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Inhomogeneous Boundary Value Problems for the Three-Dimensional Evolutionary Navier–Stokes Equations

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Abstract. In this paper, we study the solvability of inhomogeneous boundary value problems for the three-dimensional Oseen and Navier–Stokes equations in the following formulation: given function spaces for Dirichlet boundary conditions, initial values, and right-hand side forcing functions, find function spaces for solutions such that the operator generated by the boundary value problem for the Oseen equations establishes an isomorphism between the space of solutions and the spaces of data. Existence and uniqueness results for the solution of the time-dependent, three-dimensional Navier–Stokes equations are also established. These investigations are based on a theory of extensions of time-dependent, solenoidal vector fields that is developed in this paper. The results of this paper are indispensable to the study of optimal control problems with boundary control for the three-dimensional Navier–Stokes equations.

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1. Introduction

The boundary value problem with a homogeneous Dirichlet boundary condition for the evolutionary Navier–Stokes equations has evident physical interpretations. For this reason, that problem was studied intensively and the theory was developed long ago in the works of Leray [10] and [11], Hopf [8], and Ladyzhenskaya [9], among others. It became clear in recent years that boundary value problems with inhomogeneous boundary conditions are also useful in applications such as the boundary control of incompressible flows. For instance, the formulation and solution of an optimal drag reduction problem through boundary velocity control for a body moving in an incompressible viscous fluid flow requires the theory of inhomogeneous boundary value problems for the Navier–Stokes equations. The need and the nontriviality of such a theory was clearly manifested in our work [6] on the two-dimensional optimal drag reduction problem via boundary velocity control. The purpose of this paper is to develop a theory of inhomogeneous boundary value

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problems for the three-dimensional Navier–Stokes equations.

In order to reduce the investigation of inhomogeneous boundary value problems to that of homogeneous ones, we must study the problem of traces on $\Sigma = (0,T) \times \partial\Omega$ of solenoidal vector fields defined on $Q = (0,T) \times \Omega$, where Ω is a spatial domain and $\partial\Omega$ denotes its boundary. In other words, we need to characterize the restriction of solenoidal vector fields defined on the space-time cylinder Q onto the lateral surface Σ and also study the problem of extending boundary vector fields defined on Σ into solenoidal vector fields defined in Q.

Two settings of trace theorems and related boundary value problems are possible. The first setting is the one that is used in classical theories for elliptic and parabolic boundary value problems; see, e.g., [2], [3], [12], [13]. It goes as follows.

> *Setting 1.* Given the space of desired solutions for a boundary value problem, one identifies the trace on Σ of this space.

This setting, in the case of the three-dimensional, evolutionary Navier–Stokes equations, was studied in [7]. Note that an immediate application of the results of [7] to the optimal drag reduction problem is the description of appropriate boundary terms in the cost functional and of a suitable set of admissible boundary velocity controls.

Suppose now the boundary terms of the cost functional have been chosen. Then, these terms determine the set of admissible Dirichlet boundary conditions. This leads to the second setting of trace theorems (and of boundary value problems).

> *Setting 2.* Given a space $\mathbf{G}$ of desired Dirichlet boundary conditions on Σ, find the space $\mathbf{Y}$ of solenoidal vector fields on Q whose trace space coincides with $\mathbf{G}$. Moreover, $\mathbf{Y}$ can be taken as the proper space of solutions for inhomogeneous boundary value problems for the Navier–Stokes equations or for their linearized analog, the Oseen equations.

Note that we call $\mathbf{Y}$ the proper space of solutions if the direct product of the Oseen operator and the restriction operators on Σ and at $t = 0$ establishes an isomorphism between $\mathbf{Y}$ and the direct product of the space for the forcing term, the space $\mathbf{G}$, and the space for the initial condition.

In this paper we study inhomogeneous boundary value problems for the three-dimensional Oseen and Navier–Stokes equations under Setting 2. Note that in order to derive optimality systems for corresponding optimal boundary control problems we need to solve a boundary value problem in Setting 2.

We suppose that the space $\mathbf{G}$ of Dirichlet boundary values is a subspace of $\mathbf{H}^1(\Sigma)$ whose normal components have zero means over $\partial\Omega$ for almost every $t \in (0,T)$, i.e.,

$$\mathbf{G} = \left\{ \mathbf{u} \in \mathbf{H}^1(\Sigma) \ : \ \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} \, ds = 0 \quad \text{for a.e. } t \in (0,T) \right\}.$$

We choose $\mathbf{L}^2(Q)$ as the space for the forcing term. Then, we construct the space

$\mathbf{Y}$ of solutions as the direct sum of two Hilbert spaces $\mathbf{Y}_1$ and $\mathbf{Y}_2$: $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$. Recall that a Hilbert space $\mathbf{Y}$ is the direct sum of the Hilbert spaces $\mathbf{Y}_1$ and $\mathbf{Y}_2$ ($\mathbf{Y}_1 \cap \mathbf{Y}_2 \neq \phi$) if each $\mathbf{y} \in \mathbf{Y}$ can be decomposed into $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathbf{Y}_1$ and $\mathbf{y}_2 \in \mathbf{Y}_2$ and

$$\|\mathbf{y}\|^2_{\mathbf{Y}} = \inf_{\mathbf{y}_1 \in \mathbf{Y}_1, \mathbf{y}_2 \in \mathbf{Y}_2, \mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2} \left(\|\mathbf{y}_1\|^2_{\mathbf{Y}_1} + \|\mathbf{y}_2\|^2_{\mathbf{Y}_2} \right) . \tag{1.1}$$

In our case, $\mathbf{Y}_2$ is the standard space of strong solutions with zero Dirichlet boundary condition and $\mathbf{Y}_1$ is a special space of solenoidal vector fields defined on Q which are extensions of boundary vector fields belonging to $\mathbf{G}$ (defined on Σ). The main part of this paper is devoted to the construction of the extension operator $E : \mathbf{G} \to \mathbf{Y}_2$. This construction consists of two steps. First, in Section 3, we construct an extension operator E_1 in a way analogous to what we did in [7], although the spaces here are different than those used there. The proofs in Section 3 are not detailed as the readers can refer to [7]. In the second step, we construct E (with the help of E_1) by solving a boundary value problem for the steady-state Stokes equations.

The results of this paper are also valid if the function space for boundary conditions is taken as

$$\mathbf{G}^\gamma = \Big\{ \mathbf{v}(t,\mathbf{x}) \in L^2(0,T;\mathbf{H}^\gamma(\partial\Omega) : \\ \partial_t \mathbf{v}(t,\mathbf{x}) \in L^2(0,T;\mathbf{H}^{\gamma-1}(\partial\Omega), \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,ds = 0 \;\text{ a.e. } t \in [0,T] \Big\}$$

with $\gamma > 1$. Moreover, the proofs for the case $\gamma > 1$ become considerably easier than those for the case $\gamma = 1$, the latter is precisely the case treated in this paper. Of utmost interest to us is the case $\gamma = 1$ since the function space $\mathbf{G} = \mathbf{G}^1$ is a natural choice of Dirichlet controls from a computational point of view for the afore-mentioned drag minimization problem (in contrast, the space $\mathbf{G}^\gamma$ for $\gamma > 1$ involves computationally cumbersome fractional derivatives).

The rest of the paper is organized as follows. In Section 2, we prove theorems on extending solenoidal vector fields that are defined on an open domain into ones defined on a larger open domain. In Section 3, we establish results on solenoidal extensions of boundary data. In Section 4, we first discuss steady-state boundary value problems which depend on t as a parameter; we then construct the final extension operator. Finally, in Section 5, we show the existence and uniqueness of solutions for boundary value problems for the three-dimensional Oseen and Navier–Stokes equations.

2. Theorems on the extension from an open set

In this section, we study the problem of extending solenoidal vector fields on a bounded domain into ones on a larger domain.

Throughout this section we assume that $\Omega \subset \mathbb{R}^3$ is a bounded open set with a C^∞ boundary $\partial\Omega$ and that $\partial\Omega$ consists of J disjoint connected components, i.e.,

$$\partial\Omega = \cup_{j=1}^{J} \Gamma_j \,, \qquad \Gamma_i \cap \Gamma_j = \emptyset \quad \forall\, i \neq j \,, \tag{2.1}$$

where each Γ_j is a C^∞ connected, compact manifold. Also, let Ψ be a domain having a C^∞ boundary that contains Ω, i.e., $\Omega \subset \Psi$. We suppose that if $\partial\Omega \cap \partial\Psi \neq \emptyset$, then $\partial\Omega \cap \partial\Psi = \cup_{i=1}^{m} \Gamma_{j_i}$ $(m < J)$, where Γ_{j_i} are the manifolds defined in (2.1). We will establish results on solenoidal extensions from Ω onto Ψ or from $(0,T) \times \Omega$ onto $(0,T) \times \Psi$.

2.1. The extension from a spatial domain

In this subsection, we prove a theorem on the extension of solenoidal vector fields defined on Ω into solenoidal vector fields defined on Ψ that preserves the smoothness of the original vector fields.

We first introduce some notations and function spaces. The Sobolev space $H^k(\Omega)$ for a natural number k is the space of all functions which possess the finite norm

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(\mathbf{x})|^2 \, d\mathbf{x} \,,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex ($\alpha_j \geq 0$ and integer), $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3})$. The Sobolev space $H^s(\Omega)$ for arbitrary $s > 0$ is defined with the help of $H^k(\Omega)$ by iterpolation; see [12]. By definition, $H^s_0(\Omega)$, $s > 0$, is the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. The space $H^{-s}(\Omega)$, $s > 0$, is defined as the dual space of $H^s_0(\Omega)$, i.e., $H^{-s}(\Omega) = (H^s_0(\Omega))'$ with the norm

$$\|f\|_{H^{-s}(\Omega)} = \sup_{\phi \in H^s_0(\Omega),\, \phi \neq 0} \frac{\langle f, \phi \rangle}{\|\phi\|_{H^s_0(\Omega)}} \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-s}(\Omega)$ and $H^s_0(\Omega)$ generated by the scalar product in $L^2(\Omega)$; see [12].

Let $\Psi \subset \mathbb{R}^3$ be a domain having C^∞ boundary $\partial\Psi$ such that $\Omega \subset \Psi$. If $\partial\Omega \cap \partial\Psi \neq \emptyset$ (this is the case of the most interest to us), we set $Cl_\Psi\Omega \equiv \overline{\Omega} \backslash (\partial\Omega \cap \partial\Psi)$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^3$. The set $Cl_\Psi\Omega$ is called the closure of Ω in Ψ. Further, a set K is called compact in Ψ if $K \subset \Psi$ and $K \cup (\partial\Omega \cap \partial\Psi)$ is compact in $\mathbb{R}^3$. If $\partial\Omega \cap \partial\Psi = \emptyset$, then $Cl_\Psi\Omega = \overline{\Omega}$ and K is compact in Ψ if K is a compact set (in $\mathbb{R}^3$) and $K \subset \Psi$.

Let K be compact in Ψ such that $Cl_\Psi\Omega \subset K \subset \Psi$. Assume that $\partial\Omega \cap \partial\Psi = \emptyset$ or $\partial\Omega \cap \partial\Psi$ is a closed manifold. An operator L is called an extension operator if L maps every function u defined on Ω into a function defined on Ψ and satisfies the property $(Lu)(\mathbf{x}) = u(\mathbf{x})$ for every $\mathbf{x} \in \Omega$. The following results for extension operators are standard.

Lemma 2.1. *Let $m > 0$ be an integer. There exists an extension operator L_m such that for each $s \in [0, m]$, the operator*

$$L_m : H^s(\Omega) \to H^s(\Psi) \tag{2.2}$$

is bounded and for each $u \in H^s(\Omega)$, $L_m u$ is supported in K.

Sketch of proof. In the case $\Omega = \mathbb{R}^3_+ = \{\mathbf{x} = (x_1, x_2, x_3) = (\mathbf{x}', x_3) \in \mathbb{R}^3 : x_3 > 0\}$ and $\Psi = \mathbb{R}^3$, the operator L_m is defined by the well-known Whitney formulas

$$L_m u(\mathbf{x}', x_3) = \begin{cases} u(\mathbf{x}', x_3) & \text{for } x_3 > 0 \\ \sum_{k=1}^m \lambda_k u(\mathbf{x}', -\frac{1}{k} x_3)\phi(\mathbf{x}) & \text{for } x_3 < 0\,, \end{cases}$$

where $\lambda_1, \ldots, \lambda_m$ are defined by the system of equations

$$\sum_{k=1}^m \left(-\frac{1}{k}\right)^j \lambda_k = 1, \quad j = 0, 1, \ldots, m-1\,,$$

and $\phi(\mathbf{x}) \in C^\infty(\mathbb{R}^3)$, $\phi(\mathbf{x}) = 1$ for $x_3 > -\varepsilon/2$, and $\phi(\mathbf{x}) = 0$ for $x_3 < -\varepsilon$. This construction can be transformed to the case of domains Ω, Ψ with the help of partition-of-unity techniques. The boundedness of the operator (2.2) for $s = 0, 1, \ldots, m$ is verified in a straightforward way; see [1]. The boundedness of (2.2) for each $s \in [0, m]$ is established by interpolation; see [12]. $\square$

We introduce, for $s \geq -1$, the space

$$\mathbf{V}^s(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^s(\Omega) : \operatorname{div} \mathbf{v} = 0\}\,, \tag{2.3}$$

where $\operatorname{div} \mathbf{v}$ is understood in the sense of distributions and, for $s \geq 0$, the space

$$\widetilde{\mathbf{V}}^s(\Omega) \equiv \left\{\mathbf{v} \in \mathbf{V}^s(\Omega) : \int_{\Gamma_j} \mathbf{v} \cdot \mathbf{n}\, ds = 0,\ j = 1, \ldots, J\right\},$$

where the Γ_j's are the components of $\partial\Omega$ (see (2.1)) and $\mathbf{n}$ is the outward-pointing unit normal vector along $\partial\Omega$. Note that the boundary integral $\int_{\Gamma_j} \mathbf{v} \cdot \mathbf{n}\, ds$ is well defined even for $s \in [0, 1/2]$ and is understood as $\langle \mathbf{v} \cdot \mathbf{n}, 1\rangle$ for such s, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{s-1/2}(\Gamma_j)$ and $H^{1/2-s}(\Gamma_j)$; see [16]. We have, by definition, that $\|\cdot\|_{\mathbf{V}^s(\Omega)} = \|\cdot\|_{\mathbf{H}^s(\Omega)}$ and $\|\cdot\|_{\widetilde{\mathbf{V}}^s(\Omega)} = \|\cdot\|_{\mathbf{H}^s(\Omega)}$.

We recall some solvability results for the following boundary value problem:

$$\mathbf{curl}\, \mathbf{v} = \mathbf{u} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \qquad \mathbf{v} \cdot \mathbf{n}\big|_{\partial\Omega} = 0\,. \tag{2.4}$$

Lemma 2.2. *Let $s \geq 0$. Then, for each $\mathbf{u} \in \widetilde{\mathbf{V}}^s(\Omega)$, there exists a solution $\mathbf{v} \in \mathbf{V}^{s+1}(\Omega)$ for (2.4).* $\square$

This result is well known. Its proof in the case of integer s can be found in, e.g., [16]. The case of noninteger s can be handled with the help of interpolation theorems; see [7, 12].

Note that ker(**curl**), i.e., the kernel of the operator

$$\mathbf{curl} : \{\mathbf{v} \in \mathbf{V}^{s+1}(\Omega) \ : \ \mathbf{v}\cdot\mathbf{n}\big|_{\partial\Omega} = 0\} \to \widetilde{\mathbf{V}}^s(\Omega),$$

is a finite-dimensional space of C^∞-vector fields depending on the topological structure of Ω. Its complete characterization can be found in, e.g., [16]. Among all solutions of (2.4), we choose the solution $\widetilde{\mathbf{v}}$ which is orthogonal to ker(**curl**) in $\mathbf{L}^2(\Omega)$. Evidently, this solution is determined uniquely and, by the Banach theorem, it satisfies the estimate

$$\|\widetilde{\mathbf{v}}\|_{\mathbf{V}^{s+1}(\Omega)} \leq C\|\mathbf{u}\|_{\widetilde{\mathbf{V}}^s(\Omega)}\,, \tag{2.5}$$

where C does not depend on $\mathbf{u}$. In the sequel, when we speak of the solution of (2.4), it will be understood to be this solution $\widetilde{\mathbf{v}}$. We denote $\widetilde{\mathbf{v}} = \mathbf{curl}^{-1}\mathbf{u}$.

We are now in a position to prove the extension result.

Theorem 2.3. *Suppose that Ω and Ψ are two domains in $\mathbb{R}^3$, Ω is bounded, $\partial\Omega\cap\partial\Psi = \emptyset$ or $\partial\Omega\cap\partial\Psi$ is a closed manifold, and K is compact in Ψ such that $Cl_\Psi\Omega \subset K \subset \Psi$. Let m be a nonnegative integer. Then there exists an extension operator $\mathcal{L}_m$ such that for each $s \in [0,m]$, the operator*

$$\mathcal{L}_m : \widetilde{\mathbf{V}}^s(\Omega) \to \mathbf{V}^s(\Psi)$$

is bounded and for each $\mathbf{u} \in \widetilde{\mathbf{V}}^s(\Omega)$, $\mathcal{L}_m\mathbf{u}$ is supported in K.

Proof. Given $\mathbf{u} \in \widetilde{\mathbf{V}}^s(\Omega)$, let $\widetilde{\mathbf{v}} = \mathbf{curl}^{-1}\mathbf{u} \in \mathbf{V}^{s+1}(\Omega)$ be the solution of (2.4) satisfying (2.5). Since $\widetilde{\mathbf{v}} \in \mathbf{H}^{s+1}(\Omega)$, we can apply Lemma 2.1. Let $L_{m+1} : \mathbf{H}^{s+1}(\Omega) \to \mathbf{H}^{s+1}(\Psi)$ be the operator determined by that lemma. Then, the operator $\mathcal{L}_m \equiv \mathbf{curl} \circ L_m \circ \mathbf{curl}^{-1}$ gives the desired extension. □

2.2. The extension from a space-time domain

We introduce the following spaces of solenoidal vector fields defined on $Q = (0,T)\times\Omega$:

$$\mathbf{V}^{1,s}(Q) = L^2(0,T;\mathbf{V}^{s+1}(\Omega)) \cap H^1(0,T;\mathbf{V}^s(\Omega)) \tag{2.6}$$

and

$$\widetilde{\mathbf{V}}^{1,s}(Q) = L^2(0,T;\widetilde{\mathbf{V}}^{s+1}(\Omega)) \cap H^1(0,T;\widetilde{\mathbf{V}}^s(\Omega))\,.$$

Theorem 2.4. *Assume Ω, K, and Ψ satisfy the conditions of Theorem 2.3 and set $\Theta = (0,T)\times\Psi$. Let $m \geq 0$ be an integer. Then there exists an extension operator $\mathcal{L}_{m+1}$ such that for each $s \in [0,m]$ the operator*

$$\mathcal{L}_{m+1} : \widetilde{\mathbf{V}}^{1,s}(Q) \to \mathbf{V}^{1,s}(\Theta) \tag{2.7}$$

is bounded and for each $\mathbf{u} \in \widetilde{\mathbf{V}}^{1,s}(Q)$, $\mathcal{L}_{m+1}\mathbf{u}$ is supported in $[0,T]\times K$.

Proof. The desired results follow directly from Theorem 2.3 by applying the operator $\mathcal{L}_{m+1}$ in that theorem to $\mathbf{u}(t,\cdot) \in \widetilde{\mathbf{V}}^{s+1}(\Omega)$ and to $\partial_t \mathbf{u}(t,\cdot) \in \widetilde{\mathbf{V}}^{s}(\Omega)$ for almost every $t \in (0,T)$. □

3. Theorems on the extension from a boundary

To establish the theory of inhomogeneous boundary value problems for the Stokes, Oseen, or Navier–Stokes equations, the main problem is to determine the proper extension of vector fields from the lateral boundary into solenoidal vector fields defined on the space-time cylinder. This section will address this problem.

3.1. Extension results for scalar functions

First, we consider the case of nonsolenoidal vector fields, which obviously is reduced to the case of scalar functions.

Let $(t,\mathbf{x}) = (t, x_1, \dots, x_d) \in \mathbb{R}^{d+1}$. We define the function space $H^{1,s}(\mathbb{R}^{d+1})$ $(s \geq 0)$ as

$$H^{1,s}(\mathbb{R}^{d+1}) = \{u(t,\mathbf{x}) \in L^2(\mathbb{R}; H^{s+1}(\mathbb{R}^d)) : \partial_t u(t,\mathbf{x}) \in L^2(\mathbb{R}; H^s(\mathbb{R}^d))\}$$

with the norm

$$\|u\|^2_{H^{1,s}(\mathbb{R}^{d+1})} = \int_{\mathbb{R}^{d+1}} \left[(1+|\boldsymbol{\xi}|^2)^{s+1} + (1+|\tau|^2)(1+|\boldsymbol{\xi}|^2)^s\right] |\widehat{\mathbf{u}}(\tau,\boldsymbol{\xi})|^2 \, d\tau d\boldsymbol{\xi}\,,$$

where $\tau \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^d$, and

$$\widehat{u}(\tau,\boldsymbol{\xi}) = \int_{\mathbb{R}^{d+1}} e^{-i(t\tau + \mathbf{x}\cdot\boldsymbol{\xi})} u(t,\mathbf{x}) \, d\mathbf{x} dt$$

is the Fourier transform of $u(t,\mathbf{x})$. The space $H^{1,s}(\mathbb{R}^d)$ of functions $v(t,\mathbf{x}') = v(t, x_1, \dots, x_{d-1})$ is defined analogously.

The following lemma answers the question of extending functions defined on the hyperplane $\{(t,\mathbf{x}) = (t,\mathbf{x}', x_d) \in \mathbb{R}^{d+1} : x_d = 0\}$ into functions defined on $\mathbb{R}^{d+1}$.

Lemma 3.1. *There exists a continuous linear operator*

$$L : H^{1,1}(\mathbb{R}^d) \times H^{1,0}(\mathbb{R}^d) \to H^{1,3/2}(\mathbb{R}^{d+1}) \tag{3.1}$$

such that for each $w_0(t,\mathbf{x}') \in H^{1,1}(\mathbb{R}^d)$ and $w_1(t,\mathbf{x}') \in H^{1,0}(\mathbb{R}^d)$, the function $L(w_0,w_1)(t,\mathbf{x})$ satisfies the conditions

a) $L(w_0,w_1)(t,\mathbf{x})\big|_{x_d=0} = w_0(t,\mathbf{x}')$, $\dfrac{\partial}{\partial x_d} L(w_0,w_1)(t,\mathbf{x})\big|_{x_d=0} = w_1(t,\mathbf{x}')$,

$$\frac{\partial^2}{\partial x_d^2} L(w_0,w_1)(t,\mathbf{x})\big|_{x_d=0} = 0;$$

b) $\operatorname{supp} L(w_0, w_1) \subset \{(t, \mathbf{x}) \in \mathbb{R}^{d+1} : |x_d| \le \varepsilon\}$, *where* $\varepsilon > 0$ *is a given (small) number.*

Proof. We use the well-known approach; see, e.g., [12] and also [7]. Denote

$$a(\boldsymbol{\xi}') = 1 + |\boldsymbol{\xi}'|^2 \quad \text{and} \quad b(\tau) = 1 + |\tau|^2 .$$

Let $\phi_k(r) \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \phi_k \subset [-\varepsilon, \varepsilon]$ and $\phi_k(r) = r^k/k!$ for $|r| \le \varepsilon/2$, $k = 0, 1$. For $u(t, \mathbf{x}) \in H^{1,3/2}(\mathbb{R}^{d+1})$, we denote by

$$\widetilde{u}(\tau, \boldsymbol{\xi}', x_d) = (\widetilde{F}u)(\tau, \boldsymbol{\xi}', x_d) = \widetilde{u}(\tau, \xi_1, \cdots, \xi_{d-1}, x_d)$$

the Fourier transform of $u(t, \mathbf{x})$ with respect to the variables $(t, \mathbf{x}')$. We define the extension operator L by the formula

$$L(w_0, w_1) = \beta_0 w_0 + \beta_1 w_1 , \tag{3.2}$$

where the operators β_k are defined by

$$(\beta_k w_k)(t, \mathbf{x}', x_d) = \widetilde{F}^{-1}\big[\phi_k(a^{1/2}(\boldsymbol{\xi}')x_d)\, \widehat{w}_k(\tau, \boldsymbol{\xi}')\, a^{-k/2}(\boldsymbol{\xi}')\big], \qquad k = 0, 1; \tag{3.3}$$

here $\widetilde{F}^{-1}$ denotes the inverse Fourier-transform in the variables $(\tau, \boldsymbol{\xi}')$ for functions $\eta(\tau, \boldsymbol{\xi}', x_d)$ defined on $\mathbb{R}^{d+1}$ and $\widehat{w}_k(\tau, \boldsymbol{\xi}')$ denotes the Fourier transform of $w_k(t, \mathbf{x}')$, $k = 0, 1$. Using the definition of ϕ_k we may easily verify that

$$(\beta_0 w_0)\big|_{x_d=0} = \widetilde{F}^{-1}\big[\widehat{w}_0\big] = w_0, \qquad \frac{\partial(\beta_0 w_0)}{\partial x_d}\Big|_{x_d=0} = 0, \qquad \frac{\partial^2(\beta_0 w_0)}{\partial x_d^2}\Big|_{x_d=0} = 0,$$

$$(\beta_1 w_1)\big|_{x_d=0} = 0, \qquad \frac{\partial(\beta_1 w_1)}{\partial x_d}\Big|_{x_d=0} = \widetilde{F}^{-1}\big[\widehat{w}_1\big] = w_1, \qquad \frac{\partial^2(\beta_1 w_1)}{\partial x_d^2}\Big|_{x_d=0} = 0 .$$

These equalities and (3.2) imply assertions a) and b). Taking the Fourier transform on (3.3) we obtain:

$$\widehat{\beta_k w_k}(\tau, \boldsymbol{\xi}) = \frac{\widehat{w}_k(\tau, \boldsymbol{\xi}')}{a^{(1+k)/2}(\boldsymbol{\xi}')} \widehat{\phi}_k(\xi_d a^{-1/2}(\boldsymbol{\xi}'))$$

so that

$$\begin{aligned}
&\|\beta_k w_k\|^2_{H^{1,3/2}(\mathbb{R}^{d+1})} \\
&= \int_{\mathbb{R}^{d+1}} \Big[(a(\boldsymbol{\xi}') + \xi_d^2)^{\frac{5}{2}} + b(\tau)(a(\boldsymbol{\xi}') + \xi_d^2)^{\frac{3}{2}}\Big] \frac{|\widehat{w}_k|^2}{a^{1+k}(\boldsymbol{\xi}')}\, \widehat{\phi}_k(\xi_d a^{-\frac{1}{2}}(\boldsymbol{\xi}'))\, d\tau d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^{d+1}} \Big[(a(\boldsymbol{\xi}') + a(\boldsymbol{\xi}')y^2)^{5/2} + b(\tau)(a(\boldsymbol{\xi}') + a(\boldsymbol{\xi}')y^2)^{3/2}\Big] \\
&\qquad \cdot \frac{|\widehat{w}_k|^2}{a^{k+1/2}(\boldsymbol{\xi}')}\, \widehat{\phi}_k(y)\, d\tau d\boldsymbol{\xi}' dy \\
&\le C \int_{\mathbb{R}^d} [a^{2-k}(\boldsymbol{\xi}') + b(\tau)a^{1-k}(\boldsymbol{\xi}')]\, |\widehat{w}_k|^2\, d\tau d\boldsymbol{\xi}' = \|w_k\|^2_{H^{1,1-k}(\mathbb{R}^d)}, \qquad k = 0, 1.
\end{aligned}$$

These inequalities imply that the operator L defined in (3.2) is bounded from $H^{1,1}(\mathbb{R}^d) \times H^{1,0}(\mathbb{R}^d)$ to $H^{1,3/2}(\mathbb{R}^{d+1})$. $\square$

The operator in (3.1) satisfying conditions a) and b) is called an extension operator.

Having established the extension result for functions defined on the half space with an unbounded boundary, we now consider similar results for functions defined on domains with a compact boundary. Let Ω be a bounded domain with a C^∞ boundary or be the exterior of such a domain. Recall that we denote $Q = (0,T)\times\Omega$ and $\Sigma = (0,T) \times \partial\Omega$. For $s \geq 0$, define the spaces

$$H^{1,s}(Q) = \{y(t, \mathbf{x}) \in L^2(0,T; H^{s+1}(\Omega) : \partial_t y \in L^2(0,T; H^s(\Omega)\}$$

and

$$H^{1,s}(\Sigma) = \{y(t, \mathbf{x}) \in L^2(0,T; H^{s+1}(\partial\Omega) : \partial_t y \in L^2(0,T; H^s(\partial\Omega)\}.$$

Theorem 3.2. *For every $\varepsilon > 0$, there exist continuous linear operators*

$$E_k : H^{1,1-k}(\Sigma) \to H^{1,3/2}(Q), \qquad k = 0,1$$

such that the restriction operators written below are well defined for $E_k w$, $k = 0,1$, and

$$\begin{aligned} E_0 w\big|_\Sigma &= w, & \partial_n(E_0 w)\big|_\Sigma &= 0, & \partial_{nn}(E_0 w)\big|_\Sigma &= 0,\\ E_1 w\big|_\Sigma &= 0, & \partial_n(E_1 w)\big|_\Sigma &= w, & \partial_{nn}(E_1 w)\big|_\Sigma &= 0\,, \end{aligned}$$

where ∂_n and ∂_{nn} are the first and second-order normal derivatives, respectively. Moreover, $E_k w$, $k = 0,1$, have support in an ε-neighborhood of Σ.

Proof. Using a partition of unity and a rectification of Σ, one can easily derive the results of this theorem from Lemma 3.1. $\square$

3.2. An equation for differential forms

For now (until Theorem 3.6), we only consider spatial (i.e., time-independent) vector fields. In order to extend a vector field defined on $\partial\Omega$ into a solenoidal vector field on Ω, we use special local coordinates in a neighborhood of $\partial\Omega$. We suppose now that the domain $\Omega \subset \mathbb{R}^3$.

Let Γ_i be a connected component of $\partial\Omega$. For notational convenience, we dispense with the subscript i for Γ_i, i.e., we use Γ in place of Γ_i. We consider the bounded domain

$$\Theta(= \Theta_{i,\delta}) \equiv \{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x}, \Gamma) < \delta\}\,,$$

where $\delta > 0$ is a sufficiently small number.

We will need the following well-known lemma, a complete proof of which can be found in, e.g., [7].

Lemma 3.3. *Define*

$$y_3(\mathbf{x}) = dist(\mathbf{x}, \Gamma), \qquad \mathbf{x} \in \Theta\,. \tag{3.4}$$

Then, there exists a finite covering $\{U_j\}$ *of* Θ *such that in each* U_j *there exists a local coordinate system* $(y_1(\mathbf{x}), y_2(\mathbf{x}), y_3(\mathbf{x}))$*, with* y_3 *defined by* (3.4)*, which is oriented as* (x_1, x_2, x_3) *and satisfies*

$$\nabla y_j(\mathbf{x}) \cdot \nabla y_k(\mathbf{x}) = \delta_{jk}, \qquad j,k = 1,2,3\,, \tag{3.5}$$

where δ_{jk} *is the Kronecker symbol.* □

Using (3.5), one can calculate, in the local coordinates (y_1, y_2, y_3), the metric tensor $g_{kl}(\mathbf{y})$ generated by the Euclidean metric of the enveloping space $\mathbb{R}^3$. The calculation yields

$$g_{kl}(\mathbf{y}) = \delta_{kl}\,. \tag{3.6}$$

As is well known (see [4, Ch.4, §29.3] and [14, Ch.VI, §4]), relation (3.6) implies that the formulas for the div and **curl** operators in local coordinates (y_1, y_2, y_3) have the usual form:

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{3} \frac{\partial v_j}{\partial y_j} \qquad \text{and} \qquad \mathbf{curl}\,\mathbf{v} = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)\,.$$

The plan for proving the extension theorem is as follows. First, for a given vector field $\mathbf{u}$ on Γ, we solve the system of equations

$$\mathbf{curl}\,\mathbf{w}\big|_{\Gamma} = \mathbf{u} \tag{3.7}$$

for each component Γ of $\partial\Omega$ and obtain the boundary value $\mathbf{w}\big|_{\Gamma}$ and hence $\mathbf{w}\big|_{\partial\Omega}$. Then, we extend $\mathbf{w}\big|_{\partial\Omega}$ into a $\mathbf{w}$ in Ω with the help of standard extension theorems such as Theorem 3.2. The desired extension $\mathbf{v}$ for the boundary data $\mathbf{u}$ is obtained by setting $\mathbf{v} = \mathbf{curl}\,\mathbf{w}$. This plan is carried out below.

Let

$$\mathbf{u} = \sum_{i=1}^{3} u_i \frac{\partial}{\partial y_i} = (\mathbf{u}_\tau, u_n) \tag{3.8}$$

be a given vector field defined on Γ where $\mathbf{u}_\tau = u_1 \frac{\partial}{\partial y_1} + u_2 \frac{\partial}{\partial y_2}$ is the tangential component of $\mathbf{u}$ and $u_n = u_3 \frac{\partial}{\partial y_3}$ is the normal component of $\mathbf{u}$. Applying to (3.8) the operation of lowering the indices (see [4, p. 170]) and then applying the operation $*$ (see [4, p. 175]) and using (3.6), we can express the vector field (3.8) in the exterior differential form:

$$\widehat{\mathbf{u}} = u_1 dy_2 \wedge dy_3 + u_2 dy_3 \wedge dy_1 + u_3 dy_1 \wedge dy_2\,. \tag{3.9}$$

By the operations of lowering indices we may express $\mathbf{w}$ in (3.7) as a differential form $\widehat{\mathbf{w}}$ on Θ which in local coordinates (y_1, y_2, y_3) takes the form

$$\widehat{\mathbf{w}} = w_1(y_1, y_2, y_3)dy_1 + w_2(y_1, y_2, y_3)dy_2 + w_3(y_1, y_2, y_3)dy_3.$$

We introduce the following differential form $\check{\mathbf{w}}$ on Γ which depends on y_3 as a parameter:

$$\check{\mathbf{w}} = w_1(y_1, y_2, y_3)dy_1 + w_2(y_1, y_2, y_3)dy_2.$$

Now we rewrite (3.7) as the equation $d\widehat{\mathbf{w}} = \widehat{\mathbf{u}}$ for differential forms, which in local coordinates (y_1, y_2, y_3) can be written as follows:

$$(\partial_{y_1} w_2 - \partial_{y_2} w_1)dy_1 \wedge dy_2 = u_3 dy_1 \wedge dy_2 \qquad \text{at } y_3 = 0\,, \tag{3.10}$$

$$(\partial_{y_2} w_3 - \partial_{y_3} w_2)dy_2 \wedge dy_3 = u_1 dy_2 \wedge dy_3 \qquad \text{at } y_3 = 0\,, \tag{3.11}$$

and

$$(\partial_{y_1} w_3 - \partial_{y_3} w_1)dy_1 \wedge dy_3 = -u_2 dy_1 \wedge dy_3 \qquad \text{at } y_3 = 0\,. \tag{3.12}$$

To find the restrictions $w_i\big|_{y_3=0}$ $(i = 1, 2, 3)$ from (3.10)–(3.12), we set

$$w_3\big|_{y_3=0} = 0\,. \tag{3.13}$$

Then, (3.11)–(3.13) imply

$$-\partial_{y_3} w_2 = u_1 \qquad \text{and} \qquad \partial_{y_3} w_1 = u_2 \qquad \text{at } y_3 = 0\,. \tag{3.14}$$

To find the traces for w_1 and w_2 at $y_3 = 0$ we have to solve (3.10) defined on the manifold Γ. This equation can be rewritten in the invariant form

$$d\boldsymbol{\omega} = \widetilde{\mathbf{u}}\,, \tag{3.15}$$

where $\boldsymbol{\omega} = \omega_1 dy_1 + \omega_2 dy_2$ is an unknown differential form of the first order which in local coordinates can be rewritten as

$$\boldsymbol{\omega} = \omega_1 dy_1 + \omega_2 dy_2 = \check{\mathbf{w}}\big|_{y_3=0}$$

and the differential form $\widetilde{\mathbf{u}}$ of the second order has the following expression in local coordinates:

$$\widetilde{\mathbf{u}} = u_3 dy_1 \wedge dy_2\,.$$

Let $\Lambda_i(\Gamma)$, $i = 0, 1, 2$, denote the space of differential forms on Γ of order i. The operator $d : \Lambda^1(\Gamma) \to \Lambda^2(\Gamma)$ in (3.15) is the usual operator of taking differentials (see [15]). Below, we will make use of the conjugation operators $*$ (see [15]):

$$* : \Lambda^1(\Gamma) \to \Lambda^1(\Gamma)\,,$$

$$* : \Lambda^2(\Gamma) \to \Lambda^0(\Gamma)\,,$$

and

$$* : \Lambda^0(\Gamma) \to \Lambda^2(\Gamma)$$

which in local coordinates can be expressed as

$$*\boldsymbol{\omega} = -\omega_2 dy_1 + \omega_1 dy_2 \qquad \text{for } \boldsymbol{\omega} = \omega_1 dy_1 + \omega_2 dy_2 \in \Lambda^1(\Gamma)\,,$$

$$*f = f dy_1 \wedge dy_2 \qquad \text{for } f \in \Lambda^0(\Gamma)$$

and

$$*(f dy_1 \wedge dy_2) = f \qquad \text{for } f dy_1 \wedge dy_2 \in \Lambda^2(\Gamma)\,.$$

Note that (3.14) can be rewritten in the following invariant form:

$$*(\partial_{y_3} \check{\mathbf{w}})\big|_{y_3=0} = \widehat{\mathbf{u}}, \quad \text{where } \widehat{\mathbf{u}} = u_1 dy_1 + u_2 dy_2. \tag{3.16}$$

We supplement (3.15) with the following equation:

$$d(*\boldsymbol{\omega}) = 0\,. \tag{3.17}$$

Recall that the Laplace operator Δ on $f \in \Lambda^0(\Gamma)$ is defined by the formula

$$\Delta f = - * d * df \qquad \forall\, f \in \Lambda^0(\Gamma).$$

We define $L^2(\Lambda^i)$, $i = 0, 1, 2$, to be the set of $\boldsymbol{\omega} \in \Lambda^i(\Gamma)$ with the finite norm

$$\|\boldsymbol{\omega}\|^2_{L^2(\Lambda^i)} = \int_\Gamma \boldsymbol{\omega} \wedge *\boldsymbol{\omega}\,.$$

(The scalar product in $L^2(\Lambda^i)$ is defined by $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \int_\Gamma \boldsymbol{\omega}_1 \wedge *\boldsymbol{\omega}_2$.) We introduce the following subspaces of $L^2(\Lambda^1)$:

$$F = F(\Gamma) = \text{the closure of } \{df \;:\; f \in C^1(\Gamma)\} \text{ in } L^2(\Lambda^1)\,,$$

$$F^* = F^*(\Gamma) = \text{the closure of } \{*df \;:\; f \in C^1(\Gamma)\} \text{ in } L^2(\Lambda^1)\,,$$

and

$$G = G(\Gamma) = \{\boldsymbol{\omega} \in L^2(\Lambda^1) : \boldsymbol{\omega} = df \text{ where } f \in \Lambda^0(\Gamma) \text{ and } \Delta f = 0\}\,.$$

The spaces F, F^* and G are mutually orthogonal and

$$L^2(\Lambda^1) = F \oplus F^* \oplus G\,. \tag{3.18}$$

We now study the solvability of the system formed by (3.15) and (3.17).

Lemma 3.4.

a) *There exists a solution $\boldsymbol{\omega} \in G \oplus F^*$ of* (3.15) *and* (3.17) *if and only if the right-hand side $\widetilde{\mathbf{u}} \in L^2(\Lambda^2)$ satisfies the condition*

$$\int_\Gamma \widetilde{\mathbf{u}} = 0\,. \tag{3.19}$$

b) *Any solution $\boldsymbol{\omega}$ of* (3.15) *and* (3.17) *admits the representation*

$$\boldsymbol{\omega} = \mathbf{g} + *dr\,,$$

where $\mathbf{g}$ is an arbitrary element of G and $r \in L^2(\Lambda^0)$ is the unique solution of the equation

$$\Delta r = - * \widetilde{\mathbf{u}} \tag{3.20}$$

in the class $r \in L^2(\Lambda^0)$ *satisfying*

$$\int_\Gamma *r = 0\,. \tag{3.21}$$

We give here only a sketch of the proof. For a complete proof, see [7].

Sketch of proof. Let

$$D(d) = \{\boldsymbol{\omega} \in L^2(\Lambda^1) : d\boldsymbol{\omega} \in L^2(\Lambda^2)\}.$$

Integration by parts yields that each $\widetilde{\mathbf{u}} \in dD(d)$ satisfies (3.19).

Let $\widetilde{\mathbf{u}} \in dD(d)$ be given. Then, by definition there exists a $\boldsymbol{\omega} \in L^2(\Lambda^1)$ satisfying (3.15). By (3.18), $\boldsymbol{\omega} = \mathbf{g} + *dr + dr_1$, where $\mathbf{g} \in G$, $*dr \in F^*$, and $dr_1 \in F$. Substituting this decomposition into (3.15) and taking into account of the facts that $ddr_1 = 0$ and $d\mathbf{g} = 0$ (as $\mathbf{g} = df$), we have

$$d * dr = \widetilde{\mathbf{u}}\,. \tag{3.22}$$

By the definition of Δ, (3.22) yields (3.20).

Multiplying the corresponding sides of (3.22) and (3.20), we see that in any parametric circle U_i,

$$\int_{U_i} |\Delta r|^2 dy_1 \wedge dy_2 = \int_{U_i} |\widetilde{u_3}|^2 dy_1 \wedge dy_2\,.$$

This equality implies the closedness of the set $dD(d)$ in $L^2(\Lambda^2)$, which in turn leads to the solvability of (3.15) for each $\widetilde{\mathbf{u}}$ satisfying (3.19).

If $\boldsymbol{\omega}_1 = \mathbf{g} + *dr + dr_1$, where $\mathbf{g} \in G$, $*dr \in F^*$, and $dr_1 \in F$, is a solution of (3.15), then $\boldsymbol{\omega} = \mathbf{g} + *dr$ will also be a solution of (3.15), as $ddr_1 = 0$. If $\mathbf{g} \in G$, then $\mathbf{g} = df$, where $\Delta f = - * d * df = 0$. Thus, $d * \mathbf{g} = d * df = \mathbf{0}$ so that the differential form $\boldsymbol{\omega} = \mathbf{g} + *dr$ satisfies (3.17).

The uniqueness of the solution for (3.20) in the class of functions satisfying (3.21) can be proved by integration by parts. $\square$

Below, we solve the system (3.15) and (3.17) on $\partial\Omega = \cup_{j=1}^J \Gamma_j$ assuming condition (3.19) holds for each component $\Gamma = \Gamma_j$. Evidently, the solution of this system defined on $\partial\Omega$ is reduced to its solution on each component Γ of $\partial\Omega$.

Recall that the definition of the Sobolev spaces $H^s(\partial\Omega) = H^s(\Lambda^0) = \Pi_{j=1}^J H^s(\Gamma_j)$ of functions defined on $\partial\Omega = \cup_{j=1}^J \Gamma_j$ can be found in [12]. The Sobolev space $H^s(\Lambda^2)$ is the set of exterior forms $\mathbf{u} \in \Lambda^2(\partial\Omega) = \Pi_{j=1}^J \Lambda^2(\Gamma_j)$ such that $*\mathbf{u} \in H^s(\Lambda^0)$. The Sobolev space $H^s(\Lambda^1)$ is the set of exterior forms $\mathbf{u} \in \Lambda^1(\partial\Omega) = \Pi_{j=1}^J \Lambda^1(\Gamma_j)$ which, in each parametric circle U_i of Γ_j, $j = 1, \dots, J$, have the form $\mathbf{u} = q_1(\mathbf{y})dy_1 + q_2(\mathbf{y})dy_2$, where the coefficients $q_k(\mathbf{y})$, $k = 1, 2$, belong to Sobolev space $H^s(U_i)$.

Lemma 3.5. *Let* $s \geq 0$ *and assume that* $\widetilde{\mathbf{u}} \in H^s(\Lambda^2)$ *satisfies the condition* (3.19) *for each component* Γ_j *of* $\partial\Omega$. *Then, there exists a unique solution* $\boldsymbol{\omega} \in$

$H^{s+1}(\Lambda^1) \cap F^* = \Pi_{j=1}^{J}\big(H^{s+1}(\Lambda^1(\Gamma_j)) \cap F^*(\Gamma_j)\big)$ *for* (3.15) *and* (3.17) *satisfying the estimate*

$$\|\boldsymbol{\omega}\|_{H^{s+1}(\Lambda^1)} \leq C\|\widetilde{\mathbf{u}}\|_{H^s(\Lambda^2)}\,. \tag{3.23}$$

Proof. As is well known, for $\widetilde{\mathbf{u}} \in H^s(\Lambda^2)$ satisfying (3.19), there exists a unique solution $r \in H^{s+2}(\Lambda^0)$ of (3.20)–(3.21); also, the following estimate holds:

$$\|r\|_{H^{s+2}(\Lambda^0)} \leq C\|\widetilde{\mathbf{u}}\|_{H^s(\Lambda^2)}\,. \tag{3.24}$$

Set $\boldsymbol{\omega} = *dr$. Then, as in the proof of Lemma 3.4, one can establish that the exterior form $\boldsymbol{\omega} \in H^{s+1}(\Lambda^1)$ and satisfies (3.15) and (3.17). Inequality (3.23) follows from (3.24). □

Analogous to the space $\mathbf{H}^{1,s}(\Sigma)$ defined in Section 3.1, we define

$$\begin{aligned} H^{1,s}(\Lambda^i(\Sigma)) = \{\mathbf{v}(t,\mathbf{x}) \in L^2(0,T;H^{s+1}(\Lambda^i(\partial\Omega)) :& \\ \partial_t\mathbf{v}(t,\mathbf{x}) \in L^2(0,T;H^s(\Lambda^i(\partial\Omega))\}, &\qquad i=0,1,2\,. \end{aligned} \tag{3.25}$$

We may now state the main result of this subsection which is evidently a consequence of Lemma 3.5.

Theorem 3.6. *Let $s \geq 0$ and assume that $\widetilde{\mathbf{u}} \in H^{1,s}(\Lambda^2(\Sigma))$ satisfies* (3.19) *for each component Γ of $\partial\Omega$ for almost every $t \in [0,T]$. Then, there exists a unique differential form $\boldsymbol{\omega} \in H^{1,s+1}(\Lambda^1(\Sigma))$ which belongs to F^* for almost every $t \in [0,T]$ and satisfies* (3.15) *and* (3.17) *for almost every $t \in [0,T]$. Furthermore, the following estimate holds:*

$$\|\boldsymbol{\omega}\|_{H^{1,s+1}(\Lambda^1(\Sigma))} \leq C\|\widetilde{\mathbf{u}}\|_{H^{1,s}(\Lambda^2(\Sigma))}\,. \tag{3.26}$$

Proof. By Lemma 3.4, (3.15) and (3.17) with $\widetilde{\mathbf{u}}(t,\cdot)$ being the right-hand side of (3.15), admit a unique solution $\boldsymbol{\omega}(t) \in F^*$ for almost every $t \in (0,T)$. Using the results of Lemma 3.5, one can easily verify that all statements of this theorem hold for the solution $\boldsymbol{\omega}$. □

3.3. The extension result for solenoidal vector fields

Denote by $\mathcal{T}_\tau(\partial\Omega)$ (respectively $\mathcal{T}_\tau(\Sigma)$) the space of tangent vector fields on $\partial\Omega$ (respectively on Σ). Similarly, denote by $\mathcal{T}_n(\partial\Omega)$ (respectively $\mathcal{T}_n(\Sigma)$) the space of normal vector fields on $\partial\Omega$ (respectively on Σ). Let $\mathcal{T}(\partial\Omega) = \mathcal{T}_\tau(\partial\Omega) + \mathcal{T}_n(\partial\Omega)$ and $\mathcal{T}(\Sigma) = \mathcal{T}_\tau(\Sigma) + \mathcal{T}_n(\Sigma)$. Recall that the Sobolev space $H^s(\mathcal{T}(\partial\Omega))$ is the set of functions $\mathbf{u} \in \mathcal{T}(\partial\Omega)$ which in each parametric circle U_j have the form $q_1(\mathbf{y})\frac{\partial}{\partial y_1} + q_2(\mathbf{y})\frac{\partial}{\partial y_2} + q_3(\mathbf{y})\frac{\partial}{\partial y_3}$, where $q_i(\mathbf{y}) \in H^s(U_i)$, $i=1,2,3$. The Sobolev spaces $H^s(\mathcal{T}_\tau(\partial\Omega))$ and $H^s(\mathcal{T}_n(\partial\Omega))$ are defined analogously.

Remark 3.7. $\mathbf{u} \in \mathcal{T}_n(\partial\Omega)$ can be decomposed as follows: $\mathbf{u} = u_n\mathbf{n}$ and $u_n \in \Lambda^0(\partial\Omega)$. Thus, the space $H^s(\mathcal{T}_n(\partial\Omega))$ for $\mathbf{u}$ and the space $H^s(\Lambda^0(\partial\Omega)) = H^s(\partial\Omega)$ for u_n are isomorphic. □

As in (3.25), we set

$$H^{1,s}(\mathcal{T}(\Sigma)) = \{\mathbf{v}(t,\mathbf{x}) \in L^2(0,T; H^{s+1}(\mathcal{T}(\partial\Omega)) : \partial_t\mathbf{v} \in L^2(0,T; H^s(\mathcal{T}(\partial\Omega))\}\,.$$

The spaces $H^{1,s}(\mathcal{T}_n(\Sigma))$ and $H^{1,s}(\mathcal{T}_\tau(\Sigma))$ are defined analogously. Also, we have $H^{1,s}(\mathcal{T}(\Sigma)) = H^{1,s}(\mathcal{T}_n(\Sigma)) + H^{1,s}(\mathcal{T}_\tau(\Sigma))$. In other words, for each $\mathbf{u} \in H^{1,s}(\mathcal{T}(\Sigma))$, we have

$$\mathbf{u} = \mathbf{u}_\tau + u_n\mathbf{n} \quad \text{where } \mathbf{u}_\tau \in H^{1,s}(\mathcal{T}_\tau(\Sigma)) \text{ and } u_n \in H^{1,s}(\Sigma)\,. \tag{3.27}$$

Define

$$\begin{aligned}\widetilde{H}^{1,s}(\mathcal{T}(\Sigma)) = \Big\{&\mathbf{u} \in H^{1,s}(\mathcal{T}(\Sigma)) \ : \\ &\int_{\Gamma_j} u_n(t,\mathbf{x})ds = 0 \text{ for } j = 1,\ldots,J \text{ and for almost every } t \in (0,T)\Big\},\end{aligned} \tag{3.28}$$

where u_n is the normal component of $\mathbf{u}$ as in the decomposition (3.27).

The following assertion is true.

Theorem 3.8. *Suppose that $\Omega \subset \mathbb{R}^3$, that $\partial\Omega$ is of class C^∞ and is a compact set, and that $\varepsilon > 0$ is given. Then, there exists a continuous extension operator*

$$E_c : \widetilde{H}^{1,0}(\mathcal{T}(\Sigma)) \to \mathbf{V}^{1,1/2}(Q)\,, \tag{3.29}$$

i.e., the operator E_c is such that for every $\mathbf{u} \in \widetilde{H}^{1,0}(\mathcal{T}(\Sigma))$ the restrictions $E_c\mathbf{u}|_\Sigma$ and $\partial_n(E_c\mathbf{u})|_\Sigma$ are well defined and $E_c\mathbf{u}\big|_\Sigma = \mathbf{u}$ and $\partial_n(E_c\mathbf{u})\big|_\Sigma \in \mathbf{L}^2(\Sigma)$. Moreover, for each $\mathbf{u} \in \widetilde{H}^{1,0}(\mathcal{T}(\Sigma))$, the vector field $E_c\mathbf{u}$ is supported in the ε-neighborhood of Σ:

$$\operatorname{supp}(E_c\mathbf{u}) \subset \{(t,\mathbf{x}) \in Q \,:\, \operatorname{dist}((t,\mathbf{x});\Sigma) < \varepsilon\}\,,$$

where $\operatorname{dist}((t,\mathbf{x});\Sigma)$ *is the Euclidean distance between the point* $(t,\mathbf{x})$ *and the set* Σ.

Proof. Let $\mathbf{u} = \mathbf{u}_\tau + u_n\mathbf{n} \in \widetilde{H}^{1,0}(\mathcal{T}(\Sigma))$ be the trace data, which in the local coordinates (y_1, y_2, y_3) introduced in Lemma 3.3 can be written in the form (3.8). We reduce the vector field (3.8) to the differential form (3.9) and consider the system of equations (3.10)–(3.12). This system can be reduced to the system (3.10), (3.13), and (3.14) supplemented by the equations

$$(\partial_{y_3}w_3)\big|_{y_3=0} = 0, \qquad (\partial^2_{y_3y_3}w_i)\big|_{y_3=0} = 0 \quad i = 1,2\,. \tag{3.30}$$

The system (3.10) can be rewritten in the invariant form (3.15) and (3.17) with $\widetilde{\mathbf{u}} \in H^{1,0}(\Lambda^2(\Sigma))$; the system (3.14) can be rewritten as (3.16); and (3.30) can be rewritten as

$$(\partial^2_{y_3y_3}\check{\mathbf{w}})\big|_{\partial\Omega} = 0\,.$$

Theorem 3.6 asserts that there exists a unique solution $\boldsymbol{\omega} \in H^{1,1}(\Lambda^1(\Sigma))$ of (3.15) and (3.17) satisfying (3.26). The existence of a solution $\partial_{y_3}\check{\mathbf{w}}\big|_{\partial\Omega} \in H^{1,0}(\Lambda^2(\Sigma))$ of (3.16) is evident.

Now we extend the obtained data from Σ into Q. We explain how to do this in a set $(0,T) \times U_j \subset \Sigma$ on which the local coordinates constructed in Lemma 3.3 are used. So we have

$$\check{\mathbf{w}}\big|_{y_3=0} = w_1\big|_{y_3=0}dy_1 + w_2\big|_{y_3=0}dy_2 = \boldsymbol{\omega} = \omega_1 dy_1 + \omega_2 dy_2, \qquad w_3\big|_{y_3=0} = 0;$$

$$\omega_i \in H^{1,1}((0,T) \times U_j) \qquad i = 1,2;$$

$$(\partial_{y_3}\check{\mathbf{w}})\big|_{y_3=0} = \widehat{\mathbf{u}} = u_1 dy_1 + u_2 dy_2, \qquad (\partial_{y_3} w_3)\big|_{y_3=0} = 0;$$

$$u_i \in H^{1,0}((0,T) \times U_j) \qquad i = 1,2;$$

$$(\partial^2_{y_3y_3}\check{\mathbf{w}})\big|_{y_3=0} = (\partial^2_{y_3y_3} w_1)\big|_{y_3=0}dy_1 + (\partial^2_{y_3y_3} w_2)\big|_{y_3=0}dy_2 \equiv 0,$$

and

$$(\partial^2_{y_3y_3} w_3)\big|_{y_3=0} = 0.$$

Using these data for $w_1\big|_{y_3=0}$, $w_2\big|_{y_3=0}$, and for their y_3-derivatives and applying Theorem 3.2 on the extension of scalar functions, we can extend w_i, $i = 1,2$, from $(0,T) \times U_j$ into functions $w_i \in H^{1,3/2}((0,T) \times U_j \times \{y_3 : 0 \le y_3 \le \varepsilon\})$, $i = 1,2$. We set $w_3 = 0$ for $(t, y_1, y_2, y_3) \in (0,T) \times U_j \times (0,\varepsilon)$. As a result, we obtain an extension of the exterior form $\boldsymbol{\omega}$ and $\widehat{\mathbf{u}}$ from $(0,T) \times U_j$ into an exterior form

$$E(\boldsymbol{\omega}, \widehat{\mathbf{u}}) = w_1 dy_1 + w_2 dy_2 + w_3 dy_3 \in \mathbf{H}^{1,3/2}((0,T) \times U_j \times \{y_3 : 0 \le y_3 \le \varepsilon\}).$$

By Theorem 3.2, the form $\boldsymbol{\omega}$ equals zero for $y_3 > \varepsilon$ and close to ε. The extension of the differential form $\boldsymbol{\omega}$ and $\widehat{\mathbf{u}}$ from Σ to Q can be achieved in the standard way using the partition of unity; see [7] for details. We denote this global extension by $E(\boldsymbol{\omega}, \widehat{\mathbf{u}}) \in H^{1,3/2}(\Lambda^1(Q))$. It is evident now that the form $dE(\boldsymbol{\omega}, \widehat{\mathbf{u}})$ which can be decomposed in global coordinates (t, x_1, x_2, x_3) as $dE(\boldsymbol{\omega}, \widehat{\mathbf{u}}) = \sum_{j=1}^3 (dE(\boldsymbol{\omega}, \widehat{\mathbf{u}}))_j dx_j$ with $(dE(\boldsymbol{\omega}, \widehat{\mathbf{u}}))_j \in \mathbf{H}^{1,1/2}(Q)$ gives us the desired extension of the differential form (3.9). Passage from the form $dE(\boldsymbol{\omega}, \widehat{\mathbf{u}})$ to the vector field $\mathbf{v} = E_c\mathbf{u}$ completes the proof. $\square$

Remark 3.9. The proof of Theorem 3.8 gives the possibility for extending vector fields from $\Sigma = (0,T) \times \partial\Omega$ to $(0,T) \times \mathbb{R}^3$, i.e., not only inside Q but outside the lateral surface of Q as well. More precisely, there exists a continuous extension operator $E_c : \widetilde{H}^{1,0}(\mathcal{T}(\Sigma)) \to \mathbf{V}^{1,1/2}((0,T) \times \mathbb{R}^3)$ such that $E_c\mathbf{u}|_\Sigma = \mathbf{u}$, $\partial_n E_c\mathbf{u}|_\Sigma \in \mathbf{L}^2(\Sigma)$, and $E_c\mathbf{u}$ is supported in a two-sided ε-neighborhood of Σ. $\square$

Analogous to (3.28), we set

$$\widehat{H}^{1,s}(\mathcal{T}(\Sigma)) = \left\{ \mathbf{u} \in H^{1,s}(\mathcal{T}(\Sigma)) \,:\, \int_{\partial\Omega} u_n(t, \mathbf{x})\, ds = 0 \text{ for almost all } t \in (0,T) \right\}. \tag{3.31}$$

We now construct the extension in the case of the unconnected boundary (2.1) when the boundary data $\mathbf{u} = \mathbf{u}_\tau + u_n\mathbf{n}$ satisfies the condition

$$\int_{\partial\Omega} u_n(t,\mathbf{x})\,ds = 0 \qquad \text{a.e. } t \in (0,T)\,. \tag{3.32}$$

However, the extension in this case no longer will be localized near $\partial\Omega$ (although it will have compact support if Ω is the exterior of bounded domains).

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain or be the exterior of a bounded domain and assume $\partial\Omega$ is of class C^∞. Then, there exists a continuous extension operator*

$$E : \widehat{H}^{1,0}(\mathcal{T}(\Sigma)) \to \mathbf{V}^{1,1/2}(Q)\,. \tag{3.33}$$

In the case where Ω is the exterior of a bounded domain, the extension $E\mathbf{u}$ for each $\mathbf{u} \in \widehat{H}^{1,0}(\mathcal{T}(\Sigma))$ has compact support belonging to the set $\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < N+1\}$ for N that satisfies the condition

$$\partial\Omega \subset \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < N\}\,. \tag{3.34}$$

Proof. Let Ω have the structure (2.1). Denote

$$\alpha_j(t) = \frac{1}{|\Gamma_j|}\int_{\Gamma_j} u_n(t,\mathbf{x})\,ds\,, \quad j = 1,\dots,J\,,$$

where $|\Gamma_j|$ is the Lebesque measure of the manifold Γ_j. We look for a desired extension $E\mathbf{u}$ in the form

$$E\mathbf{u} = \mathbf{v} + \nabla r\,, \tag{3.35}$$

where r is the solution of the Neumann problem

$$\begin{cases} \Delta r(t,\mathbf{x}) = 0 \quad \text{in } \Omega\,, \\ \partial_n r(t,\mathbf{x})\big|_{\Gamma_j} = \alpha_j(t) \end{cases} \tag{3.36}$$

for almost every $t \in (0,T)$. Since $\mathbf{u}$ satisfies (3.32), we have $\sum_{j=1}^J \alpha_j(t) \equiv 0$ so that the solution $r(t,\mathbf{x})$ of (3.36) exists and $\nabla r \in \widetilde{\mathbf{V}}^{1,1/2}(Q)$. By (3.35), the boundary condition of the desired vector field $\mathbf{v}$ is

$$\mathbf{v}\big|_\Sigma = \mathbf{u} - (\nabla r)\big|_\Sigma\,.$$

Then, evidently,

$$\frac{1}{|\Gamma_j|}\int_{\Gamma_j} v_n(t,\mathbf{x})\,ds = 0\,, \quad j = 1,\dots,J\,.$$

Hence, the construction of $\mathbf{v}$ from $\mathbf{v}\big|_\Sigma$ is reduced to Theorem 3.8. This finishes the proof in the case of bounded Ω.

When Ω is the exterior of a bounded domain, we apply Theorem 2.4. To this end, we apply the extension theorem just proved to the bounded domain

$\Omega_{N+1/2} \equiv \{\mathbf{x} \in \Omega \,:\, |\mathbf{x}| < N + 1/2\}$ with N satisfying (3.34), where we set the extended data equal to zero on the additional component $\Gamma_{J+1} = \{|\mathbf{x}| = N+1/2\}$ of $\partial\Omega_{N+1/2} = \cup_{j=1}^{J+1}\Gamma_j$. After that, we consider the extended vector field $E\mathbf{u}$ as defined in (3.35), which we now denote by $\widehat{E}\mathbf{u}$, and restrict it to $(0,T) \times \widetilde{\Omega}_N \equiv \widetilde{Q}$, where $\widetilde{\Omega}_N = \{\mathbf{x} \in \mathbb{R}^3 \,:\, N < |\mathbf{x}| < N + \frac{1}{2}\}$. By virtue of (3.32) and the relation $\operatorname{div}(\widehat{E}\mathbf{u}) = 0$, we deduce

$$\int_{|\mathbf{x}|=N} (\widehat{E}\mathbf{u})_n \, ds = 0 \qquad \text{and} \qquad \int_{|\mathbf{x}|=N+1/2} (\widehat{E}\mathbf{u})_n \, ds = 0\,.$$

Hence, $\widehat{E}\mathbf{u} \in \widetilde{\mathbf{V}}^{1,1/2}(\widetilde{Q})$. Taking $\Omega = \widetilde{\Omega}_N$, $Q = \widetilde{Q}$, $\Psi = \{\mathbf{x} \in \mathbb{R}^3 \,:\, |\mathbf{x}| > N\}$, $K = \Psi \cap \{\mathbf{x} \in \mathbb{R}^3 \,:\, N < |\mathbf{x}| \le N+1\}$, and $\Theta = (0,T) \times \Psi$, and applying Theorem 2.4 to $\widehat{E}\mathbf{u}\big|_{\widetilde{Q}}$, we obtain $\mathcal{L}\widehat{E}\mathbf{u}\big|_{\widetilde{Q}}$. We may define the final extension $E\mathbf{u}$ as follows:

$$E\mathbf{u} = \begin{cases} \widehat{E}\mathbf{u}(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,T) \times (\Omega \cap \{\mathbf{x} \in \mathbb{R}^3 \,:\, |\mathbf{x}| < N + \frac{1}{2}\}) \\ \mathcal{L}\widehat{E}\mathbf{u}(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,T) \times \{\mathbf{x} \in \mathbb{R}^3 \,:\, |\mathbf{x}| \ge N + \frac{1}{2}\}\,. \end{cases}$$

This completes the proof in the case when $\partial\Omega$ is the exterior of an unbounded domain. □

Remark 3.11. If all conditions of Theorem 3.10 are fulfilled, then there exists a continuous extension operator

$$E : \widehat{H}^s(\mathcal{T}(\partial\Omega)) \to \mathbf{V}^{s+1/2}(\Omega) \tag{3.37}$$

where $\widehat{H}^s(\mathcal{T}(\partial\Omega)) = \{u \in H^s(\mathcal{T}(\partial\Omega)) \,:\, u \text{ satisfies (3.32)}\}$. The proof of this assertion is entirely analogous to that of Theorem 3.10.

4. Final extension results

In this and the next section, Ω is the exterior of a bounded domain having C^∞ boundary. We restrict ourselves to this case simply because this case is of the most interest in applications. The case of a bounded domain can be treated similarly and more easily.

4.1. The linear steady-state problem

The extension results obtained in the previous section are not enough for establishing a proper theory of inhomogeneous boundary value problems for the Oseen and Navier–Stokes equations. We need to strengthen those extension results. To this end we set

$$\Omega_{N+k} = \Omega \cap \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < N + k\} \quad \text{and} \quad Q_{N+k} = (0,T) \times \Omega_{N+k}$$

for each $k > 0$ and for a fixed $N > 0$ satisfying (3.34) and consider the following steady-state problem depending on the parameter $t \in (0, T)$:

$$-\Delta \mathbf{v}(t, \mathbf{x}) + \nabla p(t, \mathbf{x}) = \mathbf{0}, \qquad \operatorname{div} \mathbf{v}(t, \mathbf{x}) = 0, \qquad \text{in } \Omega_{N+2}\,, \tag{4.1}$$

$$\mathbf{v}(t, \cdot)\big|_{\partial\Omega} = \mathbf{b}(t, \cdot); \qquad \text{and} \qquad \mathbf{v}(t, \mathbf{x})|_{\{\mathbf{x}\in\mathbb{R}^3 \,:\, |\mathbf{x}|=N+2\}} = \mathbf{0}\,. \tag{4.2}$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$, $\mathbf{b}(t, \cdot) \in \mathbf{H}^1(\Sigma) = H^{1,0}(\mathcal{T}(\Sigma))$ is the boundary data with its normal component $b_n(t, \mathbf{x}')$ satisfying

$$\int_{\partial\Omega} b_n(t, \mathbf{x}')\, ds = 0 \qquad \text{a.e. } t \in (0, T)\,, \tag{4.3}$$

i.e., we suppose that $\mathbf{b} \in \widehat{\mathbf{H}}^1(\Sigma) = \widehat{H}^{1,0}(\mathcal{T}(\Sigma))$; see (3.31).

We recall some definitions from [12]. Let G be a bounded domain in $\mathbb{R}^d$ with C^∞-boundary ∂G. Let $\rho(\mathbf{x}) \in C^\infty(\overline{G})$, $\rho(\mathbf{x}) > 0$ for $\mathbf{x} \in G$, and $\rho(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial G)$ in a sufficiently small neighborhood of ∂G. As in [12, Ch.1, §11.5], we set

$$H^{1/2}_{00}(G) = \{u \,:\, u \in H^{1/2}(G),\ \rho^{-1/2} u \in L^2(G)\}$$

with the norm

$$\|u\|^2_{H^{1/2}_{00}(G)} = \|u\|^2_{H^{1/2}(G)} + \|\rho^{-1/2} u\|^2_{L^2(G)}\,.$$

In [12, Ch.1, Thm.11.7] it is proved that

$$H^{1/2}_{00}(G) = [H^1_0(G), L^2(G)]_{1/2}\,,$$

where the right-hand side denotes the intermediate space between $H^1_0(G)$ and $L^2(G)$ of order 1/2; see the definition in [12, Ch.1, §2.1]. As usual, by $(H^{1/2}_{00}(G))'$ we denote the completion of $L^2(G)$ in the norm

$$\|u\|_{(H^{1/2}_{00}(G))'} = \inf_{\phi \in H^{1/2}_{00}(G),\, \phi \neq 0} \frac{\langle u, \phi\rangle}{\|\phi\|_{H^{1/2}_{00}(G)}}\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality generated by the scalar product in $L^2(G)$. By the duality theorem (see [12, Ch.1, Thm.6.2]),

$$(H^{1/2}_{00}(G))' = [L^2(G), H^{-1}(G)]_{1/2}\,.$$

Therefore, since the operators $-\Delta : H^2(G) \to H^0(G)$ and $-\Delta : H^1(G) \to H^{-1}(G)$ are continuous, we obtain, using the interpolation theorem (see [12, Ch.1, §5.1]), that the Laplace operator $-\Delta : H^{3/2}(G) \to (H^{1/2}_{00}(G))'$ is continuous.

We now prove the following assertion regarding the existence of a solution to (4.1)–(4.2).

Theorem 4.1. *Let* $\mathbf{b}(t, \mathbf{x}') \in \mathbf{H}^1(\Sigma)$ *be given and assume that* (4.3) *holds. Then, there exists a unique solution* $(\widehat{\mathbf{v}}, \widehat{p})$ *of* (4.1)–(4.2) *satisfying*[1]

$$\widehat{\mathbf{v}} \in \mathbf{V}^{1,1/2}(Q_{N+2}) \qquad \textit{and} \qquad \nabla \widehat{p} \in L^2(0, T; (\mathbf{H}^{1/2}_{00}(\Omega_{N+2}))')\,.$$

[1] For the definition of $\mathbf{V}^{1,1/2}(Q)$, see (2.6).

Moreover, there exists a $C > 0$ *such that*

$$\|\widehat{\mathbf{v}}\|_{\mathbf{V}^{1,1/2}(Q_{N+2})} + \|\nabla\widehat{p}\|_{L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))')} \le C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}\,. \tag{4.4}$$

Proof. Let E be the extension operator constructed in Theorem 3.10. We look for the solution $\widehat{\mathbf{v}}$ of (4.1)–(4.2) in the form

$$\widehat{\mathbf{v}} = E\mathbf{b} + \mathbf{v}\,. \tag{4.5}$$

Substitution of (4.5) into (4.1)–(4.2) implies that $\mathbf{v}$ (along with a p) must satisfy the following relations:

$$-\Delta\mathbf{v}(t,\mathbf{x}) + \nabla p(t,\mathbf{x}) = \Delta(E\mathbf{b})(t,\mathbf{x}), \qquad \operatorname{div}\mathbf{v}(t,\mathbf{x}) = 0, \qquad \text{in } \Omega_{N+2}\,, \tag{4.6}$$

$$\mathbf{v}(t,\mathbf{x})\big|_{\partial\Omega} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{v}|_{\{\mathbf{x}\in\mathbb{R}^3\,:\,|\mathbf{x}|=N+2\}} = \mathbf{0}\,. \tag{4.7}$$

Using Theorem 3.10, we have that $\Delta(E\mathbf{b}) \in L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))')$ and $\Delta(E\mathbf{b}) = \mathbf{0}$ for $|\mathbf{x}| > N+1$. We claim that (4.6)–(4.7) have a unique solution $(\mathbf{v}(t),\nabla p(t)) \in \mathbf{V}^{3/2}(\Omega_{N+2})\times(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))'$ for almost every $t \in (0,T)$. Indeed, in [9], it is proved that if the right-hand side $\Delta(E\mathbf{b})(t,\cdot)$ of (4.6)–(4.7) belongs to $\mathbf{H}^{-1}(\Omega_{N+2})$, then there exists a unique solution $(\mathbf{v},\nabla p) \in \mathbf{V}^1(\Omega_{N+2}) \times \mathbf{H}^{-1}(\Omega_{N+2})$ satisfying

$$\|\mathbf{v}(t,\cdot)\|_{\mathbf{H}^1(\Omega_{N+2})} + \|\nabla p(t,\cdot)\|_{\mathbf{H}^{-1}(\Omega_{N+2})} \le C\|\Delta(E\mathbf{b})(t,\cdot)\|_{\mathbf{H}^{-1}(\Omega_{N+2})};$$

if $\Delta(E\mathbf{b})(t,\cdot) \in \mathbf{L}^2(\Omega_{N+2})$, then the unique solution $(\mathbf{v},\nabla p)$ belongs to $\mathbf{V}^2(\Omega_{N+2})\times \mathbf{L}^2(\Omega_{N+2})$ and satisfies

$$\|\mathbf{v}(t,\cdot)\|_{\mathbf{H}^2(\Omega_{N+2})} + \|\nabla p(t,\cdot)\|_{\mathbf{L}^2(\Omega_{N+2})} \le C\|\Delta(E\mathbf{b})(t,\cdot)\|_{\mathbf{L}^2(\Omega_{N+2})}.$$

Applying to these results the interpolation theorem of [12, Ch.1, §5.1] we obtain that if $\Delta(E\mathbf{b})(t,\cdot) \in (\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))' = [\mathbf{L}^2(\Omega_{N+2}),\mathbf{H}^{-1}(\Omega_{N+2})]_{1/2}$, then

$$\mathbf{v} \in \mathbf{V}^0(\Omega_{N+2}) \cap [\mathbf{H}^2(\Omega_{N+2}),\mathbf{H}^1(\Omega_{N+2})]_{1/2} = \mathbf{V}^{3/2}(\Omega_{N+2})$$

and $\nabla p \in [\mathbf{L}^2(\Omega_{N+2}),\mathbf{H}^{-1}(\Omega_{N+2})]_{1/2} = (\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))'$. Moreover, there exists a $C > 0$, independent of $\mathbf{v}$ and p, such that

$$\|\mathbf{v}(t,\cdot)\|_{\mathbf{V}^{3/2}(\Omega_{N+2})} + \|\nabla p(t,\cdot)\|_{(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))'} \le C\|\Delta(E\mathbf{b})(t,\cdot)\|_{(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))'}\,. \tag{4.8}$$

The fact that $\Delta(E\mathbf{b}) \in L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))')$ further implies that

$$(\mathbf{v},\nabla p) \in L^2(0,T;\mathbf{V}^{3/2}(\Omega_{N+2}) \times L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))')\,.$$

We claim that $\partial_t\mathbf{v} \in L^2(0,T;\mathbf{V}^{1/2}(\Omega_{N+2}))$. To prove this, we need to estimate $|\langle\partial_t\mathbf{v},\mathbf{f}\rangle|$ for arbitrary, appropriately chosen $\mathbf{f}$. Let a function $\mathbf{f}$ be given with $\mathbf{f} \in C^1(0,T;\mathbf{C}_0^\infty(\Omega_{N+2}))$ and $\mathbf{f}(0,\mathbf{x}) = \mathbf{f}(T,\mathbf{x}) \equiv \mathbf{0}$. Let $\mathbf{w} \in C^1(0,T;\mathbf{C}^\infty(\Omega_{N+2}) \cap \mathbf{V}^1(\Omega_{N+2}))$, together with some $q \in C^1(0,T;\mathbf{C}^\infty(\Omega_{N+2}))$, be a solution of

$$\begin{aligned} &-\Delta\mathbf{w} + \nabla q = \mathbf{f}, \qquad \operatorname{div}\mathbf{w} = 0, \qquad \text{in } \Omega_{N+2} \\ &\mathbf{w}\big|_{\partial\Omega} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{w}|_{\{\mathbf{x}\in\mathbb{R}^3\,:\,|\mathbf{x}|=N+2\}} = \mathbf{0}\,. \end{aligned} \tag{4.9}$$

Then, $(\partial_t \mathbf{w}, \partial_t \nabla q)$ satisfies

$$\begin{array}{lll} -\Delta\partial_t \mathbf{w} + \nabla\partial_t q = \partial_t \mathbf{f}, & \operatorname{div} \partial_t \mathbf{w} = 0, & \text{in } \Omega_{N+2} \\ \partial_t \mathbf{w}\big|_{\partial\Omega} = \mathbf{0}, & \text{and} & \partial_t \mathbf{w}|_{\{\mathbf{x}\in\mathbb{R}^3\,:\,|\mathbf{x}|=N+2\}} = \mathbf{0}\,. \end{array} \tag{4.10}$$

We rewrite (3.35) as follows:

$$E\mathbf{b} = E_c\widehat{\mathbf{b}} + \nabla r\,,$$

where ∇r is defined by (3.36) with $\alpha_j(t) = \frac{1}{|\Gamma_j|}\int_{\Gamma_j} \mathbf{b}(t,\mathbf{x}')\cdot\mathbf{n}\,ds$, E_c is the extension operator constructed in Theorem 3.8, and $\widehat{\mathbf{b}}(t,\mathbf{x})$ is the vector field defined on Σ which is constructed from $\mathbf{b}$ as follows:

$$\widehat{\mathbf{b}}(t,\mathbf{x}) = \mathbf{b}(t,\mathbf{x}) - (\nabla r)|_\Sigma\,.$$

By virtue of (3.36),

$$\|\widehat{\mathbf{b}}\|_{\mathbf{H}^1(\Sigma)} \le C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}$$

and

$$\Delta(E\mathbf{b}) = \Delta(E_c\widehat{\mathbf{b}})\,. \tag{4.11}$$

Now we estimate $|\langle\partial_t \mathbf{v}, \mathbf{f}\rangle|$. Using (4.11), integrating by parts repeatedly, and taking into account (4.6)–(4.7), (4.10), and the boundary conditions of $\mathbf{f}$, we obtain

$$\begin{aligned} &\int_{Q_{N+2}} \mathbf{v}\cdot\partial_t \mathbf{f}\,d\mathbf{x}dt = -\int_{Q_{N+2}} \mathbf{v}\cdot\Delta\partial_t \mathbf{w}\,d\mathbf{x}dt \\ &= -\int_{Q_{N+2}} \Delta\mathbf{v}\cdot\partial_t \mathbf{w}\,d\mathbf{x}dt = \int_{Q_{N+2}} \Delta(E_c\widehat{\mathbf{b}})\cdot\partial_t \mathbf{w}\,d\mathbf{x}dt \\ &= \int_{Q_{N+2}} (E_c\widehat{\mathbf{b}})\cdot\Delta\partial_t \mathbf{w}\,d\mathbf{x}dt - \int_\Sigma \widehat{\mathbf{b}}\cdot\partial_n(\partial_t \mathbf{w})\,dsdt \\ &= \int_{Q_{N+2}} (E_c\widehat{\mathbf{b}})\cdot(\nabla\partial_t q - \partial_t \mathbf{f})\,d\mathbf{x}dt - \int_\Sigma \widehat{\mathbf{b}}\cdot\partial_n(\partial_t \mathbf{w})\,dsdt \\ &= -\int_{Q_{N+2}} (E_c\widehat{\mathbf{b}})\cdot\partial_t \mathbf{f}\,d\mathbf{x}dt + \int_\Sigma \big((\widehat{\mathbf{b}}\cdot\mathbf{n})\partial_t q - \widehat{\mathbf{b}}\cdot\partial_n\partial_t \mathbf{w}\big)\,dsdt \\ &= \int_{Q_{N+2}} \partial_t(E_c\widehat{\mathbf{b}})\cdot\mathbf{f}\,d\mathbf{x}dt + \int_\Sigma \partial_t\widehat{\mathbf{b}}\cdot(\partial_n \mathbf{w} - q\mathbf{n})\,dsdt\,. \end{aligned} \tag{4.12}$$

Note that in the last step we used the equalities $\mathbf{w}(0,\cdot) = \mathbf{w}(T,\cdot) = \mathbf{0}$ which are trivial consequences of (4.9) and the assumptions $\mathbf{f}(0,\cdot) = \mathbf{f}(T,\cdot) = \mathbf{0}$. Multiplying the first equation of (4.9) by $\partial_t(E_c\widehat{\mathbf{b}})$ and integrating by parts, we deduce

$$\int_\Sigma \partial_t\widehat{\mathbf{b}}\cdot(q\mathbf{n} - \partial_n \mathbf{w})\,dsdt = \int_{Q_{N+2}} \big[\mathbf{f}\cdot\partial_t(E_c\widehat{\mathbf{b}}) - \nabla\mathbf{w} : \nabla\partial_t(E_c\widehat{\mathbf{b}})\big]\,d\mathbf{x}dt\,. \tag{4.13}$$

Substitution of (4.13) into (4.12) yields

$$\begin{aligned} \int_{Q_{N+2}} \mathbf{v}\cdot\partial_t \mathbf{f}\,d\mathbf{x}dt &= \int_{Q_{N+2}} \partial_t(E_c\widehat{\mathbf{b}})\cdot\mathbf{f}\,d\mathbf{x}dt - \int_{Q_{N+2}} \big(\mathbf{f}\cdot\partial_t(E_c\widehat{\mathbf{b}}) - \nabla\mathbf{w} : \nabla\partial_t(E_c\widehat{\mathbf{b}})\big)d\mathbf{x}dt \\ &= \int_{(0,T)\times\mathbb{R}^3} \nabla\widetilde{\mathbf{w}} : \nabla\partial_t(E_c\widehat{\mathbf{b}})\,d\mathbf{x}dt\,, \end{aligned}$$

where $\widetilde{\mathbf{w}}$ is the extension of $\mathbf{w}$ from Q_{N+2} to $(0,T)\times\mathbb{R}^3$ by zero on $\big((0,T)\times\mathbb{R}^3\big)\setminus Q_{N+2}$ and $E_c\widehat{\mathbf{b}}$ is the natural extension of $\widehat{\mathbf{b}}$ to $(0,T)\times\mathbb{R}^3$ described in Remark 3.9. Therefore,

$$\begin{aligned}\left|\int_{Q_{N+2}}\mathbf{v}\cdot\partial_t\mathbf{f}\,d\mathbf{x}\,dt\right| &\le \|\nabla\partial_t(E_c\widehat{\mathbf{b}})\|_{L^2(0,T;\mathbf{H}^{-1/2}(\mathbb{R}^3))}\|\nabla\mathbf{w}\|_{L^2(0,T;\mathbf{H}^{1/2}(\Omega_{N+2}))}\\ &\le C\|\partial_t(E_c\widehat{\mathbf{b}})\|_{L^2(0,T;\mathbf{H}^{1/2}(\mathbb{R}^3))}\|\mathbf{f}\|_{L^2(0,T;(\mathbf{H}^{1/2}_{00}(\Omega_{N+2}))')}\,,\end{aligned}$$

where in the last step we used the well-known estimates for the solution $\mathbf{w}$ of the problem (4.9) (see the derivation of (4.8)). This bound, [12, Ch.1, Eq.11.53], and Remark 3.9 imply

$$\|\partial_t\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{1/2}(\Omega_{N+2}))}\le\|\partial_t\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{1/2}_{00}(\Omega_{N+2}))}\le C_{N+2}\|E\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}. \tag{4.14}$$

Thus, the estimate (4.4) follows from (4.5), (4.8), (4.14), and Theorem 3.10. □

4.2. Extension to Ω

Now we intend to extend the solution $\widehat{\mathbf{v}}$ of (4.1)–(4.2) obtained in Theorem 4.1 to a sufficiently smooth vector field defined on Ω that equals zero at infinity. To this end, we first derive some further estimates for $\widehat{\mathbf{v}}$.

For $0<\alpha<\beta$, we denote

$$K_{N+\alpha,N+\beta}=\{\mathbf{x}\in\mathbb{R}^3\,:\,N+\alpha<|\mathbf{x}|<N+\beta\}\,.$$

Lemma 4.2. *Let $\widehat{\mathbf{v}}(t,\mathbf{x})$ be the solution of (4.1)–(4.2) obtained in Theorem 4.1. Then, there exists $C>0$ such that*

$$\|\widehat{\mathbf{v}}(t,\cdot)\|_{\mathbf{V}^{5/2}(K_{N+5/4,N+3/2})}\le C\|\mathbf{b}(t,\cdot)\|_{\mathbf{H}^1(\partial\Omega)}\ \textit{for almost all } t\in(0,T). \tag{4.15}$$

Proof. Let $\mathbf{v}$ be defined by (4.5), i.e., $\mathbf{v}=\widehat{\mathbf{v}}-E\mathbf{b}$, and let $\mathbf{u}(t,\cdot)\equiv\mathbf{curl}^{-1}\mathbf{v}(t,\cdot)$, where $\mathbf{curl}^{-1}$ is the operator that transforms $\mathbf{v}$ into the solution $\mathbf{u}$ of the system

$$\mathbf{curl}\,\mathbf{u}(t,\mathbf{x})=\mathbf{v}(t,\mathbf{x}),\qquad \operatorname{div}\mathbf{u}(t,\mathbf{x})=0\qquad\text{in }\Omega_{N+2}\,, \tag{4.16}$$

$$(\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega_{N+2}}=0 \tag{4.17}$$

such that $\mathbf{u}$ is orthogonal in $\mathbf{L}^2(\Omega_{N+2})$ to the kernel of the problem (4.16)–(4.17), i.e., to the following set of vector fields:

$$\ker\mathbf{curl}=\{\mathbf{u}_0\in\mathbf{V}^0_0(\Omega_{N+2})\,:\,\mathbf{curl}\,\mathbf{u}_0=\mathbf{0}\}\,.$$

It is well known (see, e.g., [16]) that $\mathbf{curl}^{-1}\mathbf{v}$ is well defined if

$$\int_{\Gamma_j}\mathbf{v}\cdot\mathbf{n}\,ds=\int_{|\mathbf{x}|=N+2}\mathbf{v}\cdot\mathbf{n}\,ds=0\,,\quad j=1,\dots,J\,, \tag{4.18}$$

where $\partial\Omega=\cup_{j=1}^J\Gamma_j$. Equalities (4.18) are indeed true by virtue of (4.7). Moreover (see [7]),

$$\|\mathbf{u}(t,\cdot)\|_{\mathbf{V}^{s+1}(\Omega_{N+2})}\le C\|\mathbf{v}(t,\cdot)\|_{\mathbf{V}^s(\Omega_{N+2})},\quad s>0\,. \tag{4.19}$$

We apply the **curl** operator to the first equation in (4.6) and substitute $\mathbf{v} = \mathbf{curl}\,\mathbf{u}$ into the resulting equation. Since $\mathbf{curl}\,\nabla p = 0$ and $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = -\Delta\mathbf{u}$, we obtain

$$\Delta^2\mathbf{u}(t,\mathbf{x}) = \mathbf{curl}\,\Delta(E\mathbf{b})(t,\mathbf{x}) \qquad \text{in } \Omega_{N+2}\,. \tag{4.20}$$

Let $\varphi(\mathbf{x}) \in C_0^\infty(\Omega_{N+2})$ satisfy the conditions

$$\varphi(\mathbf{x}) = 1 \quad \text{for } \mathbf{x} \in K_{N+5/4,N+3/2} \quad \text{and} \quad \varphi(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \notin K_{N+1,N+7/4}. \tag{4.21}$$

By Theorem 3.10, $(E\mathbf{b})(t,\mathbf{x}) = 0$ if $|\mathbf{x}| > N+1$. Hence, by (4.21),

$$\varphi(\mathbf{x})\,\mathbf{curl}\,\Delta(E\mathbf{b})(t,\mathbf{x}) = \mathbf{0}.$$

Taking into account this equality and using Leibnitz's formula, we obtain from (4.20)

$$\Delta^2(\varphi\mathbf{u})(t,\mathbf{x}) = \sum_{1\le|\alpha|,\,|\beta|\le 3} C_{\alpha\beta}(D^\alpha\varphi)\big(D^\beta\mathbf{u}(t,\mathbf{x})\big) \quad \text{for } \mathbf{x} \in K_{N+1,N+7/4}, \tag{4.22}$$

$$\big(\varphi\mathbf{u}(t,\cdot)\big)|_{\partial K_{N+1,N+7/4}} = 0, \qquad \partial_n\big(\varphi\mathbf{u}(t,\cdot)\big)|_{\partial K_{N+1,N+7/4}} = 0\,, \tag{4.23}$$

where $C_{\alpha\beta}$ are certain constants. For the solution $\varphi\mathbf{u}$ of the elliptic boundary value problem (4.22)–(4.23), the following estimate is well known (see, e.g., [12]):

$$\begin{aligned}\|\varphi\mathbf{u}(t,\cdot)\|_{\mathbf{H}^{s+4}(K_{N+1,N+7/4})} &\le C\Big\|\sum_{1\le|\alpha|\le 3,\,|\beta|\le 3} C_{\alpha\beta}(D^\alpha\varphi)\big(D^\beta\mathbf{u}(t,\cdot)\big)\Big\|_{\mathbf{H}^s(K_{N+1,N+7/4})}\\ &\le C_1\|\mathbf{u}(t,\cdot)\|_{\mathbf{H}^{s+3}(K_{N+1,N+7/4})}, \quad s \ge -2.\end{aligned} \tag{4.24}$$

Since $\mathbf{v} = \mathbf{curl}\,\mathbf{u}$, we obtain, using (4.21), (4.24), (4.19), and (4.8),

$$\begin{aligned}\|\mathbf{v}(t,\cdot)\|_{\mathbf{V}^{5/2}(K_{N+5/4,N+3/2})} &\le C\|\varphi\mathbf{u}(t,\cdot)\|_{\mathbf{H}^{7/2}(K_{N+1,N+7/4})}\\ &\le C_1\|\mathbf{u}(t,\cdot)\|_{\mathbf{H}^{5/2}(K_{N+1,N+7/4})}\\ &\le C_2\|\mathbf{v}(t,\cdot)\|_{\mathbf{V}^{3/2}(\Omega_{N+2})}\\ &\le C_3\|(E\mathbf{b})(t,\cdot)\|_{\mathbf{V}^{3/2}(\Omega_{N+2})}.\end{aligned} \tag{4.25}$$

Since, by Theorem 3.10, $E\mathbf{b}(t,\mathbf{x}) = \mathbf{0}$ for $|\mathbf{x}| > N+1$, we obtain by (4.25) and (3.37), the inequality (4.15). □

Let $N > 0$ be fixed and satisfy (3.34). We introduce the space

$$\begin{aligned}\mathbf{V}_N^{1,1/2}(Q) = \{\mathbf{v} \in \mathbf{V}^{1,1/2}(Q)\,:\, &\operatorname{supp}\mathbf{v} \subset [0,T]\times\Omega_{N+2},\\ &\Delta\mathbf{v} \in \mathbf{L}^2(Q_{N+2}) + \mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+2}))\}\,,\end{aligned} \tag{4.26}$$

where $\nabla H^{1/2}(\Omega_{N+2}) = \{\nabla p(\mathbf{x}) : p \in H^{1/2}(\Omega_{N+2})\}$ equipped with the norm

$$\|\nabla p\|_{\nabla H^{1/2}(\Omega_{N+2})} \equiv \|\nabla p\|_{(\mathbf{H}_{00}^{1/2}(\Omega_{N+2}))'}\,.$$

The space (4.26) is equipped with the norm[2]

$$\|\mathbf{v}\|^2_{\mathbf{V}_N^{1,1/2}(Q)} = \|\mathbf{v}\|^2_{\mathbf{V}^{1,1/2}(Q)} + \|\Delta\mathbf{v}\|_{\mathbf{L}^2(Q_{N+2})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+2}))}, \tag{4.27}$$

where $\mathbf{V}^{1,1/2}(Q)$ is the space (2.6) with $s = 1/2$.

We now prove the main result of this section.

Theorem 4.3. *There exists a continuous extension operator*

$$\mathcal{E} : \widehat{\mathbf{H}}^1(\Sigma) \to \mathbf{V}_N^{1,1/2}(Q), \tag{4.28}$$

where $\widehat{\mathbf{H}}^1(\Sigma) = \widehat{H}^{1,0}(\mathcal{T}(\Sigma))$ *is the space defined in* (3.31) *with* $s = 0$ *and* $\mathbf{V}_N^{1,1/2}(Q)$ *is defined in* (4.26).

Proof. Let $\widehat{\mathbf{v}}(t,\mathbf{x})$ be the solution of problem (4.1)–(4.2). We consider $\widehat{\mathbf{v}}(t,\mathbf{x})$ on the set $(0,T) \times K_{N+5/4,N+3/2}$ and apply to $\widehat{\mathbf{v}}(t,\mathbf{x})$ Theorem 2.4 with $\Omega = K_{N+5/4,N+3/2}$, $\Psi = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| > N + 5/4\}$, $K = \{\mathbf{x} \in \mathbb{R}^3 : N + 5/4 < |\mathbf{x}| \leq N + 2\}$, and $m = 1$. Then, we define the extension operator $\mathcal{E}$ as follows:

$$\mathcal{E}\mathbf{b} = \begin{cases} \widehat{\mathbf{v}}(t,\mathbf{x}) & \text{for } (t,\mathbf{x}) \in (0,T) \times \Omega_{N+3/2} \\ (\mathcal{L}_2\widehat{\mathbf{v}})(t,\mathbf{x}) & \text{for } (t,\mathbf{x}) \in (0,T) \times \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| > N + 3/2\}. \end{cases} \tag{4.29}$$

Taking $\nabla p = \mathbf{0}$ in the decomposition $\mathbf{w} = \mathbf{a} + \nabla p$ we obtain

$$\|\mathbf{w}\|_{\mathbf{L}^2(Q_{N+\alpha})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+\alpha}))} \leq \|\mathbf{w}\|_{\mathbf{L}^2(Q_{N+\alpha})}$$

for any $\alpha > 0$. Thus, by virtue of (4.29), (4.27), (4.26), and (2.7) we obtain

$$\begin{aligned} \|\mathcal{E}\mathbf{b}\|_{\mathbf{V}_N^{1,1/2}(Q)} &\leq \|\mathcal{L}_2\widehat{\mathbf{v}}\|_{\mathbf{V}^{1,1/2}((0,T)\times\{\mathbf{x}\in\mathbb{R}^3 : |\mathbf{x}|>N+3/2\})} + \|\widehat{\mathbf{v}}\|_{\mathbf{V}^{1,1/2}(Q_{N+3/2})} \\ &\quad + \|\Delta\mathcal{L}_2\widehat{\mathbf{v}}\|_{\mathbf{L}^2((0,T)\times\{\mathbf{x}\in\mathbb{R}^3 : |\mathbf{x}|>N+3/2\})} \\ &\quad + \|\Delta\widehat{\mathbf{v}}\|_{\mathbf{L}^2(Q_{N+3/2})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+3/2}))} \\ &\leq C\Big(\|\widehat{\mathbf{v}}\|_{\mathbf{V}^{1,1/2}(Q_{N+3/2})} + C\|\Delta\widehat{\mathbf{v}}\|_{\mathbf{L}^2((0,T)\times K_{N+5/4,N+3/2})} \\ &\quad + \|\Delta\widehat{\mathbf{v}}\|_{\mathbf{L}^2(Q_{N+5/4})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+5/4}))}\Big). \end{aligned} \tag{4.30}$$

Note that by Theorem 4.1, the pair $(\widehat{\mathbf{v}}, \nabla\widehat{p})$ satisfies (4.1) and, by virtue of (4.5) and (4.6), $\nabla\widehat{p}$ equals ∇p from (4.6). This and (4.8) and (3.29) imply that

$$\begin{aligned} &\|\Delta\widehat{\mathbf{v}}\|_{\mathbf{L}^2(Q_{N+5/4})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega)_{N+5/4})} \leq \|\nabla\widehat{p}\|_{\mathbf{L}^2(0,T;(H^{1/2}_{00}(\Omega_{N+2}))')} \\ &\quad \leq C\|\Delta(E\mathbf{b})\|_{\mathbf{L}^2(0,T;(H^{1/2}_{00}(\Omega_{N+2}))')} \leq C\|E\mathbf{b}\|_{\mathbf{L}^2(0,T;\mathbf{H}^{3/2}(\Omega_{N+2}))} \leq C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}. \end{aligned} \tag{4.31}$$

---

[2] Recall that by virtue of (1.1),

$$\|\Delta\mathbf{v}\|^2_{\mathbf{L}^2(Q_{N+2})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{N+2}))} = \inf\big(\|\mathbf{u}\|_{\mathbf{L}^2(Q_{N+2})} + \|\nabla p\|^2_{\mathbf{L}^2(0,T;(\mathbf{H}^{1/2}_{00}(\Omega_{N+2}))')}\big),$$

where the infimum is taken over all $\mathbf{u}, \nabla p$ such that $\Delta\mathbf{v} = \mathbf{u} + \nabla p$.

Now, we obtain from (4.30), (4.31), (4.4), and (4.15) the estimate

$$\|\mathcal{E}\mathbf{b}\|_{\mathbf{V}_N^{1,1/2}(Q)} \le C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}\,. \tag{4.32}$$

This proves the boundedness of the operator (4.28). □

5. Evolution problems

In this section, we study inhomogeneous boundary value problems for the Oseen equations and the Navier–Stokes equations on exterior domains. As in Section 4, we assume Ω is the exterior of a bounded domain having a boundary $\partial\Omega$ of class C^∞.

5.1. The Oseen equations

Let $\mathbf{z}(t,\mathbf{x})$ be a given solenoidal vector field. The system of Oseen equations is defined as follows:

$$\partial_t\mathbf{w} - \Delta\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} + \nabla p = \mathbf{f} \qquad \text{in } (0,T)\times\Omega \tag{5.1}$$

and

$$\operatorname{div}\mathbf{w} = 0 \qquad \text{in } (0,T)\times\Omega\,. \tag{5.2}$$

Evidently, (5.1)–(5.2) can be treated as the linearization of the Navier–Stokes equations at the vector field $\mathbf{z}$. We supplement (5.1)–(5.2) with the initial condition

$$\mathbf{w}(0,\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) \qquad \text{in } \Omega \tag{5.3}$$

and the boundary conditions

$$\mathbf{w}\big|_\Sigma = \mathbf{b} \qquad \text{and} \qquad \mathbf{w}|_{|\mathbf{x}|\to\infty} = \mathbf{0}\,. \tag{5.4}$$

We assume that

$$\mathbf{f}\in\mathbf{L}^2(Q), \qquad \mathbf{w}_0\in\mathbf{V}^1(\Omega), \qquad \text{and} \qquad \mathbf{b}\in\widehat{\mathbf{H}}^{1,0}(\Sigma)\,,$$

where the spaces $\mathbf{V}^1(\Omega)$ and $\widehat{\mathbf{H}}^{1,0}(\Sigma)\equiv\widehat{H}^{1,0}(\mathcal{T}(\Sigma))$ are as defined in (2.3) and (3.31), respectively.

It is well known that finding the solution $(\mathbf{w},\nabla p)$ of (5.1)–(5.4) can be reduced to finding $\mathbf{w}$ only, for the following de Rham Lemma (see, e.g., [16]) will allow us to obtain ∇p once we have found $\mathbf{w}$.

Lemma 5.1. *A vector field $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, where each $f_i(\mathbf{x})$ is a distribution on Ω, has the form*

$$\mathbf{f} = \nabla p \quad \textit{for some } p$$

if and only if

$$\langle\mathbf{f},\mathbf{v}\rangle = 0 \qquad \forall\,\mathbf{v}\in\mathcal{V}(\Omega)\,,$$

where

$$\mathcal{V}(\Omega) = \{\mathbf{v}\in\mathbf{C}_0^\infty(\Omega)\,:\,\operatorname{div}\mathbf{v} = 0\}$$

and $\langle \mathbf{f}, \mathbf{v} \rangle$ *denotes the value of the distribution* $\mathbf{f}$ *at the test function* $\mathbf{v} \in \mathcal{V}(\Omega)$ *generated by the scalar product in* $\mathbf{L}^2(\Omega)$. $\square$

With a fixed N satisfying (3.34) we look for a solution $\mathbf{w}$ of (5.1)–(5.4) in the space

$$\mathbf{Y} = \mathbf{V}_N^{1,1/2}(Q) + \mathbf{V}_0^{(1,2)}(Q) \tag{5.5}$$

which is the direct sum of the space (4.26) and the space

$$\mathbf{V}_0^{(1,2)}(Q) = \{\mathbf{v} \in L^2(0,T;\mathbf{V}^2(\Omega)) \cap H^1(0,T;\mathbf{V}^0(\Omega)) : \mathbf{v}\big|_\Sigma = \mathbf{0}\}\,. \tag{5.6}$$

(Recall that the direct sum of two Hilbert spaces and its norm are defined in (1.1) and above.) If $\mathbf{w} \in \mathbf{Y}$ is a solution of (5.1)–(5.4), then the data $\mathbf{w}_0$ and $\mathbf{b}$ must satisfy the compatibility condition

$$\mathbf{w}_0\big|_{\partial\Omega} = \mathbf{b}\big|_{t=0}\,. \tag{5.7}$$

The following theorem asserts the existence of a desired $\mathbf{w}$.

Theorem 5.2. *Assume that* $\mathbf{f} \in \mathbf{L}^2(Q)$*,* $\mathbf{z} \in \mathbf{Y}$*,* $\mathbf{w}_0 \in \mathbf{V}^1(\Omega)$*, and* $\mathbf{b} \in \mathbf{H}^1(\Sigma)$ *and that* $\mathbf{w}_0$ *and* $\mathbf{b}$ *satisfy* (4.3) *and* (5.7)*. Then, there exists a unique solution* $(\mathbf{w}, \nabla p)$ *for* (5.1)–(5.4) *such that* $\mathbf{w} \in \mathbf{Y}$ *and*

$$\|\mathbf{w}\|_{\mathbf{Y}}^2 + \|\nabla p\|^2_{L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega))')} \le C\left(\|\mathbf{f}\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{w}_0\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{b}\|^2_{\mathbf{H}^1(\Sigma)}\right)\,.$$

Proof. Let $\mathcal{E}\mathbf{b} \in \mathbf{V}_N^{1,1/2}(Q)$ be the extension of $\mathbf{b} \in \widehat{\mathbf{H}}^{1,0}(\Sigma)$ constructed in Theorem 4.3. By virtue of (4.28),

$$\|\mathcal{E}\mathbf{b}\|_{V_N^{1,1/2}(Q)} \le C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}\,. \tag{5.8}$$

Now we look for the solution $\mathbf{w}$ for (5.1)–(5.4) in the form

$$\mathbf{w} = \mathbf{v} + \mathcal{E}\mathbf{b}\,, \tag{5.9}$$

where $\mathbf{v}$ is a new unknown vector field. The substitution of (5.9) into (5.1)–(5.4) yields

$$\partial_t\mathbf{v}(t,\mathbf{x}) - \Delta\mathbf{v} + (\mathbf{z}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{z} + \nabla p_1 = \mathbf{f}_1, \qquad \operatorname{div}\mathbf{v} = 0, \qquad \text{in } Q\,, \tag{5.10}$$

$$\mathbf{v}\big|_{t=0} = \mathbf{v}_0, \qquad \mathbf{v}\big|_\Sigma = 0, \qquad \mathbf{v}\big|_{|\mathbf{x}|\to\infty} = \mathbf{0}\,, \tag{5.11}$$

where

$$\mathbf{v}_0(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) - \mathcal{E}\mathbf{b}(0,\mathbf{x})$$

and

$$\mathbf{f}_1 = \mathbf{f} - \partial_t(\mathcal{E}\mathbf{b}) - (\mathbf{z}\cdot\nabla)(\mathcal{E}\mathbf{b}) - ((\mathcal{E}\mathbf{b})\cdot\nabla)\mathbf{z} + \mathbf{g}\,, \qquad \nabla p_1 = \nabla p + \nabla q$$

with $\mathbf{g} + \nabla q = \Delta(\mathcal{E}\mathbf{b})$ where $\mathbf{g} \in \mathbf{L}^2(Q_{N+2})$, $\nabla q \in L^2(0,T;\nabla H^{1/2}(\Omega_{N+2}))$. By virtue of (5.8), the definition of $\mathcal{E}\mathbf{b}$, and (5.7), we see that

$$\mathbf{v}_0 \in \mathbf{V}_0^1(\Omega) \equiv \{\mathbf{v} \in \mathbf{V}^1(\Omega) \,:\, \mathbf{v}\big|_{\partial\Omega} = \mathbf{0}\}\,, \qquad \|\mathbf{v}\|^2_{\mathbf{V}_0^1(\Omega)} = \|\nabla\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\,.$$

Using (4.26), (5.8), the inclusion $\mathbf{z} \in \mathbf{V}_N^{1,1/2}(Q)$, and Sobolev embedding theorems, we easily deduce that $\mathbf{f}_1 \in \mathbf{L}^2(Q)$. Thus, (5.10)–(5.11) constitutes a boundary value problem with a homogeneous boundary condition wherein (5.10) differs from the Stokes equations by the terms $(\mathbf{z}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{z}$ only. With the help of the standard Galerkin method we may prove, as in [9] and [16], that (5.10)–(5.11) possesses a unique generalized solution $(\mathbf{v}, \nabla p_1)$, where $\mathbf{v}$ satisfies the energy estimate

$$\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{v}(s)\|^2_{\mathbf{V}^1(\Omega)}\, ds \le C\left(\|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{f}_1(s)\|^2_{\mathbf{H}^{-1}(\Omega)}\, ds \right)$$

with C depending on $\|\nabla \mathbf{z}\|_{L^2(0,T;\mathbf{V}^{1/2}(\Omega))}$.

Recall that $\mathbf{V}_0^0(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,:\, \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$ and define $\mathbf{V}_0^{-1}(\Omega)$ as the completion of $\mathbf{V}_0^0(\Omega)$ in the norm

$$\|\mathbf{f}\|_{\mathbf{V}_0^{-1}(\Omega)} = \sup_{\boldsymbol{\phi}\in\mathbf{V}_0^1(\Omega)} \frac{\langle \mathbf{f}, \boldsymbol{\phi}\rangle}{\|\boldsymbol{\phi}\|_{\mathbf{V}_0^1(\Omega)}}\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality generated by the scalar product in $\mathbf{V}_0^0(\Omega)$. Note that $\mathbf{V}_0^{-1}(\Omega)$ is not a space of distributions; it is some abstract space. By interpolation between $\mathbf{V}_0^1(\Omega)$ and $\mathbf{V}_0^0(\Omega)$ (see [12]), we define $\mathbf{V}_0^s(\Omega)$ for $s \in (0,1)$. Evidently $\mathbf{V}_0^s(\Omega) \subset \mathbf{V}^s(\Omega)$. Analogous to $\mathbf{V}_0^{-1}(\Omega)$, we define $\mathbf{V}_0^{-s}(\Omega)$ for $s \in (0,1)$ as the dual space of $\mathbf{V}_0^s(\Omega)$. Since $\mathbf{V}_0^1(\Omega) \subset \mathbf{V}_0^s(\Omega) \subset \mathbf{V}_0^0(\Omega)$, we have that $\mathbf{V}_0^{-s}(\Omega) \subset \mathbf{V}_0^{-1}(\Omega)$ if $s \in (0,1)$. We define the operator

$$\begin{aligned} &P : \mathbf{H}^{-s}(\Omega) \to \mathbf{V}_0^{-s}(\Omega)\,, \qquad s \in [0,1]\,, \quad s \ne 1/2\\ &P : (\mathbf{H}_{00}^{1/2}(\Omega))' \to \mathbf{V}_0^{-1/2}(\Omega)\,, \qquad s = 1/2\,, \end{aligned}$$

by the formula

$$\langle P\mathbf{f}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in \mathbf{V}_0^s(\Omega)\,, \qquad s \in [0,1]\,,$$

where on the left, $\langle\cdot,\cdot\rangle$ denotes the duality generated by the scalar product in $\mathbf{V}_0^0(\Omega)$ and on the right the duality generated by the scalar product in $\mathbf{L}^2(\Omega)$.

Since, by virtue of Lemma 5.1, $P\nabla H^{1-s}(\Omega) = \mathbf{0}$, one can reduce the boundary value problem for the Stokes operator:

$$\begin{gathered} \partial_t \mathbf{v} - \Delta\mathbf{v} + \nabla p = \mathbf{f}\,, \qquad \operatorname{div}\mathbf{v} = 0 \qquad \text{in } Q\\ \mathbf{v}|_{t=0} = \mathbf{v}_0\,, \qquad \mathbf{v}|_\Sigma = \mathbf{0}\,, \qquad \mathbf{v}|_{|\mathbf{x}|\to\infty} = \mathbf{0} \end{gathered}$$

to the solution of the problem

$$P(\partial_t \mathbf{v} - \Delta\mathbf{v}) = P\mathbf{f} \quad \text{in } \mathbf{V}_0^{-s}(\Omega)\,, \qquad \mathbf{v}|_{t=0} = \mathbf{v}_0\,.$$

Similarly to [17, Ch.1, §3], one can prove[3] that the solution operator $R : R(\mathbf{f}, \mathbf{v}_0) =$

[3] The difference between the proof here and that in [17] is that now Ω is the complement of a bounded domain instead of a bounded domain as it was in [17]. We now take as the basic operator for constructing interpolation spaces the operator $\mathcal{A} = P(-\Delta\mathbf{u} + \mathbf{u})$, where $P : \mathbf{L}^2(\Omega) \to \mathbf{V}_0^0(\Omega)$ is the orthogonal projector instead of $\widetilde{\mathcal{A}} = P(-\Delta\mathbf{u})$ as in [17]. In an unbounded domain, the operator $\mathcal{A}$ has a continuous spectrum and we must use general theorems on spectral decompositions instead of decompositions in terms of eigenfunctions as were used in [17]. In all other respects, the proof of (5.12) is the same as the analogous proof in [17].

$\mathbf{v}$ of this problem (and, hence, of the Stokes problem) is well defined and

$$\begin{aligned} R\,:\, & L^2(0,T;\mathbf{V}_0^{s-2}(\Omega))\times \mathbf{V}_0^1(\Omega) \\ & \to L^2(0,T;\mathbf{V}^s(\Omega)\cap \mathbf{V}_0^1(\Omega))\cap H^1(0,T;\mathbf{V}^{s-2}(\Omega)), \quad s\in[1,2] \end{aligned} \tag{5.12}$$

is bounded. Applying the composition operator RP to (5.10) yields

$$\mathbf{v}+RP\big((\mathbf{z}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{z}\big)=RP\mathbf{f}_1\,. \tag{5.13}$$

Since $\mathbf{f}_1\in\mathbf{L}^2(Q)$, we deduce that $RP\mathbf{f}_1\in L^2(0,T;\mathbf{V}^2(\Omega))\cap H^1(0,T;\mathbf{V}^0(\Omega))$.

We easily obtain

$$\|P\big((\mathbf{z}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{z}\big)\|_{L^2(0,T;\mathbf{V}_0^{-1/2}(\Omega))}\le C\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^1(\Omega))}\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))}\,. \tag{5.14}$$

From (5.12) with $s=3/2$ and (5.13)–(5.14), we obtain

$$\mathbf{v}\in L^2(0,T;\mathbf{V}^{3/2}(\Omega))\cap H^1(0,T;\mathbf{V}_0^{-1/2}(\Omega))$$

and for arbitrarily small $\varepsilon>0$" we obtain

$$\begin{aligned} &\|(\mathbf{z}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{z}\|_{L^2(0,T;\mathbf{V}_0^{-\varepsilon}(\Omega))} \\ &\le C\Big(\|\mathbf{z}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Omega_N))}+\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^1(\Omega))}\Big)\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Omega))\cap H^1(0,T;\mathbf{V}_0^{-1/2}(\Omega))}\,. \end{aligned} \tag{5.15}$$

From (5.13), (5.15), (5.12) with $s=2-\varepsilon$, and the boundary condition $\mathbf{v}|_\Sigma=\mathbf{0}$, we deduce

$$\mathbf{v}\in L^2(0,T;\mathbf{V}^{2-\varepsilon}(\Omega)\cap\mathbf{V}_0^1(\Omega))\cap H^1(0,T;\mathbf{V}^{-\varepsilon}(\Omega)).$$

Using this inclusion we obtain:

$$\begin{aligned} &\|(\mathbf{z}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{z}\|_{\mathbf{L}^2(Q)} \\ &\le C\Big(\|\mathbf{z}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Omega_N))}+\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^1(\Omega))}\Big)\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^{2-\varepsilon}(\Omega))\cap H^1(0,T;\mathbf{V}_0^{-\varepsilon}(\Omega))}\,. \end{aligned}$$

From this equality, (5.13) and (5.12) with $s=2$ we deduce that $\mathbf{v}\in\mathbf{V}_0^{(1,2)}(Q)$. Repeating the iteration argument used previously we obtain the estimate

$$\|\mathbf{v}\|^2_{\mathbf{V}_0^{(1,2)}(Q)}\le C\Big(\|\mathbf{v}_0\|^2_{\mathbf{V}^1(\Omega)}+\|\mathbf{f}_1\|^2_{\mathbf{L}^2(Q)}\Big)\,.$$

Relations (5.8), (5.9), and (5.12) guarantee the existence of a solution $(\mathbf{w},\nabla p)$ of problem (5.1)–(5.4) that belongs to $\mathbf{Y}\times L^2(0,T;(\mathbf{H}_{00}^{1/2}(\Omega))')$. The proof of the uniqueness of the solution is easily reduced to the proof of the uniqueness of its component $\mathbf{v}$ in the space $\mathbf{Y}$. Such a proof is standard in the literature and can be found in, e.g., [5]. □

5.2. The nonlinear evolution problem

We now consider the nonlinear evolution problem

$$\partial_t\mathbf{w}-\Delta\mathbf{w}+([\mathbf{z}+\mathbf{w}]\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{z}+\nabla p=\mathbf{f} \qquad \text{in } Q\,, \tag{5.16}$$

$$\operatorname{div} \mathbf{w} = 0 \qquad \text{in } Q\,, \tag{5.17}$$

$$\mathbf{w}\Big|_{t=0} = \mathbf{w}_0, \qquad \mathbf{w}\big|_{\Sigma} = \mathbf{b} \qquad \mathbf{w}\Big|_{|\mathbf{x}|\to\infty} = \mathbf{0}\,. \tag{5.18}$$

This system differs from (5.1)–(5.4) by the term $(\mathbf{w}\cdot\nabla)\mathbf{w}$ only. As in the previous section, we suppose that the coefficient $\mathbf{z}$ in (5.16) belongs to the space $\mathbf{Y}$ (see (5.5)) and we look for the solution $\mathbf{w}$ of (5.16)–(5.18) in the space $\mathbf{Y}$ as well.

We will need the following lemma on analytic inverse operators (see [17]):

Lemma 5.3. *Let X_1 and X_2 be Banach spaces, $A : X_1 \to X_2$ be a linear isomorphism of X_1 and X_2, and $B(\cdot,\cdot) : X_1 \times X_1 \to X_2$ be a continuous bilinear operator. Then the equation*

$$Ax + B(x,x) = f \tag{5.19}$$

has a solution $x \in X_1$ if $\|f\|_{X_2} < \varepsilon$ for a sufficiently small ϵ. The map $x = Rf$ which maps the right hand side f to the solution x of (5.19) is defined uniquely and is analytic (i.e., $R(f)$ can be expressed as a convergent series). □

Theorem 5.4. *Let $\mathbf{f} \in \mathbf{L}^2(Q_)$, $\mathbf{v}_0 \in \mathbf{V}^1(\Omega)$, and $\mathbf{b} \in \mathbf{H}^1(\Sigma)$ satisfy (4.3) and (5.7). Assume that*

$$\|\mathbf{f}\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{w}_0\|^2_{\mathbf{V}^1(\Omega)} + \|\mathbf{b}\|^2_{\mathbf{H}^1(\Sigma)} < \varepsilon\,,$$

where $\varepsilon > 0$ is sufficiently small. Then, there exists a unique solution $(\mathbf{w}, \nabla p)$ of the problem (5.16)–(5.18) which belongs to the space $\mathbf{Y} \times L^2(0,T;(\mathbf{H}^{1/2}_{00}(\Omega))')$. Furthermore, the solution $(\mathbf{w},\nabla p)$ satisfies the estimate

$$\|\mathbf{w}\|^2_{\mathbf{Y}} + \|\nabla p\|^2_{L^2(0,T;(\mathbf{H}^{1/2}_{00}(\Omega))')} \le C(\varepsilon)\,,$$

where $C(\varepsilon)$ is a positive continuous function which is defined for all sufficiently small ε.

Proof. As in the proof of Theorem 5.2, we seek a solution of (5.16)–(5.18) in the form (5.9) with $\mathcal{E}\mathbf{b} \in \mathbf{V}^{1,1/2}_N(Q)$ which is the extension of $\mathbf{b} \in \widehat{\mathbf{H}}^{1,0}(\Sigma)$ constructed in Theorem 4.3. The substitution of (5.9) into (5.16)–(5.18) yields

$$\partial_t\mathbf{v} - \Delta\mathbf{v} + ((\mathbf{z}+\mathcal{E}\mathbf{b}+\mathbf{v})\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)(\mathbf{z}+\mathcal{E}\mathbf{b}) + \nabla p_1 = \mathbf{f}_1, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q \tag{5.20}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \mathbf{v}|_{\Sigma} = \mathbf{0}, \quad \mathbf{v}|_{|\mathbf{x}|\to\infty} = \mathbf{0}\,, \tag{5.21}$$

where

$$\mathbf{v}_0(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) - \mathcal{E}\mathbf{b}(0,\mathbf{x})$$

and

$$\mathbf{f}_1 = \mathbf{f} - \partial_t(\mathcal{E}\mathbf{b}) - (\mathcal{E}\mathbf{b}\cdot\nabla)\mathbf{z} - ((\mathbf{z}+\mathcal{E}\mathbf{b})\cdot\nabla)\mathcal{E}\mathbf{b} + \mathbf{g}\,, \qquad \nabla p_1 = \nabla p + \nabla q$$

with $\mathbf{g} + \nabla q = \Delta \mathcal{E}\mathbf{b}$. By (5.8), (5.7), and the definition of $\mathcal{E}\mathbf{b}$ as in the proof of Theorem 5.2, we see that $\mathbf{v}_0 \in \mathbf{V}_0^1(\Omega)$, $\mathbf{f}_1 \in \mathbf{L}^2(Q)$, and

$$\|\mathbf{v}_0\|_{\mathbf{V}_0^1(\Omega)} \leq \|\mathbf{w}_0\|_{\mathbf{V}_0^1(\Omega)} + C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}\,, \qquad \|\mathbf{f}_1\|_{\mathbf{L}^2(Q)} \leq \|\mathbf{f}\|_{\mathbf{L}^2(Q)} + C\|\mathbf{b}\|_{\mathbf{H}^1(\Sigma)}\,,$$

where $C > 0$ does not depend on $\mathbf{b}$. Hence, by the hypothesis of the theorem,

$$\|\mathbf{v}_0\|^2_{\mathbf{V}_0^1(\Omega)} + \|\mathbf{f}_1\|^2_{\mathbf{L}^2(Q)} \leq C_1 \varepsilon \tag{5.22}$$

with $C_1 > 0$ that does not depend on $\mathbf{b}$ and ϵ.

Applying P (defined before (5.12)) to (5.20)–(5.21), we obtain

$$P\big(\partial_t \mathbf{v} - \Delta \mathbf{v} + ((\mathbf{z} + \mathcal{E}\mathbf{b} + \mathbf{v}) \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)(\mathbf{z} + \mathcal{E}\mathbf{b})\big) = P\mathbf{f}_1\,, \qquad \operatorname{div} \mathbf{v} = 0\,. \tag{5.23}$$

Evidently, it suffices to show the solvability and uniqueness of solutions for the problem (5.21)–(5.23). We will prove this below with the help of Lemma 5.3.

To fit (5.23) and (5.21) into the framework of Lemma 5.3, we choose $X_1 = \mathbf{V}_0^{(1,2)}(Q)$ and $X_2 = L^2(0, T; \mathbf{V}_0^0(\Omega)) \times \mathbf{V}_0^1(Q)$. We define

$$A\mathbf{v} = \Big(P\big(\partial_t \mathbf{v} - \Delta \mathbf{v} + ((\mathbf{z} + \mathcal{E}\mathbf{b}) \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)(\mathbf{z} + \mathcal{E}\mathbf{b})\big)\,,\, \mathbf{v}\big|_{t=0}\Big)$$

and

$$B(\mathbf{v}, \mathbf{v}) = \Big(P\big((\mathbf{v} \cdot \nabla)\mathbf{v}\big), \mathbf{0}\Big)\,.$$

Then, the solvability of (5.21)–(5.23) is equivalent to the existence of a $\mathbf{v} \in X_1$ satisfying

$$A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0)\,,$$

where $(\mathbf{f}, \mathbf{v}_0)$ is given in X_2.

The continuity of the operators $A : X_1 \to X_2$ and $B : X_1 \to X_2$ is well known (see, e.g., [5]) and can be easily established by the Sobolev embedding theorem. The existence of the inverse operator $A^{-1} : X_2 \to X_1$ is also well known and was discussed above in Theorem 5.2. Then, by Lemma 5.3, there exists a unique solution of the problem (5.21)–(5.23) if ε in (5.22) is sufficiently small. From this the assertions of this theorem can be derived in the same way as the analogous steps in Theorem 5.2. □

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