OPTIMAL NEUMANN CONTROL FOR THE 2D STEADY-STATE NAVIER-STOKES EQUATIONS

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Abstract. An optimal control problem, the minimization of drag, is considered for the 2D stationary Navier-Stokes equations. The control is of Neumann kind and acts at a part of the boundary which is contiguous to the rigid boundary where the no-slip condition holds. Further, certain constraints are imposed on the control and the phase variable. We derive an existence theorem as well as the corresponding optimality system

To the memory of Alexander Vasil'evich Kazhikhov.

1. Introduction

This paper is devoted to the study of an optimal control problem for the Navier-Stokes equations defined in a bounded domain Ω. We are interested in the existence of optimal solutions as well as in the derivation of the corresponding "optimality system", i.e. the first-order optimality conditions. These problems have been studied already for the stationary Navier-Stokes equations (see [GHS1],[GHS2], [CH], [A], [ALT]) and the non-stationary Navier-Stokes equations (see [F1], [F2], [AT], [S] [F], [FGH]) for small as well as large Reynolds numbers. However, not all aspects of these optimization problems have been completely investigated, yet.

In this paper, we concentrate on the following questions arisen in optimal control problems. First of all the extremal problem we study contains restrictions not only on the control but on the phase variable as well. The restriction is imposed that the component $v_1(x)$ of the fluid velocity should be nonnegative on a certain subdomain ω of Ω.

The derivation of the optimality system in such a situation needs a specific Lagrange principle. A general Lagrange principle of such kind was worked out by I. V. Girsanov [G] and A. A. Milutin, A. V. Dmitruk, N. P.

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Osmolovskiy [MDO]. In this paper, we have to adapt the approach from [G], [MDO] to the optimal control problem for the Navier-Stokes equations.

Usually in applications the boundary control is acting not on the whole boundary $\partial\Omega$ but only on a certain part Γ. Besides, often it is more reasonable to use Neumann control on Γ instead of Dirichlet control. Moreover Γ is contiguous with the part of boundary where the adhesion condition is posed. In such a situation Neumann control causes a local singularity of the state at $\partial\Gamma$. This effect was studied in many papers beginning by V. A. Kondrat'ev's work [Kon1]. This effect is not essential in the proof of the existence theorem for the optimal control problem, but it becomes important in the derivation of the optimality system.

In this paper, we derive the optimality system for an optimal control problem in which all the aforementioned complications take place. In order to focus on the essential aspects, we minimize all other possible difficulties by only considering an optimal control problem for the 2D steady-state Navier-Stokes equations. However we are sure that the results of this paper can be extended to the 3D case as well as to the nonstationary Navier-Stokes equations.

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2. Setting of the optimal control problem

Let Ω be the two-dimensional domain shown in Figure 1, i.e.rectangle without the set bounded by the curve S. We introduce the following notation for parts of the boundary $\partial\Omega$: $AH = \Gamma_{\text{in}}$, $DE = \Gamma_{\text{out}}$, $AB \cup CD \cup FE \cup HG = S'$, $BC = \Gamma_1$, $GF = \Gamma_2$, $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\partial\Omega = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma \cup S \cup S'$. We shall use the abbreviated notation $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ for the L^2 scalar product over Ω and $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ for the associated norm. For subdomains $D \subset \Omega$ and $\Gamma \subset \partial\Omega$, we write $\|\cdot\|_D = \|\cdot\|_{L^2(D)}$ and $\|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}$, respectively, and similarly for the corresponding scalar products. We will distinguish notations of norms and scalar products for scalar functions and corresponding vector fields: this should not lead to misunderstandings.

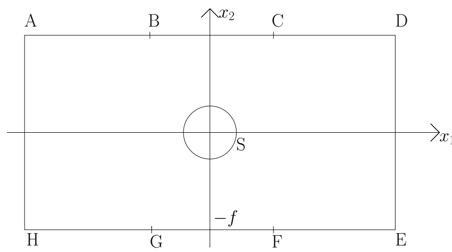

FIGURE 1. Domain

On Ω, we consider the Navier-Stokes equations

$$-\Delta v + v\cdot\nabla v + \nabla p = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla\cdot v = 0 \quad \text{in } \Omega, \tag{2.2}$$

where $v = (v_1, v_2)$ is the velocity vector field, $\nabla p = (\partial_1 p, \partial_2 p)$ the pressure gradient, $v\cdot\nabla v = \sum_{j=1}^2 v_j \partial_j v$, and $\nabla\cdot v = \sum_{j=1}^2 \partial_j v_j$. The system (2.1), (2.2) is supplemented by the boundary conditions

$$v_{|\Gamma_{\rm in}} = v^{\rm in}, \quad (\partial_n v - pn)_{|\Gamma_{\rm out}} = 0, \quad v_{|S\cup S'} = 0, \tag{2.3}$$

where $v^{\rm in}$ is a given inflow vector field, and $n = n(x)$, $x\in\partial\Omega$, is the outside normal unit vector field to $\partial\Omega$. The goal is to minimize the drag functional of S,

$$J = \int_S n\cdot\sigma\cdot e_1\,dx \quad\to\quad \inf, \tag{2.4}$$

under the action of a control $u(x_1) = (u^1(x_1), u^2(x_1))$ at the horizontal boundary component $\Gamma = \Gamma_1\cup\Gamma_2$,

$$(\partial_n v - pn)_{|\Gamma_1} = u^1, \quad (\partial_n v - pn)_{|\Gamma_2} = u^2. \tag{2.5}$$

Here e_1 is the unit vector in the x_1 direction, and

$$n\cdot\sigma = -pn + 2\mathcal{D}(v)n, \quad 2\mathcal{D}(v) = \left(\partial_j v_i + \partial_i v_j\right)_{i,j=1,2}. \tag{2.6}$$

This control problem is supplemented by the following additional constraint on the phase variable $v_1(x)$:

$$v_1(x) \geq 0, \quad x\in\omega, \tag{2.7}$$

where $\omega \subset \Omega$ is a prescribed closed subset. Further, we impose the following restriction on the controls $u = (u^1, u^2)$:

$$\|u^1\|_{\Gamma_1}^2 + \|u^2\|_{\Gamma_2}^2 \le \gamma^2, \tag{2.8}$$

where $\gamma > 0$ is a given constant.

Our goal is to prove an existence theorem for the optimal control problem (2.1)-(2.8) and to derive the corresponding optimality system.

3. Boundary value problems

In this section, we prove an existence theorem for several boundary value problems that will be used to prove the existence theorem for the optimal control problem (2.1)-(2.8).

3.1. **The Stokes boundary value problem.** On the domain Ω, we consider the Stokes system

$$-\Delta v + \nabla p = f, \quad \nabla \cdot v = 0, \quad \text{in } \Omega, \tag{3.1}$$

supplemented by the boundary condition (2.3), (2.5). For simplicity let the coordinates of the points $A, B, \dots, H$ in Figure 1 be as follows:

$$\begin{aligned} &A = (0, \pi), \quad B = (b, \pi), \quad C = (c, \pi), \quad D = (d, \pi), \\ &H = (0, 0), \quad G = (b, 0), \quad F = (c, 0), \quad E = (d, 0). \end{aligned} \tag{3.2}$$

We suppose that

$$u^1(x_1) \in L^2(\Gamma_1)^2, \quad u^2(x_1) \in L^2(\Gamma_2)^2, \quad v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})^2 \tag{3.3}$$

where

$$H_0^1(\Gamma_{\text{in}}) = \Big\{ w \in L^2(\Gamma_{\text{in}}) \,\Big|\, \|\partial_2 w\|_{\Gamma_{\text{in}}} < \infty,\, w(0) = w(\pi) = 0 \Big\}. \tag{3.4}$$

It is convenient for us to suppose that in (3.1)

$$f(x) \in L^{3/2}(\Omega)^2. \tag{3.5}$$

In this subsection, we prove an existence and uniqueness theorem for the generalized solution of the boundary value problem (3.1), (2.3), (2.5). To define the notion of "generalized solution", we introduce the space

$$\Phi = \Big\{ v \in H^1(\Omega)^2 \,\Big|\, \nabla \cdot v = 0,\, v_{|S \cup S'} = 0 \Big\}, \tag{3.6}$$

where as above $H^1(\Omega)^2 = H^1(\Omega) \times H^1(\Omega)$ and $H^1(\Omega)$ is the usual Sobolev space over Ω. We recall that for natural k the Sobolev space $H^k(\Omega)$ is defined as follows: $H^k(\Omega) = W_2^k(\Omega)$, and for each integer $k \geq 1$ and $1 \leq p < \infty$:

$$W_p^k(\Omega) = \Big\{\varphi \in L^p(\Omega) \,\Big|\, \|\varphi\|^p_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha \varphi\|^p_{L^p(\Omega)} < \infty\Big\},$$

with $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, α_i nonnegative integers. For arbitrary $s > 0$ the Sobolev space $H^s(\Omega)$ can be defined by interpolation (see [LM]). Further, we introduce the space

$$\Phi_0 := \Big\{v \in H^1(\Omega)^2 \mid \nabla \cdot v = 0,\, v_{|\Gamma_{\text{in}}} = 0,\, v_{|S \cup S'} = 0\Big\}, \tag{3.7}$$

and supply the spaces Φ and Φ_0 with the norms

$$\|\varphi\|_\Phi := \|\varphi\|_{H^1(\Omega)}, \quad \|\varphi\|_{\Phi_0} := \|\varphi\|_{H^1(\Omega)}.$$

Definition 3.1. Let $u^1 \in L^2(\Gamma_1)^2$, $u^2 \in L^2(\Gamma_2)^2$, $f \in L^{3/2}(\Omega)^2$, and $v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})^2$. The vector function $v \in \Phi$ satisfying $v|_{\Gamma_{\text{in}}} = v^{\text{in}}$ and

$$(\nabla v, \nabla \varphi) - (u^2, \varphi)_{\Gamma_2} - (u^1, \varphi)_{\Gamma_1} = (f, \varphi) \quad \forall \varphi \in \Phi_0, \tag{3.8}$$

is called a "generalized solution" of problem (3.1), (2.3), (2.5).[1]

The following result clarifies the connection between the generalized solution satisfying (3.8) and the solution of problem (3.1), (2.3), (2.5).

Proposition 3.1. *Let $v \in \Phi$ be the generalized solution of problem* (3.1), (2.3), (2.5). *Then, there exists a $p \in L^{3/2}(\Omega)$ such that the pair (v, p) satisfies* (3.1). *Moreover, if $(v, p) \in W^2_{3/2}(\Omega)^2 \times W^1_{3/2}(\Omega)$, then p is unique, and this pair satisfies* (2.3), (2.5)[2].

Proof. Integration by parts in (3.8) with $\varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2$ implies

$$(\Delta v + f, \varphi) = 0 \quad \forall \varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2. \tag{3.9}$$

Then, by the De Rham theorem (see [T]) there exists $p \in L^{3/2}(\Omega)$ such that (v, p) satisfies (3.1) in the distributional sense. Notice that though ∇p is defined uniquely in (3.1), p is determined only up to a constant. To define it uniquely, we substitute $f = -\Delta v + \nabla p$ in the right hand side of (3.8)

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[1]As we will show, a generalized solution exists even under weaker assumptions on u^1, u^2, v^{in}.

[2]Notice that in virtue of the ellipticity of the system (3.1) in the Douglas-Nirenberg sense the inclusion $(v, p) \in W^2_{3/2}(\Omega')^2 \times W^1_{3/2}(\Omega')$ holds for an arbitrary subdomain $\Omega' \Subset \Omega$.

and integrate by parts in this term. As a result, we get

$$\sum_{i=1}^{2}(\partial_n v - pn - u^i, \varphi)_{\Gamma_i} + (\partial_n v - pn, \varphi)_{\Gamma_{\rm out}} = 0 \quad \forall \varphi \in \Phi_0. \tag{3.10}$$

Equality (3.10) implies that

$$(\partial_n v - pn - u^i + cn)_{|\Gamma_i} = 0,\ i = 1, 2, \quad (\partial_n v - pn + cn)_{|\Gamma_{\rm out}} = 0, \tag{3.11}$$

where the constant c in all the equalities is the same. We choose the constant component of the pressure p such that c in equations (3.11) becomes zero.[3] □

Theorem 3.2. *Let* $u^i \in L^2(\Gamma_i)^2$, $i = 1, 2$, $f \in L^{3/2}(\Omega)^2$. *Then, there exists a unique generalized solution of problem* (3.1), (2.3), (2.5).

Proof. Let us consider the extremal problem

$$J_0(v) := \frac{1}{2}\|\nabla v\|^2 - (f, v) - \sum_{j=1}^{2}(u^j, v)_{\Gamma_j} \ \rightarrow\ \inf, \tag{3.12}$$

$$v \in \Phi, \qquad v|_{\Gamma_{\rm in}} = v^{\rm in}, \tag{3.13}$$

for $v \in \Phi$ with $v_{|\Gamma_{\rm in}} = v^{\rm in}$, where Φ is defined in (3.6). The functional $J_0(v)$ is convex and continuous on $H^1(\Omega)^2$. Therefore it is semi-continuous on $H^1(\Omega)^2$ with respect to the weak convergence in $H^1(\Omega)^2$. Besides, being a closed convex subset of $H^1(\Omega)^2$, the set of restrictions (3.13) is sequentially weakly closed in $H^1(\Omega)^2$. At last, $J_0(v_k) \to \infty$, as $v_k \in \Phi$, $\|v_k\|_{H^1(\Omega)} \to \infty$. Therefore (see [F]) there exists a unique solution $\hat{v} \in \Phi$ of problem (3.12), (3.13). The conditions $\hat{v} \in \Phi$, $\hat{v} + \varphi \in \Phi$, and $\widehat{v}|_{\Gamma_{\rm in}} = (\widehat{v} + \varphi)|_{\Gamma_{\rm in}} = v^{\rm in}$ imply the inclusion $\varphi \in \Phi_0$. Since $\hat{v}$ is a solution of (3.12), (3.13),

$$0 = \lim_{\lambda \to 0} \frac{J_0(\hat{v} + \lambda\varphi) - J_0(\hat{v})}{\lambda} = (\nabla\hat{v}, \nabla\varphi) - (f, \varphi) - \sum_{j=1}^{2}(u^j, \varphi)_{\Gamma_j},$$

for all $\varphi \in \Phi_0$. □

3.2. **An extension result.** We recall a well-known extension result using the notation

$$V^1(\Omega) = \Big\{v \in H^1(\Omega)^2 \,|\, \nabla \cdot v = 0\Big\}, \quad V_0^1(\Omega) = \Big\{v \in V^1(\Omega) \,|\, v_{|\partial\Omega} = 0\Big\}.$$

[3]The uniqueness of p without the additional assumption $(v, p) \in W^2_{3/2}(\Omega)^2 \times W^1_{3/2}(\Omega)$ will be proved below in Theorem 4.1.

Lemma 3.3. *For each function $g \in H^{1/2}(\partial\Omega)^2$ satisfying $(g,n)_S = 0$ and $(g,n)_{\partial\Omega\setminus S} = 0$, where n is the outer normal to $\partial\Omega$, there exists $u \in V^1(\Omega)$ such that $u_{|\partial\Omega} = g$. Moreover*

$$\inf_{v\in V_0^1} \|u+v\|_{H^1(\Omega)} \le c\|g\|_{H^{1/2}(\partial\Omega)}, \tag{3.14}$$

where the constant c does not depend on g.

Proof. For the proof of this lemma we refer to [GR], [ALT]. □

We introduce the space

$$\Psi^1 := \Big\{ v \in H^1(G)^2 \Big| \nabla\cdot v = 0,\, v_{|S\cup S'\cup\Gamma_1\cup\Gamma_2} = 0 \Big\}, \tag{3.15}$$

For each $v \in \Psi^1$ only the components $v_{|\Gamma_{\rm in}} = v^{\rm in}$ and $v_{|\Gamma_{\rm out}} = v^{\rm out}$ of the restriction $v_{|\partial\Omega}$ can differ from zero and

$$(v^{\rm in}, n)_{\Gamma_{\rm in}} + (v^{\rm out}, n)_{\Gamma_{\rm out}} = 0. \tag{3.16}$$

We set

$$\widehat{H}^{1/2}(\Gamma_{\rm in}\cup\Gamma_{\rm out}) = \Big\{ v^{\rm in} \in H_{00}^{1/2}(\Gamma_{\rm in})^2,\, v^{\rm out} \in H_{00}^{1/2}(\Gamma_{\rm out})^2 \;\Big|\; (3.16) \text{ holds} \Big\},$$

where $H_{00}^{1/2}$ is the space defined in [LM], Chapter 1, Theorem 11.7 [4]

Lemma 3.4. *There exists a bounded extension operator*

$$E : \widehat{H}^{1/2}(\Gamma_{in}\cup\Gamma_{out}) \to \Psi^1,$$

i.e., the operator satisfying $E(v^{in}, v^{out})_{|\Gamma_{in}} = v^{in}$, $E(v^{in}, v^{out})_{|\Gamma_{out}} = v^{out}$.

Proof. This lemma follows directly from Lemma 3.3. □

Corollary 3.5. *There exists a bounded extension operator*

$$E : H_0^1(\Gamma_{in}) \to \Psi^1.$$

Proof. Since the embedding $H_0^1(\Gamma_{\rm in}) \subset H_{00}^{1/2}(\Gamma_{\rm in})$ is continuous, for each $v^{\rm in} \in H_0^1(\Gamma_{\rm in})$, we have to choose $v^{\rm out} \in H_0^1(\Gamma_{\rm out})$ satisfying (3.16) and to apply Lemma 3.4. □

3.3. **Estimates for the solution of the Stokes problem.** We introduce the solution operator

$$R : L^{3/2}(\Omega)^2 \times H_{00}^{1/2}(\Gamma_{\rm in})^2 \times H_{00}^{-1/2}(\Gamma_1)^2 \times H_{00}^{-1/2}(\Gamma_2)^2 \;\to\; \Phi \subset H^1(\Omega)^2$$

[4] Actually, $H_{00}^{1/2}(a,b)$ consists of restrictions on $[a,b]$ of functions from the space $\{f \in H^{1/2}(\mathbb{R}) : \operatorname{supp}\ f \subseteq [a,b]\}$

where $H_{00}^{-1/2}=(H_{00}^{1/2})'$, that maps the data $(f,v^{\rm in},u^1,u^2)$ to the generalized solution $\hat v$ of problem (3.1), (2.3), (2.5), i.e., $R\Big(f,v^{\rm in},u^1,u^2\Big)(x)=\hat v\,(x)$. (Proof of Theorem 3.2 does not change if data belong to aforementioned spaces.)

Lemma 3.6. *The solution operator R is bounded,*

$$\|R(f,v^{in},u^1,u^2)\|^2_{H^1(\Omega)^2}\le c\Big(\|f\|^2_{L^{3/2}(\Omega)^2}+\|v^{in}\|^2_{H_{00}^{1/2}(\Gamma_{in})^2}+\sum_{i=1}^2\|u^i\|^2_{H_{00}^{-1/2}(\Gamma_i)^2}\Big),\tag{3.17}$$

where $c>0$ is independent of the data (f,v^{in},u^1,u^2).

Proof. In virtue of Lemma 3.4, the following decomposition is true for the solution $\hat v(x)=R(f,v^{\rm in},u^1,u^2)(x)$:

$$\hat v=Ev^{\rm in}+\hat\varphi,\quad \hat\varphi=\hat v-Ev^{\rm in}\in\Phi_0.\tag{3.18}$$

The equalities (3.18) and (3.8) imply

$$\begin{aligned}\|\nabla\hat v\|^2&=(\nabla\hat v,\nabla Ev^{\rm in})+(\nabla\hat v,\nabla\hat\varphi)\\&=(\nabla\hat v,\nabla Ev^{\rm in})+\sum_{i=1}^2(u^i,\hat\varphi)_{\Gamma_i}+(f,\hat\varphi).\end{aligned}\tag{3.19}$$

In virtue of Lemma 3.4, we get

$$|(\nabla\hat v,\nabla Ev^{\rm in})|\le c\|\nabla\hat v\|\|v^{\rm in}\|_{H_{00}^{1/2}(\Gamma_{\rm in})}\le\epsilon\|\nabla\hat v\|^2+\frac{c}{\epsilon}\|v^{\rm in}\|^2_{H_{00}^{1/2}(\Gamma_{\rm in})}.\tag{3.20}$$

By means of the trace theorem and the Poincaré inequality,

$$\begin{aligned}\Big|\sum_{i=1}^2(u^i,\hat\varphi)_{\Gamma_i}\Big|&\le c\Big(\sum_{i=1}^2\|u^i\|_{H_{00}^{-1/2}(\Gamma_i)}\Big)\|\nabla\hat\varphi\|\\&\le\frac{c}{\epsilon}\Big(\sum_{i=1}^2\|u^i\|^2_{H_{00}^{-1/2}(\Gamma_i)}\Big)+\epsilon\|\nabla\hat\varphi\|^2.\end{aligned}\tag{3.21}$$

Using the Sobolev embedding theorem $H^1(\Omega)\subset L^3(\Omega)$ and the Poincaré inequality, we get

$$|(f,\hat\varphi)|\le c\|f\|_{L^{3/2}(\Omega)}\|\nabla\hat\varphi\|\le\frac{c}{\epsilon}\|f\|^2_{L^{3/2}(\Omega)}+\epsilon\|\nabla\hat\varphi\|^2.\tag{3.22}$$

At last, (3.18) and Lemma 3.4 imply

$$\|\nabla\hat\varphi\|^2\le c\Big(\|\nabla\hat v\|^2+\|v^{\rm in}\|^2_{H_{00}^{1/2}(\Gamma_{\rm in})}\Big).\tag{3.23}$$

After substituting inequalities (3.20)-(3.23) into (3.19), we obtain that

$$\|\nabla \hat{v}\|^2 \le c\Big(\|f\|^2_{L^{3/2}(\Omega)} + \|v^{\rm in}\|^2_{H^{1/2}_{00}(\Gamma_{\rm in})} + \sum_{i=1}^{2}\|u^i\|^2_{H^{-1/2}_{00}(\Gamma_i)}\Big).$$

This bound and again the Poincaré inequality imply the asserted estimate (3.17). □

3.4. **The Navier-Stokes boundary value problem.** Now, we consider the Navier-Stokes equations

$$-\Delta v + v\cdot\nabla v + \nabla p = 0, \quad \nabla\cdot v = 0, \quad \text{in } \ \Omega, \tag{3.24}$$

with boundary conditions (2.3), (2.5).

Definition 3.2. Let $u^i \in L^2(\Gamma_i)^2,\ i = 1,2$, and $v^{\rm in} \in H^1_0(\Gamma_{\rm in})^2$. The vector field $v \in \Phi$ is called "generalized solution" of problem (3.24), (2.3), (2.5) if $v|_{\Gamma_{\rm in}} = v^{\rm in}$ and the following equality holds:

$$(\nabla v, \nabla\varphi) + (v\cdot\nabla v, \varphi) - \sum_{i=1}^{2}(u^i,\varphi)_{\Gamma_i} = 0 \quad \forall\varphi\in\Phi_0, \tag{3.25}$$

where Φ and Φ_0 are defined in (3.6), (3.7).

Our goal is now to prove the following theorem.

Theorem 3.7. *Suppose that*

$$\|v^{\rm in}\|^2_{H^1_0(\Gamma_{\rm in})} + \sum_{i=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)} \le \epsilon, \tag{3.26}$$

where $\epsilon > 0$ is sufficiently small. Then, there exists a unique generalized solution $v \in \Phi$ of problem (3.24)*,* (2.3)*,* (2.5)*. This solution satisfies the inequality*

$$\|v\|^2_{H^1(\Omega)} \le \alpha\Big(\|v^{\rm in}\|^2_{H^1_0(\Gamma_{\rm in})} + \sum_{i=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)}\Big), \tag{3.27}$$

with function $\alpha(\lambda) = c\left(\lambda^2+\lambda\right)$.

Proof. We look for a generalized solution v of (3.24), (2.3), (2.5) in the form $v = R\,(f, v^{\rm in}, u^1, u^2)$ where R is the solution operator of the Stokes boundary value problem, and $f \in L^{3/2}(\Omega)^2$ is an unknown vector field. We substitute $v = R(f, v^{\rm in}, u^1, u^2)$ into (3.25) and take into account that $v = R(\cdot)$ satisfies (3.8). As a result we get the equation

$$(\varphi, f + R\cdot\nabla R) = 0 \quad \forall\varphi\in\Phi_0. \tag{3.28}$$

Since $H^1(\Omega) \subset L^6(\Omega)$, we get using the Lipschitz inequality:

$$\int_\Omega \left|(R,\nabla)\,R\right|^{3/2}\,dx \le \|\,R\,\|_{L^6}^{3/2}\|\,\nabla R\,\|_{L^2}^{3/2}, \le c\|\,\nabla R\,\|_{L^2}^3 \tag{3.29}$$

and therefore $R\cdot\nabla R \in L^{3/2}(\Omega)^2$. We set, for $p>1$,

$$\hat{L}^p\,(\Omega) = \{\,\varphi \in L^p\,(\Omega)^2 \mid \operatorname{div}\varphi = 0,\ \varphi\cdot n\Big|_{\Gamma_{\text{in}}\cup S\cup S'} = 0\,\} \tag{3.30}$$

and define the projection operator $P: L^{3/2}\,(\Omega)^2 \to \hat{L}^{3/2}\,(\Omega)$ as follows: for each $f\in L^{3/2}\,(\Omega)^2$ the function $Pf \in \hat{L}^{3/2}(\Omega)$ is defined as the unique solution of the equation

$$(\varphi, Pf) = (\varphi, f) \quad \forall \varphi \in \hat{L}^3(\Omega). \tag{3.31}$$

Since the space Φ_0 defined in (3.7) is dense in $\hat{L}^3(\Omega)$, for each $f\in\hat{L}^{3/2}(\Omega)$ equation (3.28) is equivalent to the equality

$$f + P\Big(R(f)\cdot\nabla R(f)\Big) = 0. \tag{3.32}$$

We use the notation $R(f) = R(f, v^{\text{in}}, u^1, u^2)$ since v^{in}, u^1, u^2 are given and fixed. To prove the theorem, we have to check that the operator

$$S(f) = -P\Big(R(f)\cdot\nabla R(f)\Big)\ :\ \hat{L}^{3/2}(\Omega) \to \hat{L}^{3/2}(\Omega) \tag{3.33}$$

is a contraction operator. Using (3.29) and (3.17), we have

$$\begin{aligned}\|S(f_1)-S(f_2)\|_{L^{3/2}} &\le \|(R(f_1-f_2,0,0,0),\nabla)R(f_1)\|_{L^{3/2}} \\ &\qquad + \|(R(f_2),\nabla)R(f_1-f_2,0,0,0)\|_{L^{3/2}} \\ &\le c\|\nabla R(f_1-f_2,0,0,0)\|_{L^2}\Big(\|\nabla R(f_1)\|_{L^2} \\ &\qquad + \|\nabla R(f_2)\|_{L^2}\Big) \le \hat{c}\|f_1-f_2\|_{L^{3/2}},\end{aligned} \tag{3.34}$$

where, in virtue of (3.17),

$$\begin{aligned}\hat{c} &= c\Big(\|\nabla R(f_1)\|_{L^2} + \|\nabla R(f_2)\|_{L^2}\Big) \\ &\le 2c\Big(\tfrac{1}{2}\Big(\|f_1\|_{L^{3/2}} + \|f_2\|_{L^{3/2}}\Big) + \sum_{i=1}^2 \|u^i\|_{L^2(\Gamma_i)} + \|v^{\text{in}}\|_{H_0^1(\Gamma_{\text{in}})}\Big).\end{aligned} \tag{3.35}$$

By the assumption of the theorem the right hand side of (3.35) is small enough if $\|f_j\|_{L^{3/2}}$, $j=1,2$ are sufficiently small. Therefore $\hat{c}<1$ and the operator in(3.33) is a contraction. Hence equation (3.32) has a unique solution $f\in\hat{L}^{3/2}(\Omega)$.

As is well-known, the solution of (3.32), i.e. of the equation $f = S(f)$, has the form $f = \lim_{k\to\infty} f_k$ where $f_1 = S(0),\dots,f_k = S(f_{k-1})$. Since

$f_k = \sum_{j=1}^{k}(f_j - f_{j-1})$,

$$\begin{aligned}\|f\|_{L^{3/2}} &\le \lim_{k\to\infty}\sum_{j=1}^{k}\|f_j - f_{j-1}\|_{L^{3/2}}\\ &\le \sum_{j=1}^{\infty}\hat{c}^j\|S\,(0)\,\|_{L^{3/2}} \le \frac{\hat{c}}{1-\hat{c}}\|R(0,v^{\mathrm{in}},u^1,u^2)\|^2_{H^1}.\end{aligned} \tag{3.36}$$

This completes the proof. □

4. Existence theorem for the optimal control problem

In this section, we prove the existence of the solution for the extremal problem (2.1)-(2.4), (2.7), (2.8). For this, we need a smoothness result for the solution of the Navier-Stokes equations, which we recall in subsection 4.1.

4.1. **The smoothness theorem.** For small enough $\delta > 0$ denote by $\partial\Omega_\delta$ the curve belonging to Ω which is the rectangle with sides parallel to the sides AD, EH, HA of $\partial\Omega$ placed with distance δ from them and extending up to the side DE. Denote by Ω_δ the open subset of Ω with boundary $\partial\Omega_\delta \cup S$. Let $\chi(x) \in C^\infty(\overline{\Omega})$ be a corresponding cut-off function satisfying

$$\chi(x) = \begin{cases} 1, & x \in \Omega_\delta, \\ 0, & x \in \Omega \setminus \Omega_{\delta/2}. \end{cases} \tag{4.1}$$

and near DE $\chi(x_1,x_2) \equiv \chi(x_2)$. The following theorem holds.

Theorem 4.1. *Let v be the generalized solution constructed in Theorem 3.7. Then, $v \in W^2_{3/2}(\Omega_\delta)^2$ and there exists unique $p \in L^2(\Omega)$ satisfying (3.24) and $p \in W^1_{3/2}(\Omega_\delta)$. Moreover*

$$\|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)} \le \rho\Big(\|v^{\mathrm{in}}\|_{H^1_0(\Gamma_{\mathrm{in}})} + \sum_{j=1}^{2}\|u^j\|_{L^2(\Gamma_j)}\Big), \tag{4.2}$$

where $\rho(\lambda)$ is a continuous function for $\lambda > 0$ and $\rho(0) = 0$.

Proof. Since v is a generalized solution of the Navier-Stokes equations, (3.25) implies

$$(-\Delta v + v\cdot\nabla v, \varphi) = 0 \quad \forall \varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2.$$

This equality, identity $v\cdot\nabla v = \sum_{j=1}^{2}\partial_j(v_j v)$, inclusions $v_j v \in L^2(\Omega)$, $j = 1, 2$, and the De Rham theorem (see [T]) yield that there exists $p \in L^2(\Omega)$

such that

$$-\Delta v + \nabla p = -v \cdot \nabla v, \quad \nabla \cdot v = 0 \quad \text{in } \Omega. \tag{4.3}$$

Since $v \cdot \nabla v \in L^{3/2}(\Omega)$, and the Stokes system with right hand side $-v \cdot \nabla v$ and boundary condition $\partial_n v - np = 0$ on $\Gamma_{\text{out}} \cap \Omega_{\delta/4}$ is a Douglas-Nirenberg elliptic system, we get from (4.3) that $v \in W^2_{3/2}(\Omega_{\delta/4})$ and $p \in W^1_{3/2}(\Omega_{\delta/4})$. To prove (4.2), we note that equations (4.3) imply

$$-\Delta(\chi v) + \nabla(\chi p) = g, \quad \nabla \cdot (\chi v) = g_1, \tag{4.4}$$

where

$$g = -\chi v \cdot \nabla v - 2(\nabla \chi \cdot \nabla) v - v \Delta \chi + p \nabla \chi, \quad g_1 = \nabla \chi \cdot v. \tag{4.5}$$

Using (3.27), we obtain that

$$\|g_1\|_{H^1(\Omega)} \le c\|v\|_{H^1(\Omega)} \le c\alpha^{1/2}\Big\{\|v^{\text{in}}\|^2_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^i\|^2_{L^2(\Gamma_i)}\Big\}, \tag{4.6}$$

and

$$\|g\|_{L^{3/2}(\Omega)} \le c_1\alpha\Big\{\|v^{\text{in}}\|^2_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^i\|^2_{L^2(\Gamma_i)}\Big\} + c\|p\nabla\chi\|_{L^2(\Omega)}. \tag{4.7}$$

Below, we will prove that

$$\|p\nabla\chi\|_{L^2(\Omega)} \le c_2\beta\Big\{\|v^{\text{in}}\|^2_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^i\|^2_{L^2(\Gamma_i)}\Big\}, \tag{4.8}$$

where the function $\beta(\lambda) > 0$ is continuous, and $\beta(0) = 0$. Let us identify the sides AD and HE of the rectangle $ADEH$ (see FIGURE 1). Then this rectangle turns into a lateral area LC of a cylinder with boundary $\partial LC = \hat{\Gamma}_{\text{in}} \cup \hat{\Gamma}_{\text{out}} \cup S$ where $\hat{\Gamma}_{\text{in}} = \Gamma_{\text{in}}$ with points A and H being identified, and $\hat{\Gamma}_{\text{out}} = \Gamma_{\text{out}}$ with points D and E being identified. In virtue of the properties of the cut-off function in (4.1), we can consider (4.4) as a system defined on LC. Evidently, the pair $(\chi v, \chi p)$ from (4.4) satisfies the following boundary conditions:

$$\chi v|_{\hat{\Gamma}_{\text{in}}} = 0, \quad \chi v|_S = 0, \quad \left(\partial_1 v(x_1, x_2) - p(x_1, x_2) n\right) \chi(x_2)|_{\hat{\Gamma}_{\text{out}}} = 0. \tag{4.9}$$

Since this boundary value problem is elliptic in the Douglas-Nirenberg sense, inequalities (4.6), (4.7), (4.8), and the evident bound

$$\|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)} \le \|\chi v\|_{W^2_{3/2}(LC)} + \|\chi p\|_{W^1_{3/2}(LC)}$$

imply the asserted estimate (4.2).

Let us prove estimate (4.8). The following bound holds (see inequality (6.12) of Chapter 1 in [T]):

$$\|p\partial_j\chi\|_{L^2(\Omega)} \le c\Big\{\Big|\int_\Omega p\partial_j\chi\,dx\Big| + \|\nabla(p\partial_j\chi)\|_{H^{-1}(\Omega)}\Big\}, \quad j=1,2. \tag{4.10}$$

We estimate the first term in right side of (4.10). Let $\psi \in C^\infty(\overline{\Omega})$ be a function satisfying $\psi(x)\partial_2\chi = \partial_2\chi$, and $\psi(x) \equiv 0$ outside a small neighborhood of $\operatorname{supp}(\partial_2\chi)$. Then integrating by parts and using (4.3), we get

$$\begin{aligned}\int_\Omega p\partial_2\chi\,dx &= -\int_\Omega \partial_2 p\chi\psi\,dx = \int_\Omega (\Delta v_2 - v\cdot\nabla v_2)\chi\psi\;dx\\ &= \int_{\Gamma_{\mathrm{out}}} \partial_1 v_2\chi\psi\,dx - \int_\Omega \nabla v_2\cdot\nabla(\chi\psi)\;dx - \int_\Omega (v\cdot\nabla v_2)\chi\psi\,dx.\end{aligned} \tag{4.11}$$

The boundary condition $(\partial_n v - pn)|_{\Gamma_{\mathrm{out}}} = 0$ implies $\partial_1 v_2|_{\Gamma_{\mathrm{out}}} = 0$. Therefore estimation of other terms in the right side of (4.11) yields

$$\Big|\int_\Omega p\partial_2\chi\,dx\Big| \le c\Big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big). \tag{4.12}$$

Since $\operatorname{supp}(\partial_1\chi) \Subset \Omega$, we can choose $\psi \in C_0^\infty(\Omega)$ such that $\psi\partial_1\chi \equiv \partial_1\chi$. Thereore the inequality

$$\Big|\int_\Omega p\partial_1\chi\,dx\Big| \le c\Big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big) \tag{4.13}$$

can be obtained similarly to (4.11), (4.12), but now without any boundary term. In virtue of (4.3) there holds

$$\begin{aligned}\|\nabla(p\partial_j\chi)\|_{H^{-1}(\Omega)} &\le \|\Delta v\partial_j\chi\|_{H^{-1}(\Omega)} + \|v\cdot\nabla v\partial_j\chi\|_{H^{-1}(\Omega)}\\ &\quad + \|p\nabla\partial_j\chi\|_{H^{-1}(\Omega)}\\ &\le c(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}) + \|p\nabla\partial_j\chi\|_{H^{-1}(\Omega)}.\end{aligned} \tag{4.14}$$

Now we estimate the last term on the right side of (4.14). We choose a function $\psi \in C^\infty(\overline{\Omega})$ satisfying $\psi(x)\partial_i\partial_j\chi \equiv \partial_i\partial_j\chi$, $\psi(x) \equiv 0$ outside a small neighborhood of $\operatorname{supp}(\partial_i\partial_j\chi)$. Besides, we take an arbitrary function $\varphi \in W_3^1(\Omega)$ satisfying $\varphi|_{\partial\Omega} = 0$, and set $w := \varphi\psi\partial_i\partial_j\chi$. Then, for any fixed point $x^0 = (x_1^0, x_2^0) \in \Omega \setminus \mathrm{supp}(\psi)$, there holds

$$\begin{aligned}\int_\Omega p\partial_i\partial_j\chi\varphi\,dx &= \int_\Omega\int_{x_1^0}^{x_1}\partial_y p(y,x_2)\,dyw(x)\,dx\\ &= \int_\Omega\int_{x_1^0}^{x_1}\Delta v_1\ dyw\,dx - \int_\Omega\int_{x_1^0}^{x_1} v\cdot\nabla v_1\,dy\,w\,dx\\ &= \int_\Omega\Big(\partial_1 v_1 - \int_{x_1^0}^{x_1} v\cdot\nabla v_1\,dy\Big)w\,dx - \int_\Omega\Big(\int_{x_1^0}^{x_1}\partial_2 v_1\,dy\Big)\partial_2 w\,dx.\end{aligned}$$

Estimating the right side of this equality, we get

$$\Big|\int_\Omega p\partial_i\partial_j\chi\varphi\,dx\Big| \le c\Big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\Big)\|\varphi\|_{H^1(\Omega)}. \tag{4.15}$$

Estimates (4.10), (4.12)-(4.15) imply

$$\|p\nabla\chi\|_{L^2(\Omega)} \le c(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}). \tag{4.16}$$

Finally, the bound (4.8) follows from (4.16) and (3.27). □

Remark 4.1. The generalized solution $v \in H^1(\Omega)$ of problem (3.24),(2.3), (2.5) constructed in Theorem 3.7 together with the function $p \in L^2(\Omega)$ constructed in Theorem 4.1 possess enough smoothness in order to define traces $(\partial_n v - pn)|_{\Gamma_1}$ and $(\partial_n v - pn)|_{\Gamma_2}$. Moreover, the relations (2.5) hold. To prove this assertion, one has to use methods of [LM], [F] Chapter 2.5, and of Theorem 4.1 proved above.

4.2. **Existence theorem for the extremal problem.** In order to prove the existence theorem for problem (2.1)-(2.5), (2.7), (2.8), we have to describe the set of admissible elements for this problem. First of all, for given boundary condition $v^{\mathrm{in}} \in H_0^1(\Gamma_{\mathrm{in}})$ and controls $u^i \in L^2(\Gamma_i)$, $i = 1, 2$, satisfying

$$\sum_{i=1}^2 \|u^i\|^2_{L^2(\Gamma_i)} \le \gamma^2; \quad \gamma^2 + \|v^{\mathrm{in}}\|^2_{H_0^1(\Gamma_{\mathrm{in}})} \le \epsilon, \tag{4.17}$$

where ϵ is small enough, we have to define uniquely the pair (v,p) that is the solution of the boundary value problem (2.1)-(2.3),(2.5). In virtue of the second condition in (4.17), by Theorem 3.7 there exists a unique generalized solution $v \in \Phi$ of the Navier-Stokes equation. In virtue of the De Rham Theorem and the argument in the proof of Theorem 4.1 there exists a unique $p \in L^2(\Omega)$ that together with v satisfies equation (4.3). Assuming that $v^{\mathrm{in}} \in H_0^1(\Gamma_{\mathrm{in}})$ is fixed and small enough, we define the map $NR_\delta(u^1, u^2)$ that maps the pair (u^1, u^2) to the corresponding generalized solution (v,p) of problem (2.1)-(2.5),

$$NR_\delta(u^1,u^2) = (v(u^1,u^2), p(u^1,u^2)) \in \Phi \times L^2(\Omega). \tag{4.18}$$

We introduce the following notation:

$$B := \Big\{(u^1,u^2) \,\Big|\, \|u^1\|^2_{L^2(\Gamma_1)} + \|u^2\|^2_{L^2(\Gamma_2)} \le \gamma^2\Big\}, \tag{4.19}$$

$$VP(\Omega) := \Big\{(v,p) \in \Phi \times L^2(\Omega) : (v,p)|_{\Omega_\delta} \in W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)\Big\}, \tag{4.20}$$

$$\|(v,p)\|_{VP(\Omega)} := \|v\|_{H^1(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)} + \|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)}. \tag{4.21}$$

Lemma 4.2. *Let $\|v^{in}\|_{H^1_0(\Gamma_{in})} + \gamma$ be small enough. Then, the mapping*

$$NR_\delta : B \to VP(\Omega) \tag{4.22}$$

is continuous and its range $NR_\delta(B)$ is a bounded and closed set.

Proof. Using the estimate (3.27) and expressing ∇p by (3.24) with the following application of (3.27), (3.29), we get the inequality

$$\|v\|_{H^1(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)} \le c\Big(\|v^{\text{in}}\|_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^j\|_{L^2(\Gamma_j)}\Big), \tag{4.23}$$

where $c(\lambda) = c_1(\lambda + \lambda^2)$. The bounds (4.23), (4.2) imply the boundedness of the operator $NR_\delta(B)$ in (4.22). Let us prove the closedness of $NR_\delta(B)$. We only prove the closedness of $NR_\delta(B)|_{\Omega_\delta}$ in $W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ because the closedness of $NR_\delta(B)$ in $\Phi \times L^2(\Omega)$ can be established in the same way. Let $(v_k,p_k) \in NR_\delta(B)$ with

$$(v_k,p_k)|_{\Omega_\delta} \to (\hat v, \hat p)|_{\Omega_\delta} \quad \text{in } W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta) \quad (k \to \infty). \tag{4.24}$$

Inclusion $(v_k,p_k) \in NR_\delta(B)$ implies relation $(v_k,p_k) = NR_\delta(u^1_k,u^2_k)$ for some $(u^1_k,u^2_k) \in B$. Since B is a bounded set, passing if necessary to a subsequence, we can assume that $(u^1_k,u^2_k) \rightharpoonup (\hat u^1, \hat u^2)$ weakly in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. Hence, $(\hat u^1,\hat u^2) \in B$ because the set B is convex. Now, we substitute (v_k,u^1_k,u^2_k) into (3.25). Evidently one can pass to the limit in (3.25). Since $\nabla p_k = \Delta v_k - v_k \cdot \nabla v_k$, then, $\nabla p_k \to \nabla \hat p$ weakly in the space $\nabla W^1_{3/2}(\Omega_\delta) = \{\nabla p \,|\, p \in W^1_{3/2}(\Omega_\delta)\}$. In virtue of (4.2) the functions p_k are bounded with respect to k. That is why, passing if necessary to a subsequence, we get that $p_k \to \hat p$ in $W^1_{3/2}(\Omega_\delta)$ weakly. So $(\hat v,\hat p) = NR_\delta(\hat u^1,\hat u^2)$ and the set $NR_\delta(B)$ is closed in $W^2_{3/2}(\Omega_\delta) \times W^1_{3/2}(\Omega_\delta)$. □

Since $\omega \subset \Omega$ in (2.7) is a given closed subset of domain Ω, there exists $\delta > 0$ so small that

$$\omega \subset \Omega_\delta \subset \Omega. \tag{4.25}$$

We choose $\delta > 0$ such that (4.25) holds and from now on assume it as fixed.

Definition 4.1. Let $v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})$ be fixed. The collection $(v, p, u^1, u^2) \in VP(\Omega) \times B$ is called "admissible" for problem (2.1)-(2.5), (2.7), (2.8) if $(v, p) = NR_\delta(u^1, u^2)$ and inequality (2.7) is fulfilled.

Notice that the equality $(v, p) = NR_\delta(u^1, u^2)$ means that (v, p) is a generalized solution of the boundary value problem (2.1)-(2.3), (2.5). Besides, the integral in (2.4) is well defined because in virtue of the inclusion $(v, p) \in W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ all traces used in (2.4) are well defined. An inequality for each $x \in \omega$, as in (2.7), is well defined for $(v, p) = NR_\delta(u^1, u^2)$ because such $v = (v_1, v_2)$ belong to $W^2_{3/2}(\Omega_\delta)$ that is embedded into $C(\overline{\Omega}_\delta)$ by the Sobolev embedding theorem. The set of all admissible collections, i.e. the admissible set for the extremal problem (2.1)-(2.5), (2.7), (2.8) is denoted by $\mathfrak{A}$. We impose the following important condition.

Condition 1. The admissible set of the extremal problem (2.1)-(2.5), (2.7), (2.8) is not empty,

$$\mathfrak{A} \neq \emptyset. \tag{4.26}$$

Remark 4.2. The situation with Condition 1 is not trivial at all. Calculations show that this condition is fulfilled rather often. Indeed, if $\|v^{\text{in}}\|_{\widehat{H}_0^1}$ and γ^2 in (2.8) are sufficiently small, and the part S of the boundary is convex, then, as numerical calculations show (see Section 8), $v_1 \geq 0$ on certain subdomains $\omega \subset \Omega$ (see also [VD]). Moreover, the calculated steady flow is stable and therefore this is the case of small Reynolds number which we consider in this paper.

Recall that by definition the collection $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2)$ is the solution of problem (2.1)-(2.5), (2.7), (2.8) if $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2) \in \mathfrak{A}$ and

$$J(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2) = \inf_{(v,p,u^1,u^2)\in\mathfrak{A}} J(v, p, u^1, u^2), \tag{4.27}$$

where $J(v, p, u^1, u^2)$ is the functional in (2.4), (2.6). Notice that the dependence of J on (u^1, u^2) is implicit and connected with the domain $\mathfrak{A}$ for the functional J.

Theorem 4.3. *If* $\|v^{in}\|_{H_0^1(\Gamma_{in})}$ *and* γ^2 *in* (2.8) *are small enough, then there exists a solution* $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2)$ *of problem* (2.1)-(2.8).

Proof. i) First, we prove that the projection $\Pi\mathfrak{A}$ of the admissible set $\mathfrak{A} \subset PV_1(\Omega) \times B$ into $W^2_{3/2}(\Omega_\delta{}^2) \times W^1_{3/2}(\Omega_\delta)$ is closed in this space. Since

$$\mathfrak{A} = \Big\{(v, p, u^1, u^2) \Big| (v, p) = NR_\delta(u^1, u^2),\, v_1(x) \geq 0,\, x \in \omega\Big\},$$

by virtue of Lemma 4.2, it is enough to prove that if $(v_k, p_k) \to (\hat{v}, \hat{p})$ as $k \to \infty$ in $W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ and $v_1^k \geq 0$ on ω for each k, then $\hat{v}_1 \geq 0$ on ω. But this assertion immediately follows from the embedding $v \in W^2_{3/2}(\Omega_\delta) \subset C(\overline{\Omega}_\delta)$ and the inclusion $\omega \subset \Omega_\delta$.

ii) Next, we consider the direct product of the Besov spaces $W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$ [5] and introduce the trace operator $\hat{\gamma}_S(v, p) = n \cdot \sigma_{|S} := (-np + 2\mathcal{D}(v)n)_{|S}$ (see (2.4)). Then, the well-known Besov theorem ([BIN]) implies that the operator

$$\hat{\gamma}_S : W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta) \to W^{1/6}_{3/2}(S)^2 \tag{4.28}$$

is continuous. Since the embedding $W^{1/6}_{3/2}(S) \subset L^1(S)$ is continuous, the functional in (2.4),

$$J(v, p) = \int_S n \cdot \sigma \cdot e_1 \, ds = \int_S \hat{\gamma}_S(v, p) \cdot e_1 \, ds, \tag{4.29}$$

is continuous on the space $W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$. As is well-known, the embedding $W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta) \subset W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$ is compact. Therefore, in virtue of part i) of this proof, the set $\Pi\mathfrak{A}$ is a compact subset of the space $W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$. Evidently the extremal problem (2.1)-(2.8) is equivalent to the problem

$$J = \int_S \hat{\gamma}_S(v, p) \cdot e_1 \, ds \to \inf, \quad (v, p) \in \Pi\mathfrak{A}. \tag{4.30}$$

Problem (4.30) is a minimization problem for a continuous function on a compact set. Therefore it possesses a solution, which completes the proof. □

[5]When the upper index is not integer, the Besov space coincides with the corresponding Sobolev space. Therefore, we use the notation of Sobolev spaces. We use the Besov spaces because the trace theorem is not always true for Sobolev spaces

5. Abstract Lagrange principle

To derive the optimality system for problem (2.1)-(2.8), we use the abstract Lagrange principle. For problems without phase constraints one can recall the Lagrange principle from [ATF, F]. The essential peculiarity of the extremal problem studied here is just the phase constraint (2.7). For such extremal problems the Lagrange principle has been established as well [DM, G, MDO]. We recall some abstract notion (for details we refer to [MDO]).

5.1. **Sub-linear functionals.** Let Y be a Banach space. A functional $\varphi : Y \to \mathbb{R}$ is called "sub-linear" if it satisfies

a) $\varphi(\lambda y) = \lambda\varphi(y)$, $\forall y \in Y$, $\lambda > 0$ (positive homogeneity)
b) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ (subadditivity)

Notice that for a functional satisfying a) condition b) is equivalent to a convexity condition. The sub-linear functional φ is called "bounded" if there exists a constant $c > 0$ such that

c) $|\varphi(y)| \leq c\|y\| \quad \forall y \in Y$.

Lemma 5.1. *If the sub-linear functional φ satisfies $\varphi(y) \leq c\|y\|\ \forall y \in Y$ with a certain $c > 0$, then $|\varphi(y)| \leq c\|y\|$ and $|\varphi(y_1) - \varphi(y_2)| \leq c\|y_1 - y_2\|$, i.e. φ is a Lipschitz functional with the same constant c.*

Proof. The proof can be found in [MDO], p.75. □

A linear functional $l \in Y^*$ is called "supported by a sub-linear functional $\varphi(y)$" if $l(y) \leq \varphi(y)\ \forall y \in Y$. The set of all functionals supported by φ is called "subdifferential of φ" (at zero) and is denoted by $\partial\varphi$ (at zero). If φ is a bounded sub-linear functional, then $\partial\varphi$ is a non-empty convex closed set and $\forall l \in \partial\varphi : \|l\| \leq c$. Let $f : Y \to \mathbb{R}$ be a functional. If for $y_0, y_1 \in Y$ there exists the limit

$$f'(y_0, y_1) := \lim_{\lambda\to 0} \frac{f(y_0 + \lambda y_1) - f(y_0)}{\lambda},$$

then $f'(y_0, y_1)$ is called "derivative" of f at the point y_0 in the direction y_1.

5.2. **Formulation of the Lagrange principle.** Consider an abstract extremal problem of the form of problem (2.1)-(2.8),

$$f_0(y) \to \inf, \quad F(y) = 0, \quad f_1(y) \leq 0, \quad G(y) \leq 0, \tag{5.1}$$

where $f_i : Y \to \mathbb{R}$, $i = 0, 1$, $G : Y \to \mathbb{R}$ are functionals defined on a Banach space Y, and $F : Y \to Z$ is a map to another Banach space Z. Suppose that there exists a solution $\hat{y} \in Y$ of problem (5.1) and the mappings $f_i : Y \to \mathbb{R}$, $i = 0, 1$, and $F : Y \to Z$ are continuously differentiable in a neigborhood of $\hat{y}$. Assume also that

a) The image $F'(\hat{y})Y$ of Y is closed in Z.
b) $G(y)$ posseses a derivative $G'(\hat{y}, y_1)$ at $\hat{y}$ in each direction $y_1 \in Y$, and the map $Y \ni y \to G'(\hat{y}, y)$ is a bounded sub-linear functional on Y.

The Lagrange function for problem (5.1) has the form

$$\mathcal{L}(\hat{y}, \lambda_0, \lambda_1, z^*, \alpha) = \sum_{j=0}^{1} \lambda_j f_j(\hat{y}) + \langle F(\hat{y}), z^* \rangle + \alpha G(\hat{y}). \tag{5.2}$$

The following theorem holds (see [DM, G, MDO]).

Theorem 5.2. *Let the conditions formulated above be fulfilled. Then, there exist Lagrange multipliers* $(\lambda_0, \lambda_1, z^*, \alpha) \in \mathbb{R}^2 \times Z^* \times \mathbb{R}$ *satisfying*
i) Non-triviality condition:

$$|\lambda_0| + |\lambda_1| + \|z^*\| + |\alpha| > 0. \tag{5.3}$$

ii) Condition of sign concordance:

$$\lambda_0 \geq 0, \quad \lambda_1 \geq 0, \quad \alpha \geq 0. \tag{5.4}$$

iii) Condition of complementary slackness:

$$\lambda_i f_i(\hat{y}) = 0, \; i = 0, 1; \quad \alpha G(\hat{y}) = 0. \tag{5.5}$$

iv) Euler-Lagrange equation: there exists $\mu^* \in \partial G'(\hat{y}, \cdot)$ *such that*

$$\begin{aligned} \Big\langle \mathcal{L}'_y(\hat{y}, \lambda_0, \lambda_1, z^*, \alpha, \mu^*), h \Big\rangle &= \sum_{j=0}^{1} \lambda_j \Big\langle f'_j(\hat{y}), h \Big\rangle \\ &\quad + \Big\langle F'(\hat{y}), h \Big\rangle + \alpha \Big\langle \mu^*, h \Big\rangle = 0, \end{aligned} \tag{5.6}$$

for all $h \in Y$.

In the remaining of this section, we briefly recall some properties of concrete functionals used for defining the phase constraints. Details can be found in [MDO]. These properties will be used in the derivation of the optimality system for the extremal problem (2.1)-(2.8).

5.3. **The functional** $\max_{x \in M} y(x)$ **and its support functionals.** Let $M \subset \Omega$ be an arbitrary closed subset. We consider the functional Θ :

$C(\overline{\Omega}) \to \mathbb{R}$:

$$\Theta(y) = \max_{x\in M} y(x), \quad y \in C(\overline{\Omega}), \tag{5.7}$$

where $C(\overline{\Omega})$ is the space of continuous functions defined on $\overline{\Omega}$. This is a Lipschitz functional with constant one because

$$|\Theta(y_1) - \Theta(y_2)| \le \max_{x\in M} |y_1(x) - y_2(x)| \le \|y_1 - y_2\|_{C(\overline{\Omega})}.$$

By the Riesz Theorem $C(\overline{\Omega})^*$ consists of functionals of the form

$$\lambda(y) = \int_\Omega y(x)\,\mu(dx), \tag{5.8}$$

where $\mu(dx)$ is a measure that can have positive as well as negative values. Evidently the functional in (5.7) is sub-linear (see subsection 5.1).

Lemma 5.3. *The functional* $\lambda(y) = \int_\Omega y(x)\,\mu(dx)$ *from* $C(\overline{\Omega})^*$ *is supported by the functional* $\Theta(y)$ *from* (5.7) *if and only if* $\mu(dx)$ *satisfies*

1) $\mu(dx)$ *is supported on* M *, i.e.* $\forall\, y \in C(\overline{\Omega}) :\ y(x) = 0,$ *for* $x \in M$*, we have* $\lambda(y) = \int_\Omega y(x)\,\mu(dx) = 0$.
2) $\mu(dx) \ge 0$.
3) $\int_\Omega \mu(dx) = 1$.

Proof. For the proof see [MDO], p.95. □

5.4. **Directional derivatives of** $\max y(x)$**.** Let $y_0 \in C(\overline{\Omega})$, $y_1 \in C(\overline{\Omega})$ and $\Theta(y)$ be the functional in (5.7). We calculate the derivative $\Theta'(y_0, y_1)$ in the direction y_1. Without loss of generality, we suppose that $\Theta(y_0) = 0$. Then, the closed set

$$M_0 = \{\, x \in M \mid y_0(x) = 0 \,\}$$

is not empty.

Lemma 5.4. *For all* $y_1 \in C(\overline{\Omega})$ *the functional in* (5.7) *possesses a derivative in the direction* y_1 *at* $y_0(x) \in C(\overline{\Omega})$*, which is defined by the equality*

$$\Theta'(y_0, y_1) = \max_{x\in M_0} y_1(x). \tag{5.9}$$

Evidently (5.9) defines a sub-linear functional. By Lemma 5.3 the set $\partial\Theta'(y_0, y_1)$ of linear functionals supported by $\Theta'(y_0, y_1)$ consists of all probability measures $\mu(dx)$ concentrated (supported) on the set M_0.

5.5. **Directional derivative of the functional** $\max \Phi(x,y)$**.** Let $\Phi(x,y) : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. On $C(\overline{\Omega})$, we consider the functional

$$G(y) = \max_{x\in M} \Phi(x, y(x)), \tag{5.10}$$

where M is a closed subset of Ω. Let $y_0 \in C(\overline{\Omega})$ be such that $G(y_0) = 0$ and

$$M_0 := \Big\{ x \in \overline{\Omega} \,\Big|\, \Phi(x, y_0(x)) \Big\} = 0. \tag{5.11}$$

Evidently $G(y) = \Theta(N(y))$ where Θ is the functional in (5.7) and

$$N \,:\, C(\overline{\Omega}) \to C(\overline{\Omega}) \quad y \to \Phi(x, y(x))$$

is a Nemytskiy operator. Since N is differentiable in the Fréchet sense and $N'(y)h = \Phi'_y(x, y(x))h(x)$ (see [ATF]), and Θ possesses a derivative at $N(y_0)$ in an arbitrary direction $y_1 \in (\overline{\Omega})$, then, by the theorem on the derivative of the superposition of functions the functional $G(y) = \Theta(N(y))$ at y_0 also possesses a derivative in an arbitrary direction y_1 that is defined by the equality

$$G'(y_0, y_1) = \max_{x\in M_0} \Big(\Phi'_y(x, y_0(x)) y_1(x) \Big). \tag{5.12}$$

Evidently, the functional in (5.12) is sub-linear in y_1.

Lemma 5.5. *The set of linear functionals supported by a sub-linear functional* $y_1 \to G'(y_0, y_1)$ *consists of the functionals* $l \in C(\overline{\Omega})^*$*, which have the representation*

$$l(y_1) = \int_\Omega \Phi'_y\Big(x, y_0(x)\Big) y_1(x)\, \mu(dx), \tag{5.13}$$

where $\mu(dx)$ *is a probability measure concentrated on the set* M_0 *(see* (5.11)*).*

6. Application of the abstract Lagrange principle

After some preliminaries related to checking the condition a) in Subsection 5.2, we check that all conditions of the Lagrange principle are satisfied for problem (2.1)-(2.8) and apply the Lagrange principle to this situation.

6.1. **On the smoothness of solutions for the Oseen problem.** Let $(\hat{v}, \hat{p}) \in \Phi \times L^2(\Omega)$ be the solution of the extremal problem (2.1)-(2.8) constructed in Section 4. We consider the Oseen problem, i.e. the linearization of the Navier-Stokes problem at $(\hat{v},\hat{p})$:

$$-\Delta v + \hat{v}\cdot\nabla v + v\cdot\nabla\hat{v} + \nabla p = f, \quad \nabla\cdot v = 0 \quad \text{in}\, \Omega, \tag{6.1}$$

$$v|_{\Gamma_0} = 0, \quad (\partial_n v - pn)|_{\Gamma} = g, \tag{6.2}$$

where Ω is the domain introduced in Section 2. We have

$$\Gamma_0 = \Gamma_{\text{in}} \cup S' \cup S \cup A \cup H, \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out}}, \tag{6.3}$$

and therefore g consists of three components g_i on Γ_i, $i = 1, 2$, and g_{out} on Γ_{out}. We suppose that $g_{\text{out}} \equiv 0$. Since $\|\hat{v}\|_{H^1}$ is small enough one can prove, as in Section 3, that for each $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ there exists a unique generalized solution $v \in \Phi_0$ of problem (6.1), (6.2) where Φ_0 is the space in (3.7). Since problem (6.1), (6.2) is elliptic in the Agmon-Douglas sense, the solution possesses additional smoothness: in each subdomain Ω_0 of Ω, such that $\overline{\Omega}_0 \subset \Omega$, $v \in H^2(\Omega_0)$ and the pressure p exists and belongs to $H^1(\Omega_0)$. Moreover (v, p) are smooth up to $\partial\Omega$ except at the corner points A, D, E, H and the points B, C, F, G: if $B(\epsilon)$ is the union of the circles with radius ϵ centered at the indicated points, then $v \in H^2(\Omega \setminus B(\epsilon))$, $p \in H^1(\Omega \setminus B(\epsilon))$ for each $\epsilon > 0$. Actually, the solution (v, p) is smooth in a neighborhood of the corner points A, D, E, H, as well.

Lemma 6.1. *Let $B_1(\epsilon)$ be the union of the circles with radius ϵ and centers at A, D, E, H. Then, $(v, p) \in H^2(\Omega \cap B_1(\epsilon)) \times H^1(\Omega \cap B_1(\epsilon))$.*

Proof. Recall that the point H is the origin, the interval (A, H) belongs to the axis x_2 with $x_2(A) > 0$, and the interval (EH) belongs to the axis x_1 with $x_1(H) > 0$. Consider a neighborhood of the point H and extend the solution (v, p) on the domain $\{x_1 < 0,\, x_2 > 0\}$ in the odd sense:

$$\begin{aligned} \text{for}\, x_1 < 0,\, x_2 > 0: \quad & v(x_1, x_2) = -v(-x_1, x_2),\; p(x_1, x_2) = -p(-x_1, x_2), \\ & \hat{v}(x_1, x_2) = -\hat{v}(-x_1, x_2),\; f(x_1, x_2) = -f(-x_1, x_2). \end{aligned}$$

It is easy to check that this extension satisfies (6.1) not only for the set $\{x_1 < 0,\, x_2 > 0\}$ but for $\{|x_1| < \epsilon,\, x_2 > 0\}$, as well. Since the boundary of the extended domain is smooth in a neighborhood of H, the extended pair (v, p) belongs to $H^2 \times H^1$ in a neighborhood of H. Near the point E, we use the same arguments but apply the extension on the domain $\{x_2 < 0\}$. Our arguments near the points A, D are analogous: additionally we need only to do appropriate changing of the variables (x_1, x_2). $\square$

The situation of the smoothness of the solution (v, p) in a neighborhood of the points B, C, F, G where the type of boundary condition changes is different. At these points the solution (v, p) can possess a singularity. Therefore there is a reason to study problem (6.1), (6.2) in function spaces with weights near these points.

Let this weight be defined as a function $\rho(x_1, x_2) \in C^\infty(\Omega)$, $\rho(x_1, x_2) > 0$, $\forall(x_1, x_2) \in \overline{\Omega} \setminus B_2(\epsilon)$ for a certain $\epsilon > 0$, where $B_2(\epsilon)$ is the union of the circles with radius ϵ and centers at B, C, F, G, and in $\Omega \cap B_2(\epsilon)$ the weight $\rho(x_1, x_2)$ is equal to the distance of the closest points among B, C, F, G.

We introduce the following Sobolev spaces with weights. Let k be a natural number or zero and $\alpha \in \mathbb{R}$. Then,

$$H^k_\alpha(\Omega) := \left\{ u(x),\, x \in \Omega \,\middle|\, \|u\|^2_{H^k_\alpha} = \sum_{j=0}^{k} \int_\Omega \rho^{2(\alpha+j)}(x) \sum_{|\beta|=j} |D^\beta u(x)|^2 \, dx < \infty \right\},$$

where $\beta = (\beta_1, \beta_2)$, $\beta_i \geq 0$, are integer, $|\beta| := \beta_1 + \beta_2$. For $k \geq 1$, let

$$\Psi^k_\alpha(\Omega) = \left\{ v \in H^k_\alpha(\Omega)^2 \mid \nabla \cdot v = 0,\, v|_{\Gamma_0} = 0 \right\}. \tag{6.4}$$

We need also the spaces $H^k_\alpha(B, C)$, $H^k_\alpha(G, F)$ of functions defined on intervals (B, C) or (G, F) with non integer k. For this, we first define the space $H^k_\alpha(\mathbb{R}_+)$. Using the sign $\sim$ for notation of norm equivalence, we get for integers k and $\alpha \in \mathbb{R}$:

$$\begin{aligned}
\|u\|^2_{H^k_\alpha(\mathbb{R}_+)} &= \int_0^\infty \sum_{j=0}^{k} x^{2(\alpha+j)} |\partial_x^j u\,(x)\,|^2 \, dx \\
&\approx \int_0^\infty \sum_{j=0}^{k} \left| (x \cdot \partial_x)^j (x^\alpha u(x)) \right|^2 dx \\
&\approx \int_{-\infty}^\infty \sum_{j=0}^{k} \left| \partial_t^j (e^{(\alpha+1/2)t}) u(e^t)) \right|^2 dt,
\end{aligned} \tag{6.5}$$

where in the last step, we made the change of variable $x := e^t$. Applying the Melling transform

$$\hat{u}(\xi) = \int_0^\infty u(x) e^{i\xi \ln x} \frac{dx}{x} = \int_{-\infty}^\infty u\,(e^t)\, e^{it\xi} \, dt,$$

(which in fact is the Fourier transform of $u(e^t)$), to the function $x^\alpha u(x)$ and taking into account the Plancherel theorem, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\sum_{j=0}^{k}\Big|\partial_t^j(e^{(\alpha+1/2)t}u(e^t))\Big|^2\,dt \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{k}|\xi+i(\alpha+1/2)|^{2j}|\hat{u}(\xi+i(\alpha+1/2))|^2\,d\xi.\end{aligned} \tag{6.6}$$

By virtue of (6.5), (6.6), we can introduce the following equivalent norm for $H^k_\alpha(\mathbb{R}_+)$:

$$\|u\|^2_{H^k_\alpha(\mathbb{R}_+)} := \int_{-\infty}^{\infty}(1+|\xi+i(\alpha+1/2)|^2)^k|\hat{u}(\xi+i(\alpha+1/2))|^2\,d\xi. \tag{6.7}$$

But the norm in (6.7) is well-defined for arbitrary $k\in\mathbb{R}$. To define $H^k_\alpha(a,b)$ with $k\in\mathbb{R}$, $\alpha\in R$, we define a decomposition of unity, i.e., $\varphi_i(x)\in C^\infty(a,b)$, $i=1,2$, $\varphi_i(x)\geq 0$, $\varphi_1(x)+\varphi_2(x)\equiv 1$, $\varphi_1(x)=1$ for x close to a, and $\varphi_2(x)=1$ for x close to b. Then, by definition,

$$\|u\|^2_{H^k_\alpha(a,b)} = \|\widetilde{\varphi_1 u}\|^2_{H^k_\alpha(\mathbb{R}_+)} + \|\widetilde{\varphi_2 u}\|^2_{H^k_\alpha(\mathbb{R}_+)}, \tag{6.8}$$

where by definition $\widetilde{\varphi_1 u}(y)=(\varphi_1 u)(a+y)$, $y\in\mathbb{R}_+$, and $\widetilde{\varphi_2 u}(y)=(\varphi_2 u)(b-y)$, $y\in\mathbb{R}_+$.

Now, we are in the position to formulate the main theorem of this subsection.

Theorem 6.2. *Let $\|\hat{v}\|_\Phi$ be small enough (i.e., there exists a unique generalized solution of* (6.1), (6.2)*). Then, there exists a discrete set $\{\alpha_i\}=\mathfrak{a}\subset\mathbb{R}$ such that for each $\alpha\notin\mathfrak{a}$ and for every $f\in H^0_\alpha(\Omega)^2$, $g\in H^{1/2}_\alpha(\Gamma)^2$ (we suppose that $g^{out}\equiv 0$) there exists a unique solution $(v,p)\in\Psi^2_\alpha(\Omega)\times H^1_\alpha(\Omega)$ of problem* (6.1), (6.2)*, and the following a priori estimates holds true:*

$$\|v\|^2_{\Psi^2_\alpha}+\|p\|^2_{H^1_\alpha(\Omega)}\leq c\Big(\|f\|^2_{H^0_\alpha(\Omega)^2}+\|g\|^2_{H^{1/2}_\alpha(\Gamma)}\Big). \tag{6.9}$$

Proof. This theorem can be proved using the Mellin transform method of Kondrat'ev [Kon1, Kon2] (see also [BR]). □

Remark 6.1. The considerations of this subsection can be extended to the case when the assumption that the solution $(\hat{v},\hat{p})$ of the extremal problem (2.1)-(2.8) has a sufficiently small norm is not fulfilled. In this case for $\alpha\notin\mathfrak{a}$ one can prove an analog of Theorem 6.2 in which the solvability of (6.1), (6.2) is true for $(f,g)\in\mathcal{F}$ where $\mathcal{F}$ is a subspace of $H^0_\alpha(\Omega)^2\times H^{1/2}_\alpha(\Gamma)^2$

of finite codimension. This assertion is sufficient for the application of the Lagrange principle.

6.2. **First reduction of the problem.** First of all, in problem (2.1)-(2.8), we remove the unknown functions u_1, u_2 (controls) together with relations (2.5) and change condition (2.8) to

$$\sum_{i=1}^{2} \|\partial_n v - pn\|^2_{\Gamma_i} \le \gamma^2. \tag{6.10}$$

Evidently, the new problem is equivalent to the old one. In problem (2.1)-(2.3), (2.4), (2.7), (6.10), we make the following change of the dependent variables:

$$v = w + \hat{v}, \quad p = t + \hat{p}, \tag{6.11}$$

where $(\hat{v}, \hat{p})$ is a solution of the original extremal problem (2.1)-(2.8). As a result, we obtain the following extremal problem: Minimize the functional

$$J_0(w,t) = \int_S (\partial_n w - tn) \cdot e_1 \, ds \to \inf \tag{6.12}$$

on the set of pairs (w,t) satisfying

$$-\Delta w + \hat{v} \cdot \nabla w + w \cdot \nabla \hat{v} + w \cdot \nabla w + \nabla t = 0, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \tag{6.13}$$

$$w|_{\Gamma_0} = 0, \quad (\partial_n w - tn)|_{\Gamma_{\text{out}}} = 0, \tag{6.14}$$

$$J_1(w,t) = \sum_{i=1}^{2} \|\partial_n \hat{v} - \hat{p}n + \partial_n w - tn\|^2_{\Gamma_i} \le \gamma^2, \tag{6.15}$$

$$w_1(x) + \hat{v}(x) \ge 0 \quad x \in \omega. \tag{6.16}$$

Evidently, problems (6.12)-(6.16) and (2.1)-(2.8) are equivalent; moreover since $(\hat{v}, \hat{p})$ is a solution of (2.1)-(2.8), then by (6.11) the solution $(\hat{w}, \hat{t})$ of problem (6.12)-(6.16) is as follows:

$$\hat{w}(x) \equiv 0, \quad \hat{t}(x) \equiv 0. \tag{6.17}$$

6.3. **Second reduction of the problem.** We take

$$Y := \Big\{ y = (w,t) \in \Psi^2_\alpha(\Omega) \times H^1_\alpha(\Omega) \,\Big|\, (\partial_n w - tn)|_{\Gamma_{\text{out}}} = 0 \Big\}, \tag{6.18}$$

$$Z = H^0_\alpha(\Omega)^2, \tag{6.19}$$

and define the map $F(y): Y \to Z$ by the following formula:

$$F(y) = F(w,t) = -\Delta w + \hat{v}\cdot\nabla w + w\cdot\nabla\hat{v} + w\cdot\nabla w + \nabla t. \tag{6.20}$$

The functionals $f_0,\ f_1$ and G in (5.1) are defined as follows:

$$f_0(y) = J_0(w,t), \quad f_1(y) = J_1(w,t), \quad G(y) = \max_{x\in\omega}\Big(-\hat{v}(x) - w(x)\Big), \tag{6.21}$$

where $J_0,\ J_1$ are defined in (6.12), (6.15). With the help of (6.18)-(6.21), we reduced problem (6.12)-(6.16) to the abstract problem (5.1). We check now that all conditions of Theorem 5.2 are fulfilled. Let Ω_S be a neighbourhood of S. Then, evidently $H^k_\alpha(\Omega_S\cap\Omega) = H^k(\Omega_S\cap\Omega)$ and therefore the trace operators $\gamma_1 w = \partial_n w_{|S},\ \gamma_0 p = p_{|S}$ are well defined and continuous in the sense

$$\gamma_1 : \Psi^2_\alpha(\Omega) \to H^{1/2}(S)^2, \quad \gamma_0 : H^1_\alpha(\Omega) \to H^{1/2}(S).$$

Hence, the functional in (6.12) is bounded on $\Psi^2_\alpha(\Omega)\times H^1(\Omega)$, and being linear, it is continuously differentiable on this space and therefore also on Y. It is well known [Kon1] that the operators $\gamma_1 w = \partial_n w_{|\Gamma},\ \gamma_0 p = p_{|\Gamma}$ are well defined and continuous in the sense

$$\gamma_1 : H^2_\alpha(\Omega) \to H^{1/2}_\alpha(\Gamma)^2, \quad \gamma_0 : H^1_\alpha(\Omega) \to H^{1/2}_\alpha(\Gamma). \tag{6.22}$$

Therefore, if $\alpha \le 0$, then

$$\begin{aligned}\sum_{i=1}^2 \|\partial_n\hat{v} - \hat{p}n + \partial_n w - tn\|^2_{\Gamma_i} &\le c + 2\sum_{i=1}^2 \|\partial_n w + tn\|^2_{\Gamma_i}\\ &\le c + 2\sup_{x\in\Gamma}(\rho^{-2\alpha})\sum_{i=1}^2\|\rho^\alpha(\partial_n w + tn)\|^2_{\Gamma_i}\\ &\le c_1\Big(1 + \|\partial_n w + tn\|^2_{H^{1/2}_\alpha(\Gamma)}\Big)\\ &\le c_2\Big(1 + \|w\|^2_{\Psi^2_\alpha(\Omega)} + \|t\|^2_{H^1_\alpha(\Omega)}\Big).\end{aligned} \tag{6.23}$$

Relations (6.22), (6.23) imply the continuous differentiability of the functional in (6.15) on the space $\Psi^2_\alpha(\Omega)\times H^1_\alpha(\Omega)$, for $\alpha\le 0$.

The operator in (6.20) is evidently continuous and continuously differentiable in the spaces (6.18), (6.19), for each $\alpha \le 0$. In virtue of (6.17) the derivative $F'(\hat{w},\hat{t})$ at the solution $(\hat{w},\hat{t})$ is defined by the left part of equation (6.1). To check the property a) in subsection 5.2, we have to prove that the boundary value problem (6.1), (6.2) with $g\equiv 0$ for each $f\in H^0_\alpha(\Omega)^2$ possesses a solution $(v,p)\in Y$. For this, we use Theorem 6.2. We choose the parameter α in the spaces in (6.18), (6.19) as follows: if $0 \notin \mathfrak{a}$, we take $\alpha = 0$, if $0 \in \mathfrak{a}$, we take $\alpha < 0$ close enough to zero (there are no points from $\mathfrak{a}$ in the semi-interval $[\alpha, 0)$). By Theorem 6.2 property a) in subsection 5.2 is fulfilled.

In virtue of definition (6.21) of the functional $G(y)$, property b) in Subsection 5.2 is true because of Lemmas 5.1, 5.3. Hence all conditions of Theorem 5.2 are fulfilled, and we can apply this theorem to problem (6.12)-(6.16).

6.4. **Application of the Lagrange principle.** The Lagrange function for the extremal problem (6.12)-(6.16) has the following form:

$$
\begin{aligned}
\mathcal{L}(w,t,\lambda_0,\lambda_1,\alpha,z) = \lambda_0 &\int_S (\partial_n w - tn) e_1 \, ds \\
&+ \frac{\lambda_1}{2} \int_{\Gamma_1 \cup \Gamma_2} |\partial_n(\hat{v} + w) - (\hat{p} + t)n)|^2 \, dx_1 \\
&+ \alpha \sup_{x \in \omega} (-w_1(x) - \hat{v}_1(x)) \\
&+ \int_\Omega (-\Delta w + \hat{v} \cdot \nabla w + w \cdot \nabla \hat{v} + w \cdot \nabla w + \nabla t) z \, dx,
\end{aligned}
\tag{6.24}
$$

where $(\lambda_0, \lambda_1, \alpha, z) \in \mathbb{R}^3 \times H^0_{-\alpha}(\Omega)^2$ are Lagrange multipliers. In virtue of Theorem 5.2 there exists Lagrange multipliers satisfying (5.3)-(5.5) (these conditions will be discussed later). Condition (5.6) being applied to function (6.24) at $(\hat{w}, \hat{t}) = (0, 0)$ leads to the relation

$$
\begin{aligned}
&\lambda_0 \int_S (\partial_n h - \tau n) e_1 \, ds \, + \, \lambda_1 \int_{\Gamma_1 \cup \Gamma_2} (\partial_n \hat{v} - \hat{p} n) \cdot (\partial_n h - \tau n) \, dx_1 \\
&\quad -\alpha \int_\omega h_1(x) \, \mu(dx) + \int_\Omega (-\Delta h + \hat{v} \cdot \nabla h + h \cdot \nabla \hat{v} + \nabla \tau) z \, dx = \, 0,
\end{aligned}
\tag{6.25}
$$

which is true for every $(h, \tau) \in Y$ (see (6.18)). In (6.25) $\mu(dx)$ is a measure on ω. In the case of problem (6.12)-(6.16),

$$
G(w) = \max_{x \in \omega} (-\hat{v}_1(x) - w_1(x)),
\tag{6.26}
$$

and we have to find the derivative of this functional at the point $w = (w_1, w_2) = (0, 0)$ in the direction $h = (h_1, h_2)$. Recall that all functions in (6.26) and below belong to $\Psi^2_\alpha(\Omega)$ and by the Sobolev embedding theorem the restriction to ω of all these functions belong to $C(\omega)$. In virtue of (5.10), (5.12), the derivative of the functional (6.26) at zero in the direction h has the form

$$
G'(0, h) = \max_{x \in M_0} -h_1(x),
\tag{6.27}
$$

where $M_0 = \Big\{x \in \omega | -\hat{v}_1(x) = 0\Big\}$. By Lemma 5.4 the sub-differential $\partial G'(0,\cdot)$ consists of the functional

$$l(h) = -\int_\Omega h_1(x)\,\mu(dx),$$

where $\mu(dx)$ is a probability measure supported on the set M_0. Just this measure is written in equation (6.25).

7. The optimality system

In this section, we obtain the main result of this paper, the optimality system for problem (2.1) -(2.8).

7.1. **Derivation of the optimality system.** At first, we take $h \in \Psi_\alpha^2(\Omega) \cap C_0^\infty(\Omega)^2$, $\tau \in C_0^\infty(\Omega)$ in (6.25). In this way, we get

$$\int_\Omega (-\Delta h + \hat{v}\cdot\nabla h + h\cdot\nabla\hat{v} + \nabla\tau)\cdot z\,dx = \alpha\int_\omega h_1(x)\,\mu(dx). \tag{7.1}$$

If we take $h = 0$ in (7.1), the resulting equality yields

$$\nabla\cdot z = 0 \quad \text{in } \Omega, \tag{7.2}$$

which is to be understood in the distributional sense. Accordingly, taking $\tau \equiv 0$ in (7.1), we get

$$\int_\Omega (-\Delta z - \hat{v}\cdot\nabla z + \nabla\hat{v}^* z)\cdot h\,dx = \alpha\int_\omega h_1\,\mu\,(dx)\,, \tag{7.3}$$

for all $h \in \Psi_\alpha^2(\Omega) \cap C_0^\infty(\Omega)^2$, where $(\nabla\hat{v})^* z = (\partial_1\hat{v}\cdot z, \partial_2\hat{v}\cdot z)$.

This equality and the De Rham Theorem (see [T]) imply that there exists a distribution $\sigma(x)$ such that

$$-\Delta z - \hat{v}\cdot\nabla z + \nabla\hat{v}^* z - \nabla\sigma = \alpha e_1\mu(dx) \tag{7.4}$$

where $e_1 = (1,0)$.

System (7.4), (7.2) is elliptic in the sense of Douglas-Nirenberg. Therefore for each subdomain Ω_1 of Ω compactly enclosed in $\Omega\setminus\omega$, i.e. $\overline{\Omega}_1 \subset \Omega\setminus\omega$, we have $z \in H^2(\Omega_1)^2$, $\nabla\sigma \in L^2(\Omega_1)^2$.

Moreover $(z,\nabla\sigma)$ posseses enough smoothness near $\partial\Omega$ in order to define the traces of these functions on $\partial\Omega$. To prove this one has to use methods of ([F] Chapter 2.5, [LM]).

Now, we take an arbitrary $(h,\tau)\in Y$ in (6.25) and integrate by parts. Then, taking into account (7.2),(7.4), we get for all $(h,\tau)\in Y$:

$$
\begin{aligned}
&\lambda_0\int_S(-\tau n+\partial_n h)\cdot e_1\,dx+\lambda_1\int_{\Gamma_1\cup\Gamma_2}(\partial_n\hat v-\hat p n)(\partial_n h-\tau n)dx_1\\
&+\int_{\partial\Omega}\Big\{(-\partial_n h+\tau n)z+\partial_n z\cdot h+(\hat v\cdot n)(h\cdot z)\Big\}\,dx\\
&+\int_\Omega\nabla\sigma\cdot h\,dx=0.
\end{aligned}\tag{7.5}
$$

Suppose that $(h,\tau)\in Y$ and (h,τ) equals zero in a neigbourhood of $(\partial\Omega\setminus S)$. Then, recalling that $h_{|S}=0$, we obtain from (7.5) that

$$\int_S(-\tau n+\partial_n h)(\lambda_0 e_1-z)\,dx=0.\tag{7.6}$$

Since $\nabla\cdot h=0$ and $h_{|S}=0$, then $(-\tau n+\partial_n h)_{|S}=(-\tau n+\partial_n h_T)_{|S}$ where h_T is the component of vector field h that is tangent to S. Evidently the set of $(-\tau n+\partial_n h_T)_{|S}$ is dense in $L^2(S)^2$. Therefore (7.6) implies

$$z_{|S}=\lambda_0 e_1.\tag{7.7}$$

Analogously, if we take $(h,\tau)\in Y$ that equals zero in a neighbourhood of $\partial\Omega\setminus\{\Gamma_{\text{in}}\cup S'\})$, we obtain from (7.5)

$$\int_{\Gamma_{\text{in}}\cup S'}(\tau n-\partial_n h)z\,dx=0.$$

This implies the equality

$$z_{|\Gamma_{\text{in}}\cup S'}=0.\tag{7.8}$$

Taking $(h,\tau)\in Y$, $(h,\tau)=0$ in a neighborhood of $\partial\Omega\setminus\Gamma_{\text{out}}$ and using that $(-\partial_n h+\tau n)|_{\Gamma_{\text{out}}}=0$, we get from (7.5) that

$$\int_{\Gamma_{\text{out}}}\Big(\partial_n z+n\sigma+(\hat v\cdot n)z\Big)h\,dx_2=0,$$

and therefore, since $\int_{\Gamma_{\text{out}}}n\cdot h\,dx_2=0$, we get

$$\Big(\partial_n z+n\sigma+(\hat v\cdot n)z\Big)_{|\Gamma_{\text{out}}}=nc,\tag{7.9}$$

where c is a constant.

At last, for $(h,\tau)\in Y$, $(h,\tau)=0$ in neighbourhood of $\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)$, we obtain from (7.5)

$$
\begin{aligned}
\int_{\Gamma_1\cup\Gamma_2}&\Big(-\partial_n h+\tau n)z+(\partial_n z+n\sigma+(\hat v\cdot n)z\Big)\cdot h\\
&+\lambda_1(\partial_n\hat v-\hat p n)(\partial_n h-\tau n)\,dx_1=0.
\end{aligned}\tag{7.10}
$$

Taking in (7.10) $h = 0$ and $-\partial_n h + \tau n$ running through the dense set in $\left(L^2(\varGamma_1 \cup \varGamma_2)\right)^2$, we get

$$z|_{\varGamma_1 \cup \varGamma_2} = \lambda_1 (\partial_n \hat{v} - \hat{p} n)\Big|_{\varGamma_1 \cup \varGamma_2}. \tag{7.11}$$

If we take $-\partial_n h + \tau n = 0$ and h is arbitrary, then we obtain

$$\Big(\partial_n z + n\sigma + (\hat{v} \cdot n) z\Big)\Big|_{\varGamma_1 \cup \varGamma_2} = nc, \tag{7.12}$$

where c is a constant. Notice that the constant c in (7.9) and the constant c in (7.12) corresponding to $\varGamma_1$ and $\varGamma_2$ are equal. Indeed, for determining (7.9), (7.12), we can take $h = 0$ on $\partial\Omega \setminus (\varGamma_{\text{out}})_{\cup \varGamma_1 \cup \varGamma_2}$ and h arbitrary on $\varGamma_{\text{out}} \cup \varGamma_1 \cup \varGamma_2$ (Compare with (3.11)). Adding this c to σ we can take $c = 0$.

7.2. **The final form of the optimality system.** Let $(\hat{v}, \hat{p}, \hat{u}_1, \hat{u}_2)$ be the solution of problem (2.1)-(2.8). Then, the optimality system for this problem consists of equations (2.1),(2.2), (7.4),(7.2) and the boundary conditions (2.3), (7.7)-(7.9), (7.11), (7.12). We rewrite these equations in the following form:

$$-\Delta \hat{v} + \hat{v} \cdot \nabla \hat{v} + \nabla \hat{p} = 0, \quad \nabla \cdot \hat{v} = 0,\ x \in \Omega, \tag{7.13}$$

$$-\Delta z - \hat{v} \cdot \nabla z + \nabla \hat{v}^* z - \nabla \sigma = e_1 \alpha \mu(dx), \quad \nabla \cdot z = 0,\ x \in \Omega, \tag{7.14}$$

$$\hat{v}_{|\varGamma_{\text{in}}} = v^{\text{in}}, \quad (\partial_n \hat{v} - \hat{p} n)_{|\varGamma_{\text{out}}} = 0, \quad v_{|S \cup S'} = 0 \tag{7.15}$$

$$z_{|S} = \lambda_0 e_1, \quad z_{|\varGamma_{\text{in}} \cup S'} = 0, \quad z_{|\varGamma_1 \cup \varGamma_2} = \lambda_1 (\partial_n \hat{v} - \hat{p} n)_{|\varGamma_1 \cup \varGamma_2} \tag{7.16}$$

$$\Big(\partial_n z + n\sigma + (\hat{v} \cdot n) z\Big)_{|\varGamma \text{out}} = 0, \quad \Big(\partial_n z + n\sigma + (\hat{v} \cdot n) z\Big)_{|\varGamma_1 \cup \varGamma_2} = 0. \tag{7.17}$$

In virtue of (5.3), (5.5) the optimality system should be supplemented by the following condition:

1) Conditions of signs concordance:

$$\lambda_0 \geq 0, \quad \lambda_1 \geq 0, \quad \alpha \geq 0. \tag{7.18}$$

2) Conditions of complementary slackness

$$\lambda_1 (J_1 - \gamma^2) = 0, \quad \alpha \min_{x \in \omega} \hat{v}(x) = 0, \tag{7.19}$$

where $J_1 = J_1(0,0)$ and $J_1(w,t)$ is defined in (6.15).

8. Numerical calculations

The validity of the crucial Condition 1 for a given subset $\omega \subset \Omega$ (see (4.26)) can hardly be shown analytically but rather requires computational confirmationFigure 2 shows a series of plots of the velocity component v_1 for increasing strength of the control (positive pressure drop). Figure 3 shows corresponding plots for symmetric and asymmetric action of the control. The latter results demonstrate that for a large class of sub-domains $\omega \subset \Omega$ the property $v_{1|\omega} \geq 0$ can be achieved by applying appropriate controls.

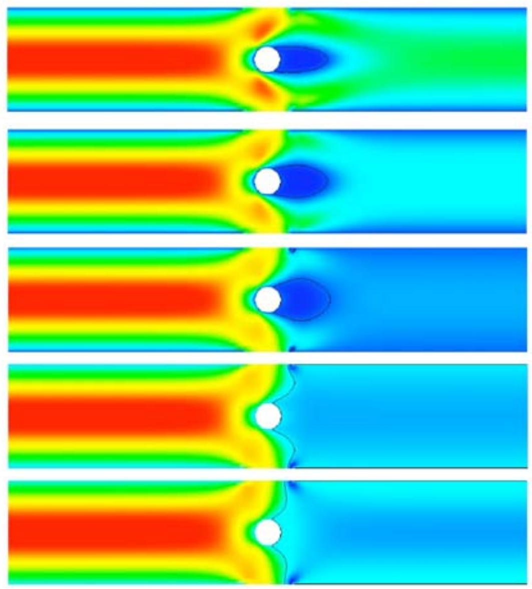

Figure 2. Velocity component v_1 for increasing strength of control pressure; area of $v_1 < 0$ dark blue

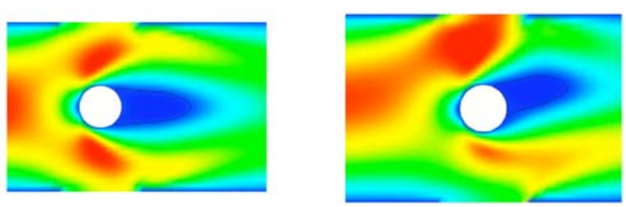

Figure 3. Velocity component v_1 for symmetric (left) and unsymmetric (right) control pressure drop; area of $v_1 < 0$ dark blue

References

[AT] Abergel F., Temam R. *On some control problems in fluid mechanics*, Theoret. Comput. Fluid Dynamics, 1, (1990) pp. 303–325.

[A] Alekseev G.V. *Solvability of Control Problems for steady-state equations of Magnetic Hydrodynamics of viscous Fluid*, Sibiryan Math Journ v.45:2, 2004, p.243-262 (in Russian).

[ALT] Alekseev G.V., Tereshko D.A. *Analysis and optimization in viscous fluid hydrodynamics*, Dalnauka, Vladivostok, 2009 (in Russian).

[ATF] Alekseev V.M., Tikhomirov V.M., Fomin S.V. *Optimal Control*, Consultants Bureau, New York 1987.

[BIN] Besov O., Il'in V., Nikol'skiy S.. *Integral representations of functions and embedding theorems*, Nauka, Moscow, 1975. (in Russian)

[BR] Blum H., Rannacher R.. *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. Appl. Sci. 2, 556–581 (1980).

[CH] Chebotarev A.Yu. *Maximum Principle in the Problem of Boundary Control for Incompressible Fluid Flow.* Sibirskiy Math. Journ. v.34:6, 1993, p.189-197 (in Russian)

[DM] Dubovitskij A.Ja., Milyutin A.A. *Extremal Problems with the presence of constraints*.- USSR Comput. Mathematics and Math. Physics, 5(3), 1965, p.1-80.

[F] Fursikov A.V. *Optimal Control of Distributed Systems. Theory and Applications* Translations of Math. Monographs v. 187. AMS, Providence, Rhode Island, 1999.

[F1] Fursikov A.V. *Control problems and theorems concerning the unique solvability of a mixed boundary value problems for the three-dimensional Navier-Stokes and Euler equations*, Math USSR Sb., 43 (1982), pp. 281–307.

[F2] Fursikov A.V. *Properties of solutions of some extremal problems connected with the Navier-Stokes system*, Math USSR Sb., 46 (1983), pp. 323–351.

[FGH] Fursikov A.V., Gunzburger M., Hou S. *Optimal Boundary Control for the evolutionary Navier-Stokes system: The three dimensional case.* SIAM J. *Control Optim.,* 42 (2005), pp.2191–2232.

[GA] GASCOIGNE: *A Finite Element Software Library*, http://www.gascoigne.uni-hd.de, 2006.

[GR] Girault V., Raviart P.A. *Finite element methods for Navier-Stokes equations. Theory and algorithms.* Springer-Verlag, Berlin, 1986.

[G] Girsanov I.V. *Lectures of mathematical theory of extremal problems.* Lecture Notes in Economics and Mathematical Systems, v.67, Springer , Berlin-Heidelberg-New York, 1972.

[GHS1] Gunzburger M., Hou L., Svobodny T. *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls*, Modél. Math. Anal. Numér., 25 (1991), pp. 711–748.

[GHS2] Gunzburger M., Hou L., Svobodny T. *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with distributed and Neumann controls*, Math. Comp., 57 (1991), pp. 123–151.

[Kon1] V. A. Kondrate'ev. *Boundary problems for elliptic equations in domain with conical or angular points*, Trans. Moscow Math. Soc. 16, 227–313 (1967).

[Kon2] V. A. Kondrate'ev. *Asymptotic of a solution of the Navier-Stokes equations near the angular part of the boundary*, J. Appl. Math. Mech. 31, 125–129 (1967).

[Lad] Ladyzhenskaya O.A. *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[LM] Lions J.-L., Magenes E. *Problems aux limites non homogenes et applications.* Dunod, Paris, 1968.

[MDO] Milyutin A.A., Dmitruk A.V., Osmolovskij N.P. *Maximum Principle in Optimal Control* Preprint (in Russian), Moscow State Univ., Mech.-Math. Faculty, 2004.

[S] Sritharan S. *An optimal control problem in exterior hydrodynamics*, in Distributed Parameter Control Systems, New Trends and Applications, Ed. by G. Chen, B. Lee, W. Littman, and L. Markus, Marcel Dekker, New York, 1991, pp.385–417.

[T] Temam R. *Navier-Stokes Equations. Theory and Numerical Analysis.* Studies in Math. and its Appl. v.2, North Holland Pub. Comp., Amsterdam, New York, Oxford, 1979.

[VD] Van Dyke M. *An Album of Fluid Motion.* The Parabolic Press, Standford, 1982.

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