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Stabilizability of Two-Dimensional Navier–Stokes Equations with Help of a Boundary Feedback Control

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Abstract. For $2D$ Navier–Stokes equations defined in a bounded domain Ω we study stabilization of solution near a given steady-state flow $\hat{v}(x)$ by means of feedback control defined on a part Γ of boundary $\partial\Omega$. New mathematical formalization of feedback notion is proposed. With its help for a prescribed number $\sigma > 0$ and for an initial condition $v_0(x)$ placed in a small neighbourhood of $\hat{v}(x)$ a control $u(t, x')$, $x' \in \Gamma$, is constructed such that solution $v(t, x)$ of obtained boundary value problem for 2D Navier–Stokes equations satisfies the inequality: $\|v(t, \cdot) - \hat{v}\|_{H^1} \leqslant ce^{-\sigma t}$ for $t \geqslant 0$. To prove this result we firstly obtain analogous result on stabilization for 2D Oseen equations.

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1. Introduction

In this paper we study so-called stabilization problem for two-dimensional (2D) Navier–Stokes equations defined in a bounded domain $\Omega \subset \mathbb{R}^2$ which is controlled by Dirichlet boundary condition for velocity vector field. Let $(\hat{v}(x), \nabla\hat{p}(x))$, $x \in \Omega$ be a steady-state solution for Navier–Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(x), \quad \operatorname{div} v = 0, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

supplied with an initial condition

$$v(t,x)|_{t=0} = v_0(x) \quad (1.2)$$

and $\hat{v} \neq v_0$. We suppose that $\hat{v}|_{\partial\Omega} = 0$ and $\hat{v}$ is an unstable singular point for the dynamic system generated by equation (1.1) supplied with zero condition $v|_{\partial\Omega} = 0$ on the boundary $\partial\Omega$ of Ω.

Hence, generally speaking the solution $(v(t,x), \nabla p(t,x))$ of (1.1)–(1.2) goes away $(\hat{v}(x), \nabla\hat{p}(x))$ as $t \to \infty$. But instead of zero boundary condition we introduce

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the boundary control

$$v|_{\partial\Omega} = u. \tag{1.3}$$

Stabilization problem is as follows: Let

$$\|\hat{v} - v_0\|_{H^1} < \varepsilon \text{ where } \varepsilon > 0 \text{ is sufficiently small.} \tag{1.4}$$

Given $\sigma > 0$ find $u(t,x)$, $t > 0$, $x \in \partial\Omega$ such that the solution $(v, \nabla p)$ of (1.1)–(1.3) satisfies the inequality:

$$\|v(t,\cdot) - \hat{v}(\cdot)\|_{H^1(\Omega)} \leqslant ce^{-\sigma t} \text{ as } t \to \infty. \tag{1.5}$$

Note that existence theorem for solution of problem (1.1)–(1.5) can be derived easily from the result of local exact controllability for Navier–Stokes equation (see [5], [7] and references there). But this existence theorem can not be considered as a complete theoretical foundation for numerical solution of problem (1.1)–(1.5). The point is that problem (1.1)–(1.5) is ill-posed and therefore unpredictable fluctuations which usually appear during numerical realization, grow with time. As a result this lead to completely wrong numerical calculations. The way how to overcome this difficulty is well-known: one has to find and to use feedback control. In other words control must react on appearing fluctuations and suppress them.

Of course, feedback control was studied in applied sciences. A mathematical formalization of feedback notion was proposed in the theory of controlled ordinary differential equations, and this notion was extended to the case of partial differential equations (PDE). In the case of problem (1.1)–(1.5) this formalization is reduced to the following definition:

A control $u(t,x)$ from (1.3) is called *feedback* if there exists a map $\mathcal{R}$ transforming vector fields defined on Ω to vector fields defined on $\partial\Omega$ such that for each $t > 0$

$$u(t,\cdot) = \mathcal{R}(v(t,\cdot)) \tag{1.6}$$

where $v(t,x)$ is velocity vector field of fluid flow from (1.1)–(1.5).

If we take

$$\mathcal{R}v(t,\cdot)) = v(t,\cdot)|_{\partial\Omega} - B(v(t,\cdot), D^\alpha_{tx} v(t,\cdot))|_{\partial\Omega} \tag{1.7}$$

where B is a nonlinear transformation and D^α_{tx} denote derivatives with respect to t, x, then (1.3), (1.6), (1.7) imply the equality

$$B(v(t,\cdot), D^\alpha_{t,x} v(t,\cdot))|_{\partial\Omega} = 0. \tag{1.8}$$

Thus, stabilization problem usually is reformulated as follows: find a boundary condition (1.8) such that the solution $v(t,x)$ of boundary value problem (1.1), (1.2), (1.8), (1.4) satisfies (1.5). Stabilization problem in this formulation was studied for a number classes of evolution PDE in many papers. (See, for instance [15], [12], [3], [1][1].) But for many linear and nonlinear parabolic equations as well as for Navier–Stokes system satisfactory results for stabilization problem were not obtained yet. From our point of view the reason of this is that formalization

[1]Here we do not pretend on completeness of references, of course.

(1.6) of feedback motion is not adopted good enough for parabolic PDE. In [6] some other mathematical formalization of feedback notion was proposed and this formalization was applied to solve stabilization problem for linear and quasilinear parabolic equations. In this paper we apply this new formalization of feedback notion to investigate stabilization problem for 2D Navier–Stokes equations. More precisely we solve stabilization problem (1.1)–(1.5) with help of feedback control, concentrated on a part of boundary $\partial\Omega$. The idea of proposed methods is explained in the next section for the case of Stokes equations.

2. Stabilizability of the Stokes system

2.1. Formulation of the problem

In this section we show the idea of method of stabilization with prescribed rate of decay. We do this for the Stokes equations, defined on a bounded domain $\Omega \subset R^2$ with C^∞-smooth connected boundary $\partial\Omega$:

$$\partial_t v(t,x) - \Delta v + \nabla p(t,x) = 0, \qquad (t,x) \in Q, \tag{2.1}$$

$$\operatorname{div} v(t,x) = 0, \qquad (t,x) \in Q, \tag{2.2}$$

$$v(t,x)|_{t=0} = v_0(x), \qquad x \in \Omega \tag{2.3}$$

with Dirichlet boundary condition

$$v(t,x') = u(t,x') \qquad (t,x') \in \Sigma \tag{2.4}$$

where $u(t,x')$ is a control, defined on the lateral surface $\Sigma = (0,\infty) \times \partial\Omega$ of space-time cylinder $Q = (0,\infty) \times \Omega$.

Here $v(t,x) = (v_1(t,x), v_2(t,x))$ is a velocity vector field defined for $(t,x) \equiv (t,x_1,x_2) \in Q$, ∇p is a pressure gradient and $v_0(x)$ is a given solenoidal vector field: $\operatorname{div} v_0(x) = 0$.

Let a magnitude $\sigma > 0$ be given. The problem of stabilization from the boundary with prescribed rate $\sigma > 0$ of a solution $v(t,x)$ for evolution problem (2.1)–(2.4) is to construct a boundary control $u(t,x')$, $(t,x') \in \Sigma$ such that the solution $(v, \nabla p)$ satisfies inequality:

$$\|y(t,\cdot)\|_{L_2(\Omega)} \leqslant ce^{-\sigma t} \tag{2.5}$$

where constant $c > 0$ depends on v_0 and σ.

The main condition on control u, that this control satisfies the feedback condition, will be formulated after presentation the whole construction of stabilization.

2.2. Method of stabilization

We choose $L > 0$ so large that the following inclusion is true:

$$\Omega \subset \{x = (x_1,x_2) \in \mathbb{R}^2 \,:\, |x_i| < L, i = 1,2\}. \tag{2.6}$$

After that we identify the opposite sides of the square from (2.6) and obtain the torus, which we denote by Π. Then (2.6) implies that

$$\Omega \subset \Pi. \tag{2.7}$$

We omit for a while condition (2.4) in problem (2.1)–(2.4) and extend this problem from Ω to Π. As a result we obtain a boundary value problem for the Stokes system with periodic boundary conditions

$$\partial_t w(t,x) - \Delta w + \nabla q(t,x) = 0, \qquad x \in \Pi, \quad t > 0 \tag{2.8}$$

$$\operatorname{div} w = 0 \tag{2.9}$$

$$w(t,x)|_{t=0} = w_0(x), \quad \operatorname{div} w_0 = 0. \tag{2.10}$$

Let

$$w_0(x) = \sum_{\xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2} \widehat{w_0}(\xi) e^{ix\cdot\xi}$$

be the decomposition of $w_0(x)$ on Fourier series, where

$$\xi = (\xi_1, \xi_2), \quad \xi_i = \frac{\Pi}{\ell} n_j, \quad n_j \in \mathbb{N}, \quad j = 1,2, \quad x \cdot \xi = x_1\xi_1 + x_2\xi_2,$$

$$\widehat{w_0}(\xi) = (2L)^{-2} \int_\Pi w_0(x) e^{-ix\cdot\xi} dx$$

is a Fourier coefficient for $w_0(x)$. Applying Fourier method we obtain that the component $w(t,x)$ of solution $(w, \nabla q)$ for (2.8)–(2.10) is defined by the formula:

$$w(t,x) = \sum_{\xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2} \widehat{w_0}(\xi) e^{-|\xi|^2 t + ix\cdot\xi}. \tag{2.11}$$

Solution (2.11) of problem (2.8)–(2.10) satisfies the inequality

$$\|w(t,\cdot)\|_{L_2(\Pi)} \leqslant ce^{-\sigma t} \tag{2.12}$$

if and only if the following equalities hold:

$$\widehat{w_0}(\xi) = 0 \quad \forall \xi : |\xi| < \sqrt{\sigma}. \tag{2.13}$$

We introduce some functions spaces. Let Θ be an open domain or a torus, then

$$V^k(\Theta) = \{v(x) = (v_1(x), v_2(x)) \in \left(H^k(\Theta)\right)^2 \,:\, \operatorname{div} v(x) = 0\}, \quad k = 0,1,2,\dots \tag{2.14}$$

where $H^k(\Theta)$ is the Sobolev space of functions which belong to $L_2(\Theta)$ together with all derivatives up to the order k; if $k = 0$ then $\operatorname{div} w = 0$ is understood in the meaning of distributions theory;

$$\|v\|^2_{V^k(\Theta)} = \|v\|^2_{(H^k(\Theta))^2} = \sum_{|\alpha| \leqslant k} \int_\Theta |D^\alpha v(x)|^2 dx \tag{2.15}$$

where $\alpha = (\alpha_1, \alpha_2)$, α_j are nonnegative integer, $|\alpha| = \alpha_1 + \alpha_2$, $D^\alpha v = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$.

We set

$$X_\sigma(\Pi) = \left\{ w(x) \in V^0(\Pi) : \quad \int_\Pi v(x) e^{ix\cdot\xi} dx = 0 \quad \text{if } |\xi| < \sqrt{\sigma} \right\}. \tag{2.16}$$

As always

$$\|\cdot\|_{X_\sigma(\Pi)} = \|\cdot\|_{V^0(\Pi)} = \|\cdot\|_{(L_2(\Pi))^2}.$$

Below we prove the following assertion:

Theorem 2.1. *For each $\sigma > 0$ there exists a continuous extension operator*

$$\mathrm{Ext} : V^0(\Omega) \longrightarrow X_\sigma(\Pi) \tag{2.17}$$

(*i.e.* $\mathrm{Ext}v(x) \equiv v(x)$ *for* $x \in \Omega$).

Theorem 2.1 implies immediately the following method of solution for stabilization problem (2.1)–(2.5):

Theorem 2.2. *Let $v_0 \in V^0(\Omega)$. For each $\sigma > 0$ the solution $(v(t,x), u(t,x))$ of stabilization problem* (2.1)–(2.5) *can be obtained by the formula*

$$(v(t,\cdot), u(t,\cdot)) = (\gamma_\Omega w(t,\cdot), \gamma_{\partial\Omega} w(t,\cdot)) \tag{2.18}$$

where γ_Ω and $\gamma_{\partial\Omega}$ are restriction operators on Ω and on $\partial\Omega$ correspondingly, $w(t,x)$ is the solution of boundary value problem (2.8)–(2.10) *with initial condition $w_0(x) = \mathrm{Ext}v_0(x)$. Here* Ext *is extension operator* (2.17) *and $v_0(x)$ is the initial condition from* (2.3).

Proof. In virtue of (2.17) vector field $w_0(x) = \mathrm{Ext}v_0(x)$ satisfies (2.13). Therefore (2.18), (2.17) imply:

$$\|v(t,\cdot)\|_{L_2(\Omega)} \leqslant \|w(t,\cdot)\|_{L_2(\Pi)} \leqslant ce^{-\sigma t}. \qquad \square$$

Definition 2.1. We say that control u from problem (2.1)–(2.5) satisfies feedback property if the solution (y, u) of this problem is defined by formula (2.18) where $w(t,x)$ is the solution of a certain artificial boundary value problem.

From the physical point of view control satisfies feedback property if it can react on unpredictable fluctuations of a system. In Section 2.4 we will show that control defined by (2.18) can react on such fluctuations. But firstly we will prove Theorem 2.1.

2.3. Proof of Theorem 2.1. Let $v(x) \in V^0(\Omega)$. Since Ω is a bounded domain with smooth boundary and $\operatorname{div} v = 0$, we have $v\big|_{\partial\Omega}(s)\nu(s) \in H^{-\frac{1}{2}}(\partial\Omega)$ (see [17]) and

$$\int_{\partial\Omega} v(s)\cdot\nu(s) ds = 0, \tag{2.19}$$

where $\nu(s)$ is the vector field of normals to $\partial\Omega$.

Introduce the stream function $F(x)$ by identities

$$\partial_{x_2} F = v_1, \quad -\partial_{x_1} F = v_2. \tag{2.20}$$

Recall that

$$H_0^1(\Omega) = \{\varphi(x) \in H^1(\Omega) : \quad \varphi|_{\partial\Omega} = 0\},$$

$$H_{-1}(\Omega) = \text{completion in } L_2(\Omega) \text{ by } \|f\|_{H^{-1}} = \sup_{\varphi \in H_0^1, \varphi \neq 0} \left(\int_\Omega f\varphi dx / \|\varphi\|_{H_0^1} \right).$$

In virtue of (2.19), (2.20)

$$\int\limits_{\partial\Omega} \nabla F|_{\partial\Omega}(s) \cdot \tau(s)\, ds = 0$$

where vector $\tau(s)$ is tangential to $\partial\Omega$, and therefore we can set

$$F|_{\partial\Omega} = \int\limits_0^s \nabla F|_{\partial\Omega}(s)(\tau)(s) d\zeta \in H^{\frac{1}{2}}(\partial\Omega) \tag{2.21}$$

(recall that $\partial\Omega$ is connected set). Relations (2.20) imply:

$$-\Delta F(x) = \mathrm{rot} v \equiv \partial_{x_1} v_2 - \partial_{x_2} v_1. \tag{2.22}$$

Evidently, $\mathrm{rot} v \in H^{-1}(\Omega)$. That is why solution F of (2.22), (2.21) exists and belongs to $H^1(\Omega)$.

Extend $F(x)$ from Ω to torus Π by the formula

$$RF(x) = (1 - \varphi(x))R_0 F(x) + \varphi(x) z(x) \tag{2.23}$$

where $R_0 : H^1(\Omega) \to H^1(\Pi)$ is an extension operator (i.e. $R_0 F(x) \equiv F(x)$, $x \in \Omega$), $\varphi(x) \in C^\infty(\Pi), \varphi(x) = 0$ for $x \in \Omega_\varepsilon \equiv \{x \in \Pi : \inf_{y\in\Omega} |x - y| < \varepsilon\}$ and $\varphi(x) = 1$ for $x \in \Pi \setminus \Omega_{2\varepsilon}$. The function $z(x)$ from (2.23) is defined by the formula

$$z(x) = \sum_{\xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2,\ |\xi| < \sqrt{\sigma}} c_\xi e^{i\xi\cdot x} \tag{2.24}$$

where c_ξ are coefficients, which are determined from the system of equations

$$\sum_{\xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2, |\xi| < \sqrt{\sigma}} a_{k\xi} c_\xi = -\int\limits_\Omega (1 - \varphi(x)) R_0 F(x) e^{-ik\cdot x} dx, \quad k \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2, \quad |k| < \sqrt{\sigma}. \tag{2.25}$$

Here

$$a_{k\xi} = \int\limits_\Pi \varphi(x) e^{ix\cdot(\xi - k)} dx.$$

Note that the matrix $A = \|a_{k\xi}\|$ of system (2.25) is positively defined. Indeed, let

$$g(x) = \sum_{\xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2, |\xi| < \sqrt{\sigma}} \alpha_\xi e^{ix\cdot\xi}.$$

Then

$$\sum_{\xi k} a_{\xi k}\alpha_\xi \overline{\alpha}_k = \int\limits_\Pi \varphi(x)|g(x)|^2 dx > 0,$$

if vector $\{\alpha_\xi,\ \xi \in \left(\frac{\pi}{L}\mathbb{Z}\right)^2, |\xi| < \sqrt{\sigma}\} \neq 0$. Hence system (2.25) has a unique solution $\{c_\xi\}$ and therefore extension (2.23) is well defined.

Relations (2.23)–(2.25) imply that

$$\int\limits_\Pi RF(x)e^{-ik\cdot x}dx = 0 \quad \forall k: \quad |k| < \sqrt{\sigma}. \tag{2.26}$$

We denote extension operator for $v_0(x),\ x \in \Omega$ as follows:

$$\mathrm{Ext}v_0(x) = \begin{cases} v_0(x), & x \in \Omega \\ \mathrm{rot}RF, & x \in \Pi \setminus \Omega \end{cases} \tag{2.27}$$

where, recall, $\mathrm{rot}RF = (\partial_{x_2}RF, -\partial_{x_1}RF)$. Since $RF \in H^1(\Pi)$, $\mathrm{rot}RF \in (L_2(\Omega))^2$ and, evidently, $\mathrm{div}\,\mathrm{rot}RF = 0$. Hence $\mathrm{rot}RF \in V^0(\Pi)$ and in virtue of definition (2.23)–(2.25), (2.27) of extension operator we get:

$$\|\mathrm{Ext}v_0(x)\|_{V^0(\Pi)} \leqslant c\|v_0\|_{V^0(\Omega)}. \tag{2.28}$$

Using (2.26) we have after integration by parts

$$\int\limits_\Pi \mathrm{rot}RF(x)e^{-ik\cdot x}dx = \left(ik_2\int\limits_\Pi RF(x)e^{-ik\cdot x}dx, -ik_1\int\limits_\Pi RF(x)e^{-ik\cdot x}dx\right) = 0, \quad |k| < \sqrt{\sigma}. \tag{2.29}$$

By (2.29) $\mathrm{Ext}v_0 \in X_\sigma(\Pi)$ and by (2.28) operator (2.17) is bounded. □

2.3. Feedback property

We show that method of stabilization proposed below can react on unpredictable fluctuations of a system. The point is that if the solution $v(t,x)$ of problem (2.1)–(2.4) satisfies inequality (2.5) and if at time moment $\tilde{t}_0$ the system (2.1)–(2.3) is subjected by certain fluctuation, then $v(t,x)$ at $t = \tilde{t}_0$ is pushed out X_σ and that is why it will not tend to zero with prescribed rate. Therefore we check when

$$\|v(t,\cdot) - \gamma_\Omega w(t,\cdot)\|_{L_2(\Omega)} \geqslant ce^{-\sigma t/2} \tag{2.30}$$

(here $w(t,x)$ is the solution of (2.8)–(2.10) with $w_0(x) = \mathrm{Ext}v_0(x)$) and at this moment, say t_1, we regard $v(t_1,x)$ as initial condition and for $t > t_0$ we construct the solution (v,u) of (2.1)–(2.4) by formula (2.18) where $w(t,x)$ is the solution of (2.8), (2.9) with initial condition

$$w|_{t=t_1} = \mathrm{Ext}v(t_1,\cdot). \tag{2.31}$$

This construction can be written briefly as impulse control for (2.8), concentrated in the artificial part $\Pi \setminus \Omega$ of domain:

$$\partial_t w(t,x) - \Delta w + \nabla q(t,x) = \delta(t-t_1)(\mathrm{Ext}v(t_1,x) - \tilde{w}(t_1,x)) \tag{2.32}$$

where $\tilde{w}(t_1,x) = v(t_1,x)$, for $x \in \Omega$ and $\tilde{w}(t,x) = w(t,x)$ for $x \in \Pi \setminus \Omega$. After next unpredictable pushing out $X_\sigma(\Pi)$ of the solution $v(t,x)$ we do the same. Thus all this process can be written by formula:

$$\partial_t w(t,x) - \Delta w + \nabla q(t,x) = \sum_{i=1}^{\infty} \delta(t-t_i)(\mathrm{Ext}v(t_i,x) - \tilde{w}(t_i,x)) \tag{2.33}$$

where t_i are moments when (2.30) became true.

Remark 2.1. Note that above we indicate only some possibility to organize reaction of control on unpredictable fluctuations. To realize this possibility the special investigations should be made. They will be made in some other place. Note that a certain previous result to this respect is obtained in [6].

3. Oseen equations

3.1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with C^∞-boundary $\partial\Omega$, $Q = \mathbb{R}_+ \times \Omega$. In space-time cylinder Q we consider the Oseen equations

$$\partial_t v(t,x) - \Delta v + (a(x),\nabla)v + (v,\nabla)a + \nabla p(t,x) = 0 \tag{3.1}$$

$$\operatorname{div} v(t,x) = 0 \tag{3.2}$$

with initial condition

$$v(t,x)|_{t=0} = v_0(x) \qquad (3.3).$$

Here $(t,x) = (t,x_1,x_2) \in Q$, $v(t,x) = (v_1,v_2)$, is a velocity vector field, $\nabla p = (\partial_{x_1}, \partial_{x_2})$ is pressure gradient, initial velocity $v_0(x) = (v_{01}(x), v_{02}(x))$ satisfies condition $\operatorname{div} v_0 = 0, a(x) = (a_1(x), a_2(x))$ is a solenoidal vector field ($\operatorname{div} a = 0$).

We suppose that the boundary $\partial\Omega$ of Ω is decomposed on two parts:

$$\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_0, \quad \Gamma \neq \emptyset \tag{3.4}$$

where Γ, Γ_0 are open sets (in topology of $\partial\Omega$). Here the line above means the closure of a set. The case $\Gamma_0 = \emptyset$ is also possible. We define $\Sigma = \mathbb{R}_+ \times \Gamma, \Sigma_0 =$

$\mathbb{R}_+ \times \Gamma_0$, and we set the following boundary conditions:

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u \tag{3.5}$$

where u is a control, concentrated on Σ.

Let a magnitude $\sigma > 0$ be given. The problem of stabilization with the decay rate σ of a solution to problem (3.1)–(3.3), (3.5) is to construct a control u on Σ such that the solution $v(t,x)$ of boundary value problem (3.1)–(3.3), (3.5) satisfies the inequality

$$\|v(t,x)\|_{L_2(\Omega)} \leqslant ce^{-\sigma t} \tag{3.6}$$

where $c > 0$ depends on v_0, σ and Γ_0. Moreover, we require that this control u satisfies the feedback property in the meaning analogous to (2.18).

Let us give the exact formulation of this feedback property. Let $\omega \subset \mathbb{R}^2$ be a bounded domain such that

$$\Omega \cap \omega = \emptyset, \quad \overline{\Omega} \cap \overline{\omega} = \overline{\Gamma}. \tag{3.7}$$

We set

$$G = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega}) \tag{3.8}$$

(the denotation $\mathrm{Int}\,A$ means, as always, the interior of the set A). We suppose that ∂G is a curve belonging to the smoothness class C^α and in all points except the set $\overline{\Gamma} \setminus \Gamma \equiv \partial\Gamma$ it possesses the C^∞ smoothness. The exact conditions on magnitude α will be imposed below in an appropriate place.

We extend problem (3.1)–(3.3) from Ω to G. Let us assume that the given vector field $a(x)$ from (3.1) actually is defined on G. Moreover, we suppose that

$$a(x) \in V^2(G) \cap \left(H_0^1(G)\right)^2. \tag{3.9}$$

That is why the extension of (3.1)–(3.3) from Ω to G can be written as follows:

$$\partial_t w(t,x) - \Delta w + (a(x), \nabla)w + (w, \nabla)a + \nabla p(t,x) = 0 \tag{3.10}$$

$$\mathrm{div}\, w(t,x) = 0 \tag{3.11}$$

$$w(t,x)|_{t=0} = w_0(x). \tag{3.12}$$

Besides, we impose on w the zero Dirichlet boundary condition

$$w|_S = 0, \tag{3.13}$$

where $S = \mathbb{R}_+ \times \partial G$. Note that, actually, w_0 from (3.12) will be some special extension of $v_0(x)$ from (3.3).

For vector fields defined on G we introduce the operator γ_Ω of restriction on Ω and the operator γ_Γ of restriction on Γ:

$$\gamma_\Omega : V^k(G) \longrightarrow V^k(\Omega),\ k \geqslant 0; \quad \gamma_\Gamma : V^k(G) \longrightarrow V^{k-1/2}(\Gamma),\ k > 1/2. \tag{3.14}$$

Evidently, operators (3.14) are bounded.

Definition 3.1. A control $u(t,x)$ in (3.1)–(3.3), (3.5) is called the feedback if for each $t>0$ the following relations are true:

$$v(t,\cdot)=\gamma_\Omega w(t,\cdot),\quad u(t,\cdot)=\gamma_\Gamma w(t,\cdot) \tag{3.15}$$

where $(v(t,\cdot),u(t,\cdot))$ is the solution of stabilization problem (3.1)–(3.3), (3.5) and $w(t,\cdot)$ is the solution of boundary value problem (3.10)–(3.13).

Below we prove that for given $\sigma>0, v_0$ the problem (3.1)–(3.3), (3.5) can be stabilized with help of feedback control in the meaning of Definition 3.1.

3.2. Preliminaries

Let G be domain (3.8) and

$$V_0^0(G)=\{v(x)\in V^0(G): v\cdot\nu|_{\partial\Omega}=0\} \tag{3.16}$$

where $V^0(G)$ is space (2.14) with $\Theta=G$, $\nu(x)$ is the vector field of outer normals to ∂G; for $v\in V^0(G)$ the function $v\cdot\nu|_{\partial\Omega}$ is well-defined in the Sobolev space $H^{-1/2}(\partial G)$ (see [17]). Denote by

$$\pi:(L_2(G))^2\longrightarrow V_0^0(G) \tag{3.17}$$

the operator of orthogonal projection. We consider the Oseen steady-state operator

$$Av\equiv-\pi\Delta v+\pi[(a(x),\nabla)v+(v,\nabla)a]: V_0^0(G)\longrightarrow V_0^0(G) \tag{3.18}$$

where $a(x)$ is vector field (3.9). This operator is closed and its domain is defined as follows:

$$\mathcal{D}(A)=V^2(G)\cap(H_0^1(G))^2 \tag{3.19}$$

and it is dense in $V_0^0(G)$.

Assuming that spaces in (3.17), (3.19) are complex we denote by $\rho(A)$ the resolvent set of operator A, i.e. the set of $\lambda\in\mathbb{C}$ such that the resolvent operator

$$R(\lambda,A)\equiv(\lambda I-A)^{-1}: V_0^0(G)\longrightarrow V_0^0(G) \tag{3.20}$$

is defined and continuous. Here I is identity operator.

Recall that a closed operator $B:X\to X$ (X is a Banach space) is called a sectorial if there exist a magnitude $\varphi\in(0,\pi/2), M\geqslant 1, a\in\mathbb{R}$ such that

$$S_{a,\varphi}=\{\lambda\in\mathbb{C}:\quad \varphi\leqslant|\arg(\lambda-a)|\leqslant\pi,\quad \lambda\neq a\}\subset\rho(A) \tag{3.21}$$

and

$$\|(\lambda I-A)^{-1}\|\leqslant M/|\lambda-a|,\quad \forall\lambda\in S_{a,\varphi}. \tag{3.22}$$

Denote by $\Sigma(A)\equiv\mathbb{C}^1\setminus\rho(A)$ the spectrum of operator A.

Lemma 3.1. *Oseen operator* (3.18) *with $a(x)$ satisfying* (3.9) *is a sectorial operator. For $\lambda\in\rho(A)$ resolvent* (3.20) *is a compact operator. That is why the spectrum $\Sigma(A)$ consists of a discrete set of points.*

The proof of this lemma can be obtained by well-known methods: compactness of resolvent and estimate (3.22) can be established by analogous estimates to that were obtained below, in Lemma's 4.7 proof. All other assertions of the Lemma follows from results of [10] and [20].

Let $\lambda_0 \in \Sigma(A)$. We decompose the resolvent $R(\lambda, A)$ in a neighbourhood of λ_0 in a Laurent series:

$$R(\lambda, A) = \sum_{k=-m}^{\infty} (\lambda - \lambda_0)^k R_k, \quad R_k = (2\pi i)^{-1} \int\limits_{|\lambda-\lambda_0|=\varepsilon} (\lambda - \lambda_0)^{-k-1} R(\lambda, A) d\lambda. \tag{3.23}$$

Lemma 3.2. ([20, Ch YII Sect.8]).

a) *All operators R_k from* (3.23) *commutate among themselves and with A;*

b) *R_{-1} is a projector, i.e. $R_{-1}^2 = R_{-1}$.*

c) *The following formulas hold:*

$$R_{-(k+1)} + (\lambda_0 I - A) R_{-k} = 0 \quad \textit{for} \quad k \geqslant 1 \tag{3.24}$$

$$R_{-1} + (\lambda_0 I - A) R_0 = E. \tag{3.25}$$

Lemma 3.3.

a) *The magnitude m from decomposition* (3.23) *is finite;*

b) *The images of each operator R_k with $k < 0$ from* (3.23) *are finite dimensional:* $\dim \mathrm{Im} R_k < \infty$ *for* $k < 0$.

Moreover

$$\mathrm{Im} R_{-1} = \ker(\lambda_0 E - A)^m, \quad R_{-k-1} = (\lambda_0 E - A)^k R_{-1}, \quad k = 1, 2, \ldots, m-1. \tag{3.26}$$

This lemma is derived in [6] from well-known results of [4], [20].

Let us consider the adjoint operator A^* to Oseen operator (3.18):

$$A^* w \equiv -\pi \Delta w - \pi[(a(x), \nabla) w - (\nabla a)^* w] : V_0^0(G) \longrightarrow V_0^0(G) \tag{3.27}$$

where

$$(\nabla a)^* w = ((\partial_1 a, w), (\partial_2 a, w)), \quad (\partial_i a, w) = \sum_{j=1}^{2} \partial_i a_j w_j. \tag{3.28}$$

Evidently, A^* is a closed operator with domain coinciding to $\mathcal{D}(A)$

$$\mathcal{D}(A^*) = \mathcal{D}(A) = V^2(G) \cap (H_0^1(G))^2.$$

Moreover

$$\rho(A^*) = \overline{\rho(A)} \quad \text{and} \quad R(\lambda, A)^* = R(\bar{\lambda}, A^*) \quad \forall\, \lambda \in \rho(A). \tag{3.29}$$

Below we always assume that

$$\text{vector field } a(x) \text{ from (3.9), (3.18), (3.27) is real valued.} \tag{3.30}$$

That is why we have

$$\rho(A) = \overline{\rho}(A) = \rho(A^*) = \overline{\rho}(A^*). \tag{3.31}$$

Since B and B^* are compact operators simultaneously, (3.29), (3.31) imply that A^* is a sectorial operator with a compact resolvent and $\Sigma(A) = \overline{\Sigma}(A) = \Sigma(A^*) = \overline{\Sigma}(A^*)$. Hence if $\lambda_0 \in \Sigma(A)$ then $\overline{\lambda}_0 \in \Sigma(A^*)$. Decomposition $R(\lambda, A^*)$ in Laurent series around $\overline{\lambda}_0$ yields:

$$R(\lambda, A^*) = \sum_{k=-m}^{\infty} (\lambda - \bar{\lambda}_0)^k R_k^*, \tag{3.32}$$

where, evidently poles orders in (3.32) and in (3.23) coincides and R_k^* from (3.32) are adjoint operators to the corresponding operators R_k from (3.23). Therefore by Lemma 3.2 $(R_{-1}^*)^2 = R_{-1}^*$,

$$R_{-(k+1)}^* + (\bar{\lambda}_0 I - A^*)R_{-k}^* = 0 \quad \text{for } k \geqslant 1 \tag{3.33}$$

$$R_{-1}^* + (\bar{\lambda}_0 I - A^*)R_0^* = I. \tag{3.34}$$

Moreover, Lemma 3.3 implies that $\dim \mathrm{Im} R_k^* = \dim \mathrm{Im} R_k$, $k < 0$ and

$$\mathrm{Im} R_{-1}^* = \ker(\bar{\lambda}_0 E - A^*)^m, \quad R_{-k-1}^* = (\bar{\lambda}_0 E - A^*)^k R_{-1}^*, \quad k = 1, 2, \ldots, m-1. \tag{3.35}$$

3.3. Structure of R_k, R_k^* with $k < 0$

Recall some definitions. If $\ker(\lambda_0 I - A) \neq 0$ then $\lambda_0 \in \mathbb{C}^1$ is called eigenvalue of A, and a vector $e \neq 0$ which belongs to $\ker(\lambda_0 E - A)$ is called eigenvector. Vector e_k is called associated vector of order k to an eigenvector e if e_k can be obtained after solving the chain of equations:

$$(\lambda_0 I - A)e = 0, \quad e + (\lambda_0 I - A)e_1 = 0, \ldots, \quad e_{k-1} + (\lambda_0 I - A)e_k = 0. \tag{3.36}$$

We say that $e, e_1, e_2, \ldots$ form a chain of associated vectors. If the maximal order of vectors, associated to e equals m then the number $m+1$ is called multiplicity of the eigenvector e.

Definition 3.2. The set of eigenvectors and associated vectors

$$e^{(k)}, e_1^{(k)}, \ldots, e_{m_k}^{(k)} \quad (k = 1, 2, \ldots, N) \tag{3.37}$$

corresponding to an eigenvalue λ_0 is called canonical system which corresponds to the eigenvalue λ_0 if set (3.37) satisfies the properties:

i) Vectors $e^{(k)}, k = 1, 2, \ldots, N$ form a basis in the space of eigenvectors corresponding to the eigenvalue λ_0.
ii) $e^{(1)}$ is an eigenvector with maximal possible multiplicity.
iii) $e^{(k)}$ is an eigenvector which can not be expressed by a linear combination of $e^{(1)}, \ldots, e^{(k-1)}$ and multiplicity of $e^{(k)}$ achieves a possible maximum.

iv) Vectors (3.37) with fixed k form a maximal chain of associated elements. Evidently, numbers $m_1, m_2, \dots, m_N$ do not depend on a choice of canonical system.

The number $N(\lambda_0) = m_1 + 1 + m_2 + 1 + \cdots + m_N + 1$ is called multiplicity of the eigenvalue λ_0.

Besides canonical system (3.37) which corresponds to an eigenvalue λ_0 of operator A we consider a canonical system

$$\varepsilon^{(k)}, \varepsilon_1^{(k)}, \dots, \varepsilon_{m_k}^{(k)} \quad (k = 1, 2, \dots, N) \tag{3.38}$$

that corresponds to the eigenvalue $\overline{\lambda_0}$ of the adjoint operator A^*. Definition of canonical system (3.38) is absolutely analogous to Definition 3.2 of canonical system (3.37). Particularly the set (3.38) with fixed k satisfies the relations which are analogous to (3.36):

$$(\bar{\lambda_0} I - A^*)\varepsilon^{(k)} = 0, \varepsilon^{(k)} + (\bar{\lambda_0} I - A^*)\varepsilon_1^{(k)} = 0, \dots, \varepsilon_{m_k-1}^{(k)} + (\bar{\lambda_0} I - A^*)\varepsilon_{m_k}^{(k)} = 0. \tag{3.39}$$

The assertion which we formulate below is very close to well-known assertion from [10], although their formulations are differ. That is why we give also a proof. Let H be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. For $u, v^* \in H$ denote by uv^* the operator $Bx = \langle x, v^* \rangle u$. Then $v^* u$ is adjoint operator $B^* g = \langle u, g \rangle v^*$.

Theorem 3.1. *Let $A : H \to H$ be a sectorial operator with a compact resolvent and the operator $A^* : H \to H$ is adjoint to A. Suppose that $\lambda_0 \in \Sigma(A)$, $\overline{\lambda_0} \in \Sigma(A^*)$ and therefore $\lambda_0, \overline{\lambda_0}$ are the poles of $R(\lambda, A)$, $R(\lambda, A^*)$ correspondingly which have the equal multiplicity m. Then*

i) *λ_0 is an eigenvalue of A and $\overline{\lambda_0}$ is an eigenvalue of A^* and both of them have eigenvectors of maximal multiplicity m.*

ii) *Let set (3.38) be an arbitrary canonical system of A^* corresponding to the eigenvalue $\overline{\lambda_0}$. Then (3.38) defines by a unique way a canonical system (3.37) of A corresponding to λ_0 such that the main part of Laurent series (3.23) has the form*

$$\sum_k \left\{ \frac{e^{(k)}\varepsilon^{(k)}}{(\lambda - \lambda_0)^{m_k+1}} + \frac{e^{(k)}\varepsilon_1^{(k)} + e_1^{(k)}\varepsilon^{(k)}}{(\lambda - \lambda_0)^{m_k}} + \dots . + \right.$$

$$\left. + \frac{e^{(k)}\varepsilon_{m_k}^{(k)} + e_1^{(k)}\varepsilon_{m_k-1}^{(k)} + \dots .. + e_{m_k}^{(k)}\varepsilon^{(k)}}{(\lambda - \lambda_0)} \right\}. \tag{3.40}$$

Proof. As was mentioned, in virtue of (3.29) operators R_k^* from (3.32) are adjoint to corresponding operators R_k from (3.23), and by Lemma 3.3 $\dim \mathrm{Im} R_k =$

$\dim \mathrm{Im} R_k^* < \infty$ if $k < 0$. That is why

$$R_k = \sum_{j=1}^{L} f_j g_j^* \quad \text{if and only if} \quad R_k^* = \sum_{j=1}^{L} g_j^* f_j \tag{3.41}$$

where $f_j \in H$, $g_j^* \in H$, $j = 1, \ldots, L$ are certain vectors. The formula (3.40) for the main part of Laurent series (3.23) will be proved simultaneously with the following formula for the main part of (3.32):

$$\sum_k \left\{ \frac{\varepsilon^{(k)} e^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^{m_k+1}} + \frac{\varepsilon_1^{(k)} e^{(k)} + \varepsilon^{(k)} e_1^{(k)}}{((\overline{\lambda} - \overline{\lambda_0})^{m_k}} + \ldots . + \right.$$
$$\left. + \frac{\varepsilon_{(m_k)}^{(k)} e^{(k)} + \varepsilon_{(m_k-1)}^{(k)} e_1^{(k)} + \ldots .. + \varepsilon^{(k)} e_{(m_k)}^{(k)}}{((\overline{\lambda} - \overline{\lambda_0})} \right\}. \tag{3.42}$$

Let $e, e_1, \ldots, e_h$ be a chain of associated vectors for A corresponding to eigenvalue λ_0. Then for each λ from a neighbourhood of λ_0 the equality holds:

$$\frac{e}{(\lambda - \lambda_0)^{h+1}} + \frac{e_1}{(\lambda - \lambda_0)^h} + \cdots + \frac{e_h}{(\lambda - \lambda_0)} = R(\lambda, A) e_h. \tag{3.43}$$

To prove (3.43) one can apply to both parts of (3.43) operator $(\lambda I - A) = ((\lambda - \lambda_0) I + \lambda_0 I - A)$ and use (3.36). Hence multiplicity m of pole (3.23) is not less than maximal possible multiplicity of a eigenvector corresponding to λ_0. If m is the multiplicity of pole (3.23) then $R_{-(m+1)} = 0$ and by (3.33) with $k = m$ we have $\mathrm{Im} R_m \subset \mathrm{Ker}(\lambda_0 I - A)$. Therefore the multiplicity m of pole (3.23) equals to maximal possible multiplicity of eigenvectors for A corresponding λ_0. Analogously, if $\varepsilon, \varepsilon_1, \ldots, \varepsilon_h$ be a chain of associated vectors for A^* then

$$\frac{\varepsilon}{(\overline{\lambda} - \overline{\lambda_0})^{h+1}} + \frac{\varepsilon_1}{(\overline{\lambda} - \overline{\lambda_0})^h} + \cdots + \frac{\varepsilon_h}{(\overline{\lambda} - \overline{\lambda_0})} = R(\overline{\lambda}, A^*) \varepsilon_h \tag{3.44}$$

and the multiplicity m of pole (3.32) is equal to maximal possible multiplicity of eigenvectors for A^* corresponding $\overline{\lambda_0}$.

Let $\varepsilon^{(k)}, k = 1, \ldots, j_1$ be a given maximal set of linear independent eigenvectors of maximal multiplicity m for A^*. Then by (3.44) with $\varepsilon_h = \varepsilon_{m-1}^{(k)}$ for each $x \in H$

$$R_{-m}^* x = \sum_{k=1}^{j_1} < e^{(k)}, x > \varepsilon^{(k)} \tag{3.45}$$

where, evidently, vectors $e^{(k)}$ are defined by $\varepsilon^{(k)}$ by unique way. Relations (3.41) implies that

$$R_{-m} y = \sum_{k=1}^{j_1} < y, \varepsilon^{(k)} > e^{(k)} \tag{3.46}$$

and in virtue of (3.43) $e^{(k)}, k = 1, \ldots, j_1$ are eigenvectors for A of maximal possible multiplicity m. Since $\dim \mathrm{Im} R_{-m} = \dim \mathrm{Im} R_{-m}^*$, $e^{(k)}$ are linear independent and there is no any other linear independent to $e^{(k)}, k = 1, \dots, j_1$ eigenvectors for A of multiplicity m.

By (3.43) with $e_h = \sum_{k=1}^{j_1} < y, \varepsilon^{(k)} > e_{m-1}^{(k)}$ we see that the main part of (3.23) contains

$$\sum_{k=1}^{j_1} \left(\frac{< y, \varepsilon^{(k)} > e^{(k)}}{(\lambda - \lambda_0)^m} + \frac{< y, \varepsilon^{(k)} > e_1^{(k)}}{(\lambda - \lambda_0)^{m-1}} + \dots + \frac{< y, \varepsilon^{(k)} > e_{m-1}^{(k)}}{(\lambda - \lambda_0)} \right)$$

where $e_j^{(k)}$ are vectors associated to eigenvectors $e^{(k)}$. Analogously by (3.44) with $\varepsilon_h = \sum_{k=1}^{j_1} < e^{(k)}, x > \varepsilon_{m-1}^{(k)}$ the main part of (3.32) contains:

$$\sum_{k=1}^{j_1} \left(\frac{< e^{(k)}, x > \varepsilon^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^m} + \frac{< e^{(k)}, x > \varepsilon_1^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^{m-1}} + \dots + \frac{< e^{(k)}, x > \varepsilon_{m-1}^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})} \right).$$

These two expressions and relation (3.41) imply that the main parts of poles (3.23) and (3.32) contain correspondingly:

$$A \to \sum_{k=1}^{j_1} \left(\frac{e^{(k)} \varepsilon^{(k)}}{(\lambda - \lambda_0)^m} + \frac{e_1^{(k)} \varepsilon^{(k)} + e^{(k)} \varepsilon_1^{(k)}}{(\lambda - \lambda_0)^{m-1}} + \dots + \frac{e_{m-1}^{(k)} \varepsilon^{(k)} + e^{(k)} \varepsilon_{m-1}^{(k)}}{(\lambda - \lambda_0)} \right) \quad (3.47)$$

$$A^* \to \sum_{k=1}^{j_1} \left(\frac{\varepsilon^{(k)} e^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^m} + \frac{\varepsilon_1^{(k)} e^{(k)} + \varepsilon^{(k)} e_1^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^{m-1}} + \dots + \frac{\varepsilon_{m-1}^{(k)} e^{(k)} + \varepsilon^{(k)} e_{m-1}^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})} \right). \quad (3.48)$$

Now we continue this process: we supplement the sum in (3.47) using (3.43) with $e_h = \sum_{k=1}^{j_1} < y, \varepsilon_l^{(k)} > e_{m-1-l}^{(k)}, h = m - 1 - l$ for $l = 1, 2, \dots, m-1$. After that we add terms in (3.48) using new (added) sum in (3.47) and relation (3.41). As a result we obtain that the main part of (3.23) contains terms

$$\sum_{k=1}^{j_1} \left\{ \frac{e^{(k)} \varepsilon^{(k)}}{(\lambda - \lambda_0)^m} + \frac{e_1^{(k)} \varepsilon^{(k)} + e^{(k)} \varepsilon_1^{(k)}}{(\lambda - \lambda_0)^{m-1}} + \frac{e_2^{(k)} \varepsilon^{(k)} + e_1^{(k)} \varepsilon_1^{(k)} + e^{(k)} \varepsilon_2^{(k)}}{(\lambda - \lambda_0)^{m-2}} + \dots . \right.$$
$$\left. + \frac{e_{m-1}^{(k)} \varepsilon^{(k)} + e_{m-2}^{(k)} \varepsilon_1^{(k)} + \dots + e^{(k)} \varepsilon_{m-1}^{(k)}}{(\lambda - \lambda_0)} \right\}$$

and the main part of (3.32) contains terms

$$A_1 = \sum_{k=1}^{j_1} \left(\frac{\varepsilon^{(k)} e^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^m} + \frac{\varepsilon_1^{(k)} e^{(k)} + \varepsilon^{(k)} e_1^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})^{m-1}} + \dots + \dots \right.$$
$$\left. + \frac{\varepsilon_{m-1}^{(k)} e^{(k)} + \varepsilon_{m-2}^{(k)} e_1^{(k)} + \dots + \varepsilon_1^{(k)} e_{m-2}^{(k)} + \varepsilon^{(k)} e_{m-1}^{(k)}}{(\overline{\lambda} - \overline{\lambda_0})} \right). \quad (3.49)$$

It is well-known that canonical system (3.38) for A^* corresponding to eigenvalue $\bar{\lambda}_0$ is a set of linear independent vectors.[2] That is why in virtue of (3.44) sum (3.49)

[2] Amplification of this result is proved in [6] and in Lemma 3.7 below.

does not contain all main part of (3.32) but contains the operator $R^*_{-m}/(\overline{\lambda}-\overline{\lambda_0})^m$. Therefore

$$\sum_{k=-m}^{-1} \frac{R_k^*}{(\overline{\lambda}-\overline{\lambda_0})^{-k}} - A_1 \tag{3.50}$$

is a pole which order is less than m. Applying to (3.50) arguments we used above: we take the eigenvalues $\varepsilon^{(k)}, k = j_1+1, \dots, j_r$ for A^* from (3.38) which multiplicity is maximal possible after excluding $\varepsilon^{(k)}$ with $k = 1, 2, \dots, j_1$. For $\varepsilon^{(k)}, k = j_1+1, \dots, j_2$ we repeat arguments written from (3.45) to (3.49). As a result we obtain a sum of terms A_2 (analogous to (3.49)) such that the order of pole

$$\sum_{k=-m}^{-1} \frac{R_k^*}{(\overline{\lambda}-\overline{\lambda_0})^{-k}} - A_1 - A_2$$

will be less than the order of pole (3.50). Repeating this process several times we complete the proof of Theorem 2.1. □

Corollary 3.1. *Let A be a sectorial operator with compact resolvent $R(\lambda, A)$ decomposed at $\lambda_0 \in \Sigma(A)$ in Laurent series* (3.23)*. Let* (3.38) *be a canonical system of A^* corresponding to the eigenvalue $\overline{\lambda_0}$, which we denote by $E(\overline{\lambda_0})$. Then*

$$R_{-k}x = 0, \quad \forall k = 1, 2, \dots, m \tag{3.51}$$

if and only if

$$\langle x, \varepsilon_j^{(k)} \rangle = 0 \qquad \forall \varepsilon_j^{(k)} \in E(\overline{\lambda_0}). \tag{3.52}$$

This assertion follows immediately from representation (3.40) for the main part of Laurent series (3.23) for $R(\lambda, A)$.

3.4. Holomorphic semigroups

We consider boundary value problem (3.10)–(3.13) for Oseen equations written in the form

$$\frac{dw(t,\cdot)}{dt} + Aw(t,\cdot) = 0, \quad w|_{t=0} = w_0 \tag{3.53}$$

where A is operator (3.18). In virtue of Lemma 3.1. Oseen operator is sectorial with compact resolvent. It is known ([8, Sect. 1.3]) that the following assertion holds.

Theorem 3.2. *For each $w_0 \in V_0^0(G)$ the solution $w(t,\cdot)$ of problem* (3.53) *is defined by $w(t,\cdot) = e^{-At}w_0$ where e^{-At} is the holomorphic semigroup that is de-*

termined by the formula

$$e^{-At} = (2\pi i)^{-1} \int_\gamma (\lambda I + A)^{-1} e^{\lambda t} d\lambda, \tag{3.54}$$

where γ is a contour belonging to $\rho(A)$ such that $\arg\lambda = \pm\theta$ *for* $\lambda \in \gamma, |\lambda| \geqslant N$ *for certain* $\theta \in (\pi/2, \pi)$ *and for sufficiently large* N*. Moreover,* γ *surrounds* $\Sigma(A)$ *from the right.*

Such contour γ exists, of course, because we can choose γ belonging to set $-S_{a,\varphi}$ from definition of sectorial operator (see (3.21)).

Let $-\sigma$ be a negative number satisfying

$$\Sigma(-A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = -\sigma\} = \emptyset. \tag{3.55}$$

The case when there are certain points of $\Sigma(-A)$ placed righter the line $\{\mathrm{Re}\lambda = -\sigma\}$ will be interesting for us.

Using contour γ described below (3.54) we define the continuous contour γ_σ that is placed in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leqslant -\sigma\}$ and constructed from an interval of the line $\{\mathrm{Re}\lambda = -\sigma\}$ and from two branches of contour γ that transform to $\{\arg\lambda = \theta\}$ and $\{\arg\lambda = -\theta\}$, $\theta \in (\pi/2, \pi)$ for sufficiently large $|\lambda|$.

Evidently, integral in (3.54) can be transformed as follows:

$$e^{-At} = (2\pi i)^{-1} \int\limits_{\gamma_\sigma} (\lambda I + A)^{-1} e^{\lambda t} d\lambda + \sum_j (2\pi i)^{-1} \int\limits_{|\lambda+\lambda_j|=\varepsilon} (\lambda I + A)^{-1} e^{\lambda t} d\lambda, \tag{3.56}$$

where summation is made over points $-\lambda_j \in \Sigma(-A)$ placed righter the line $\{\mathrm{Re}\lambda = -\sigma\}$, and $\varepsilon > 0$ is small enough.

To calculate terms in the sum from (3.56) we decompose the resolvent $R(\lambda, -A)$ at $\lambda = -\lambda_j$:

$$R(\lambda, -A) = \sum_{k=-m(-\lambda_j)}^{\infty} (\lambda + \lambda_j)^k R_k(-\lambda_j). \tag{3.57}$$

Residue of $R(\lambda, -A)e^{\lambda t}$ at $\lambda = -\lambda_j$ is calculated by the formula

$$\mathrm{Res} e^{\lambda t} R(\lambda, -A) = e^{-\lambda_j t} \sum_{n=1}^{m(-\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(-\lambda_j).$$

Hence (3.56) can be rewritten as follows:

$$e^{-At} = (2\pi i)^{-1} \int\limits_{\gamma_\sigma} (\lambda E + A)^{-1} e^{\lambda t} d\lambda + \sum_{\mathrm{Re}\lambda_j < \sigma} e^{-\lambda_j t} \sum_{n=1}^{m(-\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(-\lambda_j). \tag{3.58}$$

Lemma 3.4. *The following estimate is true:*

$$\|(2\pi i)^{-1}\int\limits_{\gamma_\sigma}(\lambda E+A)^{-1}e^{\lambda t}d\lambda\|_{V_0^0(G)}\leqslant ce^{-\sigma t}\quad as\quad t\geqslant 0 \tag{3.59}$$

where $c>0$ does not depend on t.

Proof. Definition of γ_σ and (3.21) imply that for sufficiently large R

$$\gamma_\sigma\cap\{\lambda\in\mathbb{C}:|\lambda|>R\}\subset S_{-a,\varphi}.$$

This inclusion, relation $\gamma_\sigma\subset\rho(-A)$ and (3.22) where A is changed on $-A$ and $S_{a,\varphi}$ is changed on $-S_{a,\varphi}$ lead to the inequality

$$\|(\lambda E+A)^{-1}\|\leqslant M_1/(1+|\lambda|)\quad\text{for}\quad\lambda\in\gamma_\sigma. \tag{3.60}$$

Besides, in virtue of definition of γ_σ

$$|e^{(\lambda+\sigma)t}|\leqslant ce^{\cos\theta|\lambda|t/2}\quad\text{as}\quad\lambda\in\gamma_\sigma\quad\text{and}\quad|\lambda|\to\infty \tag{3.61}$$

where $\cos\theta<0$ since $\theta\in(\frac{\pi}{2},\pi)$. By (3.60), (3.61)

$$\|(2\pi i)^{-1}\int\limits_{\gamma_\sigma}(\lambda E+A)^{-1}e^{\lambda t}d\lambda\|\leqslant e^{-\sigma t}\int\limits_{\gamma_\sigma}\frac{M_1ce^{\cos\theta\lambda t/2}}{1+|\lambda|}|d\lambda|\leqslant M_2e^{-\sigma t}\quad\text{for}\quad t>1. \tag{3.62}$$

As known, the norm of left-hand-side of (3.58) is bounded uniformly with respect to $t\in[0,1]$. Besides, the norm of the sum from right-hand-side of (3.58) is also bounded uniformly to $t\in[0,1]$. Hence

$$\|(2\pi i)^{-1}\int\limits_{\gamma_\sigma}(\lambda E+A)^{-1}e^{\lambda t}d\lambda\|\leqslant M_2e^{-\sigma t}\quad\text{for } t\in[0,1]. \tag{3.63}$$

Now (3.59) follows from (3.62), (3.63). □

Denote by

$$\varepsilon^{(k)}(-\bar\lambda_j),\varepsilon_1^{(k)}(-\bar\lambda_j),\dots,\varepsilon_{m_k}^{(k)}(-\bar\lambda_j),\quad k=1,\dots,N(-\lambda_j) \tag{3.64}$$

a certain canonical system of eigenvectors and associated vectors of operator $-A^*$ corresponding eigenvalue $-\bar\lambda_j$.

Theorem 3.3. *Suppose that A is operator* (3.18) *and $\sigma>0$ satisfies* (3.55). *Then for each $w_0\in V_0^0(G)$ that satisfies*

$$\langle w_0,\varepsilon_l^{(k)}(-\bar\lambda_j)>=0,\quad l=0,1,\dots,m_k,\ k=1,2,\dots,N(-\lambda_j),\ \mathrm{Re}(\lambda_j)<\sigma \tag{3.65}$$

(*here by definition $\varepsilon_0^{(k)}(-\lambda_j)=\varepsilon^{(k)}(-\lambda_j)$*) *the following inequality holds:*

$$\|e^{At}w_0\|_{V_0^0(G)}\leqslant ce^{-\sigma t}\|w_0\|_{V_0^0(G)}\quad for\ t\geqslant 0. \tag{3.66}$$

Proof. By Corollary 3.1 if w_0 satisfies (3.65) then $R_n(-\lambda_j)w_0 = 0$ for every operators $R_{-n}(-\lambda_j)$ from (3.58). Therefore (3.58), (3.59) imply (3.66). □

3.5. On linear independence of $\varepsilon^{(k)}(x)$

We set some strengthening of well-known result on linear independence of eigenvectors and associated vectors for operator (3.27), (3.28) conjugate to Oseen operator (3.18).

Lemma 3.5. *Let $\varepsilon^{(k)}(x)\quad k = 1, \dots, N$ be complete linear independent system of eigenvectors to an eigenvalue $\bar{\lambda}_0$. Then for an arbitrary subdomain $\omega < G$ functions $\varepsilon^{(k)}(x)$, $x \in \omega$, $k = 1, \dots, N$ are linear independent.*

Proof. Let

$$f(x) = \sum_{k=1}^{N} c_k \varepsilon^{(k)}(x) = 0,\ x \in \omega. \tag{3.67}$$

Evidently, equality $(\bar{\lambda}_0 E - A^*)f(x) = 0$ holds for $x \in G$ and therefore by (3.27) and definition of operator π

$$-\Delta f(x) - (a(x), \nabla)f(x) + (\nabla a(x))^* f(x) = \nabla p(x),\ x \in G \tag{3.68}$$

with some function $p(x) \in H^1(G)$. Applying operator $\mathrm{rot}^* g(x) \equiv \partial_1 g_2 - \partial_2 g_1$ to (3.68) we get

$$-\Delta \mathrm{rot}^* f(x) - (a(x), \nabla)\mathrm{rot}^* f(x) = 0,\ \ x \in G. \tag{3.69}$$

(This formula can be verified by straightforward calculations). By (3.67) we have

$$\mathrm{rot}^* f(x) = 0, \quad x \in \omega. \tag{3.70}$$

Relations (3.69), (3.70) imply (see [9]) that

$$\mathrm{rot}^* f(x) = 0,\ x \in G,$$

and this, as well-known (see, for example, [5, p. 204]) leads that $f(x) \equiv \mathrm{const}$. By the definition (3.67) and by relation $\varepsilon^{(k)}|_{\partial G} = 0$ we get $f(x) \equiv 0$, $x \in G$. Hence in (3.67) $c_k = 0$, $k = 1, \dots, N$. □

Theorem 3.4. *Let* (3.64) *be a certain canonical system of eigenvectors and associated vectors for $-A^*$ corresponding eigenvalue $-\bar{\lambda}_j$. Then for each σ satisfying* (3.55) *and for arbitrary subdomain $\omega \subset G$ the set of canonical systems* (3.64) *for all eigenvalues $-\bar{\lambda}_j$ satisfying*

$$\mathrm{Re}\lambda_j < \sigma \tag{3.71}$$

are linear independent.

Proof. In proof of this assertion specific character of operator A^* is used to prove Lemma 3.5 only. The last part of prove is general and has been made in [6]. $\square$

Impose on canonical systems (3.64) the following condition

$$\varepsilon^{(k)}(-\overline{\lambda}_j) = \overline{\varepsilon^{(k)}(-\lambda_j)}; \quad \varepsilon_l^{(k)}(-\overline{\lambda}_j) = \overline{\varepsilon_l^{(k)}(-\lambda_j)}. \tag{3.72}$$

Indeed, since by (3.30) vector field $a(x)$ is real valued, if we act the operation of complex conjugation to functions (3.64), we evidently get canonical system of eigenvectors and associated vectors for A^* corresponding to eigenvalue $-\lambda_j$. That is why condition (3.72) can be realized easily.

In virtue of (3.72) canonical system corresponding to real $-\lambda_j$ consists of real valued vector fields. If $\mathrm{Im}\lambda_j \neq 0$, instead of vector fields $\varepsilon^{(k)}(-\overline{\lambda}_j)$, $\overline{\varepsilon_l^{(k)}(-\lambda_j)}$, $l = 0, 1, \ldots$, we consider real valued vector fields

$$\mathrm{Re}\varepsilon_l^{(k)}(-\overline{\lambda}_j), \mathrm{Im}\varepsilon_l^{(k)}(-\overline{\lambda}_j), \quad l = 1, \ldots, \quad k = 1, 2 \ldots . \tag{3.73}$$

(As above $\varepsilon_0^{(k)}(-\bar{\lambda}_j) = \varepsilon^{(k)}(-\bar{\lambda}_j)$ by definition). We renumber all functions (3.73) with $\mathrm{Re}\lambda_j < \sigma$ (including fields with $\mathrm{Im}\lambda_j = 0$) as follows:

$$\varepsilon_1(x), \ldots, \varepsilon_K(x). \tag{3.74}$$

Lemma 3.6. *For an arbitrary subdomain $\omega \subset G$ vector fields* (3.74) *restricted on ω are linear independent over the field $\mathbb{R}$ of real numbers.*

Lemma 3.6 follows easily from Theorem 3.4. (see [6])

Note that Theorem 3.3 implies immediately the following assertion.

Corollary 3.2. *Assume that A is operator* (3.18) *and $\sigma > 0$ satisfies* (3.55). *Then for each $w_0 \in V_0^0(G)$ satisfying*

$$\int_G (w_0(x), \varepsilon_j(x))\partial x = 0, \qquad j = 1, \ldots, K \tag{3.75}$$

with ε_j from (3.74), *inequality* (3.66) *is true.*

4. Stabilization of the Oseen equation

4.1. Theorem on extension

The key step in stabilization method that we propose is construction of special extension for vector fields from Ω to G (see (3.8)). First of all we make more precise the conditions imposed on Ω and G. Recall that

$$G = \mathrm{Int}(\bar{\Omega} \cup \bar{\omega}) \tag{4.1}$$

where Ω and ω are open subsets of $\mathbb{R}^2$, $\Omega \cap \omega = \emptyset$ and $\partial\Omega$ is a closed curve of C^∞-class and

$$\partial\Omega = \Gamma \cup \Gamma_0 \cup \partial\Gamma, \quad \partial G \cap \partial\Omega = \Gamma_0 \cup \partial\Gamma \tag{4.2}$$

where Γ, Γ_0 are open subsets of $\partial\Omega$, $\Gamma \neq \emptyset$, and $\partial\Gamma$ is a finite number of points, or $\partial\Gamma = \emptyset$.

We suppose that

$$\partial\Omega = \bigcup_{j=1}^{N} \partial\Omega_j \tag{4.3}$$

where $\partial\Omega_j$ are connected components of $\partial\Omega$. We assume that the following conditions are true:

Condition 4.1. For each $j = 1, \dots, N$ the set $\partial\Omega_j \cap \Gamma_0$ is connected or it is empty.

Condition 4.1. implies that for each j the set $\partial\Omega_j \cap \Gamma$ also is connected or empty.

We impose the following smoothness condition on $\partial\Omega$ and ∂G:

Condition 4.2. Let $\partial\Omega \in C^\infty$, $\partial G \setminus \partial\Gamma \in C^\infty$ and for each point $P \in \partial\Gamma$ there exist local coordinates (x, y) such that P is origin; $P = (0, 0)$, $\{(x, 0),\, x \in (0, \varepsilon)\} \subset \Gamma$, $\{(x, 0),\, x \in (-\varepsilon, 0)\} \subset \Gamma_0$ and in a neighbourhood of P ∂G can be represented as follows:

$$\partial G \supset \{(x, y) = (x, x^\alpha),\ x \in (0, \varepsilon)\} \cup \{(x, y) = (x, 0),\ x \in (-\varepsilon, 0)\}, \quad \alpha > 1. \tag{4.4}$$

Here $\varepsilon > 0$ is a small magnitude.

Remind that Ω is a given domain where the Oseen system which should be stabilized is determined. The domain ω we choose ourselves. That is why Condition 4.2 is not restrictive.

We introduce the following spaces

$$\begin{gathered} V_0^1(G) = \{u(x) = (u_1(x), u_2(x)) \in V^1(G) \,:\, u|_{\partial G} = 0\} \\ V^1(\Omega, \Gamma_0) = \{u(x) = (u_1(x), u_2(x)) \in V^1(\Omega) \,: \\ u|_{\Gamma_0} = 0,\ \exists\, v \in V_0^1(G) \text{ that } u(x) = \gamma_\Omega v(x)\} \end{gathered} \tag{4.5}$$

where $V^1(G)$ is defined in (2.14), γ_Ω is the operator of restriction on Ω for functions from $V_0^1(G)$. The space $V^1(\Omega, \Gamma_0)$ is supplied with the following norm:

$$\|u\|_{V^1(\Omega,\Gamma_0)} = \inf_L \|Lu\|_{V_0^1(G)}$$

where infimum is taken over all bounded extension operators $L : V^1(\Omega, \Gamma_0) \to V_0^1(G)$.

Definition (4.5) is convenient to prove the extension result but it is not constructive. Below, in Subsection 4.2 we give simple condition on vector field which guarantees its belonging to $V^1(\Omega,\Gamma_0)$.

We will use the following subspaces of Sobolev space $H^2(G)$ of scalar functions:

$$\hat{H}^2(G) = \{F \in H^2(G) : \int\limits_G F(x)\,dx = 0\}, \tag{4.6}$$

$$\hat{H}^2_\nabla(G) = \{F(x) \in \hat{H}^2(G) : \nabla F|_{\partial G} = 0\} \tag{4.7}$$

where, recall, $\nabla F = (\partial_{x_1}F, \partial_{x_2}F)$.

Remind that for a scalar function $F(x)$, $x \in G \subset \mathbb{R}^2$

$$\mathrm{rot}F(x) = (\partial_2 F(x), -\partial_1 F(x)). \tag{4.8}$$

Lemma 4.1. *The operator*

$$\mathrm{rot} : \hat{H}^2_\nabla(G) \to V^1_0(G) \tag{4.9}$$

is an isomorphism.

Proof. It is known (see [5, p. 204]) that the operator

$$\mathrm{rot} : \hat{H}^2(G) \to V^1(G)$$

is isomorphism. If $\nu = (\nu_1, \nu_2)$ is the vector field of normals to ∂G, then $\tau = (\tau_1, \tau_2) = (\nu_2, -\nu_1)$ is the field of vectors, tangent to G along ∂G. Let $v \in V^1_0(G)$ and $F = \mathrm{rot}^{-1}v \in \hat{H}^2(G)$. Then

$$\begin{aligned} \partial_n F|_{\partial G} &= (\mathrm{rot}F, \tau)|_{\partial G} = (v, \tau)|_{\partial G} = 0, \\ \partial_\tau F|_{\partial G} &= -(\mathrm{rot}F, \nu)|_{\partial G} = -(v, \nu)|_{\partial G} = 0. \end{aligned} \tag{4.10}$$

Therefore $\nabla F|_{\partial G} = 0$ and $F \in \hat{H}^2_0(G)$. If $F \in \hat{H}^2_0(G)$ and $v = \mathrm{rot}F$ then in virtue of (4.10) and condition $\nabla F|_{\partial G} = (\partial_n F, \partial_\tau F)|_{\partial G} = 0$ we get that $v \in V^1_0(G)$. $\square$

For $v \in V^1(G)$ denote by $\mathrm{rot}^* v$ the operator formally conjective to (4.8). It is easy to see that

$$\mathrm{rot}^* v(x) = \partial_1 v_2(x) - \partial_2 v_1(x) \quad \text{where } v(x) = (v_1(x), v_2(x)). \tag{4.11}$$

Lemma 4.2. *Let ω_1 be a subdomain of G such that* $\mathrm{Int}(\partial\omega_1 \cap \partial G) \neq \emptyset$. *Suppose that*

$$f(x) \equiv \sum_{j=1}^{K} c_j \mathrm{rot}^* \varepsilon_j(x) = 0, \quad x \in \omega_1 \tag{4.12}$$

where c_j are constants and $\varepsilon_j(x)$ are vector fields (3.74). *Then $c_j = 0$, $j = 1, \dots, K$.*

Proof. Denote scalar functions

$$\theta_j(x) = \mathrm{rot}^{-1}\varepsilon_j(x). \tag{4.13}$$

This definition is correct in virtue of Lemma 4.1 since $\varepsilon_j(x) \in V_0^1(G)$. Let

$$g(x) = \sum_{j=1}^{N} c_j\theta_j(x). \tag{4.14}$$

Since by (4.8), (4.11) $-\Delta = \mathrm{rot}^* \circ \mathrm{rot}$ we get by (4.12), (4.13), (4.14) that

$$\Delta g(x) \equiv f(x) = 0, \quad x \in \omega. \tag{4.15}$$

By (4.13), Lemma 4.1 and (4.14) $\nabla g|_{\partial G} = 0$ and therefore

$$\nabla g|_{\partial G \cap \partial\omega_1} = 0. \tag{4.16}$$

Since $\mathrm{Int}(\partial G \cap \partial\omega_1) \neq \emptyset$, by uniqueness of a solution for Cauchy problem for Laplace operator (see [9]) relations (4.15), (4.16) imply that

$$g(x) \equiv \mathrm{const}, \quad x \in \omega_1,$$

and therefore applying to (4.14) operator rot we get in virtue of (4.13) that

$$\sum_{j=1}^{N} c_j\varepsilon_j(x) \equiv 0, \quad x \in \omega_1.$$

This inequality and Lemma 3.6 imply that $c_1 = \cdots = c_N = 0$. $\square$

We prove now the extension theorem. In the space of real valued vector fields $V_0^1(G)$ we introduce the subspace

$$X_K^1(G) = \{v(x) \in V_0^1(G) \,:\, \int_G v(x) \cdot \varepsilon_j(x)\, dx = 0, \quad j = 1, \dots, K\} \tag{4.17}$$

where $\varepsilon_j(x)$ are functions (3.76).

Theorem 4.1. *There exists a linear bounded extension operator*

$$E_K^1 : V^1(\Omega, \Gamma_0) \to X_K^1(G) \tag{4.18}$$

(*i.e.* $E_K(v)(x) \equiv v(x)$ *for* $x \in \Omega$).

Proof. By definition (4.5) of $V^1(\Omega, \Gamma_0)$ there exists a linear continuous extension operator

$$L : V^1(\Omega, \Gamma_0) \to V_0^1(G). \tag{4.19}$$

Set

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 \,:\, \mathrm{dist}(x, \Omega) < \varepsilon\}$$

where $\mathrm{dist}(x,\Omega)$ is the distance from x to Ω. Let $\psi_0(x)\in C^\infty(\bar G)$, $0\leqslant\psi_0(x)\leqslant 1$,

$$\psi_0(x)=\begin{cases}1, & x\in G\cap\Omega_{\varepsilon/2},\\ 0, & x\in G\setminus\Omega_\varepsilon,\end{cases}\qquad \psi_1(x)=1-\psi_0(x)\in C^\infty(\bar G). \tag{4.20}$$

We look for extension operator E_K^1 in a form

$$E_K^1v(x)=\mathrm{rot}(\psi_0\mathrm{rot}^{-1}Lv)(x)+\mathrm{rot}\Big[\psi_1(x)\sum_{j=1}^K c_j\mathrm{rot}^*\varepsilon_j(x)\Big], \tag{4.21}$$

where c_j are constants which we have to determine. Evidently, $E_K^1v(x)=v(x)$ if $x\in\Omega$ for any c_j. To define constants c_j we note that by (4.17) inclusion $E_Kv\in X_K(G)$ are fulfilled if

$$\int\limits_G\varepsilon_k(x)\mathrm{rot}\Big[\psi_1(x)\sum_{j=1}^K c_j\mathrm{rot}^*\varepsilon_j(x)\Big]\,dx=-\int\limits_G\varepsilon_k(x)\mathrm{rot}(\psi_0\mathrm{rot}^{-1}Lv)(x)\,dx \tag{4.22}$$

where $k=1,\dots,K$.

Let

$$a_{kj}=\int\varepsilon_k(x)\cdot\mathrm{rot}[\psi_1(x)\mathrm{rot}^*\varepsilon_j(x)]\,dx=\int(\mathrm{rot}^*\varepsilon_k(x))\psi_1(x)\mathrm{rot}^*\varepsilon_j(x)\,dx. \tag{4.23}$$

Then the matrix $A=\|a_{kj}\|$ is positively defined. Indeed, let $\alpha=(\alpha_1,\cdots,\alpha_K)$, $f(x)=\sum_{j=1}^K\alpha_j\mathrm{rot}^*\varepsilon_j(x)$. Then

$$(A\alpha,\alpha)=\sum_{k,j=1}^K\alpha_k\alpha_j\int\psi_1(\mathrm{rot}^*\varepsilon_k(x))\mathrm{rot}^*\varepsilon_j(x)\,dx=\int\limits_G\psi_1(x)|f(x)|^2dx\geqslant 0.$$

Moreover, if for some α $(A\alpha,\alpha)=\int\psi_1(x)|f(x)|^2dx=0$ then $f(x)=0$ for $x\in\omega_1=G\setminus\Omega_\varepsilon$ and by Lemma 4.2 $\alpha_1=\cdots=\alpha_K=0$. That is why $\det A\neq\emptyset$.

Equations (4.22) can be rewritten as follows:

$$Ac=g$$

where $c=(c_1,\dots,c_K)$ and g is the vector with components defined by right size of (4.22). Thus $c=A^{-1}g$ and definition of operator (4.21) is completed. Boundedness of operator (4.18) follows immediately from its definition (4.21). □

4.2. On the space $V^1(\Omega,\Gamma_0)$

In this subsection we show that each vector field $v(x)\in V^1(\Omega)$ satisfying $v|_{\Gamma_0}=0$ belongs to $V^1(\Omega,\Gamma_0)$ if $v(x)$ is smooth enough in a neighbourhood of $\partial\Gamma_0$. This result is not used directly for solution of the stabilization problem given below in Subsection 4.4.

We introduce the spaces:

$$V^1_{\Gamma_0} = \{u(x) = (u_1, u_2) \in V^1(\Omega) \,:\, v|_{\Gamma_0} = 0\} \tag{4.24}$$

$$\hat{H}^2(\Omega, \Gamma_0) = \{F(x) \in \hat{H}^2(\Omega) \,:\, \nabla F|_{\Gamma_0} = 0\} \tag{4.25}$$

where $\hat{H}^2(\Omega)$ is defined analogously to (4.6).

Lemma 4.3. *Operator*

$$\mathrm{rot}: \ \hat{H}^2(\Omega, \Gamma_0) \to V^1_{\Gamma_0}(\Omega) \tag{4.26}$$

is an isomorphism.

The proof of this lemma is completely analogous to the proof of Lemma 4.1.

We want to find out what vector field from $V^1_{\Gamma_0}(\Omega)$ can be extended to a vector field belonging to $V^1_0(G)$. Firstly we consider the simplest case when

$$\forall j \quad \partial\Omega_j \subset \Gamma \ \text{ or } \ \partial\Omega_j \cap \Gamma = \emptyset. \tag{4.27}$$

Proposition 4.1. *If condition* (4.27) *is true then there exists an extension operator*

$$L : V^1_{\Gamma_0}(\Omega) \to V^1_0(G) \tag{4.28}$$

and therefore $V^1_{\Gamma_0}(\Omega) = V^1(\Omega, \Gamma_0)$.

Proof. Let $v \in V^1_{\Gamma_0}(\Omega)$, $F = \mathrm{rot}^{-1} v \in \hat{H}^2(\Omega, \Gamma_0)$ where rot^{-1} is constructed in Lemma 4.3. Since by (4.27) Γ is closed manifold we can extend F through Γ by means of usual Witney extension theorem and this operator we denote by L_0. Then operator (4.28) can be defined as follows:

$$L = \mathrm{rot} \circ \psi_0 \circ L_0 \circ \mathrm{rot}^{-1} \tag{4.29}$$

where ψ_0 is the operator of multiplication on function (4.20). Operator (4.29), evidently, is bounded in spaces (4.28) □

Condition (4.27) actually means that $\partial\Gamma = \emptyset$. Now we consider the case when $\partial\Gamma \neq \emptyset$. In this case we also look for an extension operator in the form (4.29) where, however, L_0 is not usual Witney extension operator. Below we construct new operator L_0.

Lemma 4.4. *Let domains* Ω *and* G *satisfy* (4.1), (4.2), (4.3) *and Conditions 4.1, 4.2 with* $\alpha \geqslant 2$. *Suppose that* $F(x) \in \hat{H}^2(\Omega, \Gamma_0)$ *satisfies the condition*

$$F(x) \in H^k(\Omega \cap \mathcal{O}(\partial\Gamma)) \ \text{ with } k > \frac{3\alpha + 1}{2} \tag{4.30}$$

where $\mathcal{O}(\partial\Gamma)$ *is a neighbourhood of* $\partial\Gamma$ *and* α *is magnitude from* (4.4). *Then* $F(x)$ *can be extended up to a function* $L_0 F(x) \in H^2_\nabla(G)$ *where*

$$H^2_\nabla(G) = \{f(x) \in H^2(G) \,:\, \nabla f|_{\partial G} = 0\}. \tag{4.31}$$

Proof. Step 1. If $F \in \hat{H}^2(\Omega, \Gamma_0)$ then by (4.25) and Condition 4.1 $F|_{\Gamma_0 \cap \partial\Omega_j} = c_j =$ const. Define f by the equality

$$F = f + c_j.$$

We extend the constant c_j through $\Gamma \cap \partial\Omega_j$ by just the same constant c_j. That is why to extend F through $\Gamma \cap \partial\Omega_j$ we have to extend $f(x)$. The problem is to extend $f(x)$ to a neighbourhood of the point $P \in \partial\Omega_j \cap \partial\Gamma$ because outside this neighbourhood we can use Witney extension method.

By Condition 4.2 we can suppose that $f(x,y) \in H^k(Q_-)$ with $Q_- = \{(x,y) \in \mathbb{R}^2 \,:\, y \leqslant 0,\, x^2 + y^2 < 1\}$ [3]

$$f(x,0) = \partial_y f(x,0) = 0 \quad \text{for } x \in (-1,0). \tag{4.32}$$

We must extend f on $Q = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 < 1,\, y \leqslant \gamma(x)$ where $\gamma(x) = x^\alpha$ for $x \geqslant 0$, $\gamma(x) = 0$ for $x \leqslant 0\}$ such that extended function $L_0 f \in H^2(Q)$ and $L_0 f|_{(x,\gamma(x))} = \nabla L_0 f|_{(x,\gamma(x))} = 0$.

Step 2. We define

$$\varphi(\lambda) = \begin{cases} 1 - 2^{\beta-1}\lambda^\beta, & \lambda \in (0, 1/2), \\ 2^{\beta-1}(1-\lambda)^\beta, & \lambda \in (1/2, 1), \end{cases} \quad \beta \geqslant 2. \tag{4.33}$$

It is clear that $\varphi(\lambda) \in H^2(0,1)$

$$\varphi(1) = \varphi'(1) = \varphi'(0) = 0, \quad \varphi(0) = 1 \tag{4.34}$$

and

$$|\varphi(\lambda)| \leqslant c \;\; |\varphi'(\lambda)| \leqslant c\lambda^{\beta-1}, \;\; |\varphi''(\lambda)| \leqslant c\lambda^{\beta-2}. \tag{4.35}$$

We determine the extension operator L_0 as follows:

$$L_0 f(x,y) = \begin{cases} f(x,y), & y < 0, \\ g(x,y)\varphi(y/x^\alpha), & y > 0, \; x > 0, \end{cases} \tag{4.36}$$

where $g(x,y) = 4f(x, -\frac{y}{2}) - 3f(x,-y)$.

In virtue of (4.34) $L_0 f$ and $\nabla L_0 f$ satisfy desired zero boundary condition on $\{(x,\gamma(x)\}$. Moreover the function

$$h(x,y) = \begin{cases} f(x,y), & y < 0, \\ g(x,y), & y > 0, \; x > 0 \end{cases}$$

belongs to $H^2(Q)$. Therefore we have to prove only that

$$g(x,y)\varphi(y/x^\alpha) \in H^2(Q \setminus Q_-). \tag{4.37}$$

[3] In Condition 4.2 we have $x^2 + y^2 < \varepsilon$ instead of $x^2 + y^2 < 1$. By multiplication f on a cut function equal 1 for $x^2 + y^2 < \varepsilon/2$ and equal 0 outside $x^2 + y^2 \leqslant \varepsilon$ we can reduce the deal to the case $x^2 + y^2 < 1$.

Step 3. We prove that the following estimates for function g defined below (4.36) are true:

$$|g(x,y)| \leqslant c(x^{k-1-\varepsilon} + x^{k-2-\varepsilon}y + y^2), \quad (x,y) \in Q \setminus Q_-, \ \forall \varepsilon > 0 \tag{4.38}$$

$$|\partial_x g(x,y)| \leqslant c(x^{k-2-\varepsilon} + x^{k-3-\varepsilon}y + y^2), \quad |\partial_y g(x,y)| \leqslant c(x^{k-2-\varepsilon} + y), \tag{4.39}$$

$$(x,y) \in Q \setminus Q_-, \ \forall \varepsilon > 0.$$

Indeed, (4.32) implies that

$$g(x,0) = \partial_y g(x,0) = 0 \ \text{ for } x \in (-1,0). \tag{4.40}$$

By Sobolev embedding theorem and by (4.30)

$$g(x,y) \in H^k(Q \setminus Q_-) \subset C^{k-1-\varepsilon}(Q \setminus Q_-). \tag{4.41}$$

In virtue of Lagrange theorem, (4.40), (4.41) (4.30) we have for $(x,y) \in Q \setminus Q_-$:

$$\begin{aligned} |g(x,y)| &\leqslant |g(x,0)| + |g(x,y) - g(x,0)| \leqslant cx^{k-1-\varepsilon} + c|\partial_y g(x,\theta y)|y \leqslant \\ &\leqslant cx^{k-1-\varepsilon} + cy(|\partial_y g(x,0)| + |\partial_y g(x,\theta y) - \partial_y g(x,0)|) \leqslant \\ &\leqslant c(x^{k-1-\varepsilon} + x^{k-2-\varepsilon}y + y^2). \end{aligned}$$

This proves (4.38). Inequalities (4.39) are proved analogously.

Step 4. We prove (4.37). We have

$$\begin{aligned} &\int_0^1 dx \int_0^{x^\alpha} |\partial_{yy}(g(x,y)\varphi(y/x^2))|^2 dy \leqslant \\ &\leqslant c\int_0^1 dx \int_0^{x^\alpha} |g\partial_{yy}\varphi|^2 + |(\partial_y g)\partial_y\varphi|^2 + |\partial_{yy} g\varphi|^2.)\, dx \end{aligned} \tag{4.42}$$

Since $\partial_{yy} g \in L_2$ and $|\varphi| \leqslant c$ we have to estimate only the first and second terms from right side. Taking into account (4.35), (4.38), (4.39) we get

$$\begin{aligned} &\int_0^1 dx \int_0^{x^\alpha} |g\partial_{yy}\varphi|^2 + |(\partial_y g)\partial_y\varphi|^2\, dy \leqslant \\ &\leqslant c\int_0^1 dx \int_0^{x^\alpha} \big[(x^{2(k-1-\varepsilon)} + x^{2(k-2-\varepsilon)}y^2 + y^4)y^{2(\beta-2)}x^{-2\alpha(\beta-2)-4\alpha} + \\ &\quad +(x^{2(k-2-\varepsilon)} + y^2)y^{2(\beta-1)}x^{-2\alpha(\beta-1)-2\alpha}\big]\, dy \leqslant \\ &\leqslant c\int_0^1 \big(x^{2(k-1-\varepsilon)-2\alpha\beta+\alpha(2\beta-3)} + \\ &\quad +x^{2(k-2-\varepsilon)-2\alpha\beta+\alpha(2\beta-1)} + x^{-2\alpha\beta+\alpha(2\beta+1)}\big)dx = \end{aligned} \tag{4.43}$$

$$= c \int_0^1 (x^{2(k-1-\varepsilon)-3\alpha} + x^{2(k-2-\varepsilon)-\alpha} + x^{\alpha})\, dx.$$

The right side of (4.43) is finite because of (4.30). The terms $\partial_{xy}(g\varphi)$, $\partial_{xx}(g\varphi)$, $\partial_x(g\varphi)$, $\partial_y(g\varphi)$ can be estimated analogously. $\square$

Now we can prove the final result:

Theorem 4.2. *Let domains Ω and G satisfy* (4.1), (4.2), *Conditions 4.1, 4.2 and $\partial\Gamma \neq \emptyset$. Suppose that $v(x) \in V^1_{\Gamma_0}(\Omega)$ satisfies the condition*

$$v(x) \in \left(H^m(\Omega \cap \mathcal{O}(\partial\Gamma))\right)^2 \quad \text{with} \quad m > \frac{3\alpha - 1}{2},\ \alpha \geqslant 2, \tag{4.44}$$

where $\mathcal{O}(\partial\Gamma)$ is a neighbourhood of $\partial\Gamma$ and α is a magnitude from (4.4). *Then $v(x) \in V^1(\Omega, \Gamma_0)$, i.e. it can be extended up to a vector field $Lv \in V^1_0(G)$.*

Proof. Let $v(x)$ satisfy conditions of the Theorem. By Lemma 4.3 the function $F(x) = \mathrm{rot}^{-1} v(x)$ belongs to $\hat{H}^2(\Omega, \Gamma_0)$ and by (4.44) $F(x)$ satisfies (4.30) with $k = m + 1$. Then by Lemma 4.4 $F(x)$ can be extended up to a function $L_0 F(x) \in H^2_{\nabla}(G)$. Evidently, the vector field $Lv(x) \equiv \mathrm{rot}(\psi_0(x) L_0 F(x)) \equiv \mathrm{rot}(\psi_0(x) L_0 \mathrm{rot}^{-1} v(x))$ belongs to $V^1_0(G)$. $\square$

4.3. Extension Theorem for V^0

Extension operator constructed in Subsections 4.1, 4.2 is the main in our theory because it can be used for Navier–Stokes equations. Nevertheless there are some reasons to construct analogous extension operator for more wide function spaces of V^0-type. This construction is analogous but easier than that was written above. That is why we expound it briefly.

Set

$$V^0(\Omega, \Gamma_0) = \{u(x) \in V^0(\Omega) \,:\, u \cdot \nu|_{\Gamma_0} = 0,\ \exists v \in V^0_0(G) \text{ that } u(x) = \gamma_\Omega v(x)\} \tag{4.45}$$

where γ_Ω is the operator of restriction on Ω. Besides,

$$\|u\|_{V^0(\Omega,\Gamma_0)} = \inf_{\gamma_\Omega v = u} \|v\|_{V^0_0(G)}.$$

Denote

$$\hat{H}^1_\tau(G) = \{F(x) \in H^1(G) \,:\, \partial_\tau F|_{\partial G} = 0,\ \int_G F(x) dx = 0\}, \tag{4.46}$$

∂_τ is the derivative along the vector field τ tangent to ∂G.

Lemma 4.5. *The operator*

$$\mathrm{rot} : \hat{H}^1_\tau(G) \to V^0_0(G)$$

defined in (4.8) *is an isomorphism.* ($V^0_0(G)$ *is defined in* (3.16)).

This lemma is proved as Lemma 4.1.

Analogously to (4.17) we define

$$X^0_K(G) = \{v(x) \in V^0_0(G) \, : \, \int\limits_G v(x) \cdot \varepsilon_j(x)\, dx = 0, \; k = 1, \dots, K\}, \tag{4.47}$$

where $\varepsilon_j(x)$ are functions (3.74).

Theorem 4.3. *There exists a linear bounded extension operator*

$$E^0_K : V^0(\Omega, \Gamma_0) \to X^0_K(G). \tag{4.48}$$

The proof is analogous to Theorem's 4.1 proof.

At last let us give a condition on vector field $v \in V^0_{\Gamma_0} = \{v \in V^0(\Omega) \colon \; v \cdot \nu|_{\Gamma_0} = 0\}$, which guarantees that v belongs to space (4.45) when $\partial\Gamma \neq \emptyset$.

We do it firstly for functions

$$F(x) \in \hat{H}^1(\Omega, \Gamma_0) \equiv \{F(x) \in H^1(\Omega) \, : \, \partial_\tau F|_{\Gamma_0} = 0, \; \int_G F(x) dx = 0\}. \tag{4.49}$$

Lemma 4.6. *Let domains* Ω *and* G *satisfy* (4.1), (4.2), (4.3) *and Conditions 4.1, 4.2. Suppose that* $F(x)$ *satisfies* (4.49) *and satisfies the following condition:*

$$F(x) \in H^{1+\beta}(\Omega \cap \mathcal{O}(\partial\Gamma)) \tag{4.50}$$

and parameter α *from* (4.4) *satisfies* $1 < \alpha < 1 + \beta$, $0 < \beta < 1$.

Then $F(x)$ *can be extended to a function* $L_0 F(x) \in H^1_\tau(G)$ *where*

$$H^1_\tau(G) = \{f(x) \in H^1(G) \, : \, \partial_\tau f|_{\partial G} = 0\}$$

and ∂_τ *is derivative along vector field* τ *tangential to* ∂G.

Draft of proof. By (4.49) any $F \in \hat{H}^1(\Omega, \Gamma_0)$ equals constant c_j on $\partial\Omega_j \cap \Gamma_0$ for each connected component $\partial\Omega_j$ of $\partial\Omega$, $f(x, y) \equiv F - c_j \in H^{1+\beta}(Q_-)$, $f(x, 0) = 0$ for $x \in (-1, 0)$ and we must extend $f(x, y)$ on Q, where the sets Q_- and Q are defined near (4.32).

We set $\varphi(\lambda) = 1 - \lambda$ for $\lambda \in (0, 1)$ and define the extension operator as follows:

$$L_0 f(x, y) = \begin{cases} f(x, y), & y < 0, \\ f(x, -y)\varphi(y/x^\alpha), & y > 0, x \in (0, 1). \end{cases}$$

As well as in Lemma 4.4 all proof is reduced to establishing of the inclusions

$$\partial_x\big(f(x, -y)\varphi(y/x^\alpha)\big) \in L_2(Q \backslash Q_-), \quad \partial_y\big(f(x, -y)\varphi(y/x^\alpha)\big) \in L_2(Q \backslash Q_-). \tag{4.51}$$

To do it we take into account that by Sobolev embedding theorem $H^{1+\beta}(Q_-) \subset C^{\beta_1}(Q_-)$ and therefore the inequality is true:

$$|f(x,y)| \leqslant c(x^{\beta_1} + y^{\beta_1}) \quad \forall\,(x,y) \in Q \setminus Q_-, \quad \text{where } 1 < \beta_1 < \beta.$$

Now using this inequality and do estimates analogous to (4.42), (4.43) we prove (4.51). □

The following assertion succeeds from Lemma 4.6 as Theorem 4.2 was derived from Lemma 4.4.

Theorem 4.4. *Let Ω and G satisfy* (4.1), (4.2), *Conditions 4.1, 4.2, and $\partial\Gamma \neq \emptyset$. Suppose that $v(x) \in V^0_{\Gamma_0}(\Omega)$, $v(x) \in H^{1+\beta}(\Omega \cap \mathcal{O}(\partial\Gamma))$ where $0 < \beta < 1$, and parameter α from* (4.4) *satisfies inequalities $1 < \alpha < 1+\beta$. Then $v(x) \in V^0(\Omega,\Gamma)$.*

4.4. Results on stabilization

We prove now the main theorem of this section on stabilizability by feedback boundary control of 2D Oseen equations.

Theorem 4.5. *Let domains Ω and G satisfy* (4.1), (4.2), *and Conditions 4.1, 4.2. Then for each initial condition $v_0(x) \in V^0(\Omega,\Gamma_0)$ and each $\sigma > 0$ there exists a feedback control u defined on the part Σ of boundary $(0,\infty) \times \partial\Omega$ such that the solution $v(t,x)$ of boundary value problem* (3.1)–(3.3), (3.5) *satisfies the inequality:*

$$\|v(t,\cdot)\|_{(L_2(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as} \quad t \to \infty. \tag{4.52}$$

Proof. Let A be operator (3.18) and the magnitude $\sigma > 0$ satisfies condition (3.55). In the case if it is not so we get satisfying (3.55) by small increasing of σ. We act on initial condition $v_0 \in V^0(\Omega,\Gamma_0)$ by the operator E^0_K from (4.48). Then by Theorem 4.3 $w_0 = E^0_K v_0$ satisfies including $w_0 \in X^0_K(G)$. By definition (4.47) of $X^0_K(G)$ and definition (3.74) of vector fields $\varepsilon_1(x),\dots,\varepsilon_K(x)$ one yields that w_0 and σ satisfy all conditions of Theorem 3.3. In virtue of this theorem inequality (3.66) is true. We define desired solution (v,u) of stabilization problem (3.1)–(3.3), (3.5) by formula (3.15), where $w(t,\cdot) = e^{-At}w_0$ is the solution of problem (3.10)–(3.13) which can be rewritten in the equivalent form (3.53). Then (3.66) and evident inequality

$$\|v(t,\cdot)\|_{L_2(\Omega)} \leqslant \|w(t,\cdot)\|_{V^0_0(G)}$$

imply (4.52). □

Theorem 4.6. *Let domains Ω and G satisfy* (4.1), (4.2), *and Conditions 4.1, 4.2 with $\alpha \geqslant 2$. Then for each initial condition $v_0(x) \in V^1(\Omega,\Gamma_0)$ and for each $\sigma > 0$ there exists a feedback control u defined on Σ such that the solution $v(t,x)$*

of (3.1)–(3.3), (3.5) *satisfies the inequality*

$$\|v(t,\cdot)\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as} \quad t \to \infty. \tag{4.53}$$

Proof. As in Theorem 4.5 we can assume that σ satisfies conditions (3.55). We act on initial condition $v_0 \in V^1(\Omega, \Gamma_0)$ by the operator E^1_K from (4.18) and by Theorem 4.1 we obtain that $w_0 = E^1_K v_0 \in X^1_K(G)$. Since $X^1_K(G) \subset V^1_0(G) \subset V^0_0(G)$, the solution $w(t,x)$ of problem (3.1)–(3.3), (3.5) can be written in the form $w(t,\cdot) = e^{-At}w_0$ where A is operator (3.18). We want to prove that

$$\|w(t,\cdot)\|_{V^1_0(G)} \leqslant ce^{-\sigma t} \quad \text{for} \quad t \geqslant 0. \tag{4.54}$$

Since by well-known results on smoothness of solutions for Oseen equations inequality (4.54) is true for $t \in [0,1]$, it sufficient to prove (4.54) for $t > 1$. In virtue of representation (3.58) for e^{-At}, inclusion $w_0 \in X^1_K(G)$ and Corollary 3.1 we have

$$w(t,\cdot) = e^{-At}w_0 = \int\limits_{\gamma_\sigma} (\lambda I + A)^{-1} w_0 e^{\lambda t}\, d\lambda \tag{4.55}$$

where the contour γ_σ is define lower formula (3.55).

To estimate (4.55) we use the following assertion which will be proved after finishing of Theorem's 4.6 proof.

Lemma 4.7. *Let A be operator* (3.18) *and $\sigma > 0$ satisfies* (3.55). *Then*

$$\|(\lambda I + A)^{-1} w\|_{V^1_0(G)} \leqslant c\|w\|_{V^0_0(G)}, \tag{4.56}$$

where constant $c > 0$ does not depend on $w \in V^0_0(G)$ and $\lambda \in \gamma_\sigma$ and the contour γ_σ is defined lower (3.55).

End of proof of Theorem 4.5. We estimate integral in (4.55) using arguments of proof of the Lemma 3.4, but applying estimate (4.56) instead of (3.60). As a result we get:

$$\|w(t,\cdot)\|_{V^1_0(G)} \leqslant ce^{-\sigma t}\|w_0\|_{V^0_0(G)}, \quad t > 1.$$

Hence we established (4.54). Now we define $(v(t,\cdot), u(t,\cdot)) = (\gamma_\Omega w(t,\cdot),\ \gamma_\Gamma w(t,\cdot))$ and derive (4.53) from (4.54) and evident inequality $\|v(t,\cdot)\|_{H^1(\Omega)} \leqslant c\|w(t,\cdot)\|_{V^1_0(\Gamma)}$. □

Proof of Lemma 4.7. In virtue if Lemma 3.1, definition of contour γ_σ, and condition (3.55), operator $(\lambda I + A)^{-1}$ is well-defined on $V^0_0(G)$ if $\lambda \in \gamma_\sigma$.[4]

We denote $v = (\lambda I + A)^{-1} w$. When to prove (4.56) we have to establish the bound

$$\|(A + \lambda I)v\|_{V^0_0(G)} \geqslant c_1\|v\|_{V^1_0(G)}. \tag{4.57}$$

[4] Well-definiteness of $(\lambda I + A)^{-1}$ on $V^0_0(G)$ for $\lambda \in \gamma_\sigma$, $|\lambda| \gg 1$ can be obtained easily also by estimates analogous to that we get below in the proof of this Lemma.

If $\lambda \in \gamma_\sigma$ and $|\lambda| \gg 1$ then by definition of γ_σ (see lower (3.55))

$$\lambda = -\mu(1+i\nu) \text{ where } \mu \gg 1 \text{ and } |\nu| < 1 \ (\nu \in \mathbb{R} \text{ is fixed}). \tag{4.58}$$

We consider firstly the Stokes operator

$$A_0 = -\pi\Delta : V_0^0(G) \longrightarrow V_0^0(G) \tag{4.59}$$

As well-known (see [17], [19]), the closure of map (4.59) is self-adjoint positive operator which possesses a compact inverse operator. By Hilbert–Shmidt theorem the spectrum $\Sigma(A_0)$ consists of a countable set of positive numbers $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \to \infty$, as $k \to \infty$, and corresponding eigenfunctions $\{e_j\}$ form an orthogonal basis in $V_0^0(G)$ and in $V_0^1(G)$ (see [5]), and for $v = \sum_{j=1}^\infty v_j e_j$ we have:

$$\|v\|^2_{V_0^0(G)} = \sum_{j=1}^\infty |v_j|^2, \qquad \|v\|^2_{V_0^1(G)} = \sum_{j=1}^\infty \lambda_j |v_j|^2. \tag{4.60}$$

In virtue of (4.58), (4.60) we get that for $\lambda \in \gamma_\sigma$, $|\lambda| \gg 1$

$$\|(\lambda I + A_0)v\|^2_{V_0^0(G)} = \sum_{j=1}^\infty |-\mu(1+i\nu)+\lambda_j|^2 |v_j|^2 = \sum_{j=1}^\infty [(\lambda_j - \mu)^2 + \nu^2\mu^2]|v_j|^2. \tag{4.61}$$

Since for function $f(x) = ((x-\mu)^2 + \nu^2\mu^2)/x$

$$\inf_{x \in \mathbb{R}_+} f(x) = f(\mu\sqrt{1+\nu^2}) = 2(\sqrt{1+\nu^2} - 1)\mu$$

we get from (4.61) that

$$\|(\lambda I + A_0)v\|^2_{V_0^0(G)} \geqslant 2(\sqrt{1+\nu^2} - 1)\mu \sum_{j=1}^\infty \lambda_j |v_j|^2 = c_0 \mu \|v\|^2_{V_0^1(G)} \tag{4.62}$$

Consider now operator A from (3.18) which can be written in the form

$$Av = A_0 v + \pi[(a(x), \nabla)v + (v, \nabla)a)] \tag{4.63}$$

Since by (3.9) and in virtue of Sobolev embedding theorem

$$a(x) \in C(\overline{G}), \quad \nabla a(x) \in L_4(G),$$

we get the following estimate:

$$\begin{gathered} \|[(a,\nabla)v + (v,\nabla)a)]\|_{V_0^0(G)} \leqslant \\ \leqslant \|a\|_{C(\overline{G})} \|v\|_{V_0^1(G)} + \|\nabla a\|_{L_4(G)} \|v\|_{L_4(G)} \leqslant c_1 \|v\|_{V_0^1(G)}. \end{gathered} \tag{4.64}$$

Taking into account (4.62), (4.63), (4.64) we obtain

$$\begin{gathered} \|(\lambda I + A)v\|_{V_0^0(G)} \geqslant \\ \geqslant \|(\lambda I + A_0)v\|_{V_0^0(G)} - \|[(a,\nabla)v + (v,\nabla)a)]\|_{V_0^0(G)} \geqslant (c_0\mu - c_1)\|v\|_{V_0^1(G)}. \end{gathered}$$

This estimate and (4.58) imply (4.57) and therefore imply (4.56) for $\lambda \in \gamma_\sigma$, $|\lambda| \gg 1$. For other $\lambda \in \gamma_\sigma$ estimate (4.56) is true because $\gamma_\sigma \subset \rho(-A)$. □

5. Stabilization of 2D Navier–Stokes equations

5.1. Formulation of the stabilization problem

As in Subsections 3.1, 4.1 we suppose that $\Omega \subset \mathbb{R}^2$ is a bounded connected domain with C^∞-boundary $\partial\Omega$, and $\partial\Omega = \bar{\Gamma} \cup \bar{\Gamma}_0$, $\Gamma \neq \emptyset$ where Γ, Γ_0 are open sets satisfying Condition 4.1. Recall denotations of the space-times sets: $Q = \mathbb{R}_+ \times \Omega$, $\Sigma_0 = \mathbb{R}_+ \times \Gamma_0$, $\Sigma = \mathbb{R}_+ \times \Gamma$. In Q we consider the Navier–Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(x), \quad (t,x) \in Q \tag{5.1}$$

$$\operatorname{div} v = 0 \tag{5.2}$$

($v = (v_1, v_2)$) with initial condition

$$v(t,x)|_{t=0} = v_0(x), \quad x \in \Omega \tag{5.3}$$

and boundary conditions

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u, \tag{5.4}$$

where $u = (u_1, u_2)$ is a control defined on Σ.

We suppose also that a steady-state solution $(\hat{v}(x), \nabla\hat{p}(x))$ of Navier–Stokes system with the same right-hand side $f(x)$ as in (5.1) is given:

$$\Delta\hat{v}(x) + (\hat{v},\nabla)\hat{v} + \nabla\hat{p} = f(x), \quad \operatorname{div}\hat{v}(x) = 0, \quad x \in \Omega \tag{5.5}$$

$$\hat{v}|_{\Gamma_0} = 0. \tag{5.6}$$

Let $\sigma > 0$ be given. The problem of stabilization with the rate σ reduced to look for a solution of problem (5.1)–(5.4) which satisfies the inequality

$$\|v(t,\cdot) - \hat{v}\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as } t \to \infty. \tag{5.7}$$

The important additional condition is that u is a feedback control. Definition of feedback notion is analogous to Definition 3.1. Nevertheless we give now exact formulation of feedback property.

We extend Ω to a domain G through Γ (see (4.1), (4.2)) such that Condition 4.2 holds. After that we extend problem (5.1)–(5.4) to analogous problem defined on $\Theta = \mathbb{R}_+ \times G$:

$$\partial_t w(t,x) - \Delta w + (w,\nabla)w + \nabla q(t,x) = g(x), \quad \operatorname{div} w(t,x) = 0 \tag{5.8}$$

$$w(t,x)|_{t=0} = w_0(x) \tag{5.9}$$

with additional condition

$$w|_S = 0 \tag{5.10}$$

where $S = \mathbb{R}_+ \times \partial G$. Moreover we assume that solution $(\hat{v}, \nabla\hat{p})$ of (5.5), (5.6) is extended on G to a pair $(a(x), \nabla\hat{q}(x))$, $x \in G$ such that

$$-\Delta a(x) + (a,\nabla)a + \nabla\hat{q}(x) = g(x), \quad \operatorname{div} a(x) = 0, \quad x \in G \tag{5.11}$$

$$a|_{\partial G} = 0 \tag{5.12}$$

where right side $g(x)$ is the same as in (5.8). (We show below how to construct such extension.)

Definition 5.1. A control $u(t,x)$ in stabilization problem (5.1)–(5.4) is called feedback if the solution $(v(t,x),u(t,x))$ of (5.1)–(5.4) is defined by the equality:

$$(v(t,x),u(t,x)) = (\gamma_\Omega w(t,\cdot),\gamma_\Gamma w(t,\cdot)) \tag{5.13}$$

where $w(t,x)$ satisfies to (5.8)–(5.10), and γ_Ω, γ_Γ are operators of restriction of a function defined on G to Ω and to Γ respectively.

5.2. Invariant manifolds

Let $g(x)$ from (5.8) satisfy the condition:

$$g(x) \in (L_2(G))^2. \tag{5.14}$$

Then as well-known (see, for instance [18]) equations (5.8) are equivalent to the following equation with respect to one unknown function $w(t,x)$:

$$\partial_t w(t,x) - \pi\Delta w + \pi(w,\nabla)w = \pi g(x) \tag{5.15}$$

where π is orthoprojector (3.17) on $V_0^0(G)$ (see (3.16)). We assume in addition that solution w of (5.15) (as well as solution w of (5.8)) belongs to the space

$$V^{1,2}(\Theta_T) \equiv \{w(t,x) \in L_2(0,T;V^2(G)\cap(H_0^1(G))^2) \,:\, \partial_t w \in L_2(0,T;V_0^0(G)\} \tag{5.16}$$

for each $T>0$, where $\Theta_T=(0,T)\times G$. It is proved (see [13], [17]) that for each $T>0$, $g(x)\in(L_2^0(G))^2$, $w_0(x)\in V_0^1(G)$ there exists unique solution $w(t,x)\in V^{1,2}(Q_T)$ of problem (5.15), (5.9). Solution $w(t,x)$ of (5.15), (5.9) taken at time moment t we denote as $S(t,w_0)(x)$:

$$w(t,x) = S(t,w_0)(x). \tag{5.17}$$

Since embedding $V^{1,2}(Q_T)\subset C(0,T;V_0^1(G))$ is continuous, the family of operators $S(t,w_0)$ is continuous semigroup on the space $V_0^1(G)\,:\,S(t+\tau,w_0)=S(t,S(\tau,w_0))$.

Note that we can rewrite (5.11) in the form analogous to (5.15):

$$-\pi\Delta a(x)+\pi(a,\nabla)a=\pi g,\quad a(x)\in V_0^1(G)\cap V^2(G). \tag{5.18}$$

Since $a(x)$ is steady-state solution of (5.15), $S(t,a)=a$ for each $t\geqslant 0$. We can decompose semigroup $S(t,w_0)$ in a neighbourhood of a in the form

$$S(t,w_0+a) = a + L_t w_0 + B(t,w_0) \tag{5.19}$$

where $L_t w_0 = S'_w(t,a)w_0$ is derivative of $S(t,w_0)$ with respect to w_0 at point a, and $B(t,w_0)$ is nonlinear operator with respect to w_0. Differentiability of $S(t,w_0)$ is proved, for instance in [2, Ch. 7. Sect. 5]. Therefore

$$B(t,0)=0,\quad B'_w(t,0)=0. \tag{5.20}$$

Moreover in [2, Ch. 7. Sect. 5] is proved that $B(t,w)$ belongs to class $C^{1+\alpha}$ with $\alpha = 1/2$ with respect to w. This means that for each $w_0 \in V_0^1(G)$

$$\|B'_w(t,w_0)\|_{C^\alpha} \equiv \sup_{0\leqslant\|u-w_0\|_{V_0^1(G)}\leqslant 1} \frac{\|B'_w(t,u) - B'_w(t,w_0)\|_{V_0^1(G)}}{\|u-w_0\|^\alpha_{V_0^1(G)}} < \infty$$

and left side is a continuous function with respect to w.

We study now semigroup $L_t w_0 = S'_w(t,a)w_0$ of linear operators. First of all note that $w(t,x) = L_t w_0$ is the solution of problem (3.10)–(3.12) in which the coefficient a is the solution of (5.18). Therefore

$$L_t w_0 = e^{-At} w_0 \tag{5.21}$$

where A is operator (3.18). We remind (see [2])

Definition 5.2. Semigroup of linear bounded operators $S_t : X \to X$ (X is a Hilbert space) is called almost stable if

i) S_t are bounded in X for $0 \leqslant t \leqslant T$ by a constant depending on T.

ii) $\langle S_t u, \varphi\rangle$ is continuous with respect to t $\forall u \in X$, $\forall \varphi \in \Phi^*$ where Φ^* is a dense set in X^*.

iii) For each $t_0 > 0$ and for each $r_0 \in (0,1)$ there exists not more than finite set $\sigma_+ = (\zeta_1, \dots, \zeta_N)$ of points belonging to spectrum of operator S_{t_0} which are placed outside the disk $|\zeta| \leqslant r_0$.

iv) Invariant subspace of operator S_{t_0} corresponding to σ_+ is finite dimensional.

Below we suppose that $r_0 \in (0,1)$ satisfies the property:

$$\{\zeta \in \mathbb{C} : |\zeta| = r_0\} \cap \Sigma(e^{-At_0}) = \emptyset \tag{5.22}$$

where, recall, $\Sigma(e^{-At})$ is the spectrum of operator (5.21).

It is clear that $\zeta_j \in \Sigma(e^{-At_0})$ if and only if $\zeta_j = e^{-\lambda_j t_0}$ and $-\lambda_j \in \Sigma(-A)$. That is why condition (5.22) is equivalent to condition (3.55) where $\sigma = -\ln r_0/t_0$. Besides, if $|\zeta_j| > r_0$ then $-\mathrm{Re}\lambda_j > -\sigma$.

The following assertion holds:

Theorem 5.1. *Family of operators $e^{-At} : V_0^1(G) \to V_0^1(G)$ where A is operator* (3.18) *is well defined for each $t \geqslant 0$ and it is almost stable semigroup* (*see Definition 5.2 where $S_t = e^{-At}$, $X = V_0^1(G)$*). *Let*

$$\sigma_+ = \{\zeta_1, \dots, \zeta_K : \zeta_j \in \Sigma(e^{-At_0}),\ \ |\zeta_j| > r_0,\ j = 1, \dots, K\} \tag{5.23}$$

where $r_0 \in (0,1)$ and satisfies (5.22). *Let $X_+ \subset V_0^1(G)$ be the invariant subspace for e^{-At_0} corresponding to σ_+, $\Pi_+ : V_0^1(G) \to X_+$ be the projector on X_+* (*i.e.*, $\Pi_+ V_0^1(G) = X_+$, $\Pi_+^2 = \Pi$) *and $X_- = (I - \Pi_+)V_0^1(G)$ be complementary invariant subspace. Let $L^+_{t_0} = e^{-At_0}|_{X_+} : X_+ \to X_+$, $L^-_{t_0} = e^{-At_0}|_{X_-} : X_- \to X_-$. Then* *operator* $L^+_{t_0}$ *has inverse operator* $(L^+_{t_0})^{-1}$. *For some* t_0 *there exist constants* $\hat r$, ε_+, $\varepsilon_- \in (0,1)$ *such that*

$$\|L^-_{t_0}\| \leqslant \hat r(1-\varepsilon_-), \quad \|(L^+_{t_0})^{-1}\| \leqslant \hat r^{-1}(1-\varepsilon_+). \tag{5.24}$$

Proof. Well posedness of operator e^{-At} and the point i) of Definition 5.2 is proved, for instance, in [13], [17]. Denote in the point ii) of Definition 5.2 by $\langle , \rangle$ the duality between $V^1_0(G)$ and $V^{-1}(G) = (V^1_0(G))^*$. Then ii) also follows from [13], [17]. Semigroup $e^{-At} : V^1_0(G) \to V^1_0(G)$ is restriction of semigroup $e^{-At} : V^0_0(G) \to V^0_0(G)$. Hence formula (3.54) is true for $e^{-At} : V^1_0(G) \to V^1_0(G)$:

$$e^{-At}w_0 = (2\pi i)^{-1}\int\limits_\gamma (\lambda I + A)^{-1} w_0 e^{\lambda t} d\lambda, \quad w_0 \in V^1_0(G), \tag{5.25}$$

where $\gamma \subset \mathbb{C}^1$ is the contour described lower (3.54). In Section 3 was established that for each $-\sigma < 0$ there exists not more than finite number of $-\lambda_j \in \Sigma(-A)$ for which $\mathrm{Re}(-\lambda_j t_0) > -\sigma t_0$. Hence, for $r_0 = e^{-\sigma t_0}$ the number of $\zeta_j = e^{-\lambda_j t_0} \in \Sigma(e^{-At_0})$ such that $|\zeta_j| > r_0$ is finite. This established iii) of Definition 5.2.

In correspondence with general definition ([4], [20]) operator Π_+ defined in formulation of Theorem 5.1 can be calculated as follows:

$$\Pi_+ = (2\pi i)^{-1}\int\limits_\Gamma (\zeta I + e^{-At_0})^{-1} d\zeta \tag{5.26}$$

where $\Gamma = \{z \in \mathbb{C}^1 : |\zeta| = r_0\} \cup \{\zeta \in \mathbb{C}^1 : |\zeta| = R\}$ and $R > 0$ is so large that outside of the disk $\{|\zeta| \leqslant R\}$ there is no eigenvalues of operator e^{-At_0}; going around external circle is counterclockwise and circuit internal circle is clockwise. In [6, Lemma 5.3] is shown that

$$\Pi_+ = \sum_{\mathrm{Re}\lambda_j < \sigma} R_{-1}(-\lambda_j) \tag{5.27}$$

where $R_{-1}(-\lambda_j)$ is the operator-coefficient $R_k(-\lambda_j)$ with $k = -1$ in Laurent decomposition of the resolvent $R(\lambda, -A)$ around eigenvalue $-\lambda_j$. Since $\dim \mathrm{Im} R_{-1}(-\lambda_j) < \infty$ (see Lemma 3.3 above), and number of λ_j with $\mathrm{Re}(-\lambda_j) > -\sigma$ is finite, we proved point iv) of Definition 5.2.

Operators Π_+ and e^{-At_0} commutate. (see Lemma 3.2) Therefore operator $L^+_{t_0} = e^{-At_0}|_{X_+}$ with $X_+ = \Pi_+ V^1_0(G)$ is defined by the formula

$$L^+_{t_0} = \Pi_+ e^{-At_0} = (2\pi i)^{-1}\int\limits_\Gamma (\zeta I - e^{-At_0})^{-1}\zeta\, d\zeta. \tag{5.28}$$

As we showed in [6, Lemma 5.3] this formula can be transformed as follows:

$$L^+_{t_0} = \sum_{\mathrm{Re}\lambda_j < \sigma} e^{-\lambda_j t_0} \sum_{n=1}^{m(-\lambda_j)} \frac{t_0^{n-1} R_{-n}(-\lambda_j)}{(n-1)!}. \tag{5.29}$$

Comparing (5.26), (5.29) and (3.58) and taking into account (3.26) we see that operator $L^-_{t_0} = e^{-At_0}|_{X_-}$ where $X_- = (I - \Pi_+)V^1_0(G)$ is defined by the formula

$$L_- = (2\pi i)^{-1} \int\limits_{\gamma_\sigma} (\lambda I + A)^{-1} e^{\lambda t_0} d\lambda \tag{5.30}$$

where γ_σ is the counter described lower formula (3.55). Integral (5.30) was estimated in Theorem 4.6 (see (4.53)). As a result we have:

$$\|L_-\| \leqslant ce^{-(\sigma+\delta)t_0} \tag{5.31}$$

where $\|\cdot\|$ is the norm of operator acting in $V^1_0(G)$.

In [6, Lemma 5.3] we showed that the inverse operator $(L^+_{t_0})^{-1}$ exists and is defined by the formula

$$(L^+_{t_0})^{-1} = \sum_{\mathrm{Re}\lambda_j < (\sigma+\delta_0)} e^{+\lambda_j t_0} \sum_{n=1}^{m(-\lambda_j)} \frac{t_0^{n-1}}{(n-1)!} R_{-n}(-\lambda_j).$$

It follows from this formula that

$$\|(L^+_{t_0})^{-1}\| \leqslant c_1 e^{+(\sigma-\delta)t_0} \tag{5.32}$$

where c_1 does not depend on t_0 and $\delta > 0$ is a sufficiently small magnitude.

In virtue of (5.31), (5.32) inequalities (5.24) are true for enough large t_0 if $\hat{r} = e^{-(\sigma-\frac{\delta}{2})t_0}$ and $\varepsilon_+, \varepsilon_- \in (0,1)$. $\square$

Generally speaking, eigenvalues of operators A and e^{-At} are complex-valued. That is why all spaces in Theorem 5.1 are complex. But to apply obtained results to (nonlinear) Navier–Stokes equations we need to have analogous results for the real spaces of the same type. Actually for this we have to define the projector of (5.27) in real spaces.

The form (3.40) of the main part for Laurent series (3.23) implies that

$$\begin{aligned} R_{-1}(-\lambda_j)v = \sum_{(k)} \big(& e^{(k)}_0(-\lambda_j)\langle v, \varepsilon^{(k)}_{m_k}(-\bar{\lambda}_j)\rangle + e^{(k)}_1(-\lambda_j)\langle v, \varepsilon^{(k)}_{m_k-1}(-\bar{\lambda}_j)\rangle + \\ & + \cdots + e^{(k)}_{m_k}(-\lambda_j)\langle v, \varepsilon^{(k)}_0(-\bar{\lambda}_j)\rangle \big) \end{aligned} \tag{5.33}$$

where $\langle \cdot, \cdot \rangle$ is scalar product in complex space $(L_2(G))$, $\{\varepsilon^{(k)}_l(-\bar{\lambda}_j)\}$ is canonical system (3.64) of operator $-A^*$ corresponding to eigenvalue $-\bar{\lambda}_j$ and $\{e^{(k)}_l(-\lambda_j)\}$ is canonical system of $-A$ corresponding to eigenvalue $-\lambda_j$, which is constructed by $\{\varepsilon^{(k)}_l(-\bar{\lambda}_j)\}$ in Theorem 3.1. We suppose also that $\{\varepsilon^{(k)}_l(-\bar{\lambda}_j)\}$ satisfy (3.72). Then $\{e^{(k)}_l(-\lambda_j)\}$ satisfy analogous property that follows from the proof of Theorem 3.1.

We define restriction of operator (5.33) on the real space $V^1_0(G)$ with help of vector fields (3.73) and

$$\mathrm{Re} e^{(k)}_l(-\lambda_j, x), \quad \mathrm{Im} e^{(k)}_l(-\lambda_j, x). \tag{5.34}$$

Lemma 5.1. *Restriction of operator* (5.33) *on the real space* $V_0^1(G)$ *can be written in the form*

$$(\Pi_+)(x) = \sum_{j=1}^{K} e_j(x) \int_G v(x)\varepsilon_j(x)\,dx \tag{5.35}$$

where $\{e_j\}$ *is the set of functions* (5.34) *which is renumbered and is renormalized by suitable away and* $\{\varepsilon_j\}$ *is renumbered and is renormalized set of functions* (3.74).

The proof of this simple lemma one can find in [6, Lemma 6.2].

The following assertion holds.

Lemma 5.2. *For an arbitrary subdomain* $\omega \subset G$ *vector fields* $\{e_j(x),\, j = 1, \dots, K\}$ *from* (5.35) *restricted on* ω *are linear independent over* $\mathbb{R}$.

Since there is evident symmetry between operators A and A^* and therefore between $\{e_j(x)\}$ and fields (3.74), Lemma 5.2 is proved similarly to Lemma 3.6.

Using (5.35) we can easily restrict spaces X_+ and X_- as well as operators $L_{t_0}^+$, $L_{t_0}^-$ defined in formulation of Theorem 5.1 on the real subspaces of $V_0^1(G)$. We denote this new real spaces and operators also by X_+, X_-, $L_{t_0}^+$, $L_{t_0}^-$. This will not lead to misunderstanding because below we do not use their complex analogs.

In a neighbourhood of steady-state solution a of (5.18) we establish existence of a manifold M_- which is invariant with respect to semigroup $S(t,w)$ (i.e., $S(t,w) \in M_-$ $\forall w \in M_-$). This manifold can be represented as the graph:

$$M_- = \{u \in V_0^1(G) \,:\, u = a + u_- + g(u_-), \quad u \in X_- \cap \mathcal{O}\} \tag{5.36}$$

where $\mathcal{O}$ is a neighbourhood of origin in $V_0^1(G)$, $g : X_- \cap \mathcal{O} \to X_+$ is an operator-function of class $C^{3/2}$ and

$$g(0) = 0, \quad g'(0) = 0. \tag{5.37}$$

Note that condition (5.37) means that manifold (5.36) is tangent to X_- at point a.

The following theorem is true.

Theorem 5.2. *Let a satisfy* (5.18), $\sigma > 0$ *satisfy* (3.55), $\mathcal{O} = \mathcal{O}_\varepsilon = \{v \in V_0^1(G) : \|v\|_{V_0^1(G)} < \varepsilon\}$ *and ε is sufficiently small. Then there exists unique operator-function* $g : X_- \cap \mathcal{O} \to X_+$ *of class* $C^{3/2}$ *satisfying* (5.37) *such that the manifold* M_- *defined in* (5.36) *is invariant with respect to semigroup* $S(t, w_0)$ *connected with* (5.15).[5] *There exists a constant* $c > 0$ *such that*

$$\|S(t, w_0) - a\|_{V_0^1(G)} \leqslant c\|w_0 - a\|_{V_0^1(G)} e^{-\sigma t} \quad \text{as } t \geqslant 0 \tag{5.38}$$

for each $w_0 \in M_-$.

[5] I.e. $S(t, w_0)$ is the resolving semigroup of equation (5.15) (see (5.17)).

This theorem follows form results of [2, Ch. 5, Sect. 2; Ch. 7, Sect. 5] and from Theorem 5.1.

5.3. Extension operator

Here we construct extension operator for Navier–Stokes equations. This operator is analog of extension operators (4.18), (4.48) constructed for Oseen equations.

Recall that the domain Ω and its extension G satisfy (4.1), (4.2) and Conditions 4.1, 4.2. Besides

$$V^1(\Omega,\Gamma_0)=\{u(x)\in V^1(\Omega)\,:\,u|_{\Gamma_0}=0,\ \exists v\in V_0^1(G)\text{ that } u(x)=\gamma_\Omega v(x)\}. \tag{5.39}$$

This space is equipped with the norm

$$\|u\|_{V^1(\Omega,\Gamma_0)}=\inf_{Eu}\|Eu\|_{V_0^1(G)} \tag{5.40}$$

where infimum is taken over all bounded extension operators

$$E: V^1(\Omega,\Gamma_0)\to V_0^1(G).$$

Let us prove the theorem on extension.

Theorem 5.3. *Let $a(x)$ be a steady-state solution of (5.18), $\hat v(x)=\gamma_\Omega a$, and M_- is the invariant manifold constructed in neighbourhood $a+\mathcal{O}$ of a in $V_0^1(G)$ in Theorem 5.2. Let $B_{\varepsilon_1}=\{v_0\in V^1(\Omega,\Gamma_0)\,:\,\|v_0-\hat v\|_{V^1(\Omega,\Gamma_0)}<\varepsilon_1\}$. Then for sufficiently small ε_1 one can construct a continuous operator*

$$\mathrm{Ext}:\hat v+B_{\varepsilon_1}\to M_-, \tag{5.41}$$

which is operator of extension for vector fields from Ω to G:

$$(\mathrm{Ext}v)(x)\equiv v(x),\quad x\in\Omega. \tag{5.42}$$

Proof. By definition of $V^1(\Omega,\Gamma_0)$ there exists a bounded linear extension operator

$$R: V^1(\Omega,\Gamma_0)\to V_0^1(G).$$

Let $\varphi(x)\in C^\infty(\bar G)$, $\varphi(x)\equiv 1$ for $x\in\Omega$ and $\varphi(x)\equiv 0$ outside a neighbourhood of Ω. Similarly to (4.21) we introduce the following operator of extension:

$$Qv(x)=\begin{cases} v(x), & x\in\Omega,\\ \mathrm{rot}(\varphi\,\mathrm{rot}^{-1}Rv)(x)+w(x), & x\in\omega,\end{cases} \tag{5.43}$$

where $w(x)$ is a vector field concentrated in ω which is constructed by $v(x)$. We describe its construction below. At last we define the desired operator Ext by the formula

$$\mathrm{Ext}v=\Pi_-Qz+g(\Pi_-Qz)+a,\quad\text{with } z=v-a, \tag{5.44}$$

where $\Pi_- = I - \Pi_+$, Π_+ is operator (5.35) of projection on $X_+ = \Pi_+ V_0^1(G)$, $X_- = \Pi_- V_0^1(G)$, and $g : X_- \to X_+$ is the operator constructed in Theorem 5.2. By definition (5.36) of M_- we have $\mathrm{Ext} v \in M_-$. Hence we have to provide that the equality

$$(\mathrm{Ext} v)(x) \equiv v(x), \quad x \in \Omega \tag{5.45}$$

is true, which shows that Ext is an extension operator. By (5.35) $\{e_j(x)\}$ generates X_+ and therefore the map $g(u)$ can be written in the form

$$g(u) = \sum_{j=1}^{N} e_j g_j(u).$$

That is why taking into account (5.35) we can rewrite (5.44) in the form

$$\mathrm{Ext} v = a(x) + Qz(x) - \sum_{j=1}^{K} e_j(x) \int\limits_Q Qz(y)\varepsilon_j(y)\, dy + \sum_{j=1}^{K} e_j(x) g_j(\Pi_- Qz), \tag{5.46}$$

$(z = v - a)$.

In virtue of Lemma 5.2 $\{e_j(x),\ x \in \Omega\}$ are linear independent and therefore (5.45), (5.46) imply

$$\int\limits_G Qz(x)\varepsilon_j(x)\, dx = g_j(\Pi_- Qz), \quad j = 1, \dots, K. \tag{5.47}$$

Let $\psi(x) \in C^\infty(\bar{G})$, $\psi(x) = 1$ outside small neighbourhood of Ω and $\psi(x) = 0$, $x \in \Omega$. Similarly to (4.21) we look for the vector field $w(x)$ from (5.43) in the form

$$w(x) = \mathrm{rot}\Big[\psi(x) \sum_{k=1}^{K} c_k \mathrm{rot}^* \varepsilon_k(x)\Big]. \tag{5.48}$$

To find coefficients $(c_1, \dots, c_K) \equiv \vec{c}$ we substitute (5.48) into (5.47) taking into account (5.43), (5.35). As a result we get

$$\vec{z} + A\vec{c} = \vec{g}\big(\Gamma z + (\vec{c}, \mathrm{rot}[\psi \mathrm{rot}^* \vec{\varepsilon}]) - (\vec{e}, \vec{z} + A\vec{c})\big), \tag{5.49}$$

where $\Gamma z = \mathrm{rot}(\varphi \mathrm{rot}^{-1} Rz)$, $\vec{z} = (z_1, \dots, z_K)$, $A = \|a_{jk}\|$ and

$$z_j = \int\limits_G \mathrm{rot}(\varphi \mathrm{rot}^{-1} Rz(x))\varepsilon_j(x)\, dx, \quad a_{jk} = \int\limits_G \mathrm{rot}[\psi(x)\mathrm{rot}^* \varepsilon_k(x)]\varepsilon_j(x)\, dx,$$

$$\vec{g}(u) = (g_1(u), \dots, g_K(u)),$$

$$\vec{\varepsilon} = (\varepsilon_1(x), \dots, \varepsilon_K(x)), \quad \vec{e} = (e_1(x), \dots, e_K(x)), \quad (\vec{c}, \vec{e}) = \sum_{j=1}^{K} c_j e_j.$$

We prove as in Theorem 4.1 that matrix $A = \|a_{jk}\|$ is positive defined and therefore it is invertible.

Applying to both parts of (5.49) the matrix A^{-1} we get the equality

$$\vec{c} = G_z(\vec{c}) \tag{5.50}$$

where the map $G_z : \mathbb{R}^K \to \mathbb{R}^K$ is defined by the relation

$$G_z(\vec{c}) = A^{-1}\vec{g}(\Gamma z - (\vec{e}, \vec{z}) + (\vec{c}, \mathrm{rot}[\psi \mathrm{rot}^* \vec{\varepsilon}]) - (\vec{e}, A\vec{c})) - A^{-1}z. \tag{5.51}$$

In virtue of Theorem 5.2 the map $A^{-1}\vec{g} : \mathbb{R}^K \to \mathbb{R}^K$ belongs to the class $C^{1+1/2}$ and $A^{-1}\vec{g}(0) = 0$, $A^{-1}\vec{g}'(0) = 0$. Therefore for sufficiently small $\|\vec{c}_1\|_{\mathbb{R}^K}$, $\|\vec{c}_2\|_{\mathbb{R}^K}$, $\|z\|_{V_0^1(G)}$ we derive from (5.51) that

$$\begin{aligned}\|G_z(\vec{c}_1) - G_z(\vec{c}_2)\| \leqslant \sup_{\beta\in[0,1]} &\|A^{-1}\vec{g}'(\Gamma z - (\vec{e}, \vec{z}) + \\ &+ (\beta\vec{c}_1 + (1-\beta)\vec{c}_2, \mathrm{rot}[\psi\mathrm{rot}^*\vec{\varepsilon}]) - (\vec{e}, A[\beta\vec{c}_1 + (1-\beta\vec{c}_2)]))\| \cdot \|\vec{c}_1 - \vec{c}_2\| \leqslant \\ &\leqslant \gamma(z, \vec{c}_1, \vec{c}_2)\|\vec{c}_1 - \vec{c}_2\|, \text{ where } \gamma(z, c_1, c_2) \leqslant \gamma_1(\|z\|^{1/2}_{V_0^1(G)} + \|c_1\|^{1/2}_{\mathbb{R}^K} + \|c_2\|^{1/2}_{\mathbb{R}^K}),\end{aligned}$$

and $\gamma_1 > 0$ is a constant. Therefore the map G_z is a contraction one. Hence by contraction mapping principle ([11]) equation (5.50) has a unique solution $\vec{c} = (c_1, \dots, c_K)$ if $\|z\|_{V_0^1(G)}$ is sufficiently small. For these $\|z\|_{V_0^1(G)}$ the operator Ext defined in (5.44), (5.43), (5.48) is the desired extension operator. □

5.4. Theorem on stabilization

First of all we make more precise conditions connected with solution $(\hat{v}, \nabla\hat{p})$ of steady-state problem (5.5), (5.6). We set

$$V^2(\Omega, \Gamma_0) = \{v(x) \in V^2(\Omega) \,:\, v|_{\Gamma_0} = 0, \ \ \exists w \in V^2(G) \cap V_0^1(G), \ v(x) = \gamma_\Omega w\} \tag{5.52}$$

where γ_Ω is the operator of restriction on Ω.

Proposition 5.1. *Let $f \in (L_2(\Omega))^2$ and a pair $(\hat{v}(x), \nabla\hat{p}(x))$ belong to $V^2(\Omega, \Gamma) \times (L_2(\Omega))^2$ and satisfies equations* (5.5), (5.6). *Then there are extension $g(x) \in (L_2(G))^2$ of $f(x)$ from Ω to G and an extension $(a(x), \nabla q(x)) \in (V^2(G) \cap V_0^1(G)) \times L_2(G)$ of $(\hat{v}(x), \nabla\hat{p}(x))$ from Ω to G such that the pair $(a(x), \nabla q(x))$ is a solution of* (5.11), (5.12).

Proof. By definition of the space $V^2(\Omega, \Gamma_0)$ there exists an extension $a(x) \in V^2(G) \cap V_0^1(G)$ of $\hat{v}(x)$ from Ω to G. Since $\nabla p \in L_2(\Omega)$, one can choose $p(x) \in H^1(\Omega)$ and after that extend $p(x)$ from Ω to G up to a function $q(x) \in H^1(G)$. We substitute $(a(x), \nabla q)$ to the left side of the first equation in (5.11) and denote the obtained right side as $g(x)$. Evidently $g(x) \in L_2(G)$. Moreover $\gamma_\Omega g(x) = f(x)$ because $\gamma_\Omega(a(x), \nabla q(x)) = (\hat{v}(x), \nabla p(x))$, and (5.5) is true. □

Remark 5.1. Condition $\hat{v} \in V^2(\Omega, \Gamma_0)$ which we impose on a given steady-state solution is not constructive. But using arguments expounded in Subsection 4.2 one can prove that vector field $\hat{v}(x)$ belongs to $V^2(\Omega, \Gamma_0)$ if it satisfies the following condition: $\hat{v}(x) \in V^2(\Omega)$, $\hat{v}|_{\Gamma_0} = 0$ and $\hat{v}(x)$ is sufficiently smooth in a neighbourhood of $\partial\Gamma$. Moreover we suppose that assumptions (4.1), (4.2), Conditions 4.1, 4.2 are fulfilled and in Condition 4.2 restriction $\alpha \geqslant 3$ holds.

We now are in position to prove the main result of this paper.

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^∞-boundary $\partial\Omega$ and $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \partial\Gamma$, where Γ, Γ_0 are open curves, $\Gamma \neq \emptyset$, $\partial\Gamma$ is a finite number of points or $\partial\Gamma = \emptyset$. Suppose that a domain $G \subset \mathbb{R}^2$ is chosen such that assumptions* (4.1), (4.2) *and Conditions 4.1, 4.2 with $\alpha \geqslant 3$ are fulfilled. Let $f(x) \in (L_2(\Omega))^2$ and $(\nabla\hat{v}(x), \nabla\hat{p}(x)) \in V^2(\Omega, \Gamma_0) \times (L_2(\Omega))^2$ satisfy* (5.5), (5.6)*. Then for an arbitrary $\sigma > 0$ there exists sufficiently small $\varepsilon_1 > 0$ such that for each $v_0 \in V^1(\Omega, \Gamma_0)$ satisfying*

$$\|\hat{v} - v_0\|_{V_0^1(\Omega)} < \varepsilon_1 \tag{5.53}$$

there exists a feedback boundary control $u(t,x) \in \Sigma \equiv \mathbb{R}_+ \times \Gamma$ which stabilized Navier–Stokes boundary value problem (5.1)–(5.4) *with the rate* (5.7)*, i.e. the solution v of* (5.1)–(5.4) *satisfies* (5.7)*.*

Proof. Using Proposition 5.1 we extend $\hat{v}(x)$ to $a(x) \in V^1(G)$, and $f(x)$ to $g(x) \in (L_2(G))^2$. As a result we get boundary value problem (5.8)–(5.10) (with certain w_0) and steady-state solution $(a(x), \nabla q(x))$ of this problem. We can suppose that $\sigma > 0$ satisfies (3.55): otherwise we increase σ a little bit and get (3.55). In virtue of Theorem 5.2 in a neighbourhood of a there exists a manifold M_- which is invariant with respect of semigroup $S(t, w_0)$ connected with equation (5.15) and such that for each $w_0 \in M_-$ inequality (5.38) holds. Let ε_1 be so small that it satisfies condition of Theorem 5.3. Then we apply extension operator Ext constructed in Theorem 5.3 to initial condition v_0 of problem (5.1)–(5.3) and take $w_0 = \mathrm{Ext} v_0$ as initial condition for problem (5.8)–(5.10) or for equation (5.15)(that is equivalent). Then since $w_0 \in M_-$, $S(t, w_0) \in M_-$ for each $t \geqslant 0$, and estimate (5.38) holds. We define solution (v, u) of stabilization problem (5.1)–(5.4) by formula (5.13) where $w(t,x) = S(t, w_0)$ is the solution of (5.15), (5.9). Then (5.7) follows from (5.13), (5.38) $\square$

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