Real Processes and Realizability of a Stabilization Method for Navier–Stokes Equations by Boundary Feedback Control

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Dedicated to Olga Aleksandrovna Ladyzhenskaya

We study a problem of stabilization of a solution $v(t,x)$ to the Navier–Stokes equations given in a bounded domain $\Omega \subset R^3$ to a steady-state solution $\widehat{v}(x)$ (that is not necessary stable). The stabilization is realized by a feedback control defined on a part of the boundary $\partial\Omega$ of the domain Ω. We analyze from the mathematical point of view the possibility of numerical and engineering realization of the stabilization method we propose. To this end, we continue to study the notion of a real process introduced in [**1**]. This notion is an abstract analog of an approximate solution obtained by computers and it is characterized by fluctuations that affect the solution at discrete time moments. In this paper, we generalize the notion of a real process to the case where fluctuations appear at each time moment. We study properties of this real process and, in particular, we propose a method of stabilization by a feedback boundary control that reacts on fluctuations damping them.

1. Introduction

For a long time, unstable systems attract considerable attention of specialists in various fields of science and, in particular, of mathematicians. Unstable systems are described by boundary-value problems admitting unstable solutions,

†The work is partially supported by the Russian Foundation for Basic Research (grants no. 00–01–00415 and no. 00–15–96109).

in particular, by ill-posed boundary-value problems. The most of mathematical investigations of ill-posed problem have been about existence or(and) uniqueness of solutions. Such statements are, naturally, initiated by the theory of well-posed boundary-value problems where existence and uniqueness of solutions usually guarantee the possibility of numerical simulation.

However, in the theory of ill-posed (and even unstable) boundary-value problems, an information about existence and(or) uniqueness of a solution is not sufficient for their successful numerical solution. It is a very important theoretical problem to study arising and growth of perturbations in the process of numerical solutions of ill-posed or unstable problems in order to develop methods of damping these perturbations.

This problem is the main object of this paper. We study this problem on an example of stabilization of a solution to the Navier–Stokes equations that describes a viscous incompressible fluid flow.

The investigation of the stabilization problem was preceded by creation of an existence theory for the problem of local exact controllability of the Navier–Stokes equations (cf., for example, [**2**, Chapt. 7]). This is a typical ill-posed problem where the Navier–Stokes equations is considered with the initial condition $v\big|_{t=0} = v_0$ and with a similar condition at $t = T$: $v\big|_{t=T} = \widehat{v}$ (where $\widehat{v}$ is a function of some special form, for example, it is a steady-state solution to the Navier–Stokes equations). However, the Dirichlet condition on the lateral surface of the space-time cylinder is not prescribed since it is an unknown control.

The stabilization problem can be regarded as regularization of the local exact controllability problem because the condition $v\big|_{t=T} = \widehat{v}$ is replaced with the condition $v(t,\cdot) \to \widehat{v}$ as $t \to \infty$. Some additional condition on the convergence rate is also imposed. We have to stress that the stabilization problem means not only to prove the corresponding existence theorem but moreover to work out a robust method of its realization (for example, of its numerical simulation).

Usually, the validity of such a condition is ensured by using of feedback control. The classical mathematical formalization of this notion goes from the theory of controlled ordinary differential equations (cf. [**3, 4**]). This approach was extensively used in the stabilization theory for evolution partial differential equations (mainly, of hyperbolic type) during the last 20-30 years.

In the case of stabilization by a control defined on a boundary, in order to apply the classical notion of a feedback, one can reduce the consideration to construct boundary operators (instead of boundary control) such that the obtained boundary-value problem is well-posed and any its solution tends to a given function $\widehat{v}$ as $t \to \infty$ with a prescribed rate.

However, we meet significant difficulties if we try to apply the classical formalization of feedback notion to the stabilization theory of a more or less general quasilinear parabolic equations or of equations of hydrodynamics. We know the following recent papers devoted to the stabilization of equations of hydrodynamics. The first one is due to Coron [**5**] who studied the feedback boundary control stabilization of the two-dimensional Euler equations describing an ideal incompressible fluid. The stabilization was done to steady-state solution $\widehat{v} \equiv 0$ as $t \to \infty$. His results are of a certain interest. However, a number of questions remains open. For example, the choice of boundary operators expressing the feedback leads to a boundary-value problem for the Euler equation for which the uniqueness of a solution is not yet established. Hence the question whether this problem is well-posed remains still open.

The second paper due to Barbu and Sritharan [**6**] deals with feedback stabilization for the two-dimensional Navier–Stokes equations by some abstract control. However, this abstract control does not cover the case of boundary control.

In author's opinion, all these difficulties are caused by the fact that the classical notion of a feedback going from the ordinary differential equations does not take into account specific features of partial differential equations. In particular, huge resources for stabilization that are contained in parabolic equations and the Navier–Stokes equations and are connected with an infinite-dimensional subspace of the phase space consisting of high-frequency harmonics is not used at all.

A new approach to the study of stabilization by feedback boundary control was proposed by the author and was realized for quasilinear parabolic equations [**7**], for the two-dimensional Navier–Stokes equations and Oseen equations [**8**] and [**9**], and the for three-dimensional Navier–Stokes equations [**10**].

Using the approach of [**7**]–[**11**], we reduce the consideration to the investigation of a mixed boundary-value problem for the Navier–Stokes equations with the initial condition belonging to some stable invariant manifold M_-. The invariance and stability of M_- mean that if an initial datum w_0 belongs to M_-, then the solution $w(t,\cdot)$ belongs to M_- for any $t > 0$ and tends to a given steady-state solution $\widehat{w}$ at a prescribed rate (that realizes the required stabilization).

However, M_- is a repelling subset of the phase space of the problem. This means that if the initial data v_0 or the solution $v(t,\cdot)$ left M_- after some perturbation, then it will go away from M_- and, consequently, the solution will not be stabilized. In other words, the stabilization problem is reduced to the question on possibility to realize a solution of an unstable problem for which an existence theorem has been already established. Realization of a solution of a stabilization problem may be engineering, as well as numerical. The first attempts to clear up in the aforementioned problem were undertaken in [**11**]. But the actual step to its solution was made in [**1**]. In [**1**], the theoretical analysis of the realization problem was based on numerical realization which is more simple for analysis. The notion of a real process $\widetilde{w}$ was introduced. It is an abstract analog of numerical solution of the stabilization problem. As in the case of the numerical simulation, the real process $\widetilde{w}(t,\cdot)$ is subjected to the action of small fluctuations at discrete time moments t_j. In [**1**], the author proposed how to construct a feedback boundary control that reacts to appearance of fluctuations at the time moments t_j damping them. As a result, a stabilization of the real process $\widetilde{w}(t,\cdot)$ was obtained and, in particular, an estimate of the stabilization rate for $\widetilde{w}(t,\cdot)$ was derived. Thereby the possibility of numerical simulation of the stabilization problem for the Navier–Stokes equations had been established from the theoretical point of view.

In the present paper, we continue to study the same realization problem for a solution to an unstable problem in the following directions:

1. In Sec.2 we construct the finite-difference analog of stabilization problem corresponding to the stabilization problem for the simplest one-dimensional parabolic equation. Further, we solve this finite-difference stabilization problem and analyze its numerical realization. In particular, for this numerical solution we present the feedback stabilization method. We wrote this section in order to make more clear and convincing for readers the axiomatics of a real process corresponding to the Navier–Stokes equations.

2. We developed the axiomatics of a real process and made it more complete and adequate. As a result we improved the construction of stabilization of a real process proposed in [**1**].

3. Together with numerical simulation, engineering realization of a mathematical solution is also of great importance. The direct analysis of this problem is much more difficult than in the case of numerical simulation. However, it is clear that real processes also play an important role. However, in this case, the real processes are subjected to the action of fluctuations at each time moment. In Sec. 7 we construct a real process possessing this property and consider the stabilization of this process. We derive an integrodifferential equation that the real process satisfies to. Due to this equation, we can immediately conclude that the real process is uniquely determined by the feedback boundary control which dampens fluctuations acting on the real process. We also obtain an estimate of the convergence rate of the stabilized real process to the steady-state solution.

In Secs. 3–6, we briefly recall the construction of stabilization proposed in [**7**]–[**10**], [**1**]. In Secs. 5 and 6, we specify this construction according to modifications of the axiomatics of the real process. This specification became possible due to the analysis done in Sec. 2.

2. Stabilization of the Simplest Parabolic Equation

In this section, considering the simplest parabolic equation, we present a method to stabilize its solutions by a control defined on the boundary of a domain. This method leads to some unstable problem. Therefore, fluctuations, which always appear during the numerical simulation of this method, easily destroy the stabilization construction. The main goal of this section is to expand this method to the case of finite-difference analog of the parabolic equation under consideration, in order to understand processes connected with the numerical solution of the stabilization problems for this finite-difference equation and, using feedback control, to develop a robust method of stabilization of a solution to the finite-difference equation. A result of the numerical solution to the finite-difference equation will be called a real process. We study properties of a real process which serve as a basis of an axiomatic definition of an abstract notion which is also referred to as a "real process." This notion is introduced in Secs. 6 and 7 below in order to develop locally robust method to stabilize solutions to the Navier–Stokes equations.

2.1. Parabolic equation and the statement of the stabilization problem. On the segment $[-\pi/2, \pi/2]$, we consider the parabolic equation

$$\partial_t v(t,x) - \partial_{xx} v(t,x) - \alpha v(t,x) = 0 \tag{2.1}$$

with the boundary conditions

$$v(t,x)\big|_{x=-\pi/2} = u_-(t), \quad v(t,x)\big|_{x=\pi/2} = u_+(t) \tag{2.2}$$

and the initial condition

$$v(t,x)\big|_{t=0} = v_0(x), \tag{2.3}$$

where $v_0(x) \in L_2(-\pi/2, \pi/2)$ is a given function. The boundary data $u_-(t)$ and $u_+(t)$ are controls, i.e. they are not assumed to be known. We suppose that $\partial_t = \partial/\partial t$, $\partial_{xx} = \partial^2/\partial x^2$, $\alpha > 1$ is a fixed number. We introduce the term with coefficient α in order that Eq. (2.1) has exponentially increasing solutions. For example, the function $v(t,x) = e^{(\alpha-1)t} \sin x$ possesses such property.

The stabilization problem for a solution to the boundary-value problem (2.1)–(2.3) is formulated as follows. Let $\sigma > 0$ be given. Given $v_0(\cdot) \in L_2(-\pi/2, \pi/2)$ find controls $u_-(t)$ and $u_+(t)$ such that the solution $v(t,x)$ to the boundary-value problem (2.1)–(2.3) satisfies the inequality

$$\|v(t,\cdot)\|_{L_2(-\pi/2,\pi/2)} \leqslant c\|v_0\|_{L_2(-\pi/2,\pi/2)} e^{-\sigma t}, \quad t > 0. \tag{2.4}$$

An important additional condition we impose on the solution to this problem is the requirement that the constructed control admits a feedback, i. e., it can react on fluctuations damping them. An exact mathematical sense of this condition will become clear while solving this problem.

2.2. Method of solution of the stabilization problem. Forgetting for a while about the boundary conditions (2.2), we extend the problem (2.1), (2.3) from the segment $[-\pi/2, \pi/2]$ to the segment $[-\pi, \pi]$ and impose periodic boundary conditions. We obtain the problem

$$\partial_t w(t,x) - \partial_{xx} w(t,x) - \alpha w(t,x) = 0, \tag{2.5}$$

$$w(t,-\pi) = w(t,\pi), \quad \partial_x w(t,-\pi) = \partial_x w(t,\pi), \tag{2.6}$$

$$w(t,x)\big|_{t=0} = w_0(x) = Ev_0(x), \tag{2.7}$$

where the initial data $w_0(x)$ is an extension of the initial data $v_0(x)$ of the problem (2.1)–(2.3) from $[-\pi/2, \pi/2]$ to $[-\pi, \pi]$. The construction of the required extension operator E is the main part of the whole of construction. To clarify what conditions on the operator E are necessary, we solve the problem (2.5)–(2.7) by the Fourier method. We expand $w(t,x)$ into the Fourier series

$$w(t,x) = \sum_{k=-\infty}^{\infty} \widehat{w}_k(t) e^{ikx}$$

and substitute this series into the relations (2.5) and (2.7). We find

$$w(t,x) = \sum_{k=-\infty}^{\infty} \widehat{w}_{0k} e^{-t(k^2-\alpha)+ikx}, \tag{2.8}$$

where $\widehat{w}_{0k}$ are the Fourier coefficients of the initial data $w_0(x)$. By the Parseval equality, the solution (2.8) satisfies the estimate

$$\|w(t,\cdot)\|^2_{L_2(-\pi,\pi)} \leqslant \|w_0\|^2_{L_2(-\pi,\pi)} e^{-2\sigma t}, \quad t > 0, \tag{2.9}$$

if and only if $\widehat{w}_{0k}$ satisfy the relations

$$\widehat{w}_{0k} = 0 \quad \text{for } |k| < \sqrt{\alpha + \sigma}.$$

We introduce the space

$$X_\sigma = \{w_0(x) \in L_2(-\pi,\pi) : \widehat{w}_{0k} = 0 \text{ for } |k| < \sqrt{\alpha+\sigma}\}, \tag{2.10}$$

where $\widehat{w}_{0k}$ are the Fourier coefficients of the function $w_0(x)$.

Assume that the following assertion is valid.

Lemma 2.1. *There exists a linear continuous operator E extending functions from the segment $[-\pi/2, \pi/2]$ onto $[-\pi, \pi]$ that acts into the space X_σ:*

$$E : L_2(-\pi/2, \pi/2) \to X_\sigma. \tag{2.11}$$

Then the solution to the original stabilization problem (2.1)–(2.4) is determined by the formula

$$(v(t,x), u_-(t), u_+(t)) = (w(t,x)\big|_{x\in[-\pi/2,\pi/2]}, w(t,x)\big|_{x=-\pi/2}, w(t,x)\big|_{x=\pi/2}), \tag{2.12}$$

where $w\big|_{x\in S}$ denotes the restriction of a function w to the set S.

Indeed, the triple (v, u_-, u_+) satisfies (2.1)–(2.3) because w satisfies (2.5)–(2.7). Moreover, by (2.9), (2.11), and (2.12), we have

$$\|v(t,\cdot)\|_{L_2(-\pi/2,\pi/2)} \leqslant \|w(t,\cdot)\|_{L_2(-\pi,\pi)} \leqslant \|E\|\|v_0\|_{L_2(-\pi/2,\pi/2)}e^{-\sigma t}, \quad t > 0,$$

which proves the inequality (2.4).

Proof of Lemma 2.1. We look for an operator E in the form

$$Ev_0(x) = \begin{cases} v_0(x), & |x| < \pi/2, \\ \sum\limits_{|j|<\sqrt{\sigma+\alpha}} c_j e^{ijx}, & \pi/2 < |x| < \pi, \end{cases} \tag{2.13}$$

where c_j are unknown constants. By the conditions in (2.10), the following relations hold:

$$\int\limits_{|x|<\pi/2} v_0(x)e^{-ikx}dx + \int\limits_{\pi/2<|x|<\pi} \sum_{|j|<\sqrt{\sigma+\alpha}} c_j e^{i(j-k)x}dx = 0 \quad \forall |k| < \sqrt{\sigma+\alpha}. \tag{2.14$'$}$$

Setting $\vec{b} = \{b_k\}$ and $A = \|a_{kj}\|$, where

$$b_k = -\int\limits_{|x|<\pi/2} v_0(x)e^{-ikx}dx, \quad a_{kj} = \int\limits_{\pi/2<|x|<\pi} e^{i(j-k)x}dx,$$

we write the system (2.14$'$) in the form

$$A\vec{c} = \vec{b}, \quad \vec{c} = \{c_j, |j| < \sqrt{\sigma+\alpha}\}.$$

Obviously, it suffices to prove that $\det\|A\| \neq 0$. We obtain this relation by proving the positive definiteness of the matrix A. Indeed, for any vector $\zeta = \{\zeta_j\}$ we have

$$\langle A\zeta, \zeta\rangle \equiv \sum_{j,k} a_{jk}\zeta_j\overline{\zeta}_k = \int\limits_{\pi/2<|x|<\pi} \Big|\sum_j \zeta_j e^{ijx}\Big|^2 dx \geqslant 0.$$

If the right-hand side contains equality instead of inequality, then polynomial $\sum\limits_j \zeta_j e^{ijx} = 0$ where x runs through the continuum $1 < |x| < \pi$. Consequently, $\zeta_j = 0$ for any j. □

Thus, we complete the construction of stabilization of a solution to the problem (2.1)–(2.3). However, this construction has a particular feature. Using this construction, we reduce the consideration to the solution of the boundary-value problem (2.5)–(2.7) with the initial data $w_0(x)$ belonging to the subspace X_σ of the space $L_2(-\pi,\pi)$. The subspace X_σ is invariant with respect to Eq. (2.5) but it is a repelling set of the dynamical system generated by Eqs. (2.5), (2.6). This means the following. If, due to some fluctuation, the solution

$w(t,\cdot)$ became outside X_σ at some time moment t_0 so that the Fourier coefficient $\int\limits_{-\pi}^{\pi} w(t_0,x)\sin x\,dx$ does not vanish, them the solution $w(t,\cdot)$ increases like $e^{(\alpha-1)t}$ as $t\to\infty$ and all the construction of stabilization destroys. Since fluctuations always appear when one solves finite-difference problem numerically, it is necessary to study this phenomenon in the case of finite-difference problem and just in this case it is necessary to develop methods that allow us to overcome these difficulties.

2.3. The statement of the stabilization problem in the case of the finite-difference equations. We consider finite-difference equations corresponding to the problem (2.1)–(2.3). Let $h=\pi/(2N)$. We divide the interval $[-\pi/2,\pi/2]$ into $2N$ parts by the points $x_j=jh$, $j=0,\pm1,\dots,\pm N$. We introduce a step in time τ and set $t_n=n\tau$. Introduce the notations $v(t_n,x_j)=v_j^n$, $v_0(x_j)=v_{0,j}$, $u_\pm(t_n)=u_\pm^n$. Then the finite-difference analog of (2.1)–(2.3) can be written as follows:

$$\frac{v_j^n-v_j^{n-1}}{\tau}-\frac{v_{j-1}^n-2v_j^n+v_{j+1}^n}{h^2}-\alpha v_j^{n-1}=0,\quad |j|<N,\ n>0, \tag{2.14}$$

$$v_{-N}^n=h_-^n,\quad v_N^n=h_+^n,\quad n\geqslant 0, \tag{2.15}$$

$$v_j^0=v_{0,j},\quad |j|<N. \tag{2.16}$$

By the space $L_{2,h}[-\pi/2,\pi/2]$ we mean the set of grid functions $\{v_j,|j|\leqslant N\}$ equipped with the norm

$$\|v_\cdot\|^2_{L_{2,h}[-\pi/2,\pi/2]}=h\sum_{j=-N}^{N}|v_j|^2. \tag{2.17}$$

In this case, the stabilization problem is formulated as follows. Given $\sigma>0$, for an initial data $v_{0,\cdot}=\{v_{0j}\}$ one have to construct controls $u_-^\cdot=\{u_-^n\}$, $u_+^\cdot=\{u_+^n\}$ such that the solution $v_\cdot^\cdot=\{v_j^n\}$ to the problem (2.14)–(2.16) satisfies the estimate

$$\|v_\cdot^n\|_{L_{2h}[-\pi/2,\pi/2]}\leqslant c\|v_{0,\cdot}\|e^{-\sigma n\pi}\quad\text{as } n\to\infty. \tag{2.18}$$

We generalize the stabilization construction presented in Subsec.2.2 to the case of finite-difference equations. Forgetting for a while about the condition (2.15), we extend the problem (2.14), (2.16) from $[-\pi/2,\pi/2]$ onto the segment $[-\pi,\pi]$

imposing periodic boundary conditions. We find

$$\frac{w_k^n - w_k^{n-1}}{\tau} - \frac{w_{k-1}^n - 2w_k^n + w_{k+1}^n}{h^2} - \alpha w_k^{n-1} = 0, \quad |k| < 2N+1, \ n > 0, \tag{2.19}$$

$$w_{-2N-1}^n = w_{2N-1}^n, \quad w_{-2N}^n = w_{2N}^n, \quad w_{-2N+1}^n = w_{2N+1}^n, \quad n \geqslant 0, \tag{2.20}$$

$$w_k^0 = w_{0,k}, \quad |k| \leqslant 2N+1, \tag{2.21}$$

where $w_{0,\cdot} = \{w_{0,j}\}$ is an extension to $[-\pi, \pi]$ of the initial data from (2.16): $w_{0,\cdot} = E_h v_{0,\cdot}$. The construction of the extension operator E_h is a central step of the whole of construction.

2.4. Construction of stabilization for the finite-difference equations. In the space $L_{2,h}(-\pi, \pi)$ equipped with the inner product

$$\langle f_\cdot, g_\cdot \rangle = h \sum_{k=-2N}^{2N-1} f_k \overline{g}_k \tag{2.22}$$

(bar means the complex conjugation), we introduce the system of functions $e_\cdot(j)$, $j = -2N, -2N+1, \dots, 2N-1$, where

$$e_\cdot(j) = \left\{ \frac{1}{\sqrt{2\pi}} e^{ijhk}, k = -2N, -2N+1, \dots, 2N-1 \right\}. \tag{2.23}$$

The system (2.23) is a basis for $L_{2,h}(-\pi, \pi)$; moreover, this basis is orthonormal with respect to the inner product (2.22). This fact can be verified by direct computation with the help of the formula expressing the sum of terms of geometric progression.

We expand the solution $w_\cdot^n = \{w_k^n\}$ to the problem (2.19)–(2.21) with respect to the basis (2.23):

$$w_k^n = \sum_{j=-2N}^{2N-1} \widehat{w}^n(j) \frac{1}{\sqrt{2\pi}} e^{ijhk}. \tag{2.24}$$

Substituting (2.24) in (2.19), we obtain the following equation for the Fourier coefficients $\widehat{w}^n(j)$:

$$\frac{\widehat{w}^n(j) - \widehat{w}^{n-1}(j)}{\tau} - \widehat{w}^n(j) \frac{e^{-ijh} - 2 + e^{ijh}}{h^2} - \alpha \widehat{w}^{n-1}(j) = 0.$$

Hence

$$\widehat{w}^n(j) = R(j) \widehat{w}^{n-1}(j), \quad \text{where } R(j) = \frac{1 + \alpha\tau}{1 + \frac{4\tau}{h^2} \sin^2 \frac{jh}{2}}. \tag{2.25}$$

Consequently,

$$\widehat{w}^n(j) = (R(j))^n \widehat{w}_0(j), \tag{2.26}$$

where $\widehat{w}_0(j)$ are the Fourier coefficients of the initial data $w_{0,\cdot}$ in (2.21). Taking into account the definition (2.26) of the function $R(j)$, we compute $(R(j))^n$ for $n = t/\tau$ and $\tau \to 0$, $h \to 0$. For the sake of definiteness, we assume that $\tau = h$.[1)] Making elementary computations we find

$$\begin{aligned}(R(j))^n &= \left(1+\tau\frac{\alpha - \frac{4}{h^2}\sin^2\frac{hj}{2}}{1+\tau\frac{4}{h^2}\sin^2\frac{hj}{2}}\right)^{t/\tau} = \left(1+\tau\frac{\alpha - j^2 + o(\tau)}{1+\tau(j^2+o(\tau))}\right)^{t/\tau}\\ &= (1+\tau(\alpha - j^2 + o(\tau)))^{t/\tau} = ((1+s)^{1/s})^{t(\alpha-j^2+o(\tau))}\\ &= e^{t(\alpha-j^2)}(1+o(\tau))\end{aligned} \tag{2.27}$$

for sufficiently small $s = \tau(\alpha - j^2 + o(\tau))$. As usual, $o(\tau)$ is a certain function of τ admitting the estimate $|o(t)| \leqslant c|\tau|$ as $\tau \to 0$. By (2.24), (2.26), and (2.27), we have

$$w_k^n = \sum_{j=-2N}^{2N-1} e^{n\tau(\alpha-j^2)+ijhk}(1+o(\tau))\widehat{w}_{ok}, \tag{2.28}$$

where $\widehat{w}_{0k}$ are the Fourier coefficients of the initial data $w_{0,\cdot}$ with respect to the basis (2.23).

In accordance with (2.10), we introduce the space

$$X_{\sigma,h} = \{w_{0,\cdot} \in L_{2,h}(-\pi,\pi) : \widehat{w}_{0,k} = 0 \text{ for } |k| < \sqrt{\alpha+\sigma}\}, \tag{2.29}$$

where $\widehat{w}_{0,k}$ are the Fourier coefficients of the grid function $w_{0,\cdot}$. As in the continuous case, (2.28) and (2.29) imply that the solution $\{w_k^n\}$ to the problem (2.19)–(2.21) satisfies the estimate

$$\|w_\cdot^n\|^2_{L_{2,h}(-\pi,\pi)} \leqslant c\|w_{0,\cdot}\|^2_{L_{2,h}(-\pi,\pi)} e^{-2\sigma n\tau} \quad \text{as } n\to\infty \tag{2.30}$$

if and only if the initial data in (2.21) satisfies the inclusion $w_{0,\cdot} \in X_{\sigma,h}$.

Lemma 2.2. *There exists an extension operator E_h that extends grid functions defined on $[-\pi/2,\pi/2]$ to grid functions defined on $[-\pi,\pi]$. Moreover, E_h is a continuous operator in the spaces*

$$E_h : L_{2,h}(-\pi/2,\pi/2) \to X_{\sigma,h}. \tag{2.31}$$

PROOF. As in the case of (2.13), we set

$$E_h v_{0,k} = \begin{cases} v_{0,k}, & |k| < N,\\ \sum\limits_{|j|<\sqrt{\sigma+\alpha}} c_j e^{ijhk}, & k = -2N \text{ and } N \leqslant |k| < 2N.\end{cases} \tag{2.32}$$

[1)] Since the finite-difference equations (2.19) are absolutely stable, this assumption is not essential.

The coefficients c_j are determined from the following system which is similar to (2.14) and is deduced from the conditions in (2.29):

$$\sum_{|j|<\sqrt{\sigma+\alpha}} c_j \Bigg(\sum_{\substack{N\leqslant|k|<2N\\ k=-2N}} e^{i(j-l)hk} \Bigg) = -\sum_{|k|<N} v_{0k} e^{-ilhk}. \tag{2.33}$$

As in Lemma 2.1, to prove that the matrix of the system (2.33) is nonsingular, we show that it is positive definite. Repeating the corresponding arguments of the proof of Lemma 2.1, we reduce the proof of the positive definiteness of this matrix to the verification of the following fact: If the equality

$$\sum_{|j|<\sqrt{\sigma+\alpha}} \zeta_j e^{ijhk} = 0 \tag{2.34}$$

holds for any $k \in K_N \equiv \{k = -2N \text{ and } N \leqslant |k| < 2N\}$, then $\zeta_j = 0$ for any j. We prove this assertion under the assumption that the number of points of the set K_N is larger than the number of points of the set $\{j : |j| < \sqrt{\sigma+\alpha}\}$ (i.e., for sufficiently small h). Indeed, replacing hk with x in (2.34) and multiplying the obtained equality by $e^{-(\sqrt{\sigma+\alpha_1})x}$, we obtain a polynomial that vanishes at the points $x_k = k \in K_N$. Since the number of these roots of the polynomial is larger than its degree, we have $\zeta_j = 0$ for all j. □

To complete the stabilization construction for a solution to the finite-difference problem (2.14), (2.16), it suffices to determine the solution to this problem by the formula

$$(v_k^n, |k| < N; u_-^n, u_+^n) = \{w_k^n, |k| < N; w_{-N}^n, w_N^n\}, \tag{2.35}$$

where $\{w_k^n\}$ is a solution to the problem (2.19)–(2.21) with the initial data $w_{0,\cdot} = E_h v_{0,\cdot}$, and $v_{0,\cdot}$ is the initial data in (2.16). It is obvious that the inequality (2.18), follows from (2.30).

2.5. Numerical solution. We study what happens in the numerical simulation of the proposed stabilization method. Thus, we reduced the consideration to the solution to the boundary-value problem (2.19)–(2.21) with the initial condition $w_{0,\cdot} = E_h v_{0,\cdot} \in X_{\sigma,h}$. This problem is solved with the help of so-called levels transition operator S_h that transforms a time level $w_\cdot^n$ of the solution to the problem (2.19)–(2.21) into the subsequent time level $w_\cdot^{n+1}$:

$$w_\cdot^{n+1} = S_h. \tag{2.36}$$

However, it should be noted that formula (2.36) fails while real computations because all quantities can be given only approximately in a computer. In practice, in fact, computations are performed in accordance with the following algorithm:

$$\widetilde{w}_\cdot^0 = w_{0,\cdot} + \tau\varphi_0^0,\ \ \widetilde{w}_\cdot^1 = S_h \widetilde{w}_\cdot^0 + \tau\varphi_\cdot^1, \dots, \widetilde{w}_\cdot^{n+1} = S_h \widetilde{w}_\cdot^n + \tau\varphi_\cdot^{n+1}. \tag{2.37}$$

Here, $\widetilde{w}^n_\cdot = \{\widetilde{w}^n_k\}$ is the grid function which is actually obtained as a result of computations. We call it a *real process* unlike an exact mathematical solution $\{w^n_k\}$ to the finite-difference problem. The grid functions $\{\tau\varphi^n_k\}$ are fluctuations that appear because computations are only approximate (the factor τ is introduced for the sake of convenience). We cannot predict their exact values. However, it is possible to estimate a priori their norms:

$$\|\varphi^n_\cdot\|_{L_{2,h}(-\pi,\pi)} \leqslant \widehat{\delta}, \tag{2.38}$$

where $\widehat{\delta}$ is a known very small number.

It is easy to deduce from (2.37) the following formula for the real process $\widetilde{w}^n_\cdot$:

$$\widetilde{w}^n_\cdot = (S_h)^n w_{0,\cdot} + (S_h)^n \varphi^0_\cdot + \tau \sum_{j=1}^{n} (S_h)^{n-j} \varphi^j_\cdot. \tag{2.39}$$

Obviously, $(S_h)^n w_{0,\cdot} = w^n_\cdot$ is an exact solution to the finite-difference problem (2.19)–(2.21) and $(S_h)^{n-j}\varphi^j_\cdot$ is an exact solution to the finite-difference problem (2.19), (2.20), where $n > j$, with the initial condition

$$w^n_k\big|_{n=j} = \varphi^j_k, \quad |k| < 2N \quad \text{and} \quad k = -2N. \tag{2.40}$$

Since $\varphi^n_\cdot$ is an arbitrary grid function satisfying only (2.38), for the Fourier coefficient of this function we have $\langle \varphi^n_\cdot, e_1(1)\rangle = \widehat{\varphi}^n(1) \neq 0$ in general. Therefore, using for $(S_h)^{n-j}\varphi^j_\cdot$ a formula similar to (2.28), we find

$$\|(S_h)^{n-j}\varphi^j_\cdot\|^2_{L_{2,h}(-\pi,\pi)} \sim ce^{2(n-j)\tau(\alpha-1)}.$$

This formula and (2.38) imply that

$$\|\widetilde{w}^n_\cdot\|^2_{L_{2mh}(-\pi,\pi)} \sim c_1 e^{-2\sigma n\tau} + c_2 e^{2n\tau(\alpha-1)} \quad \text{as } n \to \infty. \tag{2.41}$$

Although the constant c_2 in (2.41) is of order $\widehat{\delta}$ (cf. (2.38)), i.e., it is very small, but as $n \to \infty$ the real process $\widetilde{w}^n_\cdot$ grows like $e^{2n\tau(\alpha-1)}$ as $n \to \infty$. This means that the fluctuations $\varphi^j_\cdot$ appearing while computing destroy the proposed stabilization construction and no stabilization holds for the real process $\widetilde{w}^n_\cdot$.

2.6. Feedback control and stabilization of a real process. We note that among all quantities in the relations (2.37), only the real process $\widetilde{w}^n_\cdot$ is completely observable because all the information about it is contained in the computer memory. We have only partial information about the exact solution $w^n_\cdot$ and fluctuation $\tau\varphi^n_\cdot$ e.g., some estimate like (2.38). But we need to stabilize the real process. To this end, we proceed as follows. At the time moment $(n+1)\tau$, when we already obtain the time level $\widetilde{w}^{n+1}_\cdot$ of the real process, we consider its projection onto the subspace $X_{\sigma,h}$ of the space $L_{2,h}(-\pi,\pi)$. This projection Π_h is determined by the formula

$$(\Pi_h \widetilde{w}^n_{\cdot})_k = \begin{cases} \widetilde{w}^n_k, & |k| < N, \\ \widetilde{w}^n_k + \sum\limits_{|j|<\sigma+\alpha} c_j e^{ijhk}, & k = -2N \text{ and } N \leqslant |k| < 2N; \end{cases} \tag{2.42}$$

moreover, by (2.29), the constants c_j are found from the following algebraic linear system of equations, which is similar to (2.33):

$$\sum_{|j|<\sqrt{\sigma+\alpha}} c_j \Bigg(\sum_{\substack{N \leqslant |k| < 2N \\ k=-2N}} e^{i(j-l)hk} \Bigg) = - \sum_{\substack{|k|<2N \\ k=-2N}} \widetilde{w}^n_k e^{-ilhk}. \tag{2.43}$$

The matrix of this system is the same as that of the system (2.33), and the non singularity of this matrix was proved in Lemma 2.2. Therefore, the constants c_j in (2.42) and, consequently, the projection (2.42) itself are uniquely determined.

Thus, to define a stabilized real process, we use the following relations instead of (2.37):

$$\begin{aligned} &\Pi_h \widetilde{w}^0_{\cdot} = w_{0,\cdot} + \Pi_h \varphi^0_{\cdot}, \ \Pi_h \widetilde{w}^1_{\cdot} = S_h \Pi_h \widetilde{w}^0_{\cdot} + \Pi_h \tau \varphi^1_{\cdot}, \dots, \\ &\Pi_h \widetilde{w}^{n+1}_{\cdot} = S_h \Pi_h \widetilde{w}^n_{\cdot} + \Pi_h \tau \varphi^n_{\cdot}. \end{aligned} \tag{2.44}$$

Here, we take into account that

$$\Pi_h w_{0,\cdot} = w_{0,\cdot}, \dots, \Pi_h S_h \Pi_h \widetilde{w}^j_{\cdot} = S_h \Pi_h \widetilde{w}^j_{\cdot}$$

because $w_{0,\cdot} \in X_{\sigma,h}$, $S_h \Pi_h \widetilde{w}^j_{\cdot} \in X_{\sigma,h}$. From (2.44) we obtain the following analog of formula (2.39):

$$\Pi_h \widetilde{w}^n_{\cdot} = (S_h)^n w_{0,\cdot} + (S_h)^n \Pi_h \varphi^0_{\cdot} + \tau \sum_{j=1}^{n} (S_h)^{n-j} \Pi_h \varphi^j_{\cdot}. \tag{2.45}$$

Now, we can determine the real process corresponding to the stabilization problem (2.14)–(2.16), (2.18) by the following formula which is similar to (2.35):

$$(\widetilde{v}^n_k, |k| < N; \widetilde{u}^n_-, \widetilde{u}^n_u) = ((\Pi_h \widetilde{w}^n_{\cdot})_k, |k| < N, (\Pi_h \widetilde{w}^n_{\cdot})_{-N}, (\Pi_h \widetilde{w}^n_{\cdot})_N). \tag{2.46}$$

The following assertion holds.

Theorem 2.1. *The real process defined by* (2.46) *satisfies the estimate*

$$\|\widetilde{v}^n_{\cdot}\|_{L_{2,h}(-\pi/2,\pi/2)} \leqslant c\|E_h\| \left(e^{-\sigma n \tau} \|v_{0,\cdot}\|_{L_{2,h}(-\pi/2,\pi/2)} + \widehat{\delta} \left(1 + \frac{e}{\sigma(e-1)} \right) \right), \tag{2.47}$$

where c is some absolute constant and $\widehat{\delta}$ is the constant from (2.38).

PROOF. From (2.46), (2.45), and (2.28) we find

$$\|\widetilde{v}^n_\cdot\|_{L_{2,h}(-\pi/2,\pi/2)} \leqslant \|\Pi_h \widetilde{w}^n_\cdot\|_{L_{2,h}(-\pi,\pi)}$$
$$\leqslant ce^{-\sigma n\tau}(\|w_{0,\cdot}\|_{L_{2,h}(-\pi,\pi)} + \|\Pi_h\varphi^0_\cdot\|_{L_{2,h}(-\pi,\pi)})$$
$$+ c\tau\sum_{j=1}^{n} e^{-\sigma(n-j)\tau}\|\Pi_h\varphi^j_\cdot\|_{L_{2,h}(-\pi,\pi)})$$
$$\leqslant \|E_h\|\left(ce^{-\sigma n\tau}\|v_0\|^2_{L_{2,h}(-\pi/2.\pi/2)}) + \widehat{\delta}\left(1+\tau\sum_{j=1}^{n} e^{-\sigma(n-j)\tau}\right)\right). \tag{2.48}$$

Since $1-e^{-a} > a(1-e^{-1})$ for $a\in(0,1)$, we have

$$\sum_{j=1}^{n} e^{-\sigma(n-j)\tau} = \tau\frac{1-e^{-\sigma n\tau}}{1-e^{-\sigma\tau}} \leqslant \frac{\tau}{\sigma\tau(1-e^{-1})} = \frac{e}{\sigma(e-1)}. \tag{2.49}$$

From (2.48) and (2.49) we arrive at (2.47). □

We note that (2.47) is actually adequate estimate of the stabilization rate for a real process. In particular, the term containing $\widehat{\delta}\left(1+\dfrac{e}{\sigma(e-1)}\right)$ is necessary because, at each time level, there are fluctuations about which we know nothing except for the estimate (2.38).

Finally, comparing (2.14′) and (2.33), it is easy to see that the norm $\|E_h\|$ of the extension operator E_h in the estimate (2.48) can be estimated from above by a constant independent of $h\to 0$.

2.7. Conclusions. According to the above analysis of numerical solution of the stabilization problem, we can make the following conclusions.

Studying unstable problems, we have to refuse from the point of view that a solution to the mathematical model describes a simulated process absolutely accurate.[2)] In the study of the stability of a solution to unstable problem it is impossible (and it is not necessary) to study processes caused by arbitrary fluctuations. It suffices to extract a class of perturbations of the solution which appear in practice and focus only on them.

Such an actual perturbation of the solution to an unstable problem will be called a “real process.” It is reasonable to construct the correct axiomatics allowing us to introduce an abstract notion of a real process, on the basis of the analysis of the situation that occurs while the numerical solution of the original unstable problem.

Formulas (2.37) show that only a real process is completely observable object (but not an exact mathematical solution to this problem and, moreover,

[2)] This point of view is reasonable in the theory of well-posed problems because in these problems arbitrary small perturbations of data lead only to small perturbations of a solution. Therefore, one can neglect with such perturbations.

fluctuations). Therefore, when one constructs a feedback control reacting on fluctuations and dampening them, he has to keep in mind that operators determining this control should act on the real process but not on an exact solution or fluctuations.

Remark 2.1. Since the computation of the projection Π_h in (2.42) is a rather labor-consuming operation, it is not reasonable to use formula (2.44) in computations, i.e., to apply the operator Π_h at each time step. We propose the following. At the n-th step, we apply formula (2.44):

$$\Pi_h \widetilde{w}_{\cdot}^{n+1} = \Pi_h S_h \widetilde{w}_{\cdot}^{n} + \tau \Pi_h \varphi_{\cdot}^{n+1}, \tag{2.50}$$

but at several next steps, we use formula (2.37):

$$\widetilde{w}_{\cdot}^{n+k+1} = S_h \widetilde{w}_{\cdot}^{n+k} + \tau \varphi_{\cdot}^{n+k+1}. \tag{2.51}$$

Formula (2.51) should be used until, for example, the function

$$r(k) = \|\widetilde{w}_{\cdot}^{n+k+1}\|_{L_{2,h}}$$

is decreasing. But in virtue of (2.41) the decreasing of this function will come to an end at a certain instant. At this moment we (only one time) use (2.50) and, after that, we again use (2.51) until $r(k)$ is decreasing.

Note that we do not mean that just this criterion should be applied. Certainly, one can choose some other criterion determining when formula (2.50) should be used. Moreover, a reasonable criterion can be worked out only in numerical simulation.

We have described the so-called hybrid control. Since our investigation is purely theoretical, we do not use the hybrid control and restrict ourselves to the study of a pure feedback control.

3. The Navier–Stokes Equations and Stabilization Problem

In this section, we recall the statement of the stabilization problem for a solution to the Navier–Stokes equations and introduce some necessary notions. We refer the reader to [**7**, **8**] for details.

3.1. The statement of the stabilization problem. Let $\Omega \subset R^3$ be a bounded domain with C^∞ boundary $\partial\Omega$ that consists of two open subsets Γ_0, Γ and their common boundary $\partial\Gamma$:

$$\partial\Omega = \Gamma_0 \cup \Gamma \cup \partial\Gamma, \quad \Gamma \neq \varnothing. \tag{3.1}$$

The cases $\Gamma_0 = \varnothing$ and $\partial\Gamma = \varnothing$ are admissible. If $\partial\Gamma \neq \varnothing$, then we assume that $\partial\Gamma$ is a closed curve of class C^∞ that can be disconnected but has no self-intersecting points.

We introduce the cylindrical sets

$$Q = R_+ \times \Omega, \quad \Sigma = R_+ \times \Gamma, \quad \Sigma_0 = R_+ \times \Gamma_0. \tag{3.2}$$

In the time-spatial cylinder Q, we consider the Navier–Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(x), \quad (t,x) \in Q, \tag{3.3}$$

$$\operatorname{div} v(t,x) = 0 \tag{3.4}$$

with the initial condition

$$v(t,x)\big|_{t=0} = v_0(x), \quad x \in \Omega, \tag{3.5}$$

and the boundary conditions

$$v\big|_{\Sigma_0} = 0, \quad v\big|_{\Sigma} = u(t,x'), \quad (t,x') \in \Sigma. \tag{3.6}$$

Here, $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ is the vector field of velocities of a viscous incompressible fluid, Δ is the Laplace operator, $(v,\nabla)v = \sum_{j=1}^{3} v_j \partial_{x_j} v$, $\nabla p(t,x) = (\partial_{x_1} p, \partial_{x_2} p, \partial_{x_3} p)$ is the gradient of pressure, $f(x)$ is the density of the external forces. We emphasize that $f(x)$ is independent of time. The Dirichlet data $u(t,x')$ imposed on a part Σ of the lateral surface of the cylinder Q serve as control. To formulate the stabilization problem, it is necessary to assume that we are given the steady-state solution $(\widehat{v}(x), \nabla\widehat{p}(x))$ to the Navier–Stokes equations with the same density of the external forces $f(x)$ as in (3.3):

$$-\Delta\widehat{v}(x) + (\widehat{v},\nabla)\widehat{v} + \nabla\widehat{p}(x) = f(x), \quad \operatorname{div}\widehat{v}(x) = 0, \quad x \in \Omega, \tag{3.7}$$

$$\widehat{v}\big|_{\Gamma_0} = 0. \tag{3.8}$$

Let $\sigma > 0$ be given. Recall that $(v(t,x), \nabla p(t,x)$ is a solution to the boundary-value problem (3.3)–(3.6) and $(\widehat{v}, \nabla\widehat{p})$ is a steady-state solution to the problem (3.7), (3.8). Then the problem of stabilization of $v(t,x)$ to $\widehat{v}$ is formulated as follows. Given initial condition v_0, construct a control $u(t,x')$ defined on Σ such that the solution $(v(t,x), \nabla p(t,x))$ to the boundary-value problem (3.3)–(3.6) satisfies the estimate

$$\|v(t,\cdot) - \widehat{v}(\cdot)\|_{H^1(\Omega)} \leqslant c e^{-\sigma t} \quad \text{for } t > 0, \tag{3.9}$$

where the constant c depends only on the initial data $v_0(x)$. [3)]

An important additional condition we impose solving the stabilization problem is the requirement that the constructed control possesses the feedback property. We will explain the essence of this requirement below. Now, we note that we can solve the stated stabilization problem only under the additional assumption that the initial condition v_0 in (3.5) is close to the steady-state solution $(\widehat{v}, \nabla\widehat{p})$ to the problem (3.7), (3.8)

$$\|\widehat{v} - v_0\|_{H^1(\Omega)} \leqslant \delta_0, \tag{3.10}$$

[3)] The definition of the Sobolev space $H^1(\Omega)$, as well as all other function spaces, will be given later.

where δ_0 is a sufficiently small number. [4)]

Now, we give a mathematical definition of a control admitting feedback. This definition is closely connected with the construction mentioned below.

We extend the domain Ω through the part of the boundary Γ up to some domain G. More exactly, let a domain $\omega \subset R^3$ satisfy the conditions

$$\Omega \cap \omega = \varnothing, \quad \partial\Omega \cap \partial\omega = \Gamma \cup \partial\Gamma.$$

We set

$$G = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega}), \tag{3.11}$$

where, as usual, $\mathrm{Int}(S)$ denotes the interior of a set S.

We assume that the boundary ∂G is a surface of finite smoothness $\alpha > 0$ (the value of α will be specified below); moreover, $\partial G \backslash \partial\Gamma$ is infinitely smooth.

We extend the solution $(\widehat{v}(x), \nabla\widehat{p})$ to the problem (3.7), (3.8) from Ω to G as follows: $\widehat{v}(x)$ is extended to a solenoidal vector field $a(x)$, $x \in G$, satisfying the condition

$$a(x)\big|_{\partial G} = 0 \tag{3.12}$$

(methods of such an extension in Sobolev spaces are described, for example, in [**12**]). We extend the function $\widehat{p}(x)$ to a function $q(x)$, $x \in G$, so that the extended function is of the same smoothness class. Substituting $(a(x), \nabla\widehat{q}(x))$ into the left-hand side of the first equation in (3.7), which is considered in G, we find

$$-\Delta a(x) + (a(x), \nabla)a + \nabla\widehat{q}(x) = g(x), \quad \operatorname{div} a(x) = 0, \quad x \in G, \tag{3.13}$$

where $g(x)$ is the result of applying to $(a, \nabla\widehat{q})$ the Navier–Stokes operator generated by the left-hand side of the first equation in (3.7). It is obvious that $g(x)$ is an extension of $f(x)$ from Ω to G.

Now, we extend the problem (3.3)–(3.6) from Ω to G, forgetting for a while about the second boundary condition in (3.6). Then we obtain the following boundary-value problem for the Navier–Stokes equations in the cylinder $\Theta = R_+ \times G$:

$$\partial_t w(t,x) - \Delta w + (w, \nabla)w + \nabla q(t,x) = g(x), \quad \operatorname{div} w = 0, \tag{3.14}$$

$$w\big|_S = 0, \tag{3.15}$$

$$w\big|_{t=0} = w_0, \tag{3.16}$$

where $g(x)$ is the vector field defined in (3.13), $S = R_+ \times \partial G$ is the lateral surface of the cylinder Θ, and the initial data w_0 is obtained from the initial data v_0 in (3.5) by using a special extension operator $E : w_0 = E(v_0)$. The construction of the extension operator E is the main element of the construction and will be described below.

[4)] Actually, instead of the norm of the space $H^1(\Omega)$ we use some other norm (cf. formula (5.23) below).

Definition 3.1. We say that the solution $(v(t,x), u(t,x))$ to the stabilization problem (3.3)–(3.9) *possesses the feedback property* if it is determined by the formula

$$(v(t,\cdot), u(t,\cdot)) = (\gamma_\Omega w(t,\cdot), \gamma_\Gamma w(t,\cdot)), \tag{3.17}$$

where $w(t,\cdot)$ is the solution to the problem (3.14)–(3.16), in the cylinder Θ, γ_Ω is the restriction operator to Ω of vector fields defined on G, and γ_Γ is the restriction operator to Γ of vector fields defined on G. Regarding the component u of the solution (3.17) of the stabilization problem, we also say that the control u possesses the feedback property.

Thus, our next goal is to solve the stabilization problem (3.3)–(3.9) with the help of the control u possessing the feedback property in the meaning of Definition 3.1.

3.2. General scheme of construction. It is clear that the pair (v, u) defined by formula (3.17) satisfies the relations (3.3)–(3.6) provided that $w(t,x)$ on the right-hand side of (3.17) is a solution to the problem (3.14)–(3.16) with the initial condition $w_0(x)$ such that $w_0(x)\big|_\Omega \equiv v_0(x)$. Moreover, if $w(t,x)$ satisfies the inequality

$$\|w(t,\cdot) - a(\cdot)\|_{H^1(G)} \leqslant ce^{-t\sigma}, \quad t > 0, \tag{3.18}$$

then $v(t,x)$ satisfies the estimate (3.9).

Therefore, to solve the stabilization problem by feedback control, it is necessary to investigate two questions:

(i) Describe the set M_- of the initial conditions $\{w_0\}$ of the problem (3.14)–(3.16) such that for any initial condition $w_0 \in M_-$ the solution $w(t,x)$ to the problem (3.14)–(3.16) satisfies the estimate (3.18).

(ii) Construct an operator E that extend the initial data $v_0(x)$ from Ω to G so that the extended function Ev_0 belongs to M_-.

It is obvious that M_- should be a certain manifold in the phase space of the problem (3.14)–(3.16); M_- is defined in an neighborhood of the steady-state solution a to the problem (3.13), (3.12) and $a \in M_-$. Moreover, this manifold M_- is invariant with respect to dynamical system generated by boundary-value problem (3.14)–(3.16). The existence of such invariant manifolds in a neighborhood of a are well known (cf., for example, [**13, 14, 7, 8**]). In Sec. 4, we describe the required invariant manifold M_-.

We note that M_- is constructed only in some neighborhood of the steady-state solution a. This is one of the reason why the extension operator E in (ii) which we can construct is defined only in some neighborhood of the steady-state solution $\widehat{v}$.

We complete the section with introduction some function spaces. Let, as usual, $H^k(G)$ be the Sobolev space of functions $f(x) \in L_2(G)$ whose derivatives in the sense of distributions of order up to k belong to $L_2(G)$. As usual,

$(H^k(G))^3$ is the space of vector fields with components belonging to $H^k(G)$. For the definition of the Sobolev spaces $H^k(G)$ with arbitrary real exponent k we refer to [**15**]. We set

$$V^k(G) = \{w(x) \in (H^k(G))^3 : \operatorname{div} w = 0\}, \tag{3.19}$$

$$V_0^1(G) = \{w(x) \in V^1(G) : w\big|_{\partial G} = 0\}, \tag{3.20}$$

$$V_0^0(G) = \{w(x) \in V^0(G) : (\nu(x) \cdot w(x))\big|_{\partial G} = 0\}, \tag{3.21}$$

where $\nu(x)$ is the filed of outward normals to ∂G. It is well known that for $w \in V^0(G)$ the function $\nu \cdot w\big|_{\partial G}$ is well defined and belongs to $H^{-1/2}(\partial G)$.

We also introduce the space $V^{-1}(G)$ as the completion of the space $V_0^0(G)$ by the norm

$$\|v\|_{V^{-1}(G)} = \sup_{\varphi \in V_0^1(G), \varphi \neq 0} \frac{\langle v, \varphi \rangle}{\|\varphi\|_{V_0^1(G)}},$$

where $\langle \cdot, \cdot \rangle$ is the duality generated by the inner product in $(L_2(G))^3$.

4. Invariant Manifold M_-

4.1. Invariant subspaces of L_- (linear theory). We take the space $V_0^1(\Omega)$ as the space of the initial data and as the phase space of the problem (3.14)–(3.16). We assume that the steady-state solution $(a, \nabla \widehat{q})$ to the problem (3.12), (3.13) satisfies the condition

$$a(x) \in V_0^1(G) \cap V^2(G), \quad a(x) \text{ is real-valued.} \tag{4.1}$$

Before constructing M_-, we construct the space L_- that is tangent to the manifold M_- at the point a.

First of all, we make the change of the unknown function in the problem (3.14)–(3.16):

$$w(t,x) = a(x) + z(t,x). \tag{4.2}$$

Substituting (4.2) into (3.14)–(3.16), and taking into account (3.12) and (3.13), we conclude that $z(t,x)$ is a solution to the problem

$$\partial_t z(t,x) - \Delta z + (a, \nabla)z + (z, \nabla)a + (z, \nabla)z + \nabla q = 0, \quad \operatorname{div} z = 0, \tag{4.3}$$

$$z\big|_S = 0, \quad z\big|_{t=0} = z_0(x) \equiv w_0(x) - a(x). \tag{4.4}$$

To construct the linear space L_-, we pass from the nonlinear problem (4.3), (4.4) to the linear one by eliminating the nonlinear term $(z, \nabla)z$ in (4.3) and applying the orthogonal projection

$$\pi : (L_2(G))^3 \to V_0^0(G) \tag{4.4'}$$

onto the space $V_0^0(G)$ to the both sides of the obtained equation (in order to eliminate the term ∇q) .

We get a problem which is written in an abstract form as follows:

$$\partial_t y(t) + \mathcal{A}y(t) = 0, \quad y\big|_{t=0} = y_0, \tag{4.5}$$

where the initial datum $y_0 = y_0(\cdot)$ belongs to $V_0^0(G)$ and the solution $y(t) \equiv y(t,\cdot)$ belongs to the space

$$V^{1,1}(\Theta_T) \equiv L_2(0,T;V_0^1(G)) \cap H^1(0,T;V^{-1}(G)), \tag{4.5$'$}$$

where $\Theta_T = (0,T)\times G$, and $T>0$ is an arbitrary number. Moreover, $\mathcal{A}$ is the closed operator defined by the relation

$$\mathcal{A}y \equiv -\pi\Delta y + \pi[(a,\nabla)y + (y,\nabla)a] : V_0^0(G) \to V_0^0(G), \tag{4.6}$$

with domain $\mathcal{D}(\mathcal{A}) = V_0^1(G)\cap V^2(G)$. As is known, the spectrum $\Sigma(\mathcal{A})$ of the operator $\mathcal{A}$ consists of a discrete set of points; moreover, since the vector field (4.1) is real, the set $\Sigma(\mathcal{A})$ is invariant under the complex conjugation. Furthermore, $\Sigma(\mathcal{A}) = \Sigma(\mathcal{A}^*)$, where $\mathcal{A}^*$ is the adjoint of the operator (4.6). It is possible to show that the spectrum $\Sigma(-\mathcal{A})$ of the operator $-\mathcal{A}$ belongs to the set $S_{a,\varphi}$:

$$\Sigma(-\mathcal{A}) \subset S_{a,\varphi} = \{\lambda \in \mathbb{C} : \varphi < |\arg(\lambda - a)| \leqslant \pi, \lambda \neq a\} \tag{4.7}$$

for any $\varphi \subset (\pi/2,\pi)$ and some $a = a(\varphi) > 0$.

The solution $y(t)$ to the problem (4.5) for any $y_0 \in V_0^0(\Omega)$ can be found by the formula

$$y(t) = e^{-\mathcal{A}t}y_0 = (2\pi i)^{-1}\int_\gamma (\lambda I + \mathcal{A})^{-1} y_0 e^{\lambda t}\, d\lambda, \tag{4.8}$$

where γ is the contour coinciding with the boundary of th set $S_{a,\varphi}$ in (4.7) with some fixed a and φ.

Assume that the number $\sigma > 0$ in the estimate (3.9) satisfies the condition

$$\Sigma(-\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = -\sigma\} = \varnothing. \tag{4.9}$$

If this condition fails, then we can reach the validity of it by increasing σ.

We divide the integral in (4.8) into two integrals

$$y(t) = y_1(t) + y_2(t), \tag{4.10$'$}$$

where

$$y_i(t) = (2\pi i)^{-1}\int_{\gamma_i} (\lambda I + \mathcal{A})^{-1} y_0 e^{\lambda t}\, dt, \quad i = 1,2. \tag{4.10}$$

Moreover, γ_1 is a continuous contour consisting of the segment $\{\operatorname{Re}\lambda = -\sigma\}$ and the part of the contour γ locating to the left of this line, whereas the contour γ_2 consists of the same segment $\{\operatorname{Re}\lambda = -\sigma\}$ and the part of γ locating to the right of this line. Using the methods of [**16**], and taking into account (4.10) we easily get the estimate

$$\|y_1(t)\|_{V_0^0(\Omega)} \leqslant ce^{-\sigma t}\|y_0\|_{V_0^0(\Omega)}$$

and the inequality

$$\|y_1(t)\|_{V_0^1(G)} \leqslant ce^{-\sigma t}\|y_0\|_{V_0^1(G)}. \tag{4.11}$$

We will use the inequality (4.11), because, in the nonlinear case, it is reasonable to take the space $V_0^1(G)$ for the phase space. Therefore, it is obvious that the required subspace of L_- consists of the initial data $y_0 \in V_0^1(G)$ such that $y_2(t) \equiv 0$.

Let $-\overline{\lambda}_j \in \Sigma(-\mathcal{A}^*)$ be an eigenvalue of the operator $-\mathcal{A}^*$ which is the adjoint of the operator $-\mathcal{A}$, and let $d_0 \neq 0$, $d_0 \in \operatorname{Ker}(\overline{\lambda}_j I + \mathcal{A}^*)$ be the eigenvector corresponding to $-\overline{\lambda}_j$. We recall that the vector d_k is referred to be the *associated vector of order* k if

$$(\overline{\lambda}_j I + \mathcal{A}^*)d_0 = 0, \quad d_0 + (\overline{\lambda}_j I + \mathcal{A}^*)d_1 = 0, \dots, d_{k-1} + (\overline{\lambda}_j I + \mathcal{A}^*)d_k = 0.$$

The space of all eigenvectors and associated vectors of the operator $-\mathcal{A}^*$ corresponding to a fixed eigenvalue $-\overline{\lambda}_j$ is finite-dimensional. We choose the basis for this space which consists of eigenvectors and associated vectors. Denote this basis by $E^*(-\overline{\lambda}_j)$. For $\operatorname{Im}\overline{\lambda}_j \neq 0$ it is obvious that any vector field $d \in E^*(-\overline{\lambda}_j)$ is, in general, complex-valued. Since the vector field in (4.1) is real, we can choose the bases $E^*(-\overline{\lambda}_j)$ satisfying the additional condition

$$d(x, -\overline{\lambda}_j) \in E^*(-\overline{\lambda}_j) \text{ if and only if } \overline{d(x, -\overline{\lambda}_j)} = d(x, -\lambda_j) \in E^*(-\lambda). \tag{4.12}$$

We introduce the set

$$\mathbb{C}E_\sigma^* = \bigcup_{\operatorname{Re}\lambda_j < \sigma} E^*(-\overline{\lambda}_j) \tag{4.13}$$

of complex-valued vector fields which, obviously, form a linearly independent system over the field of complex numbers $\mathbb{C}$.

¿From the vector fields (4.13) we form the following set of real-valued vector fields

$$\begin{aligned} &\operatorname{Re} d(x, -\overline{\lambda}_j), \operatorname{Im} d(x, -\overline{\lambda}_j) \quad \forall d(x, -\overline{\lambda}_j) \in E^*(-\overline{\lambda}_j) : \operatorname{Im}\overline{\lambda}_j > 0,\ \operatorname{Re}\overline{\lambda}_j < \sigma, \\ &d(x, -\lambda_j) \quad \forall d(x, -\lambda_j) \in E^*(-\lambda_j) : \operatorname{Im}\lambda_j = 0,\ \operatorname{Re}\lambda_j < \sigma. \end{aligned} \tag{4.14}$$

Re-enumerating the functions (4.14), we obtain the set of real-valued vector fields

$$d_1(x), \dots, d_K(x), \quad x \in G, \tag{4.15}$$

that forms a linearly independent system over the field of real numbers $\mathbb{R}$.

Finally, we introduce the required space L_- as follows:

$$L_- = \{y(x) \in V_0^1(G) : \int\limits_G (y(x), d_j(x))\, dx = 0, \quad j = 1, \dots, K\}. \tag{4.16}$$

Theorem 4.1. *The solution* (4.8) *to the problem* (4.12) *with the initial data* $y_0 \in V_0^1(G)$ *satisfies the estimate*

$$\|y(t,\cdot)\|_{V_0^1(\Omega)} \leqslant c\|y_0\|_{V_0^1(\Omega)} e^{-\sigma t} \quad \forall t > 0 \tag{4.17}$$

if and only if $y_0 \in L_-$, *where* L_- *is the space* (4.16).

Proof. As is proved in [**7, 8**], the function $y_2(t)$ defined in (4.10) vanishes identically if and only if $y_0 \in L_-$. The estimate (4.11) for the function $y_1(t)$ defined in (4.10) is proved, for example, in [**8**]. The above facts and the equalities (4.10′) and (4.10) imply the assertion of the theorem. □

4.2. Invariant subspaces of L_+. We immediately conclude from the proof of Theorem 4.1 that the space L_- has finite codimension and that it is invariant under the action of the semigroup $e^{-\mathcal{A}t}$. It is possible to construct a finite-dimension subspace L_+ of the space $V_0^1(G)$ that is invariant under the action of the semigroup $e^{-\mathcal{A}t}$; moreover, $L_+ + L_- = V_0^1(G)$.

To this end, together with the basis $E^*(-\overline{\lambda}_j)$ of eigenfunctions and associated functions of the operator $-\mathcal{A}^*$ corresponding to the eigenvalue $-\overline{\lambda}_j$, we construct a similar basis $E(-\lambda_j)$ of eigenfunctions and associated functions of the operator $-\mathcal{A}$ corresponding to the eigenvalue $-\lambda_j$. However, the construction is fulfilled by a special way. Namely, we take as $E^*(-\overline{\lambda}_j)$ an arbitrary **canonical** basis of eigenfunctions and associated functions that satisfies condition (4.12) (the definition of the canonical basis can be found in [**8**]). Further, applying the construction of Theorem 3.1 (Keldysh) in [**8**], we uniquely determine with help of $E^*(-\overline{\lambda}_j)$ the canonical basis $E(-\lambda_j)$ of eigenfunctions and associated functions of the operator $\mathcal{A}$ corresponding to the eigenvalue $-\lambda_j$. Note that since the basis $E^*(-\overline{\lambda}_j)$ satisfies (4.12), the constructed basis $E(-\lambda_j)$ satisfies a similar property automatically. Further, as in the case of (4.13), we construct the basis

$$\mathbb{C}E_\sigma = \bigcup_{\operatorname{Re}\lambda_j < \sigma} E(-\lambda_j).$$

As in the case of (4.15), the elements of this basis are divided into real and imaginary parts and, after that, we **consistently** re-enumerate the obtained basis and the basis (4.14). Thus, together with the basis (4.15) which is relative to $-\mathcal{A}^*$, we obtain the basis

$$e_1(x), \dots e_K(x) \tag{4.18}$$

relating to $-\mathcal{A}$. Moreover, construction of Keldysh theorem implies that the bases (4.15) and (4.18) are biorthogonal, i.e.

$$\int_G (e_j(x), d_k(x))\,dx = \delta_{jk}, \quad j = 1, \dots, K, \quad k = 1, \dots, K, \tag{4.19}$$

where δ_{jk} is the Kronecker symbol. For the construction of the bases (4.15) and (4.18) in detail we refer to [**8**].

We introduce the projection P_+ by the formula

$$P_+ v(x) = \sum_{j=1}^{K} e_j(x) \int_G (v(x), d_j(x))\,dx \tag{4.20}$$

and introduce the notation

$$L_+ = P_+ V_0^1(G). \tag{4.21}$$

By Theorem 3.1 in [**7**], for any $y_0 \in V_0^1(\Omega)$ the function $y_2(t)$ defined in (4.10) belongs to L_+ for any $t > 0$. In other words, the subspace L_+ is invariant under the action of the semigroup $e^{-\mathcal{A}t}$. We set

$$P_- = I - P_+, \quad I \text{ is the unit operator}. \tag{4.22}$$

It follows from (4.22), (4.20), (4.19) and the definition (4.16) of the space L_- that

$$L_- = P_- V_0^1(G). \tag{4.23}$$

Consequently,

$$L_+ + L_- = V_0^1(G).$$

4.3. Invariant manifold M_-. Thus, we have constructed the subspaces L_- and L_+ of the space $V_0^1(G)$ which are invariant under the action of the solving semigroup $e^{-\mathcal{A}t}$ of the problem (4.5) which is the linearization of the problem (4.3), (4.4). These spaces will be used for constructing an invariant manifold M_- of the problem (4.3), (4.4). First of all, applying the projection (4.4′) to both side of the first equation in (4.3), we write the problem (4.3), (4.4) in the form

$$\partial_t z(t) + \mathcal{A}z(t) + \pi[(z, \nabla)z] = 0, \quad z(t)\big|_{t=0} = z_0, \tag{4.24}$$

where

$$\begin{aligned} z(t) &= z(t, \cdot) \in V^{1,2}(\Theta_T) \\ &= L_2(0, T; V_0^1(G) \cap V^2(G)) \cap H^1(0, T; V_0^0(G)), \quad T > 0, \end{aligned} \tag{4.25}$$

the operator $\mathcal{A}$ is defined in (4.6) and π is the operator (4.4′). As is known (cf. [**17, 18**]), for any initial data z_0 in some neighborhood of the origin of the space $V_0^1(G)$ depending on $T > 0$ there exists a unique solution $z(t)$ to the problem (4.24), (4.25). We denote by $S(t,z_0): S(t,z_0)=z(t)$ the solving operator of this problem, i.e., the semigroup of operators associating with the initial condition z_0 the solution $z(t)$ at time t. We decompose this operator at the point $z_0=0$:

$$S(t,z_0)=e^{-\mathcal{A}t}z_0+B(t,z_0). \tag{4.26}$$

This decomposition is obvious because, as is well known (cf. [**18**]), the operator $S(t,z_0)$ is analytic with respect to z_0 for any fixed $t>0$ and z_0 in a neighborhood of zero.

Moreover,

$$B(t,0)=0, \quad B'_{z_0}(t,0)=0. \tag{4.27}$$

In a neighborhood of $z_0=0$, we construct a manifold M_- that is invariant under the action of the semigroup $S(t,z_0)$ (i.e., $S(t,w)\in M_-$ for all $w\in M_-$ and $t>0$). This manifold can be given as the graph

$$M_-=\{u\in V_0^1(G): u=u_-+g(u_-), u_-\in L_-\cap\mathcal{O}^{\varepsilon_0}\}, \tag{4.28}$$

where $\mathcal{O}^{\varepsilon_0}=\{u\in V_0^1(G): \|u\|_{V_0^1(G)}<\varepsilon_0\}$, $\varepsilon_0>0$, is a neighborhood of the origin, $g: L_-\cap\mathcal{O}^{\varepsilon_0}\to L_+$ is an operator-valued function of class $C^{1+\alpha}(L_0\cap\mathcal{O}^{\varepsilon_0})$ with $\alpha=1$ (i.e., the first-order derivative g satisfies the Lipschitz condition); moreover,

$$g(0)=0, \quad g'(0)=0. \tag{4.29}$$

The condition (4.29) means that the manifold (4.28) is tangent to the subspace L_- at the origin.

The following assertion holds.

Theorem 4.2. *Let $\sigma>0$ satisfy* (4.9), *and let $\varepsilon_0>0$ be sufficiently small. Then there exists a unique operator-valued function $g: L_-\cap\mathcal{O}^{\varepsilon_0}\to L_+$, $g\in C^{1+\alpha}(L_-\cap\mathcal{O}^{\varepsilon_0})$ with $\alpha=1$ satisfying* (4.29) *such that the manifold M_- defined in* (4.28) *is invariant under the action of the solving semigroup $S(t,z_0)$ of the problem* (4.24). *There exists a constant $c>0$ such that*

$$\|S(t,z_0)\|_{V_0^1(G)}\leqslant c\|z_0\|_{V_0^1(G)}e^{-\sigma t} \quad \forall t>0. \tag{4.30}$$

This theorem is obtained in [**7, 8**] from the known abstract results (cf. [**14**]).

In the sequel, we need the following estimate for the solutions to the problem (4.24) with the initial data in the manifold M_-.

Theorem 4.3. *Let $z_{01}\in M_-$, $z_{02}\in M_-$, where M_- is the invariant manifold constructed in Theorem* 4.2 *and $z_i(t)$, $i=1,2$, are solutions to the problem* (4.24), (4.25) *with the initial conditions $z_i(t)\big|_{t=0}=z_{i0}$. Then the following estimate is valid*:

$$\|z_1(t,\cdot)-z_2(t,\cdot)\|_{V_0^1(G)}\leqslant \widehat{c}\|z_{10}-z_{20}\|_{V_0^1(G)}e^{-\sigma t} \quad \forall t\geqslant 0 \tag{4.31}$$

where $\widehat{c}$ is a constant independent of $t \geqslant 0$ and $z_{10} \in M_-$, $z_{2,0} \in M_-$.

This theorem is proved in [**1**].

5. Stabilization of Solutions to the Navier–Stokes Equations

As was mentioned in Subsec. 3.2, to solve the stabilization problem, it is necessary to construct together with an invariant manifold M_- an operator that extends the initial data $v_0(x)$ from the domain Ω to a wider domain G in such a way that the extended initial data belong to this invariant manifold M_-. In this section, we construct such an operator. We begin by solving a linear analog of this problem.

5.1. Extention Operator for Solenoidal Vector Fields. First of all, we specify the conditions on the domain G constructed by formula (3.11).

Condition 5.1. *Let $\partial G \backslash \Gamma \in C^\infty$. Suppose that for every point $P \in \partial\Gamma$ there exist local coordinates (x, y, z) such that P is the origin: $P = (0,0,0)$, $(0,0,z) \in \partial\Gamma$, $\{(x,0,z), x \in (0,\varepsilon)\} \subset \Gamma$, $\{(x,0,z), x \in (-\varepsilon, 0)\} \subset \Gamma_0$, and, in a neighborhood of P, the boundary ∂G can be represented as follows*:

$$\partial G \subset \{(x,y,z) = (x, x^\alpha, z), x \in (0,\varepsilon)\} \cup \{(x,0,z), x \in (-\varepsilon,0)\}. \tag{5.1}$$

Here, $\alpha > 2$ is some number and $\varepsilon > 0$ is sufficiently small.

We note that the condition (5.1) is not very restrictive since we construct the domain G ourselves.

We introduce the space of initial data of the stabilization problem (3.3)–(3.9) as follows:

$$V^1(\Omega, \Gamma_0) = \{v(x) \in V^1(\Omega) : v\big|_{\Gamma_0} = 0, \exists w \in V_0^1(G) : v = w\big|_\Omega\}. \tag{5.2}$$

The norm in the space $V^1(\Omega, \Gamma_0)$ is defined by the equality

$$\|v\|_{V^1(\Omega,\Gamma_0)} = \inf_{w \in V_0^1(G): w\big|_\Omega = v} \|w\|_{V_0^1(G)}. \tag{5.3}$$

Thus, the existence of a solenoidal extension is included in the definition of the space of initial data.

For our goal, it is necessary to define some canonical extension operator from $V^1(\Omega, \Gamma_0)$ into $V_0^1(G)$. To this end, we introduce the following closed subspace of the space $V_0^1(G)$:

$$V_0^1(G, \Omega) = \{w \in V_0^1(G) : w\big|_\Omega = 0\} \tag{5.3$'$}$$

and define the extension operator

$$R : V^1(\Omega, \Gamma_0) \to V_0^1(G) \tag{5.4}$$

by the formula

$$Rv = \widehat{w} \in V_0^1(G) : \widehat{w}\big|_{\Omega} = v, \quad \widehat{w} \perp_{V_0^1(G)} V_0^1(G,\Omega), \tag{5.5}$$

where $\perp_{V_0^1(G)}$ means the orthogonality relation with respect to the inner product in $V_0^1(G)$.

Proposition 5.1. *Formula* (5.5) *uniquely defines a linear extension operator R. Moreover,*

$$\|Rv\|_{V_0^1(G)} = \|v\|_{V^1(\Omega,\Gamma_0)}. \tag{5.6}$$

Furthermore, the vector field $u(x) \equiv Rv\big|_{\omega}$ is a generalized solution to the Stokes problem:

$$-\Delta u(x) + \nabla p(x) = 0, \quad \operatorname{div} u = 0, \quad x \in \omega, \quad u\big|_{\Gamma} = v\big|_{\Gamma}, \quad u\big|_{\partial G\setminus\Gamma_0} = 0 \tag{5.7}$$

(*recall that* $\omega = G\setminus\overline{\Omega}$).

Proof. Let $v \in V^1(\Omega,\Gamma_0)$, and let $w_0 \in V_0^1(G)$ be some extension of v from Ω to G. The existence of such a function w_0 follows from the definition of the space $V^1(\Omega,\Gamma_0)$. Let $\widetilde{w}$ be a solution to the extremum problem

$$f(w) = \|w_0 - w\|^2_{V_0^1(\Omega)} \to \inf, \quad w \in V_0^1(G,\Omega), \tag{5.8}$$

where $V_0^1(G,\Omega)$ is the space (5.3′). As is known, a solution to the problem (5.8) is an orthogonal projection of the vector w_0 onto the closed subspace of $V_0^1(G,\Omega)$ (which exists and is unique). We set $\widehat{w}(v) = w_0 - \widetilde{w}$.

By (5.8) and the Fermat theorem, we have

$$(\widehat{w}(v), w)_{V_0^1(G)} = (w_0 - \widetilde{w}, w)_{V_0^1(G)} = 0 \quad \forall w \in V_0^1(G,\Omega).$$

Moreover, $\widehat{w}(v)\big|_{\Omega} = w_0\big|_{\Omega} - \widetilde{w}\big|_{\Omega} = v - 0 = v$. Consequently, $\widehat{w}(v)$ satisfies all the conditions in (5.5.). Assume that there are two vectors $\widehat{w}_1(v)$ and $\widehat{w}_2(v)$ satisfying (5.5). Since $\widehat{w}_i(v) \perp V_0^1(G)$, $i = 1,2$, we have $\widehat{w}_1(v) - \widehat{w}_2(v) \perp V_0^1(G,\Omega)$ by the theorem on three perpendiculars.

Since $(\widehat{w}_1(v) - \widehat{w}_2(v))|_{\Omega} = v - v = 0$, we have $\widehat{w}_1 - \widehat{w}_2 \in V_0^1(G,\Omega)$. Consequently, $w_1(v) \equiv w_2(v)$. The fact that the mapping $v \to Rv = \widehat{w}(v)$ is linear follows from (5.5) and the theorem on three perpendiculars. Since the minimum in (5.8) is attained at $\widehat{w}(v) = w_0 - \widetilde{w}$, the equality (5.6) is a consequence of the definition (5.3) of the norm in $V^1(\Omega,\Gamma_0)$. Setting $\widehat{w}(v)\big|_{\omega} = Rv\big|_{\omega} = u(x)$, applying the Fermat theorem to (5.8), and taking into account the definition of the norm in $V_0^1(G)$, as well as the definition of the space (5.3′), we get

$$\int_{\omega} (\nabla u(x) : \nabla\varphi(x))\, dx = 0 \quad \forall\varphi(x) : \exists w \in V_0^1(G,\Omega), w\big|_{\omega} = \varphi. \tag{5.9}$$

By (5.9), $u(x)$ is a generalized solution to the Stokes problem (5.7). $\square$

Remark 5.1. Applying the operator div to the first equation in (5.7), we conclude that $p(x)$ is a harmonic function in ω. Hence $u \in C^\infty(\omega)$. Of course, the smoothness of the vector field $u(x)$ fails on Γ since $u\big|_\Gamma = v\big|_\Gamma$, and also on $\partial\Gamma$.

Remark 5.2. The definition (5.2) of the space $V^1(\Omega, \Gamma_0)$, which is important for our purposes, is not constructive. However, it is possible to give sufficient conditions on $v(x) \in V^1(\Omega)$, $v\big|_{\Gamma_0} = 0$, under which this vector belongs to the space $V^1(\Omega, \Gamma_0)$. This sufficient condition is the requirement of some additional smoothness of the vector field $v(x)$ in a neighborhood of the manifold $\partial\Gamma$. An exact formulation of this condition and its justification are given in [**8**] in the case of two-dimensional domains Ω and G, but the proof can be easily extended to the three-dimensional case.

Remark 5.3. One of possible ways to obtain more exact sufficient conditions and, possibly, criterion for a vector field to belong to the space $V^1(\Omega, \Gamma_0)$ is to study solvability of the Stokes problem (5.7) in a domain ω with boundary singularity in $\partial\Gamma$. Regarding this remark, we note that it is possible to study the stabilization problem in a domain Ω with boundary $\partial\Omega$ consisting of parts Γ_0 and Γ that intersect over a manifold $\partial\Gamma$ under angle strictly less than π at any point of $\partial\Gamma$. For such a statement, the boundary domain Ω, where the problem (5.7) is considered, have corner points with positive angle at any point $x \in \partial\Gamma$. Detailed investigation of these questions will be given in an other publication.

5.2. Extension operator to the subspace L_-. Before constructing an extension operator from Ω to G that acts into the manifold M_- and is used in the solution to the stabilization problem for the Navier–Stokes equations, we need to construct a linear analog of such an extension operator. In this subsection, we construct an operator E that extends a vector field $v_0(x)$ from Ω in G so that the extended vector field $(Ev_0)(x)$ belongs to the subspace L_- defined by formula (4.16). The operator L_- is constructed with the help of the functions (4.15) $d_1(x), \dots, d_k(x)$. We recall that these functions form a linearly independent system on the domain G. The following property of the functions $\{d_j(x)\}$ is essential for our purposes.

Lemma 5.1. *For any open subset $\omega_1 \subset G$ the system of functions $d_j(x)$, $j = 1, \dots, k$, restricted to ω_1 is linearly independent.*

The proof of this lemma is contained in [**7**]–[**10**].

Theorem 5.1. *There exists a linear bounded extension operator E of solenoidal vector fields from Ω in G such that*

$$E : V^1(\Omega, \Gamma_0) \to L_- \tag{5.10}$$

(i.e., $(Ev)(x) \equiv v(x)$ for $x \in \Omega$).

SKETCH OF THE PROOF. Consider an open set $\omega_1 \subset G\backslash\Omega$ with C^∞-smooth boundary $\partial\omega_1$ such that

$$\mathrm{Int}(\partial\omega_1 \cap \partial G) \neq \varnothing. \tag{5.11}$$

In ω_1, we consider the Stokes problem

$$-\Delta w + \nabla p = v(x), \quad \operatorname{div} w(x) = 0, \quad x \in \omega_1, \quad w\big|_{\partial\omega_1} = 0. \tag{5.12}$$

As is known, for every $v \in V^0(\omega_1)$ there exists a unique solution $(w, \nabla p) \in V_0^1(\omega_1) \cap V^2(\omega_1) \times \nabla H^1(\omega_1)$ to this problem. We denote by $(-\pi\Delta)^{-1}_{\omega_1} : V^0(\omega_1) \to V_0^1(G)$ the mapping that associate with the right-hand side $v \in V^0(\omega_1)$ the component w of the solution to the problem (5.12) extended by zero outside ω_1. Since $w \in V_0^1(\omega)$, this extension $(-\pi\Delta)^{-1}_{\omega_1} v_0$ belongs to $V_0^1(G)$.

We look for the extension operator E in the form

$$Ev = Rv + \sum_{j=1}^{k} c_j (-\pi\Delta)^{-1}_{\omega_1} d_j, \tag{5.13}$$

where R is the extension operator (5.5), $d_j(x)$ are the vector fields (4.15), and c_j are constants that are required to determine. By the definition (4.16) of the space L_-, we have $Ev \in L_-$ if and only if

$$0 = \int\limits_G (Ev, d_l)\, dx = \int\limits_G (Lv, d_l)\, dx + \sum_{j=1}^{k} c_j \int\limits_{\omega_1} (d_l, (-\pi\Delta)^{-1}_{\omega_1} d_j) dx, \quad l = 1, \dots K. \tag{5.14}$$

The system (5.14) uniquely determines the required constants c_j provided that the matrix $A = \|a_{jl}\|$ is not singular, where

$$a_{jl} = \int\limits_{\omega_1} (d_l, (-\pi\Delta)^{-1}_{\omega_1} d_j)\, dx.$$

The relation $\det A \neq 0$ is proved in [**9**], [**10**] on the basis of Lemma 5.1 and the relation (5.11). □

5.3. Extension operator to the manifold M_-. We formulate the result concerning the existence of an extension operator of vector fields from Ω onto G such that the extended vector field belongs to M_-.

The manifold M_- will be sometimes denoted by $M_-(\varepsilon_0)$ in order to emphasize that the variable u_- in the definition (4.28) belongs to the ball $L_- \cap \mathcal{O}^{\varepsilon_0}$, We consider the ball

$$B^{\delta_0} = \{v \in V^1(\Omega, \Gamma_0) : \|v\|_{V^1(\Omega,\Gamma_0)} < \delta_0\}. \tag{5.15}$$

Theorem 5.2. *There exists $\delta_0 > 0$ such that for any vector field $v_0 \in B^{\delta_0}$ there exists a unique extension to a vector field $E_0(v_0)(x)$ on G such that $E_0(v_0) \in M_-(\varepsilon_0)$. This extension operator*

$$E_0 : B^{\delta_0} \to M_-(\varepsilon_0) \tag{5.16}$$

is continuous.

This theorem is proved in [**7, 8, 10**]. We do not explain here the construction of the extension operator E_0, but give the construction of some other projection operator similar to E_0 because this construction will be used in the sequel.

5.4. Theorem on stabilization of solutions to the Navier–Stokes equations. To solve the stabilization problem (3.3)–(3.9), we make the change of the unknown function v:

$$v(t,x) = \widehat{v}(x) + y(t,x), \tag{5.17}$$

where $\widehat{v}(x)$ is the component of the steady-state solution to the problem (3.7), (3.8). Substituting the decomposition (5.17) into (3.3)–(3.6) and taking into account (3.7), we obtain the following equations with respect to $y(t,x)$:

$$\begin{aligned} &\partial_t y(t,x) - \Delta y + (\widehat{v}, \nabla)y + (y, \nabla)\widehat{v} \\ &\quad + (y, \nabla)y + \nabla p_1(t,x) = 0, \quad (t,x) \in Q, \end{aligned} \tag{5.18}$$

$$\operatorname{div} y(t,x) = 0, \tag{5.19}$$

$$y(t,x)\big|_{t=0} = y_0(x) \equiv v_0(x) - \widehat{v}(x), \tag{5.20}$$

$$y\big|_{\Sigma_0} = 0, \quad y\big|_{\Sigma} = u_1 = u(t,x') - \widehat{v}\big|_{\Sigma}. \tag{5.21}$$

It is obvious that the relation (3.9) takes the form

$$\|y(t,\cdot)\|_{V^1(\Omega)} \leqslant ce^{-\sigma t} \quad \text{for } t > 0, \tag{5.22}$$

and the condition (3.10) takes the form

$$\|y_0\|_{V^1(\Omega,\Gamma_0)} \leqslant \delta_0, \tag{5.23}$$

where δ_0 is a sufficiently small number. Passing from the problem (3.3)– (3.10) to the problem (5.18)–(5.23), we only specify the norm in (3.10) replacing it with the norm of the space $V^1(\Omega,\Gamma_0)$ in (5.23).

The following assertion concerning the stabilization of solutions to the problem (5.18)–(5.23) holds.

Theorem 5.3. *Let a vector field $a(x) \in V^2(G) \cap V_0^1(G)$ is an extension of a vector field $\widehat{v}(x) \in V^2(G) : a\big|_{\Omega} = \widehat{v}$. from Ω on G. Suppose that the initial condition $y_0(x)$ in* (5.20) *satisfies the condition* (5.23) *with δ_0 in Theorem* 5.2. *Then there exists a solution (y,u) to the stabilization problem* (5.18)–(5.22) *with the control u possessing the feedback property.*

Proof. Eliminating the second boundary condition in (5.21) for a while, we extend the problem (5.18)– (5.21) from Ω on G. Here we use the fact that the vector field $a(x)$ is an extension of the vector field $\widehat{v}(x)$ in view of the assumptions of the theorem. We obtain the boundary-value problem (4.3), (4.4), where $z(t,x)$ denotes the required vector field which is an extension of the required vector field from Ω on G. Finally, for the initial data z_0 of the problem (4.3), (4.4) we take the vector field $E_0(y_0)$, where E_0 is the extension operator (5.16) constructed in Theorem 5.2. By this theorem, $z_0 = E_0(y_0) \in M_-$ and, by the definition of the invariant manifold M_-, the solution $z(t,x)$ to the problem (4.3), (4.4) satisfies the estimate

$$\|z(t,\cdot)\|_{V_0^1(G)} \leqslant ce^{-\sigma t} \quad \text{as} \quad t \to 0. \tag{5.24}$$

By the definition 3.1, the solution $(y(t,x), u(t,x))$ to the stabilization problem (518)–(5.22) is given by the formula

$$(y(t,\cdot), u(t,\cdot)) = (z(t,\cdot)\big|_\Omega, z(t,\cdot)\big|_\Gamma). \tag{5.25}$$

From (5.25) and (5.24) it is easy to obtain the estimate (5.22). □

5.5. On the projection operator onto M_-. We recall that two numbers $\varepsilon_0 > 0$ and $\delta_0 > 0$ were introduced in Theorem 5.2 on the extension operator to M_-. The number ε_0 characterized the "size" of the manifold $M_- = M_-(\varepsilon_0)$:

$$M_-(\varepsilon_0) \equiv \{u \in V_0^1(G) : u = u_- + g(u_-), u_- \in L_- \cap \mathcal{O}^{\varepsilon_0}\}, \tag{5.26}$$

where

$$\mathcal{O}^{\varepsilon_0} = \{u \in V_0^1(G) : \|u\|_{V_0^1(G)} \leqslant \varepsilon_0\} \tag{5.27}$$

is a ball in $V_0^1(G)$ (the mapping $g : L_- \cap \mathcal{O}^{\varepsilon_0} \to L_+$ is defined in Theorem 4.2), and δ_0 is the radius of the ball (5.15) on which the extension operator E_0 in (5.16) was defined. The number $\delta_0 > 0$ was determined by the number ε_0.

We now change the situation as follows. Let δ be essentially less than δ_0:

$$0 < \delta \ll \delta_0, \tag{5.28}$$

and let $z \in M_-(\varepsilon_0/2)$ be a given vector field. Introduce the notation

$$\widehat{z} = z\big|_\Omega \tag{5.29}$$

and formulate the problem of seeking the projection operator Π_z on M_- defined in the ball

$$\mathcal{O}_z^\delta = \{\varphi \in V_0^1(G) : \|\varphi - z\|_{V_0^1(G)} < \delta\} \tag{5.30}$$

that acts in $M_-(\varepsilon_0)$ and, in addition, satisfies the condition

$$\Pi_z(\varphi)(x) \equiv \varphi(x), \quad x \in \Omega \quad (\text{i.e., } \widehat{\Pi(\varphi)} = \widehat{\varphi}) \tag{5.31}$$

and the estimate

$$\|\Pi_z(\varphi) - z\|_{V_0^1(G)} \leqslant \widehat{\gamma}\|\varphi - z\|_{V_0^1(G)} \tag{5.32}$$

with constant $\widehat{\gamma}$ independent of $z \in M_-(\varepsilon_0/2)$.

Theorem 5.4. *Let $z \in M_-(\varepsilon_0/2)$, and let the number δ satisfying* (5.28) *is sufficiently small. There exists a continuous projection operator Π_z onto M_- that satisfies the following properties*:

(i) Π_z *acts in the following sets*:

$$\Pi_z : \mathcal{O}_z^\delta \to M_-(\varepsilon_0) \tag{5.33}$$

defined by (5.30) *and* (5.26),

(ii) Π_z *satisfies the condition* (5.31) *and the estimate* (5.32) *with constant $\widehat{\gamma}$ independent of $x \in M_-(\varepsilon_0/2)$.*

Although the proof of this theorem is similar to that of Theorem 6.4 in [**1**], we sketch its proof because we will use it below.

Proof of Theorem 5.4. Let $z \in M_-(\varepsilon_0/2)$ and $v \in \mathcal{O}_z^\delta$. We introduce the projection operator Π_z on $M_-(\varepsilon_0)$ by the formula

$$\Pi_z(v) = v + (-\pi\Delta)^{-1}_{\omega_j} \sum_{j=1}^K p_j d_j, \tag{5.34}$$

where $(-\pi\Delta)^{-1}_{\omega_1}$ is the solving operator of the Stokes problem in the domain $\omega_1 \subset \omega \subset G$ (an exact definition of the operator $(-\pi\delta)^{-1}_{\omega_1}$ is given at the beginning of the proof of Theorem 5.1), d_j are the vector fields (4.15), and $(p_1, p_2, \dots, p_K) \equiv \vec{p}$ are unknown yet coefficients. It is obvious that for any coefficients $\vec{p} \in R^K$ the operator (5.34) satisfies the condition (5.31). By a suitable choice of $\vec{p}$, we obtain the inclusion $\Pi_z v \in M_-(\varepsilon_0)$ and thereby construct the required projection operator.

Let $\{e_j\}$ be the vector fields (4.18), and let P_+ be the projection (4.20) onto the subspace $L :_+$ defined by (4.21). It is obvious that the mapping $g(u)$ defining the manifold $M_-(\varepsilon_0)$ can be written in the form

$$g(u_-) = \sum_{j=1}^K e_j(x) g_j(u_-), \quad \text{where } g_j(u_-) = \int (g(u_-)(x), d_j(x))\, dx. \tag{5.35}$$

By the definition (5.26) of the invariant manifold $M_-(\varepsilon_0)$, the inclusion $\Pi_z(v) \in M_-(\varepsilon_0)$ is equivalent to the equality

$$P_+\big(z + v + (-\pi\Delta)^{-1}_{\omega_1} \sum_{j=1}^K p_j d_j\big) = g\big(P_-\big(z + v + (-\pi\Delta)^{-1}_{\omega_1} \sum_{j=1}^K p_j d_j\big)\big),$$

where P_- and P_+ are the projections (4.20) and (4.22) respectively. Taking into account (5.35), we can write this equality in the form

$$F_i(v,\vec{p}) \equiv \int_G \big(\big(z+v+(-\pi\Delta)_{\omega_1}^{-1}\sum_{j=1}^{k} p_j d_j\big)(x), d_j(x)\big)\,dx$$

$$- g_i\big(P_-\big(z+v+(-\pi\Delta)_{\omega_1}^{-1}\sum_{j=1}^{k} p_j d_j\big)\big) = 0. \qquad (5.36)$$

Thus, if for each z and v we find the coefficients $\vec{p} = \vec{p}(z,v)$ such that the equality (5.36) holds, the operator (5.34) will be the required one (i.e., $E_z v \in M_-(\varepsilon_0)$). In [**1**, Theorem 6.4], the required vector-valued function $\vec{p} = \vec{p}(v,v)$ was constructed with the help of the implicit function theorem. The inequality (5.32) was also established there. □

6. Real Processes Corresponding to the Navier–Stokes Equations. The Case of Time-Discrete Fluctuations

In this section, we follow [**1**]. We recall the definition of the real process corresponding to the Navier–Stokes equations, give a method of stabilization of a real process and formulate an estimate for the stabilization rate. In addition, we prove some properties of the stabilized real processes which will be useful in the following section.

6.1. Real processes corresponding to the Navier–Stokes equations. The invariant manifolds M_- that we use to solve the stabilization problem for solutions to the Navier–Stokes equations are repelling sets: if a solution $S(t,z_0) \in M_-$ leaves M_- under an action of some perturbation, this solution will move away from M_- as $t \to \infty$ even being located in a small neighborhood of M_- at $t = 0$. As was demonstrated by the simplest example in Sec. 2, fluctuations of the solution necessarily appear while numerical realization of the stabilization method used in this paper. Therefore, it is necessary to develop methods of overcoming this undesirable phenomenon. First of all, it is necessary to investigate processes that appear in the numerical solution of the stabilization problem. These processes are simulated by using the notion of a real process which will be introduced below.

Let us consider a discrete sequence of time moments

$$0 < t_1 < t_2 < \cdots < t_j < \cdots \to \infty \quad \text{as } j \to \infty, \qquad (6.1)$$

which are similar to the time moments appearing in the finite-difference approximations of the differential equations under consideration (cf. Subsec. 2.3).

For the sake of simplicity, we assume that

$$t_j - t_{j-1} = \Delta \quad \forall\, j = 1, 2, \ldots . \tag{6.1$'$}$$

In the numerical solution of the finite-difference equations, we obtain an approximate solution instead of an exact one because any computation is approximate. We note that this approximate solution is fully observable. Therefore, together with the exact solution $v(t, x)$ to the Navier–Stokes equations, we consider a real process $\widetilde{v}(t, x)$ that is an abstract analog of the approximate solutions obtained by computer. The real process $\widetilde{v}(t, x)$ differs from the exact solution $v(t, x)$ by a small fluctuation (i.e., a residual) $\varphi(t, x)$, which is unpredictable. These fluctuations appear just at the time moments (6.1). For simplicity, we assume that the initial data for the exact solution coincides with the initial data for the real process:

$$\widetilde{v}(t, x) = v(t, x)\big|_{t=0} = v_0(x). \tag{6.2}$$

We denote by $S(t, v_0) = v(t, \cdot)$ the solving operator of the boundary-value problem (3.3)–(3.6) (for simplicity, we do not reflect the dependence on the boundary condition in this notation). We denote the fluctuations appearing at the time moments $t_1, t_2, \ldots$ as follows:

$$\Delta v_{0,1}(x), \Delta v_{0,2}(x), \ldots, \Delta v_{0,j}(x), \ldots \tag{6.3}$$

(the fluctuation $\Delta v_{0,j}(x)$, $x \in \Omega$, appears at the time moment t_j, where the coefficient Δ is the number in (6.1$'$) introduced for normalization). We assume that the fluctuations v_{0j} satisfy the following conditions:

$$\|v_{0,j}\|_{V^1(\Omega,\Gamma_0)} \leqslant \widehat{\delta} \quad \forall j,\ 0 \leqslant \widehat{\delta} \ll \delta_0, \tag{6.4}$$

where δ_0 is the number in (5.16).

The formulas connecting the real process $\widetilde{v}(t, x)$ and the exact solution $v(t, x)$ to the problem (3.3)–(3.6) correspond to formulas (2.37) which connect the approximate solution (obtained by computer) and the exact solution of the finite-difference equations. Namely, we set

$$\widetilde{v}(t_j, \cdot) = S(t_j - t_{j-1}, \widetilde{v}(t_{j-1})) + \Delta v_{0,j}(\cdot), \tag{6.5}$$

$$\widetilde{v}(t, \cdot) = S(t - t_{j-1}, \widetilde{v}_{j-1}, \cdot)), \quad t \in (t_{j-1}, t_j), \tag{6.6}$$

where $\Delta v_{0,j}(\cdot)$ is the fluctuation appearing at the time moment t_j. By assumption (which completely corresponds to the situation appearing in the solution of the finite-difference equations; (cf. Subsec. 2.4)), the real process $\widetilde{v}(t_j, \cdot)$ is fully observable at the time moments t_j in (6.1).

Thus, we arrive at the following definition.

Definition 6.1. A vector field $\widetilde{v}(t, x)$ that is determined by formulas (6.5) and (6.6) with help of time moments (6.1) and fluctuations (6.3) is called a *real process of an open system.*

As in the case of the simplest example (cf. Subsec. 2.5 above), the fluctuations (6.3) destroy the stabilization of the real process by control (5.25) constructed in Theorem 5.3. Thus, the real process (6.5), (6.6) does not tend to the steady-state solution $\widehat{v}(x)$ as $t \to \infty$ but it will go out to a turbulent behavior. This situation occurs because the control (5.25) does not react on the fluctuations (6.3).

6.2. Stabilization of a real process. To stabilize a real process, we need to use a feedback control that reacts on the fluctuations (6.3) and suppress the negative consequences caused by these fluctuations. Let us construct such a feedback control.

According to the stabilization theory developed in Secs. 3–5, the central point is the correct construction of a solution to the boundary-value problem in the extended domain G.

Let $z(t,\cdot) = S(t, z_0)$ be the solving operator of the boundary-value problem (4.24) defined in the extended domain G. Then the problem (4.24) is an extension of the stabilization problem (5.18)–(5.23) from Ω to G. We construct a real process of the problem (4.24). However, in contrast to the real process (6.5), (6.6), the undesirable effect of fluctuations is dampened by feedback control. In the terms of our main example of the real process (i.e., the result of numerical solution of the problem by computer), the fluctuations in the case of the problem (4.24) are vector fields in the domain G with small norms. Therefore, we assume that the fluctuations has the form (6.3) but they are defined in G and, instead of (6.4), they satisfy the relation

$$\forall j \quad \|v_{0,j}\|_{V_0^1(G)} \leqslant \delta, \quad 0 \leqslant \delta \ll \delta_0. \tag{6.4$'$}$$

We define the stabilized real process corresponding to the problem (4.24) by induction:

$$z_0(0,x) = z_0(x) = E_0(y_0), \quad z_0(t,\cdot) = S(t - t_0, E_0(y_0)), \quad t > t_0 = 0, \tag{6.7}$$

where E_0 is the extension operator (5.16), y_0 is the initial data of the stabilization problem (5.18)–(5.23), and $z_0(x)$ is the initial data of the problem (4.24). Formulas (6.7) provide us with the induction base, whereas the jth step is defined by the formulas

$$z_j(t) = S(t - t_j, \Pi_{z_{j-1}(t_j)}(z_{j-1}(t_j) + \Delta v_{0,j})), \quad t > t_j, \tag{6.8}$$

where $z_j(t) = z_j(t,\cdot)$. Here, Π_w is the projection operator (5.33), (5.34), $t_j = \Delta j$, where Δ is the number (6.1$'$), and Δv_{0j} are the fluctuations (6.3) appearing at the time moments t_j.

Now, the real process corresponding to the stabilization problem (5.18)–(5.23) is defined by the usual formula

$$(\widetilde{y}(t_j), \widetilde{u}(t,\cdot)) = (z_j(t_j)\big|_\Omega, z_j(t,\cdot)\big|_\Gamma), \quad t \in (t_j, t_{j+1}). \tag{6.9}$$

In the sequel, we also need the following function $z^\Delta(t,x)$, $t \in \mathbb{R}_+$:

$$z^\Delta(t,x) = z_j(t,x), \quad t \in (t_j, t_{j+1}), \quad j = 0,1,2,\ldots,. \tag{6.10}$$

Theorem 6.1. *Let the time moments* (6.1) *and the fluctuations* (6.3) *satisfy the conditions* (6.1′), (6.4), *and let the real process corresponding to the stabilization problem* (5.18)–(5.23) *be defined by the relations* (6.7)–(6.9). *Suppose that the initial data* y_0 *in* (5.20) *and* δ *in* (6.4′) *satisfy the conditions*

$$\widehat{c\gamma}\|y_0\|_{V^1(\Omega,\Gamma_0)} \leqslant \varepsilon_0/4, \quad \delta \leqslant \min\left(\frac{\delta_0}{\Delta}, \frac{\varepsilon_0}{4\widehat{c\gamma}(\Delta+1/\sigma)}\right), \tag{6.11}$$

where $\widehat{c}$ *and* $\widehat{\gamma}$ *are the constants form the estimates* (4.31) *and* (5.32) *respectively,* $\sigma > 0$ *satisfies* (4.9), Δ *is defined in* (6.1′), *and the constants* ε_0 *and* δ_0 *are defined in Theorem* 5.2. *Then the real process* $\widetilde{y}(t,x)$ *and functions* (6.10) *satisfy the estimates*

$$\|\widetilde{y}(t,\cdot)\|_{V^1(\Omega,\Gamma_0)} \leqslant \widehat{c\gamma}(e^{-\sigma t}\|y_0\|_{V^1(\Omega,\Gamma_0)} + (1+\sigma^{-1})\delta), \tag{6.12}$$

$$\|z^\Delta(t,\cdot)\|_{V_0^1(G)} \leqslant \widetilde{c\gamma}(e^{-\sigma t}\|z_0\|_{V_0^1(G)} + (1+\sigma^{-1})\delta). \tag{6.12′}$$

The proof of this theorem can be found in [**1**]. We note that the constants $\widehat{c}$ and $\widehat{\gamma}$ are independent of $\Delta < 1$.

7. Real Processes. Time-Continuous Fluctuations

This section is devoted to the generalization of the notion of a real process to the case where fluctuations appear at each time moment and the collection of fluctuations (6.3) is transformed to the differential form $v_0(t,x)\,dt$, where

$$v_0(t,\cdot) \in C(R, V_0^1(\Omega)), \quad \sup_{t\in R}\|v_0(t,\cdot)\|_{V_0^1(G)} \leqslant \delta. \tag{7.1}$$

We restrict ourselves to the case where a real process is stabilized by feedback control. This real process is obtained by the limit passage as Δ in (6.1′) tends to 0. Further, we deduce a differential equation for the constructed real process. Owing to this differential equation, we can prove the uniqueness of the stabilized real process (for given fluctuations $v_0(t,x)dt$).

7.1. The limit passage as $\Delta \to 0$. Our goal is to pass to the limit for the family of functions $z^\Delta(t)$ defined in (6.10) as $\Delta \to 0$ We begin by proving the following assertion similar to that on the equicontinuity of the functions $z^\Delta(t)$.

Lemma 7.1. *For any small* $\alpha > 0$ *there exists* $\beta > 0$ *and* $\Delta(\alpha) > 0$ *such that for any* $\tau_1 > \tau_2 > 0$ *and* $\Delta > 0$ *satisfying the inequalities*

$$\tau_1 - \tau_2 < \beta, \quad \Delta < \Delta(\alpha), \tag{7.2}$$

the following estimate holds:

$$\|z^\Delta(\tau_1) - z^\Delta(\tau_2)\|_{L_2(G)} < \alpha. \tag{7.3}$$

PROOF. Let $\tau_1 \in (t_{j+k}, t_{j+k+1})$, $\tau_2 \in (t_j, t_{j+1})$. Using formulas (6.10), (6.8) and the semigroup property of the family of operators $S(t,\cdot)$, we find

$$
\begin{aligned}
z^\Delta(\tau_1) - z^\Delta(\tau_2) &= S(\tau_1 - t_{j+k}, \Pi_{z_{j+k-1}(t_{j+k})}(z_{j+k-1}(t_{j+k}) + \Delta v_0(t_{j+k})) - z_j(\tau_2) \\
&= S(\tau_1 - t_{j+k}, \Pi_{z_{j+k-1}(t_{j+k})}(z_{j+k-1}(t_{j+k}) + \Delta v_0(t_{j+k}))) \\
&- S(\tau_1 - t_{j+k}, z_{j+k-1}(t_{j+k})) \\
&+ S(\tau_1 - t_{j+k-1}, \Pi_{z_{j+k-2}(t_{j+k-1})}(z_{j+k-2}(t_{j+k-1}) + \Delta v_0(t_{j+k-1}))) \\
&- S(\tau_1 - t_{j+k-1}, z_{j+k-2}(t_{j+k-1})) + \dots \\
&+ \{S(\tau_1 - t_j, \Pi_{z_{j-1}(t_j)}(z_{j-1}(t_j) + \Delta v_0(t_j))) - z_j(\tau_2)\}. \qquad (7.4)
\end{aligned}
$$

Owing to the semigroup property of the family $S(t,\cdot)$ and the definition of $z_j(\tau_2)$, the difference in the curly brackets on the right-hand side of (7.4) is equal to $S(\tau_1 - \tau_2, z_j(\tau_2)) - z_j(\tau_2)$. Introduce the notation

$$
\|\cdot\| = \|\cdot\|_{L_2(G)}, \quad \|\cdot\|_1 = \|\cdot\|_{V_0^1(G)}.
$$

Using the inequalities (4.31), (5.32) (6.4′) and the obvious estimate $\|\cdot\| \leqslant c_0\|\cdot\|_1$, we bound the right-hand side of (7.4) as follows:

$$
\begin{aligned}
&\|z^\Delta(\tau_1) - z^\Delta(\tau_2)\| \leqslant c_0\widehat{c}\sum_{r=0}^{k}(\|\Pi_{z_{j+r-1}(t_{j+r})}(z_{j+r-1}(t_{j+r}) + \Delta v_0(t_{j+r})) \\
&- z_{j+r-1}(t_{j+r})\|_1 e^{-\sigma(\tau_1 - t_{j+k} + \Delta(k-r))} \\
&+ \|S(\tau_1 - \tau_2, z_j(\tau_2)) - z_j(\tau_2)\| \\
&\leqslant c_0\widehat{c}\widehat{\gamma}\Delta\sum_{r=0}^{k} e^{-\sigma\Delta(k-r)}\|v_0(t_{j+r})\|_1 \\
&+ \|S(\tau_1 - \tau_2, z_j(\tau_2)) - z_j(\tau_2)\| \\
&\leqslant c_0\widehat{c}\widehat{\gamma}\Delta\delta\sum_{j=0}^{k} e^{-\sigma\Delta j} + \|S(\tau_1 - \tau_2, z_j(\tau_2)) - z_j(\tau_2)\|. \qquad (7.5)
\end{aligned}
$$

To estimate the last term on the right-hand side of (7.5), we set $z_j(\tau_2) = b$, $\tau_1 - \tau_2 = t$. Since $S(t,\cdot)$ is the solving operator of Eq. (4.24), using (4.6) and deriving estimates usual in the theory of the Navier–Stokes equations, we find:

$$
\|S(t,b) - b\|^2 = \left(\int_0^t \partial_\tau S(\tau,b)\,d\tau, S(t,b) - b\right)_{L_2(G)}
$$

$$= -\int_0^t (\nabla S(\tau,b), \nabla S(t,b) - \nabla b)_{L_2(G)}\, d\tau$$

$$- \int_0^t ((a,\nabla)S(\tau,b) + (S(\tau,b),\nabla)a + (S(\tau,b),\nabla)S(\tau,b), S(t,b) - b)_{L_2(G)}\, d\tau$$

$$\leqslant ct(\sup_{\tau\in(0,t)} \|S(\tau,b)\|_1 + \sup_{\tau\in(0,t)} \|S(\tau,b)\|_1^3)(1 + \|b\|_1)$$

$$\leqslant c_1 t(\|b\|_1 + \|b\|_1^3)(1 + \|b\|_1) \leqslant c_2 t\varepsilon_0, \tag{7.6}$$

where $\varepsilon_0$is the constant from (5.26). The estimates (7.5) and (7.6) imply the inequality

$$\|z^\Delta(\tau_1) - z^\Delta(\tau_2)\| \leqslant c_0\widehat{c}\widehat{\gamma}\delta(k+1)\Delta + \sqrt{c_2\varepsilon_0}(\tau_1 - \tau_2)^{1/2}. \tag{7.7}$$

Since $(k-1)\Delta < \tau_1 - \tau_2 < (k+1)\Delta$, the inequality (7.7) easily implies the assertion of the lemma. □

Remark 7.1. From formula (7.7) it follows that the family of functions $z^\Delta(t)$ possesses the following property which is closed to the Hölder condition. There exists a constant $c > 0$ such that for all $0 < \Delta < 1/8$ and $\tau_1 > \tau_2 \geqslant 0$ satisfying the condition $2\Delta < \tau_1 - \tau_2 < 1$ the following inequality holds:

$$\|z(\tau_1) - z(\tau_2)\|_{L_2(G)} < c|\tau_1 - \tau_2|^{1/2} \tag{7.8}$$

We now pass to the limit of the family of functions $z^\Delta(t)$ defined by formula (6.10) as $\Delta_k \to 0$.

Theorem 7.1. *There exists a sequence $\Delta_k \to 0$ such that*

$$z^{\Delta_k}(t) \to z^0(t) \text{ in } L_2(G) \text{ as } \Delta_k \to 0 \text{ uniformly with respect to } t \in R_+. \tag{7.9}$$

Moreover, $z^0(t)$ belongs to $L_\infty(R_+; V_0^1(G)) \cap C_b^{1/2}(R_+; L_2(G))$ and satisfies the inequality

$$\|z^0(t)\|_{V_0^1(G)} \leqslant \widehat{c}\widehat{\gamma}(e^{-\sigma t}\|z_0\|_{V_0^1(G)} + (1 + \sigma^{-1})\delta), \tag{7.10}$$

where the constants $\widehat{c}$, $\widehat{\gamma}$, and δ are the same as in (6.12′).

Proof. Let $\{x_j\}$ be a countable everywhere dense set of points in R_+. Since $\|z^\Delta(t)\|_{V_0^1(G)} \leqslant \text{const}$ for every $t \in R_+$ and the embedding $V_0^1(G) \subset\subset L_2(G)$ is compact, it is possible to choose a sequence $\Delta_{k_1} \to 0$ and subsequences $\Delta_{k_2} \subset \Delta_{k_1}$, $\Delta_{k_3} \subset \Delta_{k_2}, \ldots, \Delta_{k_{j+1}} \subset \Delta_{k_j}, \ldots$ such that $z^{\Delta_{k_j}}(t_j) \to z^0(t_j)$ as $\Delta_{k_j} \to 0$ strongly in $L_2(G)$ and weakly in $V_0^1(G)$.

Taking the diagonal sequence $\Delta^k = \Delta_{k_k}$, we find

$$\begin{gathered} \forall t_r \subset \{t_j\} \quad z^{\Delta^k}(t_r) \to z^0(t_r) \quad \text{as } \Delta^k \to 0 \\ \text{strongly in } L_2(G) \text{ and weakly in } V_0^1(G). \end{gathered} \tag{7.11}$$

From (7.11) and (7.8) it is easy to derive that for any $\tau_1 \in \{t_j\}$, $\tau_2 \in \{t_j\}$, $|\tau_1 - \tau_2| < 1$, the following inequality holds:

$$\|z^0(\tau_1) - z^0(\tau_2)\|_{L_2(G)} \leqslant 3c|\tau_1 - \tau_2|^{1/2}, \tag{7.12}$$

where the constant c is the same as in (7.8). Using (7.12), we can extend by continuity the function $z^0(t)$ from $\{t_j\}$ to R_+. The extended function will be also denoted by $z^0(t)$. It is obvious that this function satisfies (7.12) for any $\tau_1, \tau_2 \in R_+$ with $|\tau_1 - \tau_2| < 1$. Since $z^\Delta(t)$ satisfies (6.12′) for any Δ, for $z^0(t)$ the inequality (7.10) holds. Moreover, from (7.8), (7.11), and (7.12) it is easy to obtain the convergence of the sequence $z^{\Delta^k}(t) \to z^0(t)$ in $L_2(G)$ and weakly in $V_0^1(G)$ as $\Delta^k \to 0$ for any $t \in R$. □

Definition 7.1. The function $z^0(t)$ constructed in Theorem 7.1 is called the *stabilized real process* corresponding to the fluctuations $v_0(t)dt$ in (7.1).

This definition of a real processes has an essential disadvantage: it is not known if $z^0(t)$ is uniquely determined by the suggested stabilization method. Moreover, it is very desirable to know how to determine the real process corresponding to fluctuations $v_0(t)dt$ appearing at each time moment but do not use the limit passage. These problems are discussed in the following subsections.

7.2. Differential equations for the real process. In this subsection, we derive a differential equation for the real process $z^0(t)$ constructed in Theorem 7.1. We begin with a differential equation for the real process $z^\Delta(t)$ defined in (6.7), (6.8), (6.10) and generated by fluctuations $\Delta v_0(t_j)$ appearing at the time moments $t_j = \Delta j$. Recall that since $S(t,\cdot)$ is the solving operator of the problem (4.24), it satisfies the equation in (4.24). Therefore in virtue of (6.8) and (6.10) the real process $z^\Delta(t,x)$ is a generalized solution to the differential equation

$$\begin{aligned} &\partial_t z^\Delta(t,x) + \mathcal{A} z^\Delta(t,x) + \pi[(z^\Delta, \nabla) z^\Delta] \\ &\quad = \sum_{j=1}^{\infty} \delta(t - t_j)\big(\Pi_{z_{j-1}(t_j)}(z_{j-1}(t_j) + \Delta v_0(t_j)) - z_{j-1}(t_j)\big). \end{aligned} \tag{7.13}$$

Moreover, $z^\Delta(t,\cdot)$ satisfies the initial condition (6.7):

$$z^\Delta(t,x)\big|_{t=0} = z_0(x). \tag{7.14}$$

The integral form (i.e., the generalized statement) of the problem (7.13), (7.14) is as follows:

$$\int_0^\infty \{-z^\Delta(t,\cdot)\cdot\partial_t\varphi(t,x) + \nabla z^\Delta\cdot\nabla\varphi + (a,\nabla)z^\Delta\cdot\varphi + (z^\delta,\nabla)a\cdot\varphi$$
$$+ (z^\Delta,\nabla)z^\Delta\cdot\varphi\}dx\,dt - \int_G \varphi(0,x)z_0(x)dx$$
$$= \sum_{j=1}^\infty \int_G \Big(\Pi_{z_{j-1}(t_j)}(z_{j-1}(t_{j-1}) + \Delta v_0(t_{j_1},\cdot))(x) - z_{j-1}(t_j,x)\Big)\cdot\varphi(t_j,x)\,dx, \tag{7.15}$$

and this equality is valid for any function $\varphi(t,x)\in C^1(R_+;V_0^1(G)\cap V^2(G))$ that vanishes identically for sufficiently large t.

Theorem 7.2. *Let $z^0(t,\cdot)$ be the real process constructed in Theorem 7.1 and corresponding to the fluctuations $v_0(t,x)\,dt$ that appear at each time moment $t>0$ and satisfy the condition (7.1). Then $z^0(t,\cdot)$ satisfies the differential equation*

$$\partial_t z^0(t,x) + \mathcal{A}z^0 + \pi[(z^0,\nabla)z^0]$$
$$= v_0(t,x) + \sum_{j=1}^K \int_G \frac{\delta p_j(z^0(t,\cdot))}{\delta z(y)} v_0(t,y)dy(-\pi\Delta)^{-1}_{\omega_1} d_j(x), \tag{7.16}$$

where the operators on the left-hand side of the equation coincide with the corresponding operators in Eq. (4.24), $p_j(z)$ are the functionals in the definition (5.34) of the projection operator Π_z, $\delta p_j/\delta z(y)$ is the variational derivative of this functional, $d_j(x)$ are the functions (4.15), and the operator $(-\pi\Delta)^{-1}_{\omega_1}$ was introduced above formula (5.13).

Proof. Taking into account the smoothness property of the functions z^Δ and z^0, it is easy to see that the left-hand side of Eq. (7.15) converges as $\Delta\to 0$ to the same expression with z^Δ replaced by z^0.

To pass to the limit on the right-hand side of (7.15), we write it, using the definition (5.34) of the projection operator Π_z:

$$\sum_{j=1}^\infty \int_G \Big(\Delta v_0(t_j,x) + \sum_{l=1}^k p_l(z_{j-1}(t_j,\cdot)$$
$$+ \Delta v_0(t_j,\cdot))(-\pi\Delta)^{-1}_{\omega_1} d_l(x)\Big)\cdot\varphi(t_j,x)\,dx.^{5)} \tag{7.17}$$

From the relation (5.36) defining the functionals $p_l(v)$ and from the condition $z_{j-1}(t_j)\in M_-(\varepsilon_0/2)$ we find that $p_l(z_{j-1}(t_j)) = 0$.

5) Recall that Δ is the factor in (6.3) but not the Laplace operator.

As was established in Theorem 4.2, the mapping g defining the manifold M_- in (4.28) belongs to class $C^{1+\alpha}$ with $\alpha = 1$. Consequently, $g_l(v) \in C^{1+\alpha}$, $\alpha = 1$. Since the functionals $p_j(v)$ are determined from the equality (5.36) by the implicit function theorem, we have $p_l(v) \in C^{1+\alpha}$, $\alpha = 1$. By the definition of the derivative of the functional p_l, the expression (7.17) can be written as follows:

$$\Delta \sum_{j=1}^{\infty} \int_G \left(v_0(t_j, x) + \sum_{l=1}^{k} \left(\int_G \frac{\delta p_l(z_{j-1}(t_j))}{\delta z(y)} v_0(t_j, y)\, dy + o_{l,j}(\Delta)\right)\right.$$
$$\left. \times (-\pi\Delta)^{-1}_{\omega_1} d_l(x)\right) \cdot \varphi(t_j, x)\, dx, \tag{7.18}$$

where

$$|o_{l,j}(\Delta)| \leqslant c\Delta$$

and the constant c is independent of l and j. But the expression (7.18) is the integral sum converging as $\Delta \to 0$ to the integral

$$\int_0^{\infty}\int_G \left(v_0(t,x) + \sum_{l=1}^{K} \int_G \frac{\delta p_l(z^0(t,\cdot))}{\delta z(y)} v_0(t,y)dy(-\pi\Delta)^{-1}_{\omega_1} d_l(x) \cdot \varphi(t,x)\right) dx\, dt.$$

(This improper integral converges because $\varphi(t,x) \equiv 0$ for sufficiently large t) This expression determines the right-hand side of Eq. (7.16). $\square$

We note that the real process $z^0(t,x)$ satisfies the initial condition

$$z^0(t,x)\big|_{t=0} = z_0(x) \tag{7.19}$$

because $z^{\Delta}(t,x)$ satisfy (7.14) and $z^{\Delta}(t,x) \to z^0(x)$ in $C(0,T; L_2(G))$.

7.3. The uniqueness theorem. The real process z^0 was obtained as a result of the limit passage. Let us show that z^0 is independent of the choice of a subsequence and is uniquely determined by the fluctuation $v_0(t,x)\,dt$, the initial data $z_0(x)$, and, of course, by the stabilization method.

Theorem 7.3. *Let the fluctuation $v_0(t,x)dt$ satisfy the condition* (7.1), *and let the initial data* (7.19) *satisfy the inclusion $z_0 \in \mathcal{O}^{\varepsilon/2}$ (cf.* (5.27)). *Then the real process $z^0(t,x)$ constructed in Theorem* 7.1 *belongs to the space $V^{1,2}(\Theta_T)$ for every $T > 0$ (defined in* (4.25)) *and is unique in this space.*

PROOF. By (4.1), the coefficient $a(x)$ in the definition (4.6) of the operator $\mathcal{A}$ satisfies the inclusion $a(x) \in V_0^1(G) \cap V^2(G)$. Since $d_j(x)$ for any $j = 1, \dots, K$ is the imaginary or real part of some eigenfunction or associated function of the operator $\mathcal{A}^*$, it is easy to see that $d_j(x) \in V_0^1(G) \cap V^4(G)$. Consequently, the right-hand side of Eq. (7.16) belongs to $C(0,T; V_0^1(G))$. As was proved in Theorem 7.1, $z^0(t,x) \in L_\infty(0,t; V_0^1(G))$. Therefore, using the known estimates for the nonlinear term of the Navier–Stokes system (cf., for example, [**2**, Chap. 3, Sec. 4, Lemma 4.6.]), it is easy to show that

$$\pi[(a,\nabla)z^0(t,x)+(z^0,\nabla)a+(z^0,\nabla)z^0]\in L_\infty(0,T;V^{-1/2-\varepsilon}(G))\quad \forall\varepsilon>0 \tag{7.18$'$}$$

(the definition of the space $V^{-s}(G)$ for $s>0$ can be found in [**2**, Chap. 3, § 4]). Transferring these terms from the left-hand side of Eq. (7.16) to the right-hand side and applying the known theorems on smoothness of a solution to the Stokes system to the obtained equation, we arrive at the inclusion $z_0(t,x)\in L_2(0,T:V^{3/2-\varepsilon}(G))$. Owing to this inclusion, we can establish that the left-hand side of the inclusion (7.18$'$) belongs to the space $L_2(0,T;V_0^0(G))$. Using the theorem on the smoothness of solutions to the Stokes equations again, we find that $z^0(t,x)\in V^{1,2}(\Theta_T)$.

Let us prove the uniqueness theorem for solutions to the problem (7.16), (7.19) in the space $V^{1,2}(\Theta_T)$. Let $z_1^0(t,x)$ and $z_2^0(t,x)$ be two solutions to this problem belonging to $V^{1,2}(\Theta_T)$, and let

$$w(t,x)=z_1^0(t,x)-z_2^0(t,x).$$

It is obvious that w satisfies the equation

$$\begin{aligned}&\partial_t w+\mathcal{A}w+\pi[(w,\nabla)w]+(z_2,\nabla)w+(w,\nabla)z_2]\\&=\sum_{j=1}^k\int_G\left(\frac{\delta p_j(z_2^0+w)}{\delta z(y)}-\frac{\delta p_j(z_2^0)}{\delta z(y)}\right)\cdot v_0(t,y)\,dy(-\pi\Delta)_{\omega_1}^{-1}d_j(x).\end{aligned}\tag{7.20}$$

As was mentioned above, the functionals $p_j(z)$ belong to the class $C^{1+\alpha}$ with $\alpha=1$. This means that there exists a constant $\widetilde{c}$ independent of z that runs a δ-neighborhood $O^\delta(M_-(\varepsilon_0/2))$ of the set $M_-(\varepsilon_0/2)$ in the space $V_0^1(G)$ such that for any $w\in O^\delta(M_-(\varepsilon_0/2))$ we have

$$\left\|\frac{\delta p_j(z+w)}{\delta z(\cdot)}-\frac{\delta p_j(z)}{\delta z(\cdot)}\right\|_{V^{-1}(G)}\leqslant\widetilde{c}\|w\|_{V_0^1(G)}.\tag{7.21}$$

Taking the inner product of the equality (7.20) and w in $(L_2(G))^3$, making usual computations, and taking into account (7.21), we obtain the inequality

$$\begin{aligned}&\frac{1}{2}\partial_t\|w(t)\|^2+\|\nabla w\|^2\leqslant(\|\nabla a\|_{L_6(G)}+\|\nabla z_2\|_{L_6(G)})\|w\|^2_{L_{12/5}(G)}\\&\quad+c\sum_{j=1}^K\|d_j\|\,\|w\|\left\|\frac{\delta p_j(z+w)}{\delta z}-\frac{\delta p_j(z)}{\delta z}\right\|_{V^{-1}(G)}\|\nabla v_0(t,\cdot)\|\\&\le c(\|a\|_{V^2(G)}+\|z_2\|_{V^2(G)})\|w\|\,\|\nabla w\|+c_1K\widetilde{c}\|\nabla w\|\,\|w\|\,\|\nabla v_0(t,\cdot)\|\\&\leqslant\frac{1}{2}\|\nabla w\|^2+r(t)\|w(t)\|^2,\end{aligned}\tag{7.22}$$

where

$$\|\cdot\| = \|\cdot\|_{(L_2(G))^3}, \quad \|\nabla w\| = \|w\|_{V_0^1(G)},$$

$$r(t) = \frac{c^2}{2}(\|a\|^2_{V^2(G)} + \|z_2(t)\|^2_{V^2(G)} + \frac{c_1^2 K^2 \widetilde{c}^2}{2}\|v_0(t,\cdot)\|^2_{V_0^1(G)}.$$

Since $\|w(t,\cdot)\|\big|_{t=0} = 0$, from (7.22) we obtain the inequality

$$\|w(t,\cdot)\|^2 \leqslant 2\int_0^t r(\tau)\|w(\tau)\|^2 d\tau,$$

which implies $w(t,x) \equiv 0$ by the Gronwall lemma. □

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