Moscow University
Mechanics Bulletin
Vol. 57, No. 3, pp. 1–1, 2002

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TENSOR INVARIANTS OF DYNAMICAL SYSTEMS AND STEADY VISCOUS FLOWS

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Chaotic steady viscous flows are studied. The following two models are considered: a barotropic compressible viscous fluid and an incompressible ideal fluid with external friction. Based on the well-known Poincaré–Kozlov theorem, we show for these models that steady fluid flows are chaotic: nontrivial integrals, fields of symmetries, and integral invariants are absent.

1. INTRODUCTION

Let us consider steady viscous flows for the case when the velocity field $\boldsymbol{v} = (u, v, w)$ does not depend explicitly on time and generates the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}) \tag{1}$$

in a domain D of Euclidean space $\mathbb{R}^3 = \{x, y, z\}$. We restrict our consideration to flows that are 2π-periodic in x and y. In this case, D is the direct product $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2 = \{x, y \bmod 2\pi\}$ is a two-dimensional torus. Let $\{g^t\}$ be the phase flow of autonomous dynamical system (1). The action of $\{g^t\}$ considered on D is the *advection* (i.e., the transport of fluid particles). Chaotic advection is called the *Lagrangian turbulence*.

The field $\boldsymbol{v}$ can be found from the continuum equations. In [1], the following models of continua are considered: fluids without pressure (the Burgers equations, i.e., a simplified system of Navier–Stokes equations), ideal fluids, and heat-conducting perfect gases. It turns out that in these models, as a rule, steady flows are irregular and possess chaotic properties.

Chaotic and regular properties in the behavior of system (1) are studied in [1]. There exist different approaches to the definition of regular dynamical systems (see, for example, [2]). What all these approaches have in common is that they include the nontrivial tensor invariants: integrals, fields of symmetries, and integral invariants. Let us recall their definitions.

Let $f : D \to \mathbb{R}$ be a locally nonconstant function. This function is called the *integral* of system (1) if

$$\dot{f} = \left(\frac{\partial f}{\partial \boldsymbol{x}}, \boldsymbol{v}\right) = 0.$$

The domain D consists of the invariant integral surfaces $N_{\mathrm{c}} = \{\boldsymbol{x} \in D : f(\boldsymbol{x}) = \mathrm{c}\}$. If a surface N_{c} is compact (i.e., bounded) and, on this surface, the velocity $\boldsymbol{v}$ is not equal to zero, then N_{c} is topologically equivalent to a two-dimensional torus.

If $[\boldsymbol{v}, \boldsymbol{w}] = 0$, then the field $\boldsymbol{w}$ is called the *field of symmetries* (here $[\cdot\,,\cdot]$ is the Jacobi bracket). Of interest is the case of nontrivial fields of symmetries when the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are not collinear. The phase flow of $\boldsymbol{w}$ transforms solutions of system (1) into solutions of the same system. If $[\boldsymbol{v}, \boldsymbol{w}] = \lambda \boldsymbol{w}$, then the field $\boldsymbol{w}$ is called the *frozen field of directions* (here λ is a function in D). The flow $\{g^t\}$ maps the family of integral curves of the field $\boldsymbol{w}$ into itself (see [3, 4]).

The differential 1-form ω generates the *integral invariant* if

$$I(t) = \int\limits_{g^t\gamma} \omega = \mathrm{const}$$

for all closed contours γ. Of course, it is necessary to assume that $d\omega \neq 0$ (i.e., ω cannot be reduced to the total differential); otherwise, $I(t) \equiv \mathrm{const}$. The steady flow of a barotropic inviscid fluid in a potential force field is an example of regular flow. This flow admits Bernoulli's integral, the field of symmetries $\boldsymbol{w} = (\operatorname{rot} \boldsymbol{v})/\rho$ (here ρ is the fluid density), and the integral invariant corresponding to the circulation of the fluid around a closed contour. The field of symmetries are found in [5] and, for the case when the fluid is homogeneous ($\rho \equiv \text{const}$), in [6].

2. STEADY FLOWS AND TENSOR INVARIANTS

It is well known that, in a small neighborhood of any nonsingular point, system (1) possesses all tensor invariants: nonconstant integrals, nontrivial fields of symmetries, and integral invariants (in order to obtain these invariants, the rectification theorem for phase trajectories should be used).

The situation abruptly changes if the existence of invariants is analyzed on the whole. It turns out that general dynamical systems do not have nontrivial global invariants everywhere over D. This fact is very important in continuum mechanics. It is shown in [1, 7] that steady viscous flows are irregular and possess chaotic properties.

The distinction between local and global formulations of the above problem was first discovered in [8]. The search for constructive conditions of regular or chaotic behavior of dynamical system (1) is a very complicated problem [9].

We restrict our consideration to the following autonomous systems in $\mathbb{T}^2 \times \mathbb{R}$:

$$\dot{x} = u_0 + \varepsilon u_1 + \cdots, \quad \dot{y} = v_0 + \varepsilon v_1 + \cdots, \quad \dot{z} = \varepsilon w_1 + \cdots . \tag{2}$$

Here u_i, v_i, and w_i are analytic functions that are 2π-periodic in x and y; u_0 and v_0 depend only on the "slow" variable z; ε is a small parameter. Such systems are often encountered in the theory of nonlinear oscillations; according to Poincaré, their study is the fundamental problem of dynamics [8, Section 13]. If $\varepsilon = 0$, then the dynamical system is totally integrable and, hence, regular.

It is reasonable to seek tensor invariants of system (2) in the form of series in ε whose coefficients are unique and analytic in the direct product $\mathbb{T}^2 \times \mathbb{R}$. For example, the integral of (2) is of the form $f_0 + \varepsilon f_1 + \cdots$, where f_k are analytic functions that are 2π-periodic in x and y.

Let

$$\sum W_{mn}(z) e^{i(mx+ny)}$$

be the Fourier series of a function w_1. The set of points $z \in \mathbb{R}$ satisfying the conditions

$$m u_0(z) + n v_0(z) = 0, \quad W_{mn}(z) \neq 0 \qquad (m^2 + n^2 \neq 0)$$

is called the *Poincaré set* $\mathbb{P}$. In the general case, $\mathbb{P}$ everywhere densely fills the number line $\mathbb{R} = \{z\}$. The unperturbed system (when $\varepsilon = 0$) is said to be *nondegenerate* if $m u_0(z) + n v_0(z) \not\equiv 0$ for all integer numbers m and n such that $m^2 + n^2 \neq 0$. If $u_0' v_0 - u_0 v_0' \not\equiv 0$, then the system is nondegenerate (the prime indicates the derivative with respect to z).

Theorem (Poincaré–Kozlov). *Let us assume that unperturbed system* (2) *is nondegenerate and that the Poincaré set has at least one finite limiting point. Then, system* (2) *does not admit nontrivial integrals and frozen fields of directions analytic in* ε. *Moreover, if* $W_{00}(z) \neq 0$, *system* (2) *does not admit nontrivial integral invariants.*

The absence of integrals follows from more general results given in [9]; these results generalize the Poincaré theory on integrability of Hamiltonian systems for the case when these systems differ little from totally integrable ones [8]. The results on nonexistence of frozen fields of directions and integral invariants are discussed in [4, 7]. Note that the hypotheses of the above theorem ensure the absence of nontrivial invariants that can be represented in the form of *formal* (not necessarily convergent) series in ε.

3. THE MODEL OF VISCOUS COMPRESSIBLE FLUIDS

In continuum mechanics, a linear relation between the stress and strain-rate tensors is postulated when deriving equations of motion for viscous compressible fluids. Under the above assumptions, the Navier–Stokes equations take the form

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \right) = -\frac{\partial p}{\partial \boldsymbol{x}} + \eta \, \triangle \boldsymbol{v} + \left(\zeta + \frac{\eta}{3} \right) \frac{\partial}{\partial \boldsymbol{x}} \operatorname{div} \boldsymbol{v}, \tag{3}$$

where ρ is the density, p is the pressure, ζ is the coefficient of second viscosity (this coefficient was studied in detail in [10]), and η is the coefficient of dynamic viscosity. Suppose these coefficients are constant and positive.

In order for system (3) to be closed, it is necessary to introduce the following two equations: the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \tag{4}$$

and the equation of state (the barotropic condition)

$$\rho = f(p). \tag{5}$$

The function f is assumed analytic.

Let us apply the above method to our model of viscous fluids. We are interested in steady flows with the velocity field $\boldsymbol{v} = (u, v, w)$. The unknown functions ρ and p are related by (5) and do not depend explicitly on time:

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w\right) &= -\frac{\partial p}{\partial x} + \eta\,\triangle u + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\\
\rho\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w\right) &= -\frac{\partial p}{\partial y} + \eta\,\triangle v + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\\
\rho\left(\frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w\right) &= -\frac{\partial p}{\partial z} + \eta\,\triangle w + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\\
&\frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.
\end{aligned}$$

Solutions to these equations can be represented in the form of series in ε:

$$u = u_0 + \varepsilon u_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots, \quad w = \varepsilon w_1 + \cdots, \quad \rho = \rho_0 + \varepsilon \rho_1 + \cdots, \quad p = p_0 + \varepsilon p_1 + \cdots .$$

Here

$$u_0 = \alpha z + \beta, \quad v_0 = \gamma z + \delta, \quad \rho_0 = \text{const}, \quad p_0 = \text{const}, \tag{6}$$

where α, β, γ, and δ are arbitrary constants.

At the first approximation in ε, we obtain the following system of partial differential equations for the above functions u_1, v_1, w_1, ρ_1, and p_1:

$$\begin{aligned}
\rho_0\left(\frac{\partial u_1}{\partial x}u_0 + \frac{\partial u_1}{\partial y}v_0 + u_0' w_1\right) &= -\frac{\partial p_1}{\partial x} + \eta\,\triangle u_1 + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right),\\
\rho_0\left(\frac{\partial v_1}{\partial x}u_0 + \frac{\partial v_1}{\partial y}v_0 + v_0' w_1\right) &= -\frac{\partial p_1}{\partial y} + \eta\,\triangle v_1 + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right),\\
\rho_0\left(\frac{\partial w_1}{\partial x}u_0 + \frac{\partial w_1}{\partial y}v_0\right) &= -\frac{\partial p_1}{\partial z} + \eta\,\triangle w_1 + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right),\\
&\frac{\partial \rho_1}{\partial x}u_0 + \frac{\partial \rho_1}{\partial y}v_0 + \rho_0\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right) = 0.
\end{aligned}$$

This system is linear. Let us solve it by the Fourier method with consideration of relations (6). By U_{mn}, V_{mn}, W_{mn}, R_{mn}, and P_{mn} we denote the Fourier coefficients for the velocity components u_1, v_1, and w_1 and for the functions ρ_1 and p_1, respectively. All these coefficients are functions of z and satisfy the system of algebro-differential equations

$$\begin{aligned}
\rho_0\big[i(mu_0+nv_0)U_{mn}+\alpha W_{mn}\big] &= -imP_{mn}+\eta\big[-(m^2+n^2)U_{mn}+U''_{mn}\big] \\
&\quad+\Big(\zeta+\frac{\eta}{3}\Big)\,[-m^2U_{mn}-nmV_{mn}+imW'_{mn}], \\
\rho_0\big[i(mu_0+nv_0)V_{mn}+\gamma W_{mn}\big] &= -inP_{mn}+\eta\big[-(m^2+n^2)V_{mn}+V''_{mn}\big] \\
&\quad+\Big(\zeta+\frac{\eta}{3}\Big)\,[-mnU_{mn}-n^2V_{mn}+inW'_{mn}], \\
\rho_0\big[i(mu_0+nv_0)W_{mn}\big] &= -P'_{mn}+\eta\big[-(m^2+n^2)W_{mn}+W''_{mn}\big] \\
&\quad+\Big(\zeta+\frac{\eta}{3}\Big)\,[imU'_{mn}+inV'_{mn}+W''_{mn}], \\
i(mu_0+nv_0)R_{mn}&+\rho_0(imU_{mn}+inV_{mn}+W'_{mn})=0.
\end{aligned} \tag{7}$$

Substituting the Taylor series expansion of f in ε about $p_0 = \mathrm{const}$ into (5), we obtain the expression

$$R_{mn} = f'(p_0)P_{mn}.$$

Now we substitute this expression into the last equation of (7) and eliminate W' and W'' from the first three equations. As a result, system (7) becomes a system of ordinary differential equations of second order in U and V and of first order in P and W. Hence, solutions to (7) are uniquely determined by the values of U_{mn}, U'_{mn}, V_{mn}, V'_{mn}, P_{mn}, and W_{mn} at a point z_{mn} and can be uniquely continued to the number axis $\mathbb{R} = \{z\}$. Let

$$z_{mn} = -(m\beta+n\delta)/(m\alpha+n\gamma). \tag{8}$$

This point is a root of the equation $mu_0 + nv_0 = 0$.

The unperturbed system is nondegenerate if

$$\alpha\delta-\gamma\beta \neq 0. \tag{9}$$

This condition ensures that the relations

$$m\beta+n\delta=0, \quad m\alpha+n\gamma=0 \qquad (m^2+n^2\neq 0)$$

are not satisfied simultaneously.

By virtue of (9), the set of points (8) everywhere densely fills the axis $\mathbb{R} = \{z\}$. If $W_{mn}(z_{mn}) \neq 0$ (a typical case), then the Poincaré set is everywhere dense; by the Poincaré–Kozlov theorem, hence, the steady flows for our model of viscous barotropic compressible fluids do not have nonconstant integrals and nontrivial frozen fields of directions. When $m = n = 0$, system (7) is nondegenerate; in the general case, therefore, we can assume that $W_{00} \neq 0$. Under this assumption, hence, nontrivial integral invariants are also absent [7].

Remark. The case of homogeneous incompressible viscous fluids was previously considered in [7], where the potential of external mass forces was added to simplified equations (3). It turns out that in this case the system does not admit nontrivial integrals, fields of symmetries, and integral invariants.

4. THE MODEL WITH VISCOUS FRICTION

In dynamics of systems with finite degrees of freedom, the viscous forces are usually modeled with the aid of the Rayleigh dissipative function [11] when the velocities are low.

Let us consider a barotropic incompressible fluid whose dynamics is described by system (3) – (5) where the following term proportional to the velocity $\boldsymbol{v} = (u, v, w)$ is used in place of the terms containing viscosity:

$$\dot{\boldsymbol{v}} = -\frac{1}{\rho}\frac{\partial p}{\partial \boldsymbol{x}} - \frac{\partial V}{\partial \boldsymbol{x}} - \varepsilon\boldsymbol{v}, \qquad \operatorname{div}\boldsymbol{v} = 0. \tag{10}$$

Here V is the energy of external mass force density, $-\varepsilon\boldsymbol{v}$ is the *external friction* [3], and ε is a small parameter. If $\varepsilon = 0$, then from (10) we obtain Euler's equation. When the fluid is incompressible, it follows from (4) that the density ρ is a first integral. If equation (1) does not admit nonconstant first integrals, then $\rho \equiv \mathrm{const}$, i.e., the fluid is homogeneous. In general, $\rho \not\equiv \mathrm{const}$ for the case of regular flows.

Example. Let us consider the case $\varepsilon = 0$. Then, from the third equation of system (2) we obtain the equation $\dot{z} = 0$. In this case, u, v, and ρ are arbitrary functions of z. By $V = mgz$ we denote the gravity energy. The third equation in (10) takes the form

$$0 = -\frac{1}{\rho(z)}\frac{\partial p(z)}{\partial z} - mg. \tag{11}$$

Now we use equation of state (5): $\rho = f(p) > 0$. Then, from (11) we conclude that $dp/dz < 0$. Solving equation (11), we get

$$\int\limits_{p_0}^{p} \frac{ds}{f(s)} = -mgz + \text{const},$$

where p_0 is a fixed value of pressure and p is a function of z. Hence, $p = p(z)$ is a nonconstant function and $\rho(z) \not\equiv \text{const}$ when $f' \neq 0$. In this case, the set $\{x, y \bmod 2\pi, z = \text{const}\}$ is topologically equivalent to a two-dimensional torus $\mathbb{T}^2$.

Let us assume that $\rho \equiv \text{const}$. Hence, solutions to system (10) can be represented in the form of series in ε:

$$u = u_0 + \varepsilon u_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots, \quad w = \varepsilon w_1 + \cdots, \quad \mathcal{F} = \mathcal{F}_0 + \varepsilon\mathcal{F}_1 + \cdots .$$

Let $\mathcal{F} = \mathcal{P} + V$, where $\mathcal{P}$ is the pressure function specified by $\mathcal{P} = -\frac{p}{\rho}$. We have $\mathcal{F}_0 = \text{const}$, whereas u_0 and v_0 are arbitrary functions of z. We restrict our consideration to the particular case when u_0 and v_0 are linear functions (as in (6)).

At the first approximation in ε, we obtain the following system of partial differential equations:

$$\begin{aligned}
&\frac{\partial u_1}{\partial x}u_0 + \frac{\partial u_1}{\partial y}v_0 + u_0' w_1 = -\frac{\partial \mathcal{F}_1}{\partial x} - u_0,\\
&\frac{\partial v_1}{\partial x}u_0 + \frac{\partial v_1}{\partial y}v_0 + v_0' w_1 = -\frac{\partial \mathcal{F}_1}{\partial y} - v_0,\\
&\frac{\partial w_1}{\partial x}u_0 + \frac{\partial w_1}{\partial y}v_0 = -\frac{\partial \mathcal{F}_1}{\partial z},\\
&\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0.
\end{aligned}$$

This system is linear. Let us solve it by the Fourier method. By U_{mn}, V_{mn}, W_{mn}, and F_{mn} we denote the Fourier coefficients for the velocity components u_1, v_1, and w_1 and for the function $\mathcal{F}_1$, respectively. All these coefficients are functions of z and satisfy the system of algebro-differential equations

$$\begin{aligned}
&i(mu_0 + nv_0)U_{mn} + u_0' W_{mn} = -imF_{mn} - u_0,\\
&i(mu_0 + nv_0)V_{mn} + v_0' W_{mn} = -inF_{mn} - v_0,\\
&i(mu_0 + nv_0)W_{mn} = -F_{mn}',\\
&imU_{mn} + inV_{mn} + W_{mn}' = 0.
\end{aligned}$$

From the first two equations of this system we can find the coefficients U_{mn} and V_{mn}. In order for the above functions to be analytic on $\mathbb{R} = \{z\}$, it is necessary and sufficient that for $z = z_{mn}$ from (8) the following relations be valid:

$$\alpha W_{mn} + m(iF_{mn}) = -u_0, \qquad \gamma W_{mn} + n(iF_{mn}) = -v_0.$$

Since the system is nondegenerate, its determinant is not equal to zero for $m^2+n^2 \neq 0$. Hence, W_{mn} and F_{mn} can uniquely be specified at the point z_{mn}; moreover, by virtue of $nu_0 - mv_0 \neq 0$, we have $W_{mn}(z_{mn}) \neq 0$. Since the Poincaré set $\mathbb{P} = \{z_{mn}\}$ is everywhere dense in $\mathbb{R}$, we can apply the Poincaré–Kozlov theorem.

Remark. The above results can be generalized to the case of compressible fluids with viscous friction. In this case, the system also does not admit nontrivial integrals, fields of symmetries, and integral invariants.

The author is grateful to Academician V.V. Kozlov for formulation of the above problems and for constant attention to this work.

The work was supported by the Russian Foundation for Basic Research (02–01–01059), by the Federal program "Leading Scientific Schools" (00–15–96146), and by the Federal program "Integration" (A–00–97).

REFERENCES

1. N.V. Denisova and V.V. Kozlov, "Steady Lagrangian turbulence", *Prikl. Matem. i Mekhan.* (to appear).

2. V.V. Kozlov, "Symmetries and regular behavior of Hamiltonian systems", *CHAOS*, **6**, 1: 1–5, 1996.

3. A.A. Fridman, *Hydromechanics of Compressible Fluids* (in Russian), Moscow–Leningrad, 1934.

4. V.V. Kozlov, "Frozen fields of directions, small denominators, and randomness of steady viscous flows", *Prikl. Matem. i Mekhan.*, **63**, 2: 237–244, 1999.

5. V.V. Kozlov, "Remarks on steady motions of continuum", *Prikl. Matem. i Mekhan.*, **47**, 2: 341–342, 1983.

6. V.I. Arnold, "Topology of three-dimensional steady ideal flows", *Prikl. Matem. i Mekhan.*, **30**, 1: 183–185, 1966.

7. V.V. Kozlov, "Dynamical systems determined by the Navier–Stokes equations", *Rus. J. Math. Phys.*, **1**, 1: 57–69, 1993.

8. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste* (in French), Volume I, Paris, 1892.

9. V.V. Kozlov, *Symmetries, Topology, and Resonances in Hamiltonian Mechanics*, Berlin–Heidelberg, 1996.

10. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, New York, 1983.

11. L. Rayleigh, *Theory of Sound*, New York, 1945.

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