Almost-Spectral Decomposition for Non-self-adjoint Operators. Matrix Model Case

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We consider nonself-adjoint, non-dissipative operators acting in a Hilbert space. Our main aim is the spectral analysis of the singular spectral subspace N_i^0, which possible presence separates non-dissipative operators from dissipative ones. In particular, we single out the class of operators with almost Hermitian spectrum in which the spectral subspace N_i^0 coincides with the whole Hilbert space H. The spectrum of such operators is automatically real and purely singular. What is more inspiring is that it turns out, that in terms of its analytic properties the almost Hermitian subspace N_i^0 behaves essentially as a singular spectral subspace of a general self-adjoint operator (hence the choice of the name).

In particular, for both self-adjoint operators with purely singular spectrum and nonself-adjoint operators with almost Hermitian spectrum a natural generalization of the Cayley identity holds.

The following result holds [3] in the self-adjoint case: a self-adjoint operator A possesses purely singular spectrum (i.e., its absolutely continuous subspace is trivial) if and only if there exists a scalar bounded analytic and outer in the upper half-plane function $\gamma(\lambda)$ such that

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0} \gamma(A+i\varepsilon)=0.$$

A function γ can be given explicitly, for example as the perturbation determinant of the pair $A, A-iV$ for any non-negative trace class self-adjoint operator V such that all the generating vectors of A belong to its closed range.

A directly analogous result [1,3] holds in the situation of nonself-adjoint operators with almost Hermitian spectrum, the only (quite natural) difference being that in this case one has to speak of two functions γ and γ_*, defined in the upper and lower half-planes of the complex plane, respectively.

Further, it turns out that a natural generalization of the spectral theorem exists for the operators with almost Hermitian spectrum, although in this situation one has to understand the corresponding spectral decomposition in the sense of generalized functions.

This spectral decomposition allows one to develop a rich functional calculus for the operators of the class considered and based on this to obtain tight estimates on the norms of functions of operators with almost Hermitian spectrum.

In the talk, the corresponding results will be presented in the case of non-self-adjoint operators belonging to the so called Matrix Model class, i.e., the class of rank 2 additive completely non-self-adjoint perturbations of self-adjoint operators, introduced by us in [2].

Despite the seemingly simple setting of this model, it reveals all the major difficulties of the general case. On the other hand, its major advantage compared to the latter is that it can be parameterized by a limited set of analytic functions which allows one to do many calculations explicitly based on results of the complex functional analysis. One should also mention that in the general situation, the results have the same form as in the Matrix Model situation.

References:

[1] *A. V. Kiselev, S. N. Naboko.* Nonself-adjoint Operators With Almost Hermitian Spectrum: Weak Annihilators// Functional Analysis & App., vol. 38, N 3. 2004.

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[3] *A. V. Kiselev, S. N. Naboko.* Non-Self-Adjoint Operators with Almost Hermitian Spectrum: Cayley Identity and Some Questions of Spectral Structure// Arkiv for Matematik, vol. 47. 2009. P. 91–125.

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