Continuum Mechanics as Particle Mechanics

(Project 2018-2 proposed by Lykov A.A. and Malyshev V.A.)

Here we want to understand what equations of continuum mechanics can be deduced directly (avoiding completely any randomness) from deterministic N-particle mechanics as $N \to \infty$.

Thus here our only axiom is Newton law. Another important axiom will be the conception of continuum media as consisting of the continuum number of particles of infinitely small mass. This conception is well known in mathematics, see for example [3], p. 56. We want to develop this rigorous conception and elaborate concrete examples, as many as possible.

We define N point particle system-to be **regular** if particles do not collide, that is for any i, j and any time t, $x_i(t) \neq x_j(t)$.

Continuum media as regular classical system with continuum number of particles (Axiom Continuum_Reg) Continuum media $\mathbf{M}_T$ (and its dynamics) is defined as a set of subsets $\Lambda_t \in R^d$ enumerated by the time moments $t \in [0, T), 0 < T \leq \infty$. Moreover, Λ_0 is assumed to be the closure of some open connected subset of R^d with piece-wise smooth boundary $\partial\Lambda_0$. Each point of this domain is considered as a "material particle" of infinitely small mass and charge. The dynamics is defined by the system of one-to-one mappings (diffeomorphisms) $U_t = U_{0,t} : \Lambda_0 \to \Lambda_t, t \in [0, T)$. All these mappings are assumed to be sufficiently smooth in x and piece-wise smooth in t, and $U_0(x)$ is the identity map. Thus, each point (particle) $x \in \Lambda_0$ has its own trajectory in R^d: $y(t, x) = U_t(x)$, where $y(0, x) = x$ is the initial coordinate of this particle. It follows from the definition, that the particles never collide, that is $y(t, x) \neq y(t, x')$ for any t and $x \neq x'$.

Also two functions are defined: mass density at point y at time t: $\rho(t, y) = \rho(t, U_t x)$ if $y = y(t, x)$ and, quite similarly, the charge density $\lambda(t, y) = \lambda(t, U_t x)$.

The main problem is how to write down equations for the trajectories $y(t, x)$. There are several possibilities:

1. Similarly to Newton equations for finite system of particles. In the following simple case this can be done directly:

1a) $\mathbf{M}_T$ is called a system without interaction, if $y(t, x)$ are the solutions of the following equations

$$\frac{d^2 y(t,x)}{dt^2} = F_x(y(t,x)), \qquad y(0,x) = x, \ \frac{dy(0,x)}{dt} = v(x) \tag{1}$$

for some given functions: initial velocity $v(x)$ and external forces $F_x(y)$, possibly different for different particles. In particular, we could assume that either $F_x(y) = F(y)$ does not depend on x or $F_x(y) = \frac{F(y)}{m(x)}$ for some functions $F(y)$ and $m(x) > 0$. It is always assumed that $v(x)$ and $m(x)$ are sufficiently smooth in $x \in \Lambda_0$, and $F(y)$ is smooth or piece-wise smooth in y. Moreover, it is always assumed that any equation (1) has a unique solution on all considered interval $[0, T)$.

Many concrete examples of such systems are studied in [1]. Another simple example – Ohm's law – is discussed in Project 4.

1b) In more general case the continuum system of trajectories can be considered as the $N \to \infty$ limit of finite particle trajectories $y_{N,k}(t), k = 1, 2, \ldots, N$, under the following assumption. if for some sequence $k(N)$

$$y_{N,k(N)}(0) \to x$$

then for any t

$$y_{N,k(N)}(t) \to y(t, x)$$

This holds for the examples in [1], mentioned above, and also for far more complicated system, called Chaplygin gas, see [2].

2. Another way is to write down Euler equations for continuum media.

For example, we consider, see [2], the particle system on the real line with a particular Lennard – Jones type potential and prove that the particle trajectories of the N-particle system, for $N \to \infty$, converge, in the sense defined below, to the trajectories of the continuum particle system. Moreover, we get the system of 3 equations of the Euler type (see [4]) for the functions: $u(t, x)$ – the velocity, $p(t, x)$ – the pressure and $\rho(t, x)$ – the density

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2}$$

$$u_t + uu_x = -\frac{p_x}{\rho}, \tag{3}$$

$$p = p(\rho), \tag{4}$$

In continuum mechanics these equations correspond to the conservation laws of mass, momentum and to the thermodynamic equation of state. In physics the first two equations are quite general. But the third one depends on the matter type and thermodynamic situation and should be given separately. In our derivation, all these equations and functions obtain simple and intuitive mechanical meaning (without probability theory and thermodynamics) for the N-particle system. In particular, the pressure can be considered as an analog of interaction potential in Hamiltonian mechanics.

References

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[4] Chorin A., Marsden J. A mathematical introduction to fluid mechanics. Springer. 1992.