_2(\Omega)}). \tag{4.2}$$

Evidently, magnitude $\int\limits_{\Omega}(\Delta p(0,x))^2dx$ can be bounded by the left-hand-side of inequality (3.9). So, estimating the term with ε in (4.2) by inequality

(3.9) and setting parameter ε sufficiently small, we obtain (4.1). Relations (3.2),(3.4) and (3.7) imply (2.18). ■

Assume, that

$$\psi \in C^\infty(\overline{Q}). \tag{4.3}$$

We intend to show, that $w \in W^{1,2(2)}(Q)$ and together with (2.18) it satisfies relations (2.19), (2.20). To prove this we firstly investigate boundary problem

$$\Delta y(t,x) = z(t,x), \qquad y|_\Sigma = 0, \tag{4.4}$$

$$-\partial_t z + \Delta z + \partial_{x_1}\psi \partial_{x_2} z - \partial_{x_2}\psi \partial_{x_1} z = g, \qquad \text{where} \quad g = f + u - B(w,\psi), \tag{4.5}$$

$$z|_\Sigma = 0, \qquad z|_{t=0} = \Delta w_0. \tag{4.6}$$

We have

LEMMA 4.2. *Let* $f \in L_2(Q, e^\eta)$, $\psi \in C^\infty(\bar{Q})$, $\psi|_\Sigma = 0$,

$$w_0 \in W_2^3(\Omega)\ , \qquad w_0|_{\partial\Omega} = 0, \qquad \Delta w_0|_{\partial\Omega} = 0, \tag{4.7}$$

and w *and* u *functions from Lemma 4.1. Then there exists the unique solution* $(y,z) \in W^{1,2(1)}(Q) \times W^{1,2(-1)}(Q)$ *of problem (4.4)-(4.6).*

Proof. Lemma's assumptions and the definition (2.12) of the operator B imply

$$g = f + u - B(w,\psi) \in L_2(0,T; W_2^{-1}(\Omega)).$$

Hence, for the solution of parabolic problem (4.5), (4.6) the inclusion $z \in W^{1,2(-1)}(Q)$ is true. Therefore the solution y of elliptic boundary problem (4.4) belongs to $L_2(0,T; W_2^3(\Omega))$. Differentiating (4.4) with respect to variable t, we obtain

$$\partial_t y \in L_2(0,T; W_2^1(\Omega)). \blacksquare$$

LEMMA 4.3. *Let all assumptions of Lemma 4.2 be fulfilled. Then function*

$$w \in W^{1,2(1)}(Q)$$

satisfies relations (2.18), (2.19), (2.20$_1$)*.*

Proof. To prove this lemma it is sufficiently to show that $w \equiv y$, where y is the function constructed in Lemma 4.2. We substitute into (4.5$_1$) $z = \Delta y$ and $g = f + y - B(w,\psi)$, then multiply obtained equality by $q \in$

$W^{1,2,(2)}(Q) \cap \Phi_\lambda$ which satisfies $q|_{t=T} = 0$ and integrate by parts in this equality taking into account (4.4_2), (4.6). As the result we obtain:

$$\int\limits_Q \left[y(\partial_t \Delta q + \Delta^2 q + B_2^*(\psi, q)) + B_1^*(q,\psi)w\right] dxdt - \int\limits_{Q^\omega} uqdxdt =$$

$$= \int\limits_Q fqdxdt - \int\limits_\Omega w_0(x)\Delta q(x)dx. \tag{4.8}$$

On the other hand we can express in (3.11) L^*p and p by w and u with help of (3.2),(3.4) and express L^*q using (3.2). This yields :

$$-\int_Q \left[w(\partial_t \Delta q + \Delta^2 q + B_2^*(\psi, q)) + B_1^*(q,\psi)w\right] dxdt + \int\limits_{Q^\omega} uqdxdt =$$

$$= -\int\limits_Q fqdxdt - \int\limits_\Omega w_0(x)\Delta q(x)dx. \tag{4.9}$$

Adding (4.8),(4.9) we get equality:

$$\int\limits_Q (y-w)(\partial_t \Delta q + \Delta^2 q + B_2^*(\psi, q))dxdt = 0. \tag{4.10}$$

The Lemma 4.4, proved below and (4.10) imply $y \equiv w$. ■

LEMMA 4.4. *For an arbitrary $h \in L_2(Q)$ there exists the unique solution $q \in W^{1,2(2)}(Q)$ of the problem*

$$\partial_t \Delta q + \Delta^2 q + B_2^*(\psi, q) = h, \quad q|_\Sigma = \Delta q|_\Sigma = 0, \quad q|_{t=T} = 0. \tag{4.11}$$

Proof. First of all, let us consider the boundary value problem

$$\partial_t \Delta q + \Delta^2 q = f, \qquad q|_\Sigma = \Delta q|_\Sigma = 0, \quad q|_{t=T} = 0. \tag{4.12}$$

To prove its unique solvability we represent problem (4.12) as a superposition of two boundary value problems

$$\Delta q = \varphi, \qquad q|_\Sigma = 0;$$

$$\partial_t\varphi+\Delta\varphi=f,\qquad \varphi|_\Sigma=0,\qquad \varphi|_{t=T}=\Delta q|_{t=T}=0$$

and use the classical results on their solvability. Note that resolving operator R of problem (4.12) act continuously from $L_2(Q)$ to $W^{1,2(2)}(Q)$. We are looking for solution of problem (4.11) in the following form $q=Rf$. Substitution this equality in (4.11) yields the equation for the function f into $L_2(Q)$:

$$f+B_2^*(\psi,Rf)=h. \tag{4.13}$$

Operator $B_2^*(\psi,\cdot):W^{1,2(2)}(Q)\to W^{1,2(-1)}(Q)$ defined in (3.6) is continuous. The compactness of imbedding $W^{1,2(-1)}(Q)\subset\subset L_2(Q)$ implies that the operator $B_2^*\circ R:L_2(Q)\to L_2(Q)$ is compact. Applying to equation (4.13) Fredholm alternative theorem, and taking into account that index of the operator $I+B_2^*\circ R:L_2(Q)\to L_2(Q)$ equals zero, we reduce the question on problem's (4.11) solvability to the proof of uniqueness only of its solution.

Scaling (4.11) with $h=0$ by function q scalarly in $L_2(\Omega)$ and taking into account (3.6) after the short calculations we obtain:

$$-\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |\nabla q(t,x)|^2\,dx+\int\limits_\Omega |\nabla q|^2\,dx=$$

$$\int\limits_\Omega\sum_{j=1}^2((\partial_{x_j}\partial_{x_1}q)\partial_{x_2}\psi-(\partial_{x_j}\partial_{x_2}q)\partial_{x_1}\psi+\partial_{x_1}q(\partial_{x_j}\partial_{x_2}\psi)-\partial_{x_2}q(\partial_{x_j}\partial_{x_1}\psi))\partial_{x_j}q)dx$$

$$\le c\int\left[(\sum_{i,j=1}^2\left|\partial_{x_i}\partial_{x_j}q\right|^2)^{\frac{1}{2}}|\nabla q|+|\nabla q|^2\right]dxdt\le \tag{4.14}$$

$$\le c(\varepsilon\int\limits_\Omega\sum_{i,j=1}^2\left|\partial_{x_i}\partial_{x_j}q\right|^2dx+\frac{1}{\varepsilon}\int\limits_\Omega|\nabla q|^2\,dx).$$

Since $q|_\Sigma=0$ then the following estimate for the Dirichlet problem of Laplace operator is true:

$$\sum_{i,j=1}^2\int\limits_\Omega\left|\partial_{x_i}\partial_{x_j}q(x,t)\right|^2dx\le c_1\int\limits_\Omega|\Delta q(x,t)|^2\,dx. \tag{4.15}$$

Substituting (4.15) into right-hand-side of (4.14) and setting parameter $\varepsilon>0$ sufficiently small we can carry out the term with Δq from right part to left part of the new inequality. Then integrating this inequality respect to variable t we have

$$\int_{\Omega} |\nabla q(t,x)|^2 \, dx \le c \int_t^T \int_{\Omega} |\nabla q(\tau,x)|^2 \, dx d\tau. \tag{4.16}$$

Applying to (4.16) the Gronwall's inequality we obtain $q \equiv 0$.■

LEMMA 4.5. *Let all assumptions of the Lemma 4.2 be fulfilled. Then* $w \in W^{1,2(2)}(Q)$.

Proof. By virtue of Lemma 4.3 $w \in W^{1,2(1)}(Q)$. So the function g defined in (4.5) belongs to $L_2(Q)$. Hence, solutions (z,y) of problem (4.5), (4.6) and (4.4) satisfy conditions $z \in W^{1,2(0)}(Q)$, $y \in W^{1,2(2)}(Q)$. By Lemma 4.3 $y = w$. Thus, $w \in W^{1,2(2)}(Q)$.■

Now we get rid of assumption (4.3).

THEOREM 4.1. *Let* $f \in L_2(Q, e^{\eta})$, $\psi \in W^{1,2(2)}(Q)$, $q|_{\Sigma} = 0$ *and* w_0 *satisfy (4.7). Then the functions* (w,u) *from Lemma 4.1 satisfy for any* $t \in (0,T)$ *estimates:*

$$\|\nabla w(t,\cdot)\|^2_{L_2(\Omega)} + \int_0^t \|\Delta w(t,\cdot)\|^2_{L_2(\Omega)} \, d\tau \le c(\|\nabla w_0\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(Q)} + \|f\|^2_{L_2(Q)}), \tag{4.17}$$

$$\|w\|^2_{W^{1,2(1)}(Q)} \le c(\|f\|^2_{L_2(Q)} + \|w\|^2_{W_2^3(\Omega)}), \tag{4.18}$$

where constant c *dependes on* $\|\psi\|_{W^{1,2(2)}(Q)}$ *only. Moreover* $w \in W^{1,2(2)}(Q)$ *and satisfies (2.18), (2.19),* (2.20_1).

Proof. Firstly we prove (4.17) for $\psi \in C^{\infty}(\bar{Q})$. In this case the statement of Lemma 4.3 holds true. Multiplying (2.18) scalarly in $L_2(Q)$ by w and integrating by parts taking into account (2.19), (2.20_1) and (2.12) we have

$$\frac{1}{2} \|\nabla w(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta w(\tau,\cdot)\|^2_{L^2(\Omega)} \, d\tau \le$$

$$\le \frac{1}{2} \|\nabla w_0\|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^t \|u + f\|_{W_2^{-1}(\Omega)} \|\nabla w\|_{L^2(\Omega)} \, d\tau + \int_0^t \int_{\Omega} B(\psi, w) w dx d\tau. \tag{4.19}$$

Taking into account (2.12), integrating by parts and using Cauchy- Bynakovskii inequality and Sobolev inequality we obtain

$$\left|\int_0^t\int_\Omega B(\psi,w)wdxdt\right| = \left|\int_0^t\int_\Omega (\partial_{x_1}w\partial_{x_2}\psi - \partial_{x_2}\psi\partial_{x_1}w)\Delta wdxdt\right| \le$$

$$\le \frac{1}{2}\int_0^t \|\Delta w\|^2_{L^2(\Omega)}\,dt + \frac{1}{2}\int_0^t\int_\Omega (|\partial_{x_1}w\partial_{x_2}\psi|^2 + |\partial_{x_2}\psi\partial_{x_1}w|^2)dxdt \le \quad (4.20)$$

$$\le \frac{1}{2}\int_0^t \|\Delta w\|^2_{L^2(\Omega)}\,dt + \frac{1}{2}\int_0^t \|\nabla\psi\|^2_{C(\bar\Omega)}\,\|\nabla w\|^2_{L^2(\Omega)}\,dt \le \frac{1}{2}\int_0^t \|\Delta w\|^2_{L^2(\Omega)}\,dt +$$

$$+c\int_0^t \|\psi(t)\|^2_{W_2^4(\Omega)}\,\|\Delta w\|^2_{W_2^4(\Omega)}\,dt.$$

Substituting (4.20) into (4.19), we will have after simple calculations

$$\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \|\Delta w(\tau,\cdot)\|^2_{L^2(\Omega)}\,d\tau \le \quad (4.21)$$

$$\le \|\nabla w_0\|^2_{L^2(\Omega)} + \int_0^t (\|u(\tau,\cdot)\|^2_{L^2(\Omega)} + \|f(\tau,\cdot)\|^2_{L^2(\Omega)})d\tau$$

$$+\, c\int_0^t (1 + \|\psi(\tau,\cdot)\|^2_{W_2^4(\Omega)})\,\|\nabla w(\tau,\cdot)\|^2_{L^2(\Omega)}\,d\tau.$$

The Gronwall's inequality and (4.21) imply (4.17).

Now let $\psi \in W^{1,2(2)}(Q)$. Let us consider the sequence of $\psi_k \in C^\infty(\bar Q)$ such that $\psi_k \to \psi$ in $W^{1,2(2)}(Q)$. Denote by p_k the generalized solution with the coefficient $\psi = \psi_k$ in definition (2.18) of the operator L. Let w_k and u_k are functions constructed by p_k with help of formulas (3.2), (3.4). Denote by

p, w and u the similar functions, which corresponds to the coefficient ψ. By virtue of (4.1) for functions w_k and u_k satisfy the inequality

$$\left\|(T-t)^{-3}e^{\eta}w_k\right\|^2_{L_2(Q)} + \left\|(T-t)^{\frac{1}{2}}e^{\eta}u_k\right\|^2_{L_2(Q^w)} \le c, \tag{4.22}$$

where constant c does not dependent on k. Inequality (3.9), written for functions p_k and w_k with inequality (4.22) yield:

$$\|p_k\|_{\Phi_\lambda} \le c, \tag{4.23}$$

where Φ_λ is the space defined in (3.10). So, without loss of generality we can assume that

$$p_k \rightharpoonup \hat{p} \qquad \text{weakly in } \Phi_\lambda. \tag{4.24}$$

By virtue of (4.24) we can pass to the limit in equality (3.11) where $p = p_k$, $\psi = \psi_k$ as $k \to \infty$. As a result we get equality (3.11) for $(\hat{p}, \psi)$. The uniqueness of generalized solution imply $\hat{p} = p$. Hence

$$p_k \rightharpoonup p \quad \text{in } \Phi_\lambda, \qquad w_k \rightharpoonup w \quad \text{in } L_2(Q, (T-t)^{-3}e^{\eta}), \tag{4.25}$$

$$u_k \rightharpoonup u \quad \text{in } L_2(Q^\omega, (T-t)^{\frac{1}{2}}e^{\eta}).$$

But (4.25), (4.17) imply that

$$w_k \rightharpoonup w \quad \text{in } L_2(0,T;W_2^2(\Omega)) \qquad \text{and} \qquad w|_\Sigma \equiv 0. \tag{4.26}$$

Since $w \in L_2(0,T;W_2^2(\Omega))$, $\psi \in W^{1,2(2)}(Q)$, then $\partial_{x_j}w \in L_2(0,T;W_2^1(\Omega))$, $\partial_{x_j}\Delta\psi \in C(0,T;L_2(\Omega))$. Thus (2.12) imply $B(w,\psi) \in L_2(0,T;W_2^{-\delta}(\Omega))$ for any $\delta > 0$, This means that function g defined in (4.5) belongs to $L_2(0,T;W_2^{-\delta}(\Omega))$. Hence, the solution (y,z) of the problem (4.4)-(4.6) is from the space $W^{1,2(2-\delta)}(Q) \times W^{1,2(-\delta)}(Q)$. Using the proof of the Lemma 4.3 with the obvious modification we get $w = y \in W^{1,2(2-\delta)}(Q)$. Again, consider the right hand side of (4.5) we have $g \in L_2(0,T;L_2(\Omega))$. This imply that $w = y \in W^{1,2(2)}(Q)$ and w satisfies (2.18), (2.19),(2.20$_1$). The arguments, used above coupled with estimates for elliptic and parabolic boundary value problems imply (4.18). ■

5. Solvability of linear control problem and estimation of it's solution.

The assertion on solvability of linear control problem (2.18)-(2.20) is a simple corollary of the Theorem 4.1 and Lemma 4.1. Really, by virtue of Theorem 4.1 the pair (w, u) satisfies (2.18), (2.19), (2.20_1) and we should prove validity of (2.20_2) only.

According to Theorem 4.1 and Lemma 4.1 the inclusions $w \in W^{1,2(2)}(Q) \subset C(0,T; L_2(\Omega))$, $w \in L_2(Q, (T-t)^{-3}e^{\eta})$ are true. Thus

$$\left\| (T-t)^{-3} e^{\eta(t,\cdot)} w(t,\cdot) \right\|^2_{L_2(\Omega)} \leq c(T-t)^{-1},$$

where $c > 0$ certain constant. This estimate yields

$$\|w(t,\cdot)\|^2_{L_2(\Omega)} \leq c\, e^{-\frac{k}{(T-t)}} \tag{5.1}$$

for some positive $k > 0$, $c > 0$. Inequality (5.1), obviously implies (2.20_2).

The aim of this section is to prove the inclusion $w \in Y(Q)$, where $Y(Q)$ is space (2.26). Below, in every statement of this section we assume, that assumptions of Theorem 4.1 hold and (w, u) are functions from Lemma 4.1 formulation.

Lemma 5.1. *For arbitrary $\varepsilon \in (0,1)$ and $t \in (0,T)$ the function w satisfies inequalities:*

$$\int_\Omega \frac{|\nabla w(t,x)|^2 e^{2\eta(t,x)}}{(T-t)^4} dx \leq \varepsilon \int_\Omega |\nabla \Delta w(t,x)|^2 e^{2\eta(t,x)} dx + \frac{c}{\sqrt{\varepsilon}} \int_\Omega \frac{|w|^2}{(T-t)^6} e^{2\eta(t,x)} dx, \tag{5.2}$$

$$\int_\Omega \frac{|\Delta w(t,x)|^2}{(T-t)^2} e^{2\eta(t,x)} dx \leq \varepsilon \int_\Omega |\nabla \Delta w(t,x)|^2 e^{2\eta(t,x)} dx + \frac{c}{\varepsilon^2} \int_\Omega \frac{|w|^2}{(T-t)^6} e^{2\eta(t,x)} dx, \tag{5.3}$$

where constant c does not depend on ε, t, w.

Proof. Integrating by parts, by virtue of (2.21) and Cauchy-Bynakovskii inequality we have

$$\int_\Omega \frac{|\nabla w(t,x)|^2}{(T-t)^4} e^{2\eta} dx = -\int_\Omega \frac{w(2(\nabla w, \nabla \eta) + \Delta w)}{(T-t)^4} e^{2\eta} dx =$$

$$= -\int\limits_{\Omega}\left(\frac{(\nabla w^2, \nabla\eta) + w\Delta w}{(T-t)^4}\right)e^{2\eta}dx = \int\limits_{\Omega}\left(\frac{w^2(2\left|\nabla\eta\right|^2 + \Delta\eta) - w\Delta w}{(T-t)^4}\right)e^{2\eta}dx \le$$

$$\le \int\limits_{\Omega}\left(\frac{\delta_1\left|\Delta w)\right|^2}{(T-t)^2} + \frac{c\left|w\right|^2}{\delta_1(T-t)^6}\right)e^{2\eta}dx. \tag{5.4}$$

By similar transformations we obtain

$$\int\limits_{\Omega}\frac{\left|\Delta w(t,x)\right|^2}{(T-t)^2}e^{2\eta}dx = -\int\limits_{\Omega}\frac{(\nabla\Delta w, \nabla w) + 2\Delta w(\nabla w, \nabla\eta)}{(T-t)^2}e^{2\eta}dx \le$$

$$\le \int\limits_{\Omega}\left(\frac{\delta_2}{2}\left|\nabla\Delta w\right|^2 + \frac{\left|\Delta w\right|^2}{2(T-t)^2} + \frac{c_1\left|\nabla w\right|^2}{2\delta_2(T-t)^4}\right)e^{2\eta}dx.$$

Carry over to the left part of this inequality the term containing Δw, we get

$$\int\limits_{\Omega}\frac{\left|\Delta w(t,x)\right|^2}{(T-t)^2}e^{2\eta}dx \le \int\limits_{\Omega}(\delta_2\left|\nabla\Delta w\right|^2 + \frac{c_1\left|\nabla w\right|^2}{\delta_2(T-t)^4})e^{2\eta}dx. \tag{5.5}$$

We estimate right side of (5.5) with the help of (5.4) and transfer the term containing $|\Delta w|^2$ from left side of the obtained inequality to the right side. As a result we get

$$\int\limits_{\Omega}\frac{\left|\Delta w(t,x)\right|^2}{(T-t)^2}e^{2\eta}dx \le \int\limits_{\Omega}\left(\frac{\delta_2\left|\nabla\Delta w\right|^2}{\left(1-\frac{c_1\delta_1}{\delta_2}\right)} + \frac{c_1 c\left|w\right|^2}{\delta_1\delta_2\left(1-\frac{c_1\delta_1}{\delta_2}\right)(T-t)^6}\right)e^{2\eta}dx. \tag{5.6}$$

Setting in (5.6) $\delta_1 = \frac{\delta_2}{2c_1}$ and denoting $\varepsilon = 2\delta_2$ imply (5.3).

Substitution (5.5) into right side of (5.4) and carry- out the term containing $|\nabla w|^2$ from right side of obtained inequality to the left side yield:

$$\int\limits_{\Omega}\frac{\left|\nabla w(t,x)\right|^2}{(T-t)^4}e^{2\eta}dx \le \int\limits_{\Omega}\left(\frac{\delta_1\delta_2}{\left(1-\frac{c_1\delta_1}{\delta_2}\right)}\left|\nabla\Delta w\right|^2 + \frac{c\left|w\right|^2}{\delta_1\left(1-\frac{c_1\delta_1}{\delta_2}\right)(T-t)^6}\right)e^{2\eta}dx. \tag{5.7}$$

Setting in (5.7) $\delta_2 = 2\delta_1 c_1$, and denoting $\varepsilon = 4c_1\delta_1^2$ we get (5.2). ■

We have

Lemma 5.2. *The function w satisfies the estimates*

$$\int_\Omega \sum_{|\alpha|\le 2} |D^\alpha w(t,x)|^2 e^{2\eta(t,x)}dx \le c\int_\Omega e^{2\eta(t,x)} \left(|\Delta w(t,x)|^2 + \frac{|w(t,x)|^2}{(T-t)^4}\right) dx, \tag{5.8}$$

$$\int_\Omega \sum_{|\alpha|\le 3} |D^\alpha w(t,x)|^2 e^{2\eta(t,x)}dx \le c\int_\Omega e^{2\eta(t,x)} \left(|\nabla\Delta w(t,x)|^2 + \frac{|w(t,x)|^2}{(T-t)^6}\right) dx, \tag{5.9}$$

where c does not depend on t, w.

Proof. Set $v = e^\eta w$. By (2.19)

$$\Delta v = q, \qquad v|_{\partial\Omega} = 0, \tag{5.10}$$

where

$$g = e^\eta(\Delta w + 2(\nabla\eta, \nabla w) + (|\eta|^2 + \Delta\eta)w). \tag{5.11}$$

As consequence of well known estimates for solutions of elliptic boundary problem applied to (5.10) and also to (5.11), (2.21), (5.4) with $\delta_1 = 1$ we get

$$\int_\Omega \sum_{|\alpha|\le 2} |D_x^\alpha v(t,x)|^2 dx \le c\int_\Omega |g|^2 dx \le c_1 \int_\Omega \left(|\Delta w|^2 + \frac{|w|^2}{(T-t)^4}\right) e^{2\eta}dx. \tag{5.12}$$

By Leibnitz formula of the product differentiation, (2.26) and (5.4) with $\delta = 1$ the inequality hold:

$$\begin{aligned}\int_\Omega \sum_{|\alpha|\le 2} |D_x^\alpha v(t,x)|^2 dx &\ge \int_\Omega \sum_{|\alpha|\le 2} e^{2\eta} |D_x^\alpha w(t,x)|^2 dx \\ &\qquad - c\int_\Omega e^{2\eta}\left(\frac{|\nabla w|^2}{(T-t)^2} + \frac{|w|^2}{(T-t)^4}\right) dx \\ &\ge \int_\Omega \sum_{|\alpha|\le 2} e^{2\eta} |D_x^\alpha w(t,x)|^2 dx - c_1\int_\Omega e^{2\eta}\left(|\Delta w|^2 + \frac{|w|^2}{(T-t)^4}\right) dx. \end{aligned} \tag{5.13}$$

Inequalities (5.12), (5.13) imply (5.8). Applying a known estimate to the solution v of the problem (5.10) we obtain by arguments similar to (5.12)

$$\int_\Omega \sum_{|\alpha|\le 3} |D_x^\alpha v(t,x)|^2 dx \le \int_\Omega \sum_{|\alpha|\le 1} |D_x^\alpha g|^2 dx \le c_1 \int_\Omega \left(|\nabla\Delta w|^2 + \frac{|w|^2}{(T-t)^6}\right) e^{2\eta}dx.$$

Similarly to (5.13) we have

$$\int_\Omega \sum_{|\alpha|\le 3} |D_x^\alpha v(t,x)|^2\,dx \ge$$

$$\ge \int_\Omega e^{2\eta} \sum_{|\alpha|\le 3} |D_x^\alpha w(t,x)|^2\,dx - c\int_\Omega e^{2\eta}\left(|\nabla\Delta w|^2 + \frac{|w|^2}{(T-t)^6}\right)dx.$$

The two last inequalities imply (5.9). ■

THEOREM 5.1. *Let $f \in L_2(Q, e^\eta)$, where $\eta = \eta^\lambda$, λ is a constant from Theorem 3.2, $\psi \in W^{1,2(2)}(Q)$, w_0 satisfy (4.7). Let p be a generalized solution (3.7),(3.8), (3.3), functions w and u defined by p using formulas (3.2), (3.4). Then the pair $(w,u) \in Y(Q)\times U_\omega(Q)$ is a solution of problem (2.18)-(2.20) and inequality*

$$\|w\|^2_{Y(Q)} + \|u\|^2_{U_\omega(Q)} \le c(\int_Q e^{2\eta}\,|f(t,x)|^2\,dxdt + \|w_0\|^2_{W_2^3(\Omega)}), \tag{5.14}$$

is true where $Y(Q)$, $U_\omega(Q)$ are Banach spaces (2.26), (2.27), and constant c does not depend on w, u, f, w_0.

Proof. Multiplying (2.18) by $-e^{2\eta}\Delta w$ and integrating in $Q_\tau = (0,\tau)\times\Omega$, $\tau\in(0,T)$ we obtain after simple transformations:

$$\int_{Q_\tau} e^{2\eta}\left(\frac{1}{2}\partial_t(\Delta w)^2 + |\nabla\Delta w|^2 - 2\Delta w(\Delta w, \eta^2 + \Delta\eta)\right)dxdt =$$

$$= \int_{Q_\tau} (f + u - B(\psi,w) - B(w,\psi))\,e^{2\eta}\Delta w dxdt. \tag{5.15}$$

We transform (5.15), bearing in mind (2.21), as follows

$$\int_{Q_\tau} e^{2\eta}\,|\nabla\Delta w|^2\,dxdt + \frac{1}{2}\int_\Omega e^{2\eta(\tau,x)}\,|\Delta w(\tau,x)|^2\,dx \le$$

$$\le \frac{1}{2}\int_\Omega e^{2\eta(0,x)}\,|\Delta w_0(x)|^2\,dx + \int_{Q_\tau}\left(\frac{c\,|\Delta w|^2}{(T-t)^2} + \frac{1}{2}\,|\nabla\Delta w|^2 + \frac{c_1\,|\Delta w|^2}{(T-t)^2} + \right.$$

$$+\varepsilon(T-t)^2\left(|f|^2+|u|^2+|B(\psi,w)|^2+|B(w,\psi)|^2\right)+\frac{c_2\,|\Delta w|^2}{\varepsilon(T-t)^2}\Bigg)e^{2\eta}dxdt. \tag{5.16}$$

Let us throw off the second term from the left part of (5.16) and pass to the limit as $\tau \to T$ in the new inequality. Then carry-out the term containing $|\nabla\Delta w|$ from right side of obtained inequality to the left side yields

$$\int_Q e^{2\eta}\,|\nabla\Delta w|^2\,dxdt \le c\,\|\Delta w_0\|^2_{L_2(\Omega)}+\int_Q\left(\left(c_{1+}\frac{c_2}{\varepsilon}\right)\frac{|\Delta w|^2}{(T-t)^2}+\right.$$

$$\left.+\varepsilon(T-t)^2\left(|f|^2+|u|^2+|B(\psi,w)|^2+|B(w,\psi)|^2\right)e^{2\eta}dxdt.\right. \tag{5.17}$$

Now let us estimate the terms containing the operator B. Taking into account (2.12) and continuity of the imbedding $W^{1,2(2)}(Q)\subset C(\bar{Q})$ we get

$$T^2\varepsilon\int_Q e^{2\eta}\,|B(\psi,w)|^2\,dxdt \le \varepsilon\,\|\nabla\psi\|^2_{C(\bar{Q})}\,T^2\int_Q e^{2\eta}\,|\nabla\Delta w|^2\,dxdt \le$$

$$\le \varepsilon c T^2\,\|\psi\|^2_{W^{1,2(2)}(Q)}\int_Q e^{2\eta}\,|\nabla\Delta w|^2\,dxdt. \tag{5.18}$$

By (2.12) and the Sobolev imbedding theorem we have

$$T^2\varepsilon\int_Q e^{2\eta}\,|B(w,\psi)|^2\,dxdt \le \varepsilon T^2\int_0^T\int_\Omega \|e^{\eta}\nabla w(t,\cdot)\|^2_{C(\Omega)}\,|\nabla\Delta\psi(t,x)|^2\,dxdt \le$$

$$\le \varepsilon T^2\,\|\nabla\Delta\psi\|^2_{L_\infty(0,T;L_2(\Omega))}\int_Q\sum_{|\alpha|\le 2}|D_x^\alpha(e^{\eta}\nabla w(t,x))|^2\,dxdt \le$$

$$\le \varepsilon T^2 c\,\|\psi\|^2_{W^{1,2(2)}(Q)}\int_Q e^{2\eta}\Bigg(\sum_{|\alpha|\le 3}|D_x^\alpha w|^2+\frac{1}{(T-t)^2}\sum_{|\alpha|\le 2}|D_x^\alpha w|^2$$

$$+\frac{1}{(T-t)^4}\,|\nabla w|^2\Bigg)dxdt. \tag{5.19}$$

Applying to the right of (5.19) inequality (5.8) multiplied on $(T-t)^{-2}$ and estimates (5.9), (5.3) we obtain

$$T^2\varepsilon \int\limits_Q e^{2\eta} |B(w,\psi)|^2 \, dxdt \le \varepsilon T^2 c_1 \|\psi\|^2_{W^{1,2(1)}(Q)} \int\limits_Q e^{2\eta} \left(|\nabla\Delta w|^2 + \frac{|w|^2}{(T-t)^6} \right) dxdt. \tag{5.20}$$

Let us set parameter $\varepsilon \in (0,1)$ such small, that coefficients in the right sides of (5.18), (5.20) satisfy conditions

$$\varepsilon c T^2 \|\psi\|^2_{W^{1,2(2)}(Q)} < \frac{1}{8}, \qquad \varepsilon T^2 c_1 \|\psi\|^2_{W^{1,2(2)}(Q)} < \frac{1}{8}. \tag{5.21}$$

Substituting (5.18), (5.20) into right side of (5.17), taking into account (5.21), (5.3), we obtain

$$\int\limits_Q e^{2\eta} |\nabla\Delta w|^2 \, dxdt \le c \|\Delta w_0\|^2_{L_2(\Omega)} + \int\limits_Q e^{2\eta} \left(\frac{1}{2} |\nabla\Delta w|^2 + |f|^2 + \right.$$

$$\left. +(T-t)|u|^2 \right) dxdt + c\|w_0\|^2_{L_2(\Omega)},$$

By (4.1) this inequality implies the estimate

$$\int\limits_Q e^{2\eta} |\nabla\Delta w|^2 \, dxdt \le c \left(\|w_0\|^2_{W_2^2(\Omega)} + \int\limits_Q e^{2\eta} |f|^2 \, dxdt \right). \tag{5.22}$$

Inequalities (5.9), (5.8) multiplied by $(T-t)^{-2}$, (5.2), (5.22) and (4.1) yield the estimate

$$\int\limits_Q e^{2\eta} \left(\sum_{k=0} (T-t)^{2k-6} \sum_{|\alpha|=k} |D_x^\alpha w|^2 \right) dxdt \le c \left(\|w_0\|^2_{W_2^2(\Omega)} + \int\limits_Q e^{2\eta} |f|^2 \, dxdt \right)$$

$$\le c \left(\int\limits_Q e^{2\eta} |f|^2 \, dxdt + \|w_0\|^2_{W_2^2(\Omega)} \right). \tag{5.23}$$

By virtue of (5.18), (5.20), (5.22), (4.1) equation (2.18) imply the inequality

$$\int\limits_Q e^{2\eta}\left(\partial_t\left(-\Delta w\right)+\Delta^2 w-u\right)^2 dxdt\le$$
$$\le c\int\limits_Q e^{2\eta}\left(|f|^2+|B(\psi,w)|^2+|B(w,\psi)|^2\right)dxdt\le \tag{5.24}$$
$$\le c\left(\int\limits_Q e^{2\eta}\,|f|^2\,dxdt+\|w_0\|^2_{W_2^2(\Omega)}\right).$$

Estimate (5.14) follows from (5.23), (5.24), (4.18), (4.1). ■

6. Proof of the main results

Proof of the Theorem 2.1. The reduction of the problem (2.11)-(2.14), (2.16) to the equation $A(x)=z$ introduced in the Theorem I.4.1 was described in details below the formulation of the Theorem 2.1. In addition condition (I.4.3) is trivially fulfilled for $x_0=(0,0), z_0=(0,0)$, continuous differentiability of the mapping (I.4.2) was checked in Proposition 2.1, and coincidence of the image of the operator (I.4.4) with the space Z was proved in Theorem 5.1. Thus, all assumptions of Theorem 2.2 are checked and according to this theorem there exists a solution $(w,u)\in X$ of the problem (2.11)-(2.14), (2.16), where X is the space defined in (2.26)-(2.28). Since by virtue of (2.26), (2.27) $W^{1,2(2)}(Q)\times\{u\in L_2(Q):\operatorname{supp}u\subset Q^\omega\}\supset X$ the Theorem 2.1 is proved ■

Remark 6.1. Since the component w of the solution of the problem (2.11)-(2.14), (2.16) belongs to space (2.27) the following estimate for the function w is true:

$$\|w(t,\cdot)\|_{W_2^3(\Omega)}\le c\exp\left(-\frac{k}{(T-t)}\right)\qquad\text{as}\quad t\to T. \tag{6.1}$$

Remark 6.2. Besides the solvability of the problem of local controllability, proved in Theorem 2.1, the statement on the convergence rate of iteration process, similar to the rate of convergence of classical Newton's method holds true.

More precisely, let $(w^1,\ u^1)\in Y(Q)\times U_\omega(Q)$ is a solution of the linear problem (2.18) - (2.20) constructed in the Theorem 5.1 with initial datum $\psi=\hat{\psi}, f\equiv 0$. We suppose that $n+1-$approximation $(w^{n+1},\ u^{n+1})$

constructed by means of $n-$approximation $(w^n,\ u^n)$ with help of formula $w^{n+1} = w^n + y^n$, where (y^n, u^{n+1}) is the solution of linear controllability problem:

$$L\left(\hat{\psi} + w^n\right) y^n \equiv \partial_t\left(-\Delta y^n\right) + \Delta^2 y^n \\ + B\left(\hat{\psi} + w^n, y^n\right) + B\left(y^n, \hat{\psi} + w^n\right) = u^{n+1} + f^n,$$

$$y^n|_\Sigma = \Delta y^n|_\Sigma = 0, \qquad y^n|_{t=0} = y^n|_{t=T} = 0.$$

Here $u^{n+1} = \chi_\omega u^{n+1}$ (χ_ω is the function defined in (1.13)) and f^n defined by formula

$$f^n = \partial_t\left(-\Delta w^n\right) + \Delta^2 w^n + B\left(\widehat{\psi} + w^n, w^n\right) + B\left(w^n, \widehat{\psi} + w^n\right).$$

Applying to indicated iterations the classic estimates of the abstract Newton's method, taking into account estimates obtained in section 5 one can prove existence of constant c which depends on $\|\hat{\psi}\|_{W^{1,2(2)}(Q)}$ such that for sufficiently small ε from (2.15) the inequality holds

$$c\left(\left\|w - w^1\right\|_{Y(Q)} + \left\|u - u^1\right\|_{U_\omega(Q)}\right) < 1$$

and

$$\left\|w - w^{n+1}\right\|_{Y(Q)} + \left\|u - u^{n+1}\right\|_{U_\omega(Q)} \leq c^{-1}\left(c\left(\left\|w - w^1\right\|_{Y(Q)} + \left\|u - u^1\right\|_{U(Q)}\right)\right)^{2^n},$$

where (w, u) is a solution of the nonlinear controllability problem (2.11)-(2.14).

To prove the Theorem 1.1 and 1.2 we need in

LEMMA 6.1. *Let functions $\hat{y} \in V^{1,2(1)}(Q)$, $y_0 \in V^2(\Omega)$ satisfy conditions (1.11). Then there exist functions $\hat{\psi} \in W^{1,2(1)}(Q)$, $\psi_0 \in W_2^3(\Omega)$, connected by (2.1) with the functions $\hat{y}, y_0$ respectively. In addition if $\hat{y}, y_0$ satisfy the conditions (1.5), (1.10) and the set Σ_0 is connected then functions ψ, ψ_0 satisfy (2.8) and*

$$\left.\widehat{\psi}\right|_{\Sigma_0} = \left.\Delta\widehat{\psi}\right|_{\Sigma_0} = 0, \qquad \psi_0|_{\Gamma_0} = \Delta\psi_0|_{\Gamma_0} = 0. \tag{6.2}$$

One can prove the statement of this Lemma by well known methods(see for example Appendix 1 in [63]).

Proof of the Theorem 1.2. Let $\hat{\psi} \in W^{1,2(2)}(Q), \psi_0 \in W_2^3(\Omega)$ are the functions, constructed by means of $\hat{y},\ y_0$ in Lemma 6.1. This fact imply that these functions satisfy (2.3), (2.5), (2.7), (2.8) (since $\Gamma_0 = \partial\Omega$). Above, equation (2.2) for the function $\hat{\psi}$ with $u \equiv 0$ $g = \mathbf{rot}\, f$ in right side was deduced from equations (1.1), (1.2) for the $\hat{y}$. Thus all assumptions of Theorem 2.1 are fulfilled for $\hat{\psi}$ and $w_0 = \hat{\psi}(0,\cdot) - \psi_0$.

Let w be solution of problem (2.11)-(2.14), (2.16), constructed in this theorem, and function ψ be defined in (2.10). Obviously ψ is a solution of (2.2),(2.3),(2.5),(2.6),(2.9). Thus $y = (y_1, y_2)$ constructed by means of ψ in (2.1) is a solution of (1.15), (1.2), (1.3), (1.5), (1.13). Inequality (1.14) follows from (6.1). ∎

The proof of the Theorem 1.1. By virtue of (1.8) we can construct stream functions $\hat{\psi} \in W^{1,2(2)}(Q)$, and $\psi_0 \in W_2^3(\Omega)$ of the vector fields $\hat{y} \in V^{1,2(1)}(Q)$ and $y_0 \in V^2(\Omega)$ respectively. Relations (1.5), (1.9) and connectedness of the Γ_0 imply (6.2) and by virtue of (1.7) functions $\psi_0,\ \hat{\psi}$ satisfy (2.8).

Let G is a bounded domain in R^2, which satisfy conditions

$$\Omega \subset G, \quad \partial G \in C^\infty, \quad \Gamma_0 \subset \partial G, \quad \Gamma_1 \cap \partial G = \emptyset.$$

(To construct G we need to extend Ω across Γ_1, preserving Γ_0 as a part of the boundary $\partial\Omega$.) If $\Theta = (0,T) \times G,\ S = (0,T) \times \partial G$ then $\Sigma_0 \subset S$. Obviously functions $\psi_0 \in W_2^3(\Omega),\ \hat{\psi} \in W^{1,2(2)}(Q)$, satisfying to (6.2) can be extended up to the functions $\psi_{0,1} \in W_2^3(G), \widehat{\psi}_1 \in W^{1,2(2)}(\Theta)$, which in turn satisfy equations

$$\psi_{0,1}|_{\partial G} = \Delta\psi_{0,1}|_{\partial G} = 0, \qquad \widehat{\psi}_1\Big|_S = \Delta\widehat{\psi}_1\Big|_S = 0,$$

Moreover

$$\|\psi_{0,1} - \psi_1(0,\cdot)\|_{W^3(G)} < c\varepsilon,$$

where c is independent on ε form (2.8).

Let us apply to the function $\hat{\psi}_1$ the operator from left-hand-side of (2.2), and denote by g_1 the function which we received as a result.

Obviously $g_1 \in L_2(\Theta)$ and g_1 is an extension of g from Q up to Θ, where function g is the result of substitution to the left side of (2.2) of the function $\hat{\psi}$.

Note, that the functions $\hat{\psi}_1$ and $w_{0,1} = \psi_{0,1} - \hat{\psi}(0,\cdot)$ satisfy to the conditions of the Theorem 2.1, where Ω is replaced by G. Let us take $\omega \subset G \setminus \Omega$. Then Theorem 2.1 implies Theorem 1.2 as it was mentioned above. The solution of problem (1.12), (1.2), (1.3), (1.5) constructed in Theorem 1.2 after the restriction of $y(t,x)$ from Θ on Q will satisfy to all assertions of Theorem 1.1, and boundary control u can be constructed by y with help (1.6).■

7. Carleman's inequalities.

7.1. Our aim in this section is to prove Theorem 3.1. For this, we get firstly Carleman inequalities for equations, more simple than (3.2). We start from heat equation with inverse time

$$\partial_t z(t,x) + \Delta z(t,x) = f(t,x), \quad (t,x) \in Q, \quad z|_\Sigma = 0, \tag{7.1}$$

where $Q = (0,T) \times \Omega, \Omega \subset R^n$ - bounded domain, with boundary $\partial\Omega \in C^\infty$, $\Sigma = (0,T) \times \partial\Omega$. Let function $\gamma(t) \in C^\infty(0,T)$ satisfy condition

$$0 < \gamma(t) \leq 1, \quad \gamma(t) = \begin{cases} t, & t \in (0,T_0) \\ T-t, & t \in (T-T_0,T) \end{cases}, \quad T_0 = \min(\frac{T}{3}, \frac{1}{2}). \tag{7.2}$$

Let $\omega' \subset \omega \subset \Omega$ is the subdomain of Ω.

We remind that by Lemma I.1.1 there exist a function

$$\beta(x) \in C^2(\Omega), \quad \beta|_{\partial\Omega} = 0, \quad (\nabla\beta, \nu) \leq 0 \;\forall x \in \partial\Omega. \tag{7.3}$$

and there are no critical points of the function $\beta(x), x \in \overline{\Omega \setminus \omega'}$ the inequality holds

$$\min_{x \in \overline{\Omega\setminus\omega}} |\nabla\beta(x)| > 0. \tag{7.4}$$

Moreover, if to the function β constructed in the Lemma I.1.1 add sufficiently large constant, the new function will be satisfied conditions (7.3),(7.4) and

$$\beta(x) \geq \ln 3, \qquad \min_{x\in\bar{\Omega}} \beta(x) > \frac{3}{4} \max_{x\in\bar{\Omega}} \beta(x). \tag{7.5}$$

We introduce functions φ, α by formulas

$$\varphi(t,x) = e^{\lambda\beta(x)}/\gamma(t), \quad \alpha = \alpha^\lambda(t,x) = \left(e^{\frac{4\lambda}{3}\|\beta\|_{C(\bar{\Omega})}} - e^{\lambda\beta(x)}\right)/\gamma(t), \tag{7.6}$$

where function $\alpha(x)$ satisfy (7.3)-(7.5),γ -satisfy (7.2), and parameter $\lambda > 0$. We have.

THEOREM 7.1. *Let functions z and f satisfy (7.1) and $s \geq -3$.* Then for $\lambda > \hat{\lambda}$, where $\hat{\lambda} \geq 1$ sufficiently large, the Carleman inequality is true*

$$\int\limits_Q \varphi^{2s-1} \left[\left(|\partial_t z|^2 + \sum_{i,j=1} \left| \partial^2_{x_i x_j} z \right|^2 \right) + \lambda^2 \varphi^2 |\nabla z|^2 + \lambda^4 \varphi^4 |z|^2 \right] e^{-2\alpha^\lambda} dxdt$$

$$\leq c \left(\int\limits_Q \varphi^{2s} |f(t,x)|^2 e^{-2\alpha^\lambda(t,x)} dxdt + \int\limits_{Q^{\omega'}} \lambda^4 \varphi^{2s+3} |z(t,x)|^2 e^{-2\alpha} dxdt \right), \tag{7.7}$$

where $Q^{\omega'} = (0,T) \times \omega'$, γ, α^λ -are functions from (7.2),(7.6), and constant $c > 0$ is independent on f, z.

Proof. After change in (7.1) of the unknown function

$$z(t,x) = \varphi^{-s} e^{\alpha} w, \tag{7.8}$$

we have equalities

$$L_1 w + L_2 w = f_\lambda(t,x), \quad (t,x) \in Q, \qquad w|_\Sigma = 0, \tag{7.9}$$

where

$$L_1 w = \Delta w + \lambda^2 \varphi^2 |\nabla \beta|^2 w + (s+\alpha)(\partial_t \ln \gamma^{-1}) w, \tag{7.10}$$

$$L_2 w = \partial_t w - 2\lambda(\varphi + s)(\nabla \beta, \nabla w), \tag{7.11}$$

$$f_\lambda = \varphi^{-s} e^{-\alpha} f + \left(\lambda (\varphi + s) \Delta \beta + (\lambda^2 \varphi (1-2s) - s^2 \lambda^2 \right) |\nabla \beta|^2 . \tag{7.12}$$

By (7.8) and properties of the function α following relations holds

$$w|_{t=0} = w|_{t=T} = 0. \tag{7.13}$$

We have by (7.9):

$$\|L_1 w\|^2_{L_2(Q)} + \|L_2 w\|^2_{L_2(Q)} + 2 (L_1 w, L_2 w)_{L_2(Q)} = \|f_\lambda\|^2_{L_2(Q)} . \tag{7.14}$$

By virtue of (7.10), (7.11) we obtain:

$$(L_1 w, L_2 w)_{L^2(Q)} = I_1 + I_2 + I_3, \tag{7.15}$$

*We use later just such s. This condition of course , by change (7.5_1) cad be weakened.

where

$$I_1 = \int\limits_Q \left(\Delta w + \lambda^2\varphi^2 |\nabla\beta|^2 w + (s+\alpha)\left(\partial_t \ln\gamma^{-1}\right) w\right) \partial_t w dxdt, \tag{7.16}$$

$$I_2 = -\int\limits_Q 2\left(\lambda^2\varphi^2 |\nabla\beta|^2 + (s+\alpha)\left(\partial_t \ln\gamma^{-1}\right)\right) w\lambda(\varphi+s)(\nabla\beta, \nabla w)\, dxdt, \tag{7.17}$$

$$I_3 = -\int\limits_Q (\Delta w)\, 2\lambda(\varphi+s)(\nabla\beta,\nabla w)\, dxdt. \tag{7.18}$$

Let us transform integrals I_1, I_2, I_3. Integration by parts in (7.16), bearing in mind (7.13), (7.9_2), gives equality

$$I_1 = \int\limits_Q \left[-\frac{1}{2}\partial_t |\nabla w|^2 + \frac{1}{2}\left(\lambda^2\varphi^2|\nabla\beta|^2 + (s+\alpha)\left(\partial_t\ln\gamma^{-1}\right)\right)\partial_t w^2\right] dxdt =$$

$$= -\int\limits_Q \left(\lambda^2\varphi\partial_t\varphi\,|\nabla\beta|^2 + \frac{1}{2}\partial_t\left((s+\alpha)\left(\partial_t\ln\gamma^{-1}\right)\right)\right)|w|^2\, dxdt. \tag{7.19}$$

Integrating by parts respect to variable x in (7.17) and taking into account (7.6) yields:

$$I_2 = -\int\limits_Q \left(\lambda^2\varphi^2|\nabla\beta|^2 + (s+\alpha)\left(\partial_t\ln\gamma^{-1}\right)\right)\lambda(\varphi+s)\left(\nabla\beta,\nabla w^2\right) dxdt =$$

$$= \int\limits_Q \Big\{(3\lambda^4\varphi^3 + 2\lambda^3 s\varphi^2)|\nabla\beta|^4 + \lambda^3\varphi^2(\varphi+s)\Big[\left(\nabla\beta,\nabla|\nabla\beta|^2\right) +$$

$$+ |\nabla\beta|^2\Delta\beta\Big] + \left(\partial_t\ln\gamma^{-1}\right)\Big[-\lambda^2\varphi(\varphi+s)|\nabla\beta|^2 +$$

$$+\lambda^2\varphi(s+\alpha)|\nabla\beta|^2 + \lambda(s+\alpha)(\varphi+s)\Delta\beta\Big]\Big\}|w|^2\, dxdt. \tag{7.20}$$

Finally, let us transform (7.18):

$$I_3 = \int\limits_Q (\nabla w, \nabla(2\lambda(\varphi+s)(\nabla\beta,\nabla w)))\, dxdt + I_{31} = \int\limits_Q \Big[(\nabla w,\nabla\beta)^2\, 2\lambda^2\varphi +$$

$$+\left[\left[\sum_{i,j=1}^{n}\partial^2_{x_ix_j}\beta\partial_{x_i}w\partial_{x_j}w+\frac{1}{2}\left(\nabla\beta,\nabla\left|\nabla w\right|^2\right)\right]2\lambda\left(\varphi+s\right)\right]dxdt+I_{31}=$$

$$=\int\limits_Q\left[\left(\nabla w,\nabla\beta\right)^2 2\lambda^2\varphi+\left[\sum_{i,j=1}^{n}\partial^2_{x_ix_j}\beta\partial_{x_i}w\partial_{x_j}w-\right.\right.$$

$$\left.\left.-\frac{\Delta\beta}{2}\left|\nabla w\right|^2\right]2\lambda\left(\varphi+s\right)-\lambda^2\left|\nabla\beta\right|^2\left|\nabla w\right|^2\varphi\right]dxdt+I_{31}+I_{32},\qquad(7.21)$$

where

$$I_{31}=-\int\limits_\Sigma\left(\nabla w,\nu\right)2\lambda\left(\varphi+s\right)\left(\nabla w,\nabla\beta\right)d\sigma dt\;\; I_{32}=\int\limits_\Sigma\lambda(\varphi+s)|\nabla w|^2(\nabla\beta,\nu)d\sigma dt.$$

Since $w|_\Sigma=0$,

$$\partial_{x_j}w=\partial_\nu w\nu_j,\quad j=1,\dots,n,$$

where, remind, $\nu=(\nu_1,\dots,\nu_n)$ is the outward normal to $\partial\Omega$.

Hence

$$I_{33}=I_{31}+I_{32}=-\int\limits_\Sigma\left(\partial_\nu w\right)^2\left(\partial_\nu\beta\right)\lambda\left(\varphi+s\right)dxdt\geq 0,\qquad(7.22)$$

where inequality follows from the definition (7.6) of function φ and inequalities (7.5), $s\geq-3$. Substitution (7.19)-(7.21) in (7.15), and next substitution of the obtained inequality in (7.14) yield relation:

$$\|L_1w\|^2_{L_2(Q)}+\|L_2w\|^2_{L_2(Q)}+2\int\limits_Q\left(3\lambda^4\varphi^3\left|\nabla\beta\right|^4\left|w\right|^2-\lambda^2\varphi\left|\nabla\beta\right|^2\left|\nabla w\right|^2+\right.$$

$$\left.+2\left(\nabla w,\nabla\beta\right)^2\lambda^2\varphi\right)dxdt+I_{33}=\|f_\lambda\|^2_{L_2(Q)}+X_1,\qquad(7.23)$$

where

$$X_1=2\int\limits_Q\left[\left(\lambda^2\varphi\partial_t\varphi\left|\nabla\beta\right|^2+\frac{1}{2}\partial_t\left((s+\alpha)\left(\partial_t\ln\gamma^{-1}\right)\right)-2\lambda^3 s\varphi^2\left|\nabla\beta\right|^4-\right.\right.$$

$$-\lambda^3\varphi^2\left(\varphi+s\right)\left(\nabla\beta,\nabla\left|\nabla\beta\right|^2\right)-\lambda^3\varphi^2\left(\varphi+s\right)\left|\nabla\beta\right|^2\Delta\beta+$$

$$+\lambda^2\varphi\left(\varphi+s\right)\left(\partial_t \ln\gamma^{-1}\right)\left|\nabla\beta\right|^2-\lambda^2\varphi\left(s+\alpha\right)\left(\partial_t \ln\gamma^{-1}\right)\left|\nabla\beta\right|^2-$$

$$\left.-\lambda\left(s+\alpha\right)\left(\varphi+s\right)\left(\partial_t \ln\gamma^{-1}\right)\Delta\beta\right)\left|w\right|^2+\lambda\left(\varphi+s\right)\left[\Delta\beta\left|\nabla w\right|^2-\right.$$

$$\left.\left.-2\sum_{i,j=1}^{n}\left(\partial^2_{x_ix_j}\beta\right)\left(\partial_{x_i}w\right)\left(\partial_{x_j}w\right)\right]\right]dxdt. \tag{7.24}$$

Estimating (7.12), we get:

$$\left\|f_\lambda\right\|^2_{L_2(Q)}\le 3\int\limits_Q\left\{\varphi^{2s}e^{-2\alpha}f^2+c\left(\lambda^2\varphi^2\left|\Delta\beta\right|^2+\ \lambda^4\varphi^2\left|\nabla\beta\right|^4\right)\left|w\right|^2\right\}dxdt. \tag{7.25}$$

The definition (7.6) of the functions ψ and α implies the inequalities

$$\left|\partial_t\varphi\right|\le c\varphi^2,\ \left|\left(s+\alpha\right)\left(\partial_t\ln\gamma^{-1}\right)\right|\le c\varphi^2,\ \left|\partial_t\left(\left(s+\alpha\right)\partial_t\ln\gamma^{-1}\right)\right|\le c\varphi^3, \tag{7.26}$$

where constant $c>0$ is independent on $(t,x)\in\bar{Q}$ and $\lambda>1$. Estimating (7.24) with the help of (7.26) we obtain

$$\left|X_1\right|\le c\int\limits_Q\left(\left(1+\lambda^3\right)\varphi^3\left|w\right|^2+\left(1+\lambda\varphi\right)\left|\nabla w\right|^2\right)dxdt. \tag{7.27}$$

Scaling (7.9) by $\lambda^2\varphi\left|\nabla\beta\right|^2 w$ scalarly in $L_2(Q)$ taking into account (7.10) and integrating by parts we get

$$\int\limits_Q f_\lambda\lambda^2\varphi\left|\nabla\beta\right|^2 wdxdt=\int\limits_Q\left(L_2w\right)w\lambda^2\varphi\left|\nabla\beta\right|^2dxdt+$$

$$+\int\limits_Q\left[\lambda^4\varphi^2\varphi\left|\nabla\beta\right|^4w^2+\lambda^2\varphi\left(s+\alpha\right)\left(\partial_t\ln\gamma^{-1}\right)\left|\nabla\beta\right|^2w^2-\right.$$

$$\left.-\lambda^2\varphi\left|\nabla\beta\right|^2\left|\nabla w\right|^2+\frac{1}{2}\Delta\left(\lambda^2\varphi\left|\nabla\beta\right|^2\right)w^2\right]dxdt.$$

One can rewrite this equality as follows

$$\int\limits_Q\lambda^2\varphi\left|\nabla\beta\right|^2\left|\nabla w\right|^2dxdt=\int\limits_Q\lambda^4\varphi^3\left|\nabla\beta\right|^4w^2dxdt-X_2, \tag{7.28}$$

where

$$X_2 = \int\limits_Q \Big[f_\lambda \lambda^2 \varphi \left|\nabla\beta\right|^2 w - (L_2 w)\, w \lambda^2 \varphi \left|\nabla\beta\right|^2 -$$

$$-\lambda^2 \varphi (s+\alpha) \left(\partial_t \ln \gamma^{-1}\right) \left|\nabla\beta\right|^2 w^2 - \frac{1}{2}\left(\lambda^2 \varphi \left(\Delta \left|\nabla\beta\right|^2\right)\right) w^2 -$$

$$-\frac{\lambda^2}{2} \varphi w^2 \left(2\lambda \left(\nabla\beta, \nabla\left|\nabla\beta\right|^2\right) + \lambda^2 \left|\nabla\beta\right|^4 + \lambda \left|\nabla\beta\right|^2 \Delta\beta\right)\Big]\, dxdt.$$

Let us estimate X_2 by means of (7.25), (7.26):

$$|X_2| \le \frac{1}{6} \left\| L_2 w \right\|^2_{L_2(Q)} +$$

$$+c \int\limits_Q \left(\varphi^{2s} e^{-2\alpha} f^2 + \left(\lambda^4 \varphi^2 + \lambda^2 \varphi^3 + \lambda^2 \left(1+\lambda^2\right) \varphi\right) w^2\right) dxdt. \tag{7.29}$$

Estimation (7.23) by means of (7.22), (7.27) we yields

$$\left\| L_1 w \right\|^2_{L_2(Q)} + \left\| L_2 w \right\|^2_{L_2(Q)} + \int\limits_Q \left(6\lambda^4 \varphi^3 \left|\nabla\beta\right|^4 |w|^2 - 2\lambda^2 \varphi \left|\nabla\beta\right|^2 \left|\nabla w\right|^2\right) dxdt$$

$$\le \int\limits_Q \gamma(t)^{2s} e^{-2\alpha} f^2 dxdt + c \int\limits_Q \left(\left(1+\lambda^4\right)\varphi^2 + \left(1+\lambda^3\right)\varphi^3\right) |w|^2 dxdt +$$

$$+ \int\limits_Q c_0 (1+\lambda)\, \varphi \left|\nabla\beta\right|^2 \left|\nabla w\right|^2 dxdt + \int\limits_{Q^{\omega'}} c (1+\lambda)\, \varphi \left|\nabla w\right|^2 dxdt. \tag{7.30}$$

In addition we include $|\nabla\beta|^2$ into penultimate term of right side of inequality (7.30). By virtue of (7.4$_1$) this is possible. We express terms in (7.30) which contains $c\varphi\left|\nabla\beta\right|^2\left|\nabla w\right|^2$ by means of (7.28) and apply (7.29) to the obtained equality. As a result we have

$$\left\| L_1 w \right\|^2_{L_2(Q)} + \left\| L_2 w \right\|^2_{L_2(Q)} + \int\limits_Q \left(4 - \frac{c_0(1+\lambda)}{\lambda^2}\right) \lambda^4 \varphi^3 \left|\nabla\beta\right|^4 |w|^2 dxdt \le$$

$$\le \left(2 + \frac{c_0(1+\lambda)}{\lambda^2}\right) \frac{1}{6} \left\| L_2 w \right\|^2_{L_2(Q)} + \int\limits_Q \{ \varphi^{2s} e^{-2\alpha} f^2 \left(1 + \left(2 + \frac{c_0(1+\lambda)}{\lambda^2}\right)\right) +$$

$$+\left(1+\left(2+\frac{c_0\left(1+\lambda\right)}{\lambda^2}\right)\left(\left(1+\lambda^4\right)\varphi^2+\left(1+\lambda^3\right)\varphi^3\right.\right.$$

$$\left.\left.+\lambda^2\left(1+\lambda^2\right)\varphi\right)\right)|w|^2\}dxdt+c\int\limits_{Q^{\omega'}}c\left(1+\lambda\right)\varphi\left|\nabla w\right|^2dxdt. \tag{7.31}$$

We take λ so large that $c_0\left(1+\lambda\right)/\lambda^2 < 1$. Taking into account, that φ dependes on λ exponentially, and increasing λ if it would be necessary, we see that (7.31) implies inequality

$$\|L_1w\|^2_{L_2(Q)}+\|L_2w\|^2_{L_2(Q)}+\int\limits_Q\lambda^4\varphi^3\left|\nabla\beta\right|^4|w|^2\,dxdt\leq$$

$$\leq c\left(\int\limits_Q\varphi^{2s}e^{-2\alpha}f^2dxdt+\int\limits_{Q^{\omega'}}\left(\lambda\varphi\left|\nabla w\right|^2+\lambda^3\varphi^3\left|w\right|^2\right)dxdt\right). \tag{7.32}$$

Multiplication of (7.9) by $\lambda\varphi w$ scalarly in $L_2(Q)$, and the simple calculations, similar to (7.28), (7.30), yield the estimate

$$\int\limits_Q\lambda\varphi\left|\nabla w\right|^2dxdt\leq\frac{1}{2c}\|L_2w\|^2_{L_2(Q)}+3\int\limits_Q e^{2s}e^{-2\alpha}f^2dxdt+c_1\int\limits_Q\lambda\varphi^3\left|w\right|^2dxdt, \tag{7.33}$$

where the constant c_1 defined in (7.32). By virtue of (7.4) inequality

$$\int\limits_Q\lambda\varphi^3\left|w\right|^2dxdt\leq c_1\int\limits_Q\lambda\varphi^3\left|\nabla\beta\right|^4|w|^2\,dxdt+\int\limits_{Q^{\omega'}}\lambda\varphi^3\left|w\right|^2dxdt \tag{7.34}$$

holds. Let us substitute (7.34) in the right part of (7.33), and new inequality in turn substitute in (7.32). As a result, increasing if it would be necessary parameter λ, we obtain

$$\|L_1w\|^2_{L_2(Q)}+\|L_2w\|^2_{L_2(Q)}+\int\limits_Q\lambda^4\varphi^3\left|\nabla\beta\right|^4|w|^2\,dxdt\leq$$

$$\leq c\left(\int\limits_Q\varphi^{2s}e^{-2\alpha}f^2dxdt+\int\limits_{Q^{\omega'}}\lambda^3\varphi^3\left|w\right|^2dxdt\right). \tag{7.35}$$

Relations (7.35), (7.4) imply estimate :

$$\|L_1 w\|^2_{L_2(Q)} + \|L_2 w\|^2_{L_2(Q)} + \int\limits_Q \lambda^4 \varphi^3 |w|^2 \, dxdt \leq$$

$$\leq c \left(\int\limits_Q \varphi^{2s} e^{-2\alpha} f^2 dxdt + \int\limits_{Q^{\omega'}} \lambda^4 \varphi^3 |w|^2 \, dxdt \right). \tag{7.36}$$

Estimation of right side of (7.33) by means of (7.36) we yields:

$$\int\limits_Q \lambda\varphi |\nabla w|^2 \, dxdt \leq c \left(\int\limits_Q \varphi^{2s} e^{-2\alpha} f^2 dxdt + \int\limits_{Q^{\omega'}} \lambda^4 \varphi^3 |w|^2 \, dxdt \right). \tag{7.37}$$

Multiplying (7.10) by $\left(\sqrt{\varphi}\right)^{-1}$ and estimating (7.36), we get the inequality

$$\int\limits_Q \varphi^{-1} |\Delta w|^2 \, dxdt \leq c \int\limits_Q (\varphi^{-1} |L_1 w|^2 + \left(\lambda^4 \varphi^2 |\nabla \beta|^4 + c\varphi^3\right) |w|^2) dxdt \leq$$

$$\leq c \left(\int\limits_Q \varphi^{2s} e^{-2\alpha} f^2 dxdt + \int\limits_{Q^{\omega'}} \lambda^4 \varphi^3 |w|^2 \, dxdt \right). \tag{7.38}$$

By similar arguments, multiplying (7.10) by $\left(\sqrt{\lambda\varphi}\right)^{-1}$ and estimating by means of (7.37) we obtain

$$\int\limits_Q \varphi^{-1} |\partial_t w|^2 \, dxdt \leq c \left(\int\limits_Q \varphi^{2s} e^{-2\alpha} f^2 dxdt + \int\limits_{Q^{\omega'}} \lambda^4 \varphi^3 |w|^2 \, dxdt \right). \tag{7.39}$$

Note that the following equations are true:

$$\Delta \left(\varphi^{-\frac{1}{2}} w\right) = \varphi^{-\frac{1}{2}} \Bigg(\Delta w - \lambda (\nabla\beta, \nabla w) +$$

$$+ \left(\frac{\lambda^2}{4} (\nabla \beta)^2 - \frac{\lambda}{2} \Delta \beta \right) w \Bigg), \qquad \left(\varphi^{-\frac{1}{2}} w\right)\Big|_{\partial\Omega} = 0. \tag{7.40}$$

Applying to solution $\varphi^{-\frac{1}{2}}w$ of elliptic boundary problem (7.40) well known estimates, and then estimating the right part of new inequality by (7.36)-(7.38) we obtain

$$\int\limits_Q \sum_{i,j=1}^n \left|\partial^2_{x_ix_j}\left(\varphi^{-\frac{1}{2}}w\right)\right|^2 dxdt \le c\left(\int\limits_Q \gamma^{2s}e^{-2\alpha}f^2dxdt + \int\limits_{Q^{\omega'}} \lambda^4\varphi^3\left|w\right|^2 dxdt\right). \tag{7.41}$$

Substitution into estimates (7.36)-(7.39),(7.41) $w = e^{-\alpha}\varphi^s z$, yields (7.7).■

Let us consider the Dirichlet problem for the Laplace operator:

$$\Delta p\,(t,x) = z\,(t,x)\,, \quad (t,x) \in Q, \qquad p|_\Sigma = 0. \tag{7.42}$$

We have

Theorem 7.2. *There exists $\widehat{\lambda} > 1$ such, that for any $\lambda > \widehat{\lambda}$ and $s \ge -3$ solution p of the problem (7.42) satisfies the Carleman estimate*

$$\int\limits_Q \left(\varphi^{2s-1}\sum_{i,j=1}^n \left|\partial^2_{x_ix_j}p\right|^2 + \lambda\varphi^{2s+1}\left|\nabla p\right|^2 + \lambda^4\varphi^{2s+3}\left|p\right|^2\right)e^{-2\alpha^\lambda(t,x)}dxdt \le$$

$$\le c\left(\int\limits_Q \varphi^{2s}e^{-2\alpha^\lambda}\left|z\,(t,x)\right|^2 dxdt + \int\limits_{Q^{\omega'}} \lambda^4\varphi^{2s+3}\left|p\,(t,x)\right|^2 e^{-2\alpha^\lambda}dxdt\right). \tag{7.43}$$

Proof. Making in (7.42) change $p = \varphi^{-s}e^\alpha w$, we get equality (7.9) where

$$L_1 w = \Delta w + \lambda^2\varphi^2\left|\nabla\beta\right|^4 w, \quad L_2 w = -2\lambda\left(\varphi + s\right)\left(\nabla\beta, \nabla w\right) \tag{7.44}$$

and f_λ is defined in (7.12). All terms of operators (7.44) are contained in operators (7.10), (7.11). So if for these terms form (7.44) to conduct estimates similar as in the proof of the Theorem 7.1, we obtain (7.43).■

7.2. Let us consider problem (3.2), (3.3) in the cylinder $Q = (0,T) \times \Omega,\ \Omega \subset R^2$:

$$\partial_t\Delta p\,(t,x) + \Delta^2 p\,(t,x) = g - B_2^*\left(\psi,p\right) - B_1^*\left(p,\psi\right), \tag{7.45}$$

$$p|_\Sigma = \Delta p|_\Sigma = 0, \tag{7.46}$$

where $\psi \in W^{1,2(2)}\left(Q\right)$ is a given function and operators B_2^*, B_1^* are defined by equalities (3.5),(3.6). The main statement of the this section is as follows.

THEOREM 7.3. *Let functions p, f satisfy (7.45),(7.46). Then there exists $\widehat{\lambda} > 0$ such that for any $\lambda > \widehat{\lambda}$ the estimate holds:*

$$J(p) \equiv \int_Q \left(\varphi^{-7} \left(|\partial_t \Delta p|^2 + \sum_{i,j=1}^{2} \left| \partial^2_{x_i x_j} \Delta p \right|^2 \right) + \varphi^{-5} |\nabla \Delta p|^2 + \right.$$

$$\left. + \varphi^{-3} |\Delta p|^2 + \sum_{k=0}^{4} \sum_{|\alpha|=k} |D^\alpha_x p|^2 \varphi^{-2k} \right) e^{-2\alpha^\lambda} dxdt \leq$$

$$\leq c \int_Q \varphi^{-6} e^{-2\alpha^\lambda} |g|^2 dxdt + \int_{Q^{\omega'}} \left(1 + \lambda^{13}\right) \varphi p^2 e^{-2\alpha^\lambda} dxdt. \tag{7.47}$$

Proof. Set

$$\Delta p(t,x) = z(t,x), \qquad f = g - B_2^*(\psi, p) - B_1^*(p, \psi). \tag{7.48}$$

Then, by virtue of (7.45),(7.46) the functions z and f satisfy (7.1) and Theorem 7.1 implies inequality (7.7) holds with $s = 3$. Since p and z satisfy (7.50),(7.46), by virtue of Theorem 7.2 the estimate (7.43) holds with $s = 3/2$. The inequalities (7.7) with $s = 3$ and (7.43) with $s = 3/2$ imply the estimate

$$J_0(p) \equiv \int_Q e^{-2\alpha} \left(\varphi^{-7} \left(\lambda^{-1} |\partial_t \Delta p|^2 + \sum_{i,j=1}^{2} \left| \partial^2_{x_i x_j} \Delta p \right|^2 \right) + \lambda \varphi^{-5} |\nabla \Delta p|^2 + \right.$$

$$\left. + \lambda^4 \varphi^{-3} |\Delta p|^2 + \varphi^{-4} \sum_{i,j=1}^{2} \left| \partial^2_{x_i x_j} \Delta p \right|^2 + \lambda \varphi^{-2} |\nabla p| + \lambda^4 |p|^2 \right) dxdt \leq$$

$$\leq c \left(\int_Q e^{-2\alpha} \left(|f|^2 \varphi^{-6} + |\Delta p|^2 \varphi^{-3} \right) dxdt \right.$$

$$\left. + \int_{Q^{\omega'}} e^{-2\alpha} \left(\lambda^4 |p|^2 + \lambda^4 \varphi^{-3} |\Delta p|^2 \right) dxdt \right). \tag{7.49}$$

Let $\rho(x) \in C_0^\infty(\omega)$, $\rho(x) \equiv 1$ for any $x \in \omega'$. The relation holds

$$\int\limits_{Q^{\omega'}} \lambda^4 \varphi^{-3} |\Delta p|^2 e^{-2\alpha} dxdt \leq \int\limits_{Q^\omega} \rho \lambda^4 \varphi^{-3} e^{-2\alpha} |\Delta p|^2 dxdt =$$

$$= \int\limits_{Q^\omega} \lambda^4 p \Delta \left(\rho \varphi^{-3} e^{-\alpha} \Delta p \right) dxdt \leq \tag{7.50}$$

$$\leq c \int\limits_{Q^\omega} \left(\lambda^6 \varphi^{-1} |p \Delta p| + \lambda^5 |p| |\nabla \Delta p| \varphi^{-2} + \left| p \Delta^2 p \right| \lambda^4 \varphi^{-3} \right) e^{-2\alpha} dxdt \leq$$

$$c \int\limits_{Q^\omega} e^{-2\alpha} \left[\varepsilon \left(\lambda^4 \varphi^{-3} |\Delta p|^2 + \lambda \varphi^{-5} |\nabla \Delta p|^2 + \varphi^{-7} \left| \Delta^2 p \right|^2 \right) + \frac{1}{\varepsilon} \left(\lambda^8 + \lambda^9 \right) \varphi p^2 \right] dxdt.$$

Substituting (7.50) to the right part of the inequality (7.49), taking in the obtained inequality parameter ε sufficiently small, and increasing parameter λ if it would be necessary we get the estimate

$$J_0(p) \leq c \left(\int\limits_Q e^{-2\alpha} \varphi^{-6} |f|^2 dxdt + \int\limits_{Q^\omega} e^{-2\alpha} \varphi \lambda^9 p^2 dxdt \right), \tag{7.51}$$

where functional J_0 defined in (7.49).

To estimate terms in last sum from the left side of the inequality (7.47) we write out the identities

$$\Delta \left(\varphi^{-3} e^{-\alpha} p \right) = f_1, \qquad \left(\varphi^{-3} e^{-\alpha} p \right)\Big|_\Sigma = 0, \tag{7.52}$$

where

$$f_1 = \varphi^{-3} e^{-\alpha} \Delta p + 2 \left(\nabla \left(\varphi^{-3} e^{-\alpha} \right), \nabla p \right) + \Delta \left(\varphi^{-3} e^{-\alpha} \right) p. \tag{7.53}$$

Applying a priori estimates for the Dirichlet boundary problem (7.52) we have

$$\int\limits_Q \sum_{|\alpha|=3} \left| D_x^\alpha \left(\varphi^{-3} e^{-\alpha} p \right) \right|^2 dxdt \leq c \, \|f_1\|^2_{L_2\left(0,T;W_2^1(\Omega)\right)} \leq c \lambda J_o(p). \tag{7.54}$$

The Leibnitz rule of the differentiation of functions product applied to the left side of (7.54) and simple calculations give inequality

$$\int_Q e^{-2\alpha}\varphi^{-6} \sum_{|\alpha|=3} |D_x^\alpha p|^2 \, dxdt \le c\lambda^2 J_o(p) . \tag{7.55}$$

By similar argument, substituting φ^{-4} instead of φ^{-3} in (7.52), and summing with respect to $|\alpha| = 4$ instead of $|\alpha| = 3$ in (7.54), we obtain

$$\int_Q e^{-2\alpha}\varphi^{-8} \sum_{|\alpha|=4} |D_x^\alpha p|^2 \, dxdt \le c\lambda^4 J_o(p) . \tag{7.56}$$

By estimates (7.51),(7.55),(7.56) for the functional $J_0(p)$, defined in (7.49), imply the inequality

$$J(p) \le c \left(\int_Q e^{-2\alpha} \left(1+\lambda^4\right) \varphi^{-6} |f|^2 \, dxdt + \int_{Q^\omega} e^{-2\alpha} \left(1+\lambda^{13}\right) \varphi p^2 dxdt \right), \tag{7.57}$$

where c depends on λ.

Now we estimate the term $B_2^* + B_1^*$ from the definition (7.48) of the function f. Differentiation of the product in (3.5),(3.6) and short calculation gives the equation

$$B_1^*(p,\psi) + B_2^*(\psi,p) = (\partial_{x_1}\Delta p)\,\partial_{x_2}\psi - (\partial_{x_2}\Delta p)\,\partial_{x_1}\psi +$$
$$+2\left((\nabla\partial_{x_1}p)(\nabla\partial_{x_2}\psi) - (\nabla\partial_{x_2}p)(\nabla\partial_{x_1}\psi)\right). \tag{7.58}$$

By virtue of imbedding theorem

$$\int_Q e^{-2\alpha}\varphi^{-6}\left(1+\lambda^4\right)|\nabla\Delta p|^2 |\nabla\psi|^2 \, dxdt \le$$
$$\le c\,\|\psi\|_{W^{1,2(2)}(Q)} \int_Q e^{-2\alpha}\varphi^{-6}\left(1+\lambda^4\right)|\nabla\Delta p|^2 \, dxdt. \tag{7.59}$$

Applying imbedding theorem again we obtain

$$\int_Q e^{-2\alpha}\varphi^{-6}\left(1+\lambda^4\right)\left|\partial^2_{x_ix_j}p\right|^2 \left|\partial^2_{x_ix_k}\psi\right|^2 dxdt \le$$

$$\leq \left(1+\lambda^4\right)\int_0^T \left\|e^{-\alpha}\varphi^{-3}\partial^2_{x_ix_j}p\right\|^2_{L_4(\Omega)}\left\|\partial^2_{x_ix_k}\psi\right\|^2_{L_4(\Omega)}dt \leq$$

$$c\left(1+\lambda^4\right)\|\psi\|^2_{L_\infty\left(0,T;W_2^3(\Omega)\right)}\int_0^T \left\|e^{-\alpha}\varphi^{-3}\partial^2_{x_ix_j}p\right\|_{W_2^1(\Omega)}\left\|e^{-\alpha}\varphi^{-3}\partial^2_{x_ix_j}p\right\|_{L_2(\Omega)}dt$$

$$\leq c\left(1+\lambda^4\right)\|\psi\|^2_{W^{1,2(2)}(Q)}\left(\varepsilon\lambda\int_Q e^{-2\alpha}\sum_{k=2}^{3}\sum_{|\alpha|=k}\left|D_x^\alpha p\right|^2\varphi^{-2k}dxdt+\right.$$

$$\left.+\frac{1}{\varepsilon}\int_Q e^{-2\alpha}\varphi^{-6}\left|\partial^2_{x_ix_j}p\right|^2dxdt\right). \tag{7.60}$$

Substitution the expression for f from (7.48) into the right side of (7.57) and then, application of (7.58) and estimates (7.59),(7.60) yield the inequality

$$J(p)\leq c\left(\int_Q e^{-2\alpha}\left(1+\lambda^4\right)\left(\varphi^{-6}|g|^2+c_1\varepsilon\lambda\sum_{k=2}^{3}\sum_{|\alpha|=k}\left|D_x^\alpha p\right|^2\varphi^{-2k}+\right.\right.$$

$$\left.\left.+\frac{c_1}{\varepsilon}\varphi^{-6}\sum_{|\alpha|=2}\left|D_x^\alpha p\right|^2\right)dxdt+\int_{Q^\omega}e^{-2\alpha}\left(1+\lambda^{13}\right)\varphi p^2dxdt\right). \tag{7.61}$$

Taking ε sufficiently small in (7.61) and keeping in mind definition (7.47) of the functional J we get:

$$J(p)\leq c_2\left(\int_Q e^{-2\alpha}\left(1+\lambda^4\right)\left(\varphi^{-6}|g|^2+\frac{c_1}{\varepsilon}\varphi^{-6}\sum_{|\alpha|=2}\left|D_x^\alpha p\right|^2\right)dxdt+\right.$$

$$\left.+\int_{Q^\omega}e^{-2\alpha}\left(1+\lambda^{13}\right)\varphi p^2dxdt\right). \tag{7.62}$$

Taking into account (7.47) function's φ definition (7.6) and increasing, if it would be necessary parameter λ in (7.62), we get from (7.62) inequality (7.47).■

We set the initial condition at $t_0 \in (0, T)$:

$$p|_{t=t_0} = p_0, \tag{7.63}$$

for the (7.45), (7.46) where $p_0 \in W_2^3(\Omega)$ satisfies compatibility conditions

$$p_0|_{\partial\Omega} = \Delta p_0|_{\partial\Omega} = 0, \tag{7.64}$$

and consider the problem (7.45),(7.46),(7.63) in the domain $(t, x) \in (0, t_0) \times \Omega$.

LEMMA 7.2. *Let $\Omega \subset R^2$, $\psi \in W^{1,2(2)}(Q)$, $f \in L_2(Q)$, $p_0 \in W_2^3(\Omega)$ and satisfy (7.64). Then there exists the unique solution $p \in W^{1,2(2)}((0, t_0) \times \Omega)$ of the problem (7.45),(7.46),(7.63) which satisfy the estimate*

$$\|p\|_{W^{1,2(2)}((0,t_0)\times\Omega)} \leq c \left(\|p_0\|_{W_2^3(\Omega)} + \|f\|_{L_2((0,t_0)\times\Omega)} \right), \tag{7.65}$$

where the constant c is independent on p_0 and f.

The proof of Lemma 7.2 is similar to Lemma's 4.4 proof.

Proof of the Theorem 3.1. Denote by $R(p)\, e^{-2\eta^\lambda}$ the expression in the left side of equality (3.9):

$$I_\lambda(p) \equiv \int\limits_Q R(p)(t, x)\, e^{-2\eta^\lambda(t,x)} dtdx, \tag{7.66}$$

where $I_\lambda(p)$ defined in (3.9). Let $T_0 \in R^1$ defined in (7.2). For $t \in (0, T - T_0)$ functions $(T - t)$ and η^λ from (2.26) are bounded from above and below by positive constants, which depends on λ only. Therefore, by virtue of (3.2), (3.3) and estimate (7.65) we have

$$I_\lambda(p) = \int\limits_{T-T_0}^{T} \int\limits_\Omega R(p)(t, x)\, e^{-2\eta^\lambda(t,x)} dtdx + \|p\|^2_{W^{1,2(2)}((0,T-T_0)\times\Omega)} \leq \tag{7.67}$$

$$\leq \int\limits_{T-T_0}^{T} \int\limits_\Omega R(p)(t, x)\, e^{-2\eta^\lambda(t,x)} dtdx + c\left(\|p(T - T_0, \cdot)\|^2_{W_2^3(\Omega)} + \|w\|^2_{L_2((0,T-T_0)\times\Omega)} \right).$$

Denote by $R_1(p)\,e^{-2\alpha^\lambda}$ the integrand in the left hand side of inequality (7.47) :

$$J(p)=\int\limits_Q R_1(p)(t,x)\,e^{-2\alpha^\lambda(t,x)}\,dxdt, \tag{7.68}$$

where the functional $J(p)$ defined in (7.47). Relation (7.64) and estimates for the Dirichlet boundary problem for the Laplace operator we obtain the inequality

$$\|p_0\|^2_{W_2^3(\Omega)}\le c\,\|\Delta p_0\|^2_{W_2^1(\Omega)}\,.$$

This estimate, trace theorem and definition (7.68) of the function R_1 yields

$$\|p(T-T_0,\cdot)\|^2_{W_2^3(\Omega)}\le\|\Delta p\|^2_{W^{1,2(2)}((T-T_0-\varepsilon,T-T_0)\times\Omega)}\le$$

$$\le\int\limits_0^{T-T_0}\int\limits_\Omega R_1(p)(t,x)\,e^{-2\alpha^\lambda(t,x)}\,dxdt. \tag{7.69}$$

By virtue of (7.67), (7.69), compearing definitions (2.26), (7.6), of the functions η and α, as well as definitions (7.66),(7.68) of the functions $R(p)$ and $R_1(p)$ we obtain

$$I_\lambda(p)\le c\left(J(p)+\|w\|^2_{L_2((0,T-T_0)\times\Omega)}\right). \tag{7.70}$$

By virtue of (7.45), (3.2) we can replace g by $-e^{2\eta^\lambda(t,x)}w/(T-t)^6$ in the estimate (7.47). Then applying the new estimate to the right-hand-side of (7.70) we get (7.49).■