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CERTAIN QUESTIONS OF FEEDBACK STABILIZATION FOR NAVIER-STOKES EQUATIONS

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Abstract. The authors study the stabilization problem for Navier-Stokes and Oseen equations near steady-state solution by feedback control. The cases of control in initial condition (start control) as well as impulse and distributed controls in right side supported in a fixed subdomain of the domain G filled with a fluid are investigated. The cases of bounded and unbounded domain G are considered.

1. **Introduction.** In this paper we continuer to study stabilization problem for Navier-Stokes system and its linearization near a steady-state solution by feedback control. The case when a control is defined on the boundary ∂G of the domain G filled with the fluid is the most interesting for application. That is why the most part of investigations in this topic were made just for control from the boundary (see [3]-[6], [7]-[13], [27]).

This article is devoted to investigation of stabilization problem with other types of feedback control such as control in initial condition (start control), impulse and distributed controls in right side. We assume that for all of these cases a control is supported in a given subdomain ω of G. There is a big reason in such studies, for instance, because anyone from these controls can be used to construct feedback stabilizing control on the boundary ∂G with help of technique developed in ([7]-[13]): actually in mentioned papers the start control already has been implicitly used for construction of control on the boundary.

The necessity to apply impulse control for feedback stabilization from the boundary was understood in [12],[13] where physical feedback property (that is the most important for application) has been studied. But there is at least one reason more to study impulse control, and this reason is as follows. In the paper [22] of A.A.Ivanchikov where approach from [7]-[13] was applied for numerical stabilization of Taylor curls to Couette flow by feedback boundary control, this stabilizing control u had big gradients for times close to zero. It was clearly seen in examples

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of numerical stabilization described in [22]. Evidently, these gradients should be an obstacle in possible attempts to increase Reynolds number. These big gradients arise because of big gradients in start control u for auxiliary stabilization problem in extended domain (see [7],[8]). It is easy to decrease gradients of u by imposing restriction in the form $\|u\|_{H^1} \leqslant \gamma$ with sufficiently small γ. However, due to this this restriction we can not push the phase variable into the corresponding invariant manifold (that is the goal of stabilization construction). Therefore, we have to act with restricted control u in several time instants. But this means that we apply just impulse control with restriction imposed on impulses (see Section 5).

Note that one can also decrease gradients of boundary control mentioned above by applying distributed control in auxiliary stabilization problem in extended domain. Besides, in exact controllability problems for Navier-Stokes equations just distributed control in right side is used as a rule (see e.g. [16],[17]). Since exact controllability and feedback stabilization are very close problems, it is useful to develop technique of feedback stabilization by distributed control: it can give opportunity to use some information from exact controllability topic for feedback stabilization.

The other important goal of this paper is to study feedback stabilization of solutions defined in unbounded domains. The Cauchy problem for Stokes and Navier-Stokes systems in $\mathbb{R}^2$ with impulse control in the right-hand side is investigated (see Section 7 below).

Unbounded domains have some features which distinguish them from bounded domains. First, in unbounded domains the rate of stabilization is slower than in bounded ones. In bounded domains solutions from stable invariant manifolds exponentially decay to stationary solutions, whereas for Cauchy problem the rate of convergence is only of power-type (see [18]).

Another one feature of Cauchy problem, which can be very useful for applications is that for Stokes system stable invariant subspaces can be constructed in explicit form. Moreover, in some cases stable invariant manifolds for Navier-Stokes system coincide with these linear subspaces.

The paper is build up as follows. In sections 2-4 we study feedback stabilization problem with start control. This topic was investigated implicitly in [7]-[11], and therefore rather often instead of proofs we give references on corresponding papers containing these proofs. Nevertheless in subsection 3.3 we give a complete proof of one important theorem that is more simple than one obtained in [7]. Using [15] we get here new result on local stabilization when initial condition belongs to an unbounded neighborhood of stabilized steady-state solution.

Section 5 is devoted to stabilization problem for Oseen equations with impulse feedback control when impulses satisfy certain restrictions on their gradients.

In Section 6 stabilization by distributed control supported in a prescribed subdomain for Oseen and Navier-Stokes equations is constructed. Note that stabilization of Navier-Stokes equations by feedback distributed control has been studied by V.Barbu, R.Triggiani [6] with help of Riccati equation. This method is rather difficult for numerical simulation. Our approach we have proposed in Section 6 is more preferable from this point of view.

In Section 7 we consider 2D Stokes and Navier-Stokes equations defined in $\mathbb{R}^2$ and investigate their power-like stabilization with impulse control in right-hand side.

Note that for each type of control considered here feedback property is realized by a connection $u(t,\cdot) = Ey(t,\cdot)$ where u is a control, y is phase variable and E is a certain map (in the case of start control $t = 0$). One of the main issue of this paper is that for different types of control (start, impulse and distributed) operators E from feedback connection are identical up to a constant.

2. **Setting of the stabilization problem with a control in initial condition (with a start control).** In this section we formulate the stabilization problem with control in initial condition.

Let $G \subset \mathbb{R}^d$, $d = 2, 3$ be a domain with the boundary $\partial G \in C^\infty$, $Q = \mathbb{R}_+ \times G$. Everywhere below until the Section 7 we assume that domain G is bounded. We consider the Navier-Stokes equations

$$\partial_t v(t,x) - \Delta v + (v, \nabla)v + \nabla p(t,x) = h(x), \quad \operatorname{div} v(t,x) = 0 \tag{2.1}$$

with boundary condition

$$v(t,x)|_{x \in \partial G} = 0 \tag{2.2}$$

and initial condition containing a control

$$v(t,x)|_{t=0} = v_0(x) + u(x). \tag{2.3}$$

Here $(t, x) = (t, x_1, \ldots, x_d) \in Q$, $v(t, x) = (v_1, \ldots, v_d)$ is a velocity of fluid flow, $p(t, x)$ is a pressure, $h(x) = (h_1, \ldots, h_d)$ is a given right side, $v_0(x)$ is a given initial condition, and $u(x)$ is a control supported in a given fixed subdomain $\omega \Subset G$.

Denote, as usually, by $H^k(G)$, $k \in \mathbb{N}$ the Sobolev space of scalar functions, defined and square integrable on G together with all its derivatives up to order k and by $(H^k(G))^d$ the analogous space of vector fields. Besides, $H_0^1(G) = \{f(x) \in H^1(G) : f(x)|_{x \in \partial G} = 0\}$. We will also use the following spaces of solenoidal vector fields:

$$V^k(G) = \left\{ v(x) = (v_1, \ldots, v_d) \in (H^k(G))^d : \quad \operatorname{div} v = 0 \right\}, k = 0, 1, 2, \ldots \tag{2.4}$$

$$V_0^1(G) = V^1(G) \cap H_0^1(G)^d, \quad V_0^0(G) = \text{closure } \ \mathcal{V}(G) \ \text{ in } \ L_2(G)^d \tag{2.5}$$

where $\mathcal{V}(G) = \{v(x) \in (C_0^\infty(G))^d : \operatorname{div} v = 0\}$. Evidently,

$$\|v\|_{V^0(G)} = \|v\|_{V_0^0(G)} := \|v\|_{(L_2(G))^d}; \quad \|v\|_{V_0^1(G)} := \|\nabla v\|_{(L_2(G))^{d^2}}.$$

Let introduce the notations:

$$V = V_0^1(G) \qquad V_{00}^1(\omega) = \{v(x) \in V : \operatorname{supp} v \subset \omega\} \tag{2.6}$$

By V we denote the phase space of dynamical system generated by boundary value problem (2.1)–(2.3).

The setting of local stabilization problem for a solution of boundary value problem (2.1)–(2.3) with start control is as follows:

Let $h \in L_2(G)^d$, $v_0 \in V_0^1(G)$ and $\mathcal{O}$ is a neighborhood of origin in the phase space $V = V_0^1(\Omega)$. Suppose that $\sigma > 0$ and an unstable steady-state solution $(\widehat{v}(x), \widehat{p}(x))$ of Navier-Stokes equations

$$-\Delta \widehat{v}(x) + (\widehat{v}, \nabla)\widehat{v} + \nabla \widehat{p}(x) = h(x), \quad \operatorname{div} \widehat{v}(x) = 0 \tag{2.7}$$

$$(\widehat{v}(x), \widehat{p}(x)) \in (V^2(G) \cap V_0^1(G)) \times H^1(G) \tag{2.8}$$

are given, and inclusion $\widehat{v} - v_0 \in \mathcal{O}$ is true. Construct a control $u(x) \in V_{00}^1(\omega)$ such that the component $v(t, x)$ of the solution $(v, \nabla p)$ to boundary value problem (2.1)–(2.3) satisfies:

$$\|v(t, \cdot) - \widehat{v}\|^2_{V_0^1(G)} \leqslant c\|v_0\|^2_{V_0^1(G)} e^{-\sigma t} \tag{2.9}$$

with a certain constant $c > 0$ independent of $v_0 \in \widehat{v} + \mathcal{O}$.

The simplest example of the neighborhood $\mathcal{O}$ is the ball $B_\rho = \{v_0 \in V_0^1(\Omega)\}$ with small enough ρ but it is possible to prove stabilization problem for some unbounded $\mathcal{O}$.

Moreover, we would like to look for feedback control. We will use the following mathematical formalization of this general physical definition. [1]

Definition 2.1. Control $u(x)$ is called feedback if there exist a continuous operator $F : \mathcal{O} \to V_{00}^1(\omega)$ such that after substitution

$$u(x) = v_0(x) + (Fv_0)(t, x) \tag{2.10}$$

into (2.3) the solution $v(t, x)$ of closed-loop system (2.1)–(2.3) [2] satisfies inequality (2.9) with constant c independent of $v_0 \in \mathcal{O}$

3. **Stabilization of Oseen equations by start control.** In this and next sections we describe stabilization construction with control by initial condition.

3.1. **Reduction to linear case.** We make change of unknown functions

$$v(t, x) = y(t, x) + \widehat{v}(x), \quad p(t, x) = q(t, x) + \widehat{p}(x) \tag{3.1}$$

in (2.1) where $(\widehat{v}, \widehat{p})$ is solution of (2.7). As a result we get

$$\partial_t y(t, x) - \Delta y + (\widehat{v}(x), \nabla)y + (y, \nabla)\widehat{v} + (y, \nabla)y + \nabla q(t, x) = 0, \quad \operatorname{div} y = 0, \tag{3.2}$$

$$y(t, x)|_{t=0} = y_0(x) + u(x) \tag{3.3}$$

where $y_0 = v_0 - \widehat{v}$. We omit in (3.2) nonlinear term $(y, \nabla)y$, and, changing notation for pressure from q on p, we obtain:

$$\partial_t y(t, x) - \Delta y + (\widehat{v}(x), \nabla)y + (y, \nabla)\widehat{v} + \nabla p(t, x) = 0, \quad \operatorname{div} y = 0, \quad y|_{\partial G} = 0 \tag{3.4}$$

Set initial condition

$$y(t, x)|_{t=0} = y_0(x) \tag{3.5}$$

Our aim now is to describe the set of initial conditions $\{y_0\}$ such that solutions $y(t, x)$ of (3.4)-(3.5) satisfy estimate

$$\|y(t, \cdot)\|_{V_0^1(G)} \leqslant c\|y_0\|_{V_0^1(G)} e^{-\sigma t} \quad \text{for } t \geqslant 0 \tag{3.6}$$

with constant $c > 0$ independent of y_0.

3.2. **Description of initial conditions generating decreasing solutions.** Denote by

$$\hat{\pi} : (L_2(G))^2 \longrightarrow V_0^0(G) \tag{3.7}$$

the operator of orthogonal projection. We consider the Oseen steady-state operator

$$Av := -\hat{\pi}\Delta v + \hat{\pi}[(\widehat{v}(x), \nabla)v + (v, \nabla)\widehat{v}] : V_0^0(G) \longrightarrow V_0^0(G) \tag{3.8}$$

and its adjoint operator A^*. These operators possess the following properties:

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[1] We will not discuss here connection of the notion given here with the notion of physical feedback (see [12],[13]).

[2] More exactly, the component v of solution (v, p)

Theorem 3.1. *Operator A defined in (3.8) and its adjoint A^* are closed and have the domain $\mathcal{D}(A) = V^2(G) \cap (H_0^1(G))^2$. They are sectorial operators, i.e. spectra $\Sigma(A), \Sigma(A^*)$ of operators A and A^* are discrete subsets of a complex plane $\mathbb{C}$ belonging to a sector S symmetric with respect to $\mathbb{R}$ and containing $\mathbb{R}_+$:*

$$S = -\gamma_0 + S_0, \quad \text{where} \quad \gamma_0 > 0,\ S_0 = \{z \in \mathbb{C} :\ |\arg z| < \theta < \pi/2\},$$

and

$$\|(A - \lambda I)^{-1}\| \leqslant M/|\lambda + \gamma|, \qquad \forall \lambda \notin S \tag{3.9}$$

Moreover $\Sigma(A) = \Sigma(A^*)$.

The proof see, for instance, in [7],[8].

We rewrite boundary value problem (3.4)-(3.5) for Oseen equations in the following form

$$\frac{dy(t,\cdot)}{dt} + Ay(t,\cdot) = 0, \quad y|_{t=0} = y_0. \tag{3.10}$$

where A is operator (3.8). (To get (3.10) we have to act operator $\hat{\pi}$ to both parts of the first equation in (3.4).) Then for each $y_0 \in V_0^0(G)$ the solution $y(t,\cdot)$ of (3.10) is defined by $y(t,\cdot) = e^{-At}y_0$ where e^{-At} is the resolving semigroup of problem (3.10).

Although the most natural phase space for problem (3.10) is the space $V = V_0^1(G)$, in order to solve so-called stabilization problem for Navier-Stokes equations with unbounded neighborhood $\mathcal{O}$ we have to study (3.10) in some other phase spaces as well. For this we recall definition of one family of spaces.

Let $\{\widehat{e}_j(x), \widehat{\lambda}_j, j = 1, 2, \dots\}$ be eigenfunctions and eigenvalues of the following spectral problem for the Stokes operator:

$$-\Delta\widehat{e}(x) + \nabla\widehat{p}(x) = \widehat{\lambda}\widehat{e}(x), \quad \text{div}\ \widehat{e} = 0, \quad x \in \Omega; \quad \widehat{e}|_{\partial\Omega} = 0 \tag{3.11}$$

As well-known, $0 < \widehat{\lambda}_1 \leqslant \widehat{\lambda}_2 \leqslant \dots, \widehat{\lambda}_j \to \infty$ as $j \to \infty$, and $\{\widehat{e}_j(x)\}$ forms orthonormal basis in $V_0^0(G)$:

$$\forall v \in V_0^0(G) \quad v(x) = \sum_{j=1}^{\infty} v_j\widehat{e}_j(x), \quad \text{where} \ \ v_j = (v, \widehat{e}_j)_{V^0(G)}, \quad \|v\|^2_{V_0^0(G)} = \sum_{j=1}^{\infty}|v_j|^2$$

For each $s \in \mathbb{R}$ we introduce the space V^s by the formula

$$V^s = \left\{ v(x) = \sum_{j=1}^{\infty} v_j\widehat{e}_j(x),\ v_j \in \mathbb{R} : \|v\|_s^2 = \sum_{j=1}^{\infty}\widehat{\lambda}_j^s|v_j|^2 < \infty \right\} \tag{3.12}$$

It is well-known (see, for instance [14], Ch.3,Sect.4) that

$$V^s = V_0^s(G), s = 1, 2, \quad V^2 = V_0^1(G) \cap H^2(G), \quad \|v\|_s = \|v\|_{V^s(G)} \quad s = 0, 1, 2 \tag{3.13}$$

where V^s is defined in (3.12), and spaces $V_0^0(G), V_0^1(G), V^2(\Omega)$ are defined in (2.5), (2.4).

Theorem 3.1 and results of [1], [25] implies

Corollary 1. *There exists a constant c_0 such that*

$$\|e^{-At}y_0\|_{V^s} \leqslant c_0\|y_0\|_{V^s}e^{\gamma_0 t} \tag{3.14}$$

for each $y_0 \in V^s$, $s \in [-1, 1]$. Besides, the bound (3.9) is true when operator $(A - \lambda I)^{-1}$ is considered in the space V^s, $s \in [-1, 1]$.

Let $\sigma > 0$ satisfy:

$$\Sigma(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = \sigma\} = \emptyset \tag{3.15}$$

The case when there are certain points of $\Sigma(A)$ which are in the left of the line $\{\mathrm{Re}\lambda = \sigma\}$ will be interesting for us. That is why everywhere below we will assume that the following condition is true:

Condition 3.1. Operator A defined in (3.8) possesses eigenvalues with negative real part.

Denote by $X_\sigma^+(A)$ the subspace of $V_0^0(G)$ generated by all eigenfunctions and generalized eigenfunctions of operator A [3] corresponding to all eigenvalues of A placed in the set $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < \sigma\}$. By $X_\sigma^+(A^*)$ we denote analogous subspace corresponding to adjoint operator A^*. We denote the orthogonal complement to $X_\sigma^+(A^*)$ in $V_0^0(G)$ by $X_\sigma(A) \equiv X_\sigma$:

$$X_\sigma = V_0^0(G) \ominus X_\sigma^+(A^*) \tag{3.16}$$

For each $s \in [-1, 1]$ we set

$$V_-^s = X_\sigma \cap V^s \text{ if } s \geqslant 0; \qquad V_-^s = \text{ closure of } X_\sigma \text{ in } V^s(\Omega) \text{ for } s < 0 \tag{3.17}$$

Let $V_+^s = X_\sigma^+(A)$, $s \in [-1, 1]$, i.e. finite-dimensional space V_+^s as set does not depend on s.

One can show that subspaces V_+^s, V_-^s are invariant with respect to the action of semigroup e^{-At}, and $V_-^s + V_+^s = V^s$.

Theorem 3.2. *Suppose that A is operator (3.8) and $\sigma > 0$ satisfies (3.15). Then for each $y_0 \in V_-^s$, $s \in [-1, 1]$ the following analog of inequality (3.6) holds:*

$$\|y(t, \cdot)\|_{V^s} \leqslant c\|y_0\|_{V^s} e^{-\sigma t} \quad \text{for } t \geqslant 0 \tag{3.18}$$

with constant $c > 0$ independent of y_0. Besides, the solution of problem (3.10) with such initial conditions is defined by the formula

$$y(t, \cdot) = e^{-At} y_0 = (2\pi i)^{-1} \int_\gamma (A - \lambda I)^{-1} e^{-\lambda t} y_0 d\lambda. \tag{3.19}$$

Here γ is a contour belonging to $\rho(A) := \mathbb{C} \setminus \Sigma(A)$ such that $\arg \lambda = \pm\theta$ for $\lambda \in \gamma, |\lambda| \geqslant N$ for certain $\theta \in (0, \pi/2)$ and for sufficiently large N. Moreover, γ encloses from the left the part of the spectrum $\Sigma(A)$ placed right to the line $\{\mathrm{Re}\lambda = \sigma\}$. The complementary part of the spectrum $\Sigma(A)$ is placed on the left to the contour γ.

Proof: See [7], [8].

3.3. **The basic property of eigenfunctions for Oseen equations.** We discuss here one property of eigenfunctions and generalized eigenfunctions for the operator A^* adjoint to (3.8)

$$A^* v := -\hat{\pi}\Delta v - \hat{\pi}[(\widehat{v}(x), \nabla)v - (\nabla\widehat{v})^* v] : V_0^0(G) \longrightarrow V_0^0(G) \tag{3.20}$$

supplied with zero Dirichlet boundary condition where

$$(\nabla\widehat{v})^* v = (\sum_{k=1}^{3} \partial_j \widehat{v}_k v_k, j = 1, 2, 3)$$

[3] Their definition see below, in Subsection 3.3 before Definition 3.3

Recall that if $\lambda_j \in \mathbb{C}$ is eigenvalue of operator A^* then by definition there exist functions $\varepsilon_0(x), \dots \varepsilon_k(x)$ with integer $k \geqslant 0$ such that

$$(A^* - \lambda_j I)\varepsilon_0 = 0, \quad \varepsilon_0 + (A^* - \lambda_j I)\varepsilon_1 = 0, \quad , \dots \varepsilon^{k-1} + (A^* - \lambda_j I)\varepsilon_k = 0, \tag{3.21}$$

and ε_0 is called eigenfunction as well as ε_m with $m \geqslant 1$ are called generalized eigenfunctions.

Definition 3.3. The set of eigenfunctions and generalized eigenfunctions

$$\varepsilon_0^{(k)}, \varepsilon_1^{(k)}, \dots, \varepsilon_{m_k}^{(k)} \quad (k = 1, 2, \dots, N) \tag{3.22}$$

corresponding to an eigenvalue λ_j is called canonical system which corresponds to the eigenvalue λ_j if set (3.22) satisfies the properties:

i) Functions $\varepsilon_0^{(k)}(x), k = 1, 2, \dots, N$ form a basis in the space of eigenfunctions corresponding to the eigenvalue λ_j.
ii) $\varepsilon_0^{(1)}$ is an eigenfunction with maximal possible multiplicity.
iii) $\varepsilon_0^{(k)}$ is an eigenfunction which can not be expressed by a linear combination of $\varepsilon_0^{(1)}, \dots, \varepsilon_0^{(k-1)}$ and multiplicity of $\varepsilon_0^{(k)}$ achieves a possible maximum.
iy) Vectors (3.22) with fixed k form a maximal chain of generalized eigenfunctions. Evidently, numbers $m_1, m_2, \dots, m_N$ do not depend on a choice of canonical system.

The number $N(\lambda_j) = m_1 + 1 + m_2 + 1 + \dots + m_N + 1$ is called multiplicity of the eigenvalue λ_j. Evidently, this number is the dimension of the space generated by canonical system.

Let $\lambda_j \in \mathbb{C}$ be the finite set of eigenvalues for operator (3.20) satisfying

$$\text{Re } \lambda_j < \sigma \tag{3.23}$$

where σ is the magnitude from (3.15), and

$$\varepsilon_0^{(k)}(\lambda_j; x), \varepsilon_1^{(k)}(\lambda_j; x), \dots, \varepsilon_{m_k}^{(k)}(\lambda_j; x) \quad (k = 1, 2, \dots, N(\lambda_j)) \tag{3.24}$$

be canonical system corresponding to λ_j. Evidently the union of all these canonical systems over λ_j satisfying (3.23) forms a basis in the subspace $X_\sigma^+(A^*)$. This basis satisfies the following property:

Theorem 3.4. *Let (3.24) be the set of canonical systems with eigenvalues λ_j of operator A^* running trough the set (3.23). Then for each open subset $\omega \Subset G$ restriction of all functions from indicated canonical systems forms linear independent family of functions.*

Remark 1. As it will be clear below, the property formulated in Theorem 3.4 is the key to construct stabilization of Oseen equation.

Proof. Step 1. First the following unique continuation property of operator $(A^* - \lambda_j I)$ should be established: Let a solution $f(x)$ of the problem

$$(A^* - \lambda_j I)f(x) = 0, \quad x \in G; \qquad f|_{\partial\Omega} = 0 \tag{3.25}$$

equals to zero on an open subset $\omega \Subset G: \quad f|_\omega \equiv 0$. Then $f(x) = 0 \ \forall x \in G$. The proof of this assertion had been obtained in [10], [11] with help of Carleman estimates.

Step 2. Show first that eigenfunctions $\varepsilon_0^{(k)}(\lambda_j;x)|_\omega$ with a fixed λ_j, restricted on ω are linear independent. Let

$$f(x) \equiv \sum_k c_k \varepsilon_0^{(k)}(\lambda_j;x) = 0 \quad \text{for} \quad x \in \omega, \tag{3.26}$$

where c_k are complex coefficients. By definition of eigenfunction the function $f(x)$ defined in (3.26) satisfies (3.25). By unique continuation property for equation (3.25) equality (3.26) implies that $f(x) \equiv 0$ for $x \in G$. Therefore in virtue of linear independence $\varepsilon^{(k)}(\lambda_j;x)$ on G we get that $c_k = 0\ \forall k$.

Step 3. Let show that eigenfunctions and generalized eigenfunctions $\varepsilon_j^{(k)}(\lambda_j;x)|_\omega$ with a fixed λ_j, restricted on ω, are linear independent. Let

$$\sum_{k=1}^{N(\lambda_j)} \sum_{i=0}^{m_k} d_i^{(k)} \varepsilon_i^{(k)}(\lambda_j x) = 0 \quad \text{for} \quad x \in \omega, \tag{3.27}$$

and eigenfunctions $\varepsilon_0^{(k)},\ k = 1, ..., k_1$ have maximal multiplicity $n_1 + 1 : m_1 = ... = m_{k_1} = n_1$, functions $\varepsilon_0^{(k)},\ k = k_1+1, k_1+2, ..., k_2$ have maximal multiplicity among the rest functions $1 + n_2 : m_{k_1+1} = ... = m_{k_2} = n_2$ and so on. Applying to (3.27) operator $(A^* - \lambda_j I)^{n_1}$ we get by (3.21) [4] that

$$\sum_{k=1}^{k_1} d_{n_1}^{(k)} \varepsilon_0^{(k)}(\lambda_j;x) = 0, \quad \text{for} \quad x \in \omega.$$

Therefore, by Step 2 of this Theorem proof $d_{n_1}^{(k)} = 0, k = 1, ..., k_1$. We continuer this process applying to rest part of (3.27) operator $(A^* - \lambda_j I)^m$ with corresponding m and conclude that new portion of $d_m^{(k)}$ equals to zero. After finite number of such steps we prove the assertion.

Step 4. The general case we prove by induction with respect to number M of eigenvalues satisfying (3.23) which we take into consideration. The case $M = 1$ has been proved in Step 3. Assume that we proved assertion when the number of choosed eigenvalues equals $M - 1$, and prove it for the case M. We numerate choosed eigenvalues in the order of decreasing of maximal possible multiplicity for corresponding eigenfunctions: if $m_1(\lambda_j)$ is maximal possible multiplicity for eigenfunctions corresponding to the eigenvalue λ_j, then

$$m_1(\lambda_1) \geqslant m_1(\lambda_2) \geqslant ... \geqslant m_1(\lambda_M).$$

Suppose that

$$\sum_{j=1}^{M} \sum_{k=1}^{N(\lambda_j)} \sum_{i=0}^{m_k(\lambda_j)} d_i^{(k)}(\lambda_j) \varepsilon_i^{(k)}(\lambda_j;x) = 0 \qquad \text{for} \quad x \in \omega. \tag{3.28}$$

We apply to (3.28) the operator $(A^* - \lambda_1 I)^{m_1(\lambda_1)+1}$. Then by (3.21) with λ_1 instead of λ_j we get that summand with $j = 1$ disappears and the rest members transform to

$$\sum_{j=2}^{M} \sum_{k=1}^{N(\lambda_j)} \sum_{i=0}^{m_k(\lambda_j)} d_i^{(k)}(\lambda_j)((\lambda_j - \lambda_1)I + (A^* - \lambda_j I))^{m_1(\lambda_1)+1} \varepsilon_i^{(k)}(\lambda_j;x) = 0 \tag{3.29}$$

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[4] In virtue of (2.8) smoothness of eigenfunctions and generalized eigenfunctions is finite ($\varepsilon_r^{(k)} \in V^2(G)$). Nevertheless application to them of operator $(A^* - \lambda_j I)^{n_1}$ is correctly defined in virtue of relations (3.21).

Applying to (3.29) binomial formula and taking into account (3.21) we get:

$$\sum_{j=2}^{M}\sum_{k=1}^{N(\lambda_j)}\sum_{i=0}^{m_k(\lambda_j)} d_i^{(k)}(\lambda_j)\sum_{l=0}^{i}\binom{m_1(\lambda_1)+1}{l}(\lambda_j-\lambda_1)^{m_1(\lambda_1)+1-l}\varepsilon_{i-l}^{(k)}(\lambda_j;x)=0, \tag{3.30}$$

where $\binom{n}{p}=\frac{n!}{p!(n-p)!}$ are binomial coefficients. Making change of variables $i-l=r$ in inner sum, and then changing the order of summation on i and r we obtain:

$$\sum_{j=2}^{M}\sum_{k=1}^{N(\lambda_j)}\sum_{r=0}^{m_k(\lambda_j)}(\Lambda_1 d(\lambda_j))_r^{(k)}\varepsilon_r^{(k)}(\lambda_j;x)=0, \tag{3.31}$$

where

$$(\Lambda_1 d(\lambda_j))_r^{(k)}=\sum_{i=r}^{m_k(\lambda_j)}\binom{m_1(\lambda_1)+1}{i-r}d_i^{(k)}(\lambda_j)(\lambda_j-\lambda_1)^{m_1(\lambda_1)+1+r-i},\ j\geqslant 2. \tag{3.32}$$

By induction assumption we have $(\Lambda_1 d(\lambda_j))_r^{(k)}=0$. Applying to (3.28) the operator $(A^*-\lambda_1 I)^{m_1(\lambda_1)+p}$ with $p=2,3,\dots,m_k(\lambda_j)-r+1$ and repeating aforementioned arguments we get the following system of equations

$$\sum_{i=r}^{m_k(\lambda_j)}\binom{m_1(\lambda_1)+p}{i-r}d_i^{(k)}(\lambda_j)(\lambda_j-\lambda_1)^{m_1(\lambda_1)+p+r-i}=0,\quad p=1,2,\dots,m_k(\lambda_j)-r+1 \tag{3.33}$$

To prove desired equalities $d_i^{(k)}(\lambda_j)=0$ we have to prove that determinant of the matrix corresponding to system (3.33) is not equal to zero.

Step 5. Let prove this assertion. In this determinant we take out common multiplier $(\lambda_j-\lambda_1)^p$ of p-th line ($p=1,2,\dots$) and after that take out common multiplier $(\lambda_j-\lambda_1)^{m_1(\lambda_j)+r-i}$ of i-th column ($i=1,2,\dots$). Then it is enough to prove that

$$\det\left\|\binom{m_1(\lambda_j)+p}{i-r}\right\|_{p,i-r+1=1}^{m_k(\lambda_j)-r+1}\neq 0,\quad r=0,1,\dots,m_k(\lambda_j) \tag{3.34}$$

Renaming $m_1(\lambda_j)+1=m,\ m_k(\lambda_j)-r=n,\ i-r=q$ we can rewrite (3.34) as follows:

$$\det\left\|\binom{m+p}{q}\right\|_{p,q=0}^{n}\neq 0,\quad n=0,1,\dots,m_k(\lambda_j) \tag{3.35}$$

Using formula $\binom{m+p}{q}=\frac{((m+p)!}{q!(m+p-q)!}$ and taking out common multiplier $q!$ of q-th column ($q=2,3,\dots$) we get that (3.35) is equivalent to the relation

$$\det\left\|\frac{(m+p)!}{(m+p-q)!}\right\|_{p,q=0}^{n}\neq 0,\quad n=0,1,\dots,m_k(\lambda_j) \tag{3.36}$$

We multiply q-th column of determinant from (3.36) on $m-q$ and subtract it from $(q+1)$-th column for $q=0,1,\dots$. After that we decompose obtained determinant with respect to upper line (it has 1 on the first place and 0 on others ones), and take out common multiplear p of p-th line ($p=1,2,\dots$). As a result we obtain

$$\det\left\|\frac{(m+p)!}{(m+p-q)!}\right\|_{p,q=0}^{n}=n!\det\left\|\frac{(m+p)!}{(m+p-q)!}\right\|_{p,q=0}^{n-1},\quad n=1,2,\dots,m_k(\lambda_j) \tag{3.37}$$

Repeating these operations we step by step will decrease the size of determinant from $n \times n$ until 1×1. As a result we prove relation (3.34)

We apply reasons written after (3.32) until (3.37) to each coefficients $(\Lambda_1 d(\lambda_j))_r^{(k)}$ from (3.31). As a result we obtain that $d_i^{(k)}(\lambda_j) = 0$ for each k, i and $j \geqslant 2$. Similar equality with $j = 1$ follows from Step 3. □

Remark 2. Steps 2-5 of Theorem 3.4 are simplification of complete version of the proof of Theorem 4.1 from [7]

Remark 3. The most deep and untrivial part of the proof of Theorem 3.4 is the Step 1. It based on Carleman estimates (see [10], [11]). Note that just here deep connection between feedback stabilization and exact controllability problems becomes apparent. Solutions of both these problems are based on Carleman estimates but for exact controllability problem more hard evolution version of Carleman estimate is used (see [16],[17]), whereas for stabilization problem it is quite enough to use Carleman estimates for steady-state equation ([10],[11]). Recall that from the point of view of ill-posed problems theory feedback stabilization can be considered as regularization of exact controllability problem.

Note that the basis $\{\varepsilon_r^{(k)}(\lambda_j; x)\}$ studied in Theorem 3.4 consists of complex valued functions. Then the set $\{\mathrm{Re}\, \varepsilon_r^{(k)}(\lambda_j; x), \mathrm{Im}\, \varepsilon_r^{(k)}(\lambda_j; x)\}$ will form a basis in the space of real valued functions $X_\sigma^+(A^*)$ and it will possess the property proved in Theorem 3.4 (see details in [7]). Let rename this real valued basis as $(d_1(x), \dots, d_K(x))$. So the following corollary of Theorem 3.4 is true:

Lemma 3.5. *There exists a basis $(d_1(x), \dots, d_K(x))$ in the space $X_\sigma^+(A^*)$ such that restriction $(d_1(x)|_\omega, \dots, d_K(x)|_\omega)$ on an arbitrary subdomain $\omega \Subset G$ forms a linear independent set of vector fields.*

It is known that one can choose the basis $(e_1(x), \dots, e_K(x))$ in the space $X_\sigma^+(A)$ that satisfies together with the basis $(d_1(x), \dots, d_K(x))$ in $X_\sigma^+(A^*)$ the following relation:

$$(e_j, d_k)_{V^0(G)} = \delta_{j,k} \qquad \forall\, j, k = 1, \dots, K \tag{3.38}$$

where $\delta_{j,k}$ is Kronecker symbol.

3.4. **Theorem on stabilization of Oseen equations.** We consider stabilization problem in the phase space $V = V^s,\ s \in [0,1]$, i.e. the problem to find a control

$$u \in V_{00}^1(\omega) := \{w \in V_0^1(G) : w(x) = 0\ \forall x \in G \setminus \omega)\} \tag{3.39}$$

such that the solution y of (3.4),(3.3) with $y_0 \in V^s$ satisfies (3.18). To complete the construction of stabilization for Oseen equations (3.4), (3.3) we have to construct the operator $E : V^s \to V_{00}^1(\omega)$ that transforms arbitrary initial condition y_0 from (3.3) to control u such that $y_0 + u \in V_-^s$. We consider here analog of construction from [11].

Using (3.38) we can define space (3.17) by the following equivalent form:

$$V_-^s = \{v(x) \in V^s\ :\ \int_G v(x) \cdot d_j(x)\, dx = 0, \quad j = 1, \dots, K\}. \tag{3.40}$$

Theorem 3.6. *([10], [11]) For each $s \in [0,1]$ there exists a linear bounded operator*

$$E : V^s \to V_{00}^1(\omega) \quad \text{such that} \quad y_0 + Ey_0 \in V_-^s. \tag{3.41}$$

Proof. Let subset $\omega_1 \subset \omega$ be a domain with C^∞- boundary $\partial\omega_1$. In this set we consider the Stokes problem:

$$-\Delta w(x) + \nabla p(x) = v(x), \quad \mathrm{div} w(x) = 0, \quad x \in \omega_1; \quad w|_{\partial\omega_1} = 0$$

As is well known, for each $v \in V^0(\omega_1)$[5] there exists the unique solution $w \in V_0^1(\omega_1)$ of this problem. The resolving operator to this problem we denote as $(-\hat{\pi}\Delta)_{\omega_1}^{-1} v = w$. Extension of $(-\hat{\pi}\Delta)_{\omega_1}^{-1} v$ from ω_1 in G by zero we also denote as $(-\hat{\pi}\Delta)_{\omega_1}^{-1} v$. Evidently, $(-\hat{\pi}\Delta)_{\omega_1}^{-1} v \in V_{00}^1(\omega_1)$.

We look for the desired operator E in the form

$$Ev(x) = \left[\sum_{j=1}^{K} c_j (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j \right](x), \tag{3.42}$$

where $c_j = c_j(v)$ are constants which should be determined. Since $d_j \in V_0^0(G)$, $Ev \in V_0^1(G)$, $\mathrm{supp}\, Ev \subset \overline{\omega}_1$ for every $v \in V^s$ and for each $s \in [0,1]$. To define constants c_j we note that by (3.40) $v + Ev \in V_-^s$ if

$$\int\limits_G d_k(x) \left[\sum_{j=1}^{K} c_j (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x) \right] dx = -\int\limits_G d_k(x) v(x))\, dx \tag{3.43}$$

for $k = 1, \ldots, K$. Lemma 3.7 (see below) implies that this system of linear equations has a unique solution. $\square$

Lemma 3.7. *The matrix $M = \{m_{kj}\}_{k,j=1}^K$ where*

$$m_{kj} = \int\limits_G d_k(x) (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x)\, dx \tag{3.44}$$

is positive defined.

Proof. Using notation $\tilde{d}_j(x) = (-\widehat{\pi}\Delta)_{\omega_1}^{-1} d_j(x)$ we get from (3.44) that

$$m_{kj} = \int\limits_G (-\widehat{\pi}\Delta)_{\omega_1} \tilde{d}_k(x) \cdot \tilde{d}_j(x)\, dx = \int\limits_{\omega_1} \nabla \tilde{d}_k(x) \cdot \nabla \tilde{d}_j(x)\, dx$$

Then for each $\overline{c} = (c_1, \ldots, c_K)$ we get using notation

$$\widehat{d}(x) = \sum_{j=1}^{K} = c_j \tilde{d}_j(x)$$

the following relations:

$$(M\overline{c}, \overline{c}) = \sum_{k,j=1}^{K} m_{k,j} c_k c_j = \sum_{k,j=1}^{K} c_k c_j \int\limits_{\omega_1} \nabla \tilde{d}_k(x) \cdot \nabla \tilde{d}_j(x)\, dx = \int_{\omega_1} |\nabla \widehat{d}(x)|^2\, dx \geqslant 0$$

Note that the last inequality transforms to equality only if $\overline{c} = 0$. This follows from Lemma 3.5. Therefore matrix M is positive defined, i.e.

$$(M\overline{c}, \overline{c}) \geqslant \alpha \|\overline{c}\|^2 \quad \forall \overline{c} = (c_1, \ldots, c_K)$$

$\square$

Theorems 3.2, 3.6 imply the final result on stabilization of problem (3.4),(3.3):

[5] For definition of this space see (2.4),(2.5) where G is changed on ω_1.

Theorem 3.8. *Let $s \in [0,1]$ and $u = Ey_0$ in (3.3) where E is operator constructed in Theorem 3.6. Then solution $y(t,x)$ of closed-loop problem (3.4),(3.3) satisfies inequality*

$$\|y(t,\cdot)\|_{V^s(G)} \leqslant c\|y_0\|_{V^s(G)}e^{-\sigma t} \tag{3.45}$$

where constant $c > 0$ does not depend on initial condition $y_0 \in V^s(G)$.

Proof. In virtue of Theorem 3.6 and inequality (3.18) proved in Theorem 3.2 solution $y(t,x)$ of problem (3.4),(3.3) with $u = Ey_0$ satisfies inequality

$$\|y(t,\cdot)\|_{V^s(G)} \leqslant c_1\|y_0 + Ey_0\|_{V^s(G)}e^{-\sigma t} \leqslant c\|y_0\|_{V^s(G)}e^{-\sigma t}$$

□

Remark 4. Note that although a function $v \in V^s(G)$ is equal to zero on ∂G only if $s > 1/2$,[6] the solution y from Theorem 3.8 satisfies $y(t,\cdot)|_{\partial G} = 0$ for almost every $t \in [0,T]$. indeed, by [1], [25] the solution of problem (3.4),(3.3) with initial condition $y_0 + Ey_0 \in V^s(G),\ s \in [0,1]$ belongs to the space $V^{1,2(s-1)(Q_T)}$ where

$$V^{1,2(-\alpha)}(Q_T) = L_2(0,T;V^{2-\alpha}(G)) \cap H^1(0,T;V^{-\alpha}(G)) \quad \alpha \in [0,1]. \tag{3.46}$$

4. **Stabilization for Navier-Stokes equations by start control.** In this section we give a construction for stabilization of problem (3.2),(3.3) obtained from Navier-Stokes system (2.1) by change of unknown function.

4.1. **Definition of stable invariant manifold.** Applying orthoprojector $\hat{\pi}$ defined in (3.7) to both parts of the first equation from (3.2) we get equivalent equation

$$\frac{dy(t,\cdot)}{dt} + Ay(t,\cdot) + B(y(t,\cdot)) = 0 \tag{4.1}$$

where A is operator (3.8), and B is the operator

$$B(y) = \hat{\pi}[(y,\nabla)y] \tag{4.2}$$

Natural space for solution of problem (4.1), (3.5) is

$$V^{1,2(0)}(Q_T) = L_2(0,T;V^2(G) \cap V_0^1(G)) \cap H^1(0,T;V_0^0(G)),$$

and in virtue of inclusion $C(0,T;V_0^1(G)) \subset V^{1,2}(Q_T)$ natural phase space V for corresponding dynamical system is $V_0^1(G)$. It is well-known (see [23], [28]), that for each $y_0 \in V$ there exists a unique solution $y(t,x) \in V^{1,2(0)}(Q_{T_{\|v_0\|}})$ of problem (4.1),(3.5), where $0 < T_{\|v_0\|} \to \infty$ as $\|v_0\| := \|v_0\|_V \to 0$. Denote by $S(t,y_0)$ the solution operator of the boundary value problem (4.1),(3.5):

$$S(t,y_0) = y(t,\cdot) \tag{4.3}$$

where $y(t,x)$ is the solution of (4.1),(3.5).

Definition of spaces given around (3.16) and relations for them imply:

$$V = V_+ + V_- \quad \text{where} \quad V = V_0^1(G),\ V_+ = X_\sigma^+(A),\ V_- = X_\sigma \cap V_0^1(G) \tag{4.4}$$

Definition 4.1. The set $\mathcal{W}_- = \mathcal{W}_-(\mathcal{O})$ defined in a neighborhood $\mathcal{O}$ of origin is called a stable invariant manifold of the dynamical system generated by problem (4.1),(3.5) if for each $y_0 \in \mathcal{W}_-$ the solution $S(t,y_0)$ is well-defined and belongs to $\mathcal{W}_-$ for each $t > 0$, and

$$\|S(t,y_0)\|_V \leqslant c\|y_0\|_V e^{-\sigma t}, \quad t \geqslant 0 \tag{4.5}$$

where quantities $c > 0, \sigma > 0$ does not depend on $y_0 \in \mathcal{O}$.

[6]This follows from definition (3.11),(3.12) of the space $V^s(G)$ (see [25])

It is clear from this definition that solution of stabilization problem (3.2), (3.3) can be reduced to projection on $\mathcal{W}_-(\mathcal{O})$. To construct this projection one can use that in a neighborhood $\mathcal{O}$ the stable invariant manifold can be defined as a graph in the phase space $V = V_+ + V_-$ by the formula

$$\mathcal{W}_- \equiv \mathcal{W}_-(\mathcal{O}) \equiv \mathcal{W}_-(\mathcal{O}(V_-), f) := \{y \in V : y = y_- + f(y_-),\ y_- \in \mathcal{O}(V_-)\} \quad (4.6)$$

where $\mathcal{O}(V_-)$ is a neighborhood of the origin in the subspace V_-, and

$$f : \mathcal{O}(V_-) \to V_+ \quad (4.7)$$

is a certain map satisfying

$$\|f(y_-)\|_{V_+}/\|y_-\|_{V_-} \to 0 \quad \text{as} \quad \|y_-\|_{V_-} \to 0. \quad (4.8)$$

Existence of such map f when its domain of definition $\mathcal{O}(V_-)$ is small ball in V_- is well-known (see [24], [26], [20], [2] and references there in).

It has been proved recently that actually domain $\mathcal{O}(V_-)$ of f is unbounded. Let formulate this result. Consider the set

$$El_\rho = \{v = \sum_{j=1}^{\infty} v_j \widehat{e}_j(x) \in V^1 : \sum_{j=1}^{\infty} \widehat{\lambda}_j^{1/2} v_j^2 < \rho\} \quad (4.9)$$

where $\rho > 0$. Since by definition (3.12) of V^s, $\|v_j \widehat{e}_j\|^2_{V^1} = \widehat{\lambda}_j v_j^2$, (4.9) can be rewritten as follows

$$El_\rho = \{v \in V^1 : \sum_{j=1}^{\infty} \|v_j \widehat{e}_j\|^2_{V^1} / (\widehat{\lambda}_j^{1/2} \rho) < 1\} \quad (4.10)$$

Therefore El_ρ^α is ellipsoid in V^1 with axises of length $\sqrt{\widehat{\lambda}_j^\alpha \rho}$ directed along $\widehat{e}_j$. Since $\sqrt{\widehat{\lambda}^\alpha \rho} \to \infty$ as $j \to \infty$, El_ρ^α is an unbounded set.

The following existence theorem for invariant manifold $\mathcal{W}_-$ holds (see [15]):

Theorem 4.2. *Let $G \subset \mathbb{R}^d, d = 2, 3$ be a bounded domain with C^∞-boundary $\partial\Omega$, steady-state solution $\widehat{v}$ from (3.8) satisfies (2.8). Then there exists unique map (4.7) defined on domain $\mathcal{O}(V_-) = El_\rho \cap V_-$ with sufficiently small ρ such that the set $\mathcal{W}_-$ defined by formula (4.6) is stable invariant manifold for family of maps $S(t, \cdot)$ defined in (4.3). Moreover,*

$$\|S(t, y_0)\|_{V^1} \leqslant ce^{-\sigma t} \|y_0\|_{V^1} \qquad \text{as} \quad t \to \infty \quad (4.11)$$

where constants $c > 0, \sigma > 0$ do not depend on $y_0 \in \mathcal{W}_-$

Note that the set (4.9) is intersection of V^1 and the ball in $V^{1/2}$ of radius ρ with the center in origin. Straightforward repeating of proof from [24], [26], [20], [2] gives Theorem 4.2 with analog of bound (4.11) where instead of norms in V^1 norms in the space $V^{1/2}$ are used. Actually in [15] it has been proved that in this situation estimate (4.11) with norms in V^1 also holds.

4.2. **Feedback operator and stabilization.** Here we construct feedback operator for Navier—Stokes equations. This operator is nonlinear analog of feedback operator (3.41) constructed for Oseen equations.

As in Theorem 3.6 we use the domain $\omega \Subset G$ and the space $V^1_{00}(\omega)$ defined in (3.39). Denote $\mathcal{O}_\varepsilon = \{v \in V : \|v\|_{V^{1/2}} < \varepsilon\} \equiv El_\varepsilon$.

Theorem 4.3. *Suppose that $\mathcal{W}_-$ is the invariant manifold constructed in a neighborhood of origin in $V = V_0^1(G)$ in Theorem 4.2. Then for sufficiently small ε there exists a continuous operator*

$$F : El_\varepsilon \to V_{00}^1(\omega), \tag{4.12}$$

such that

$$v + F(v) \in \mathcal{W}_- \quad \forall v \in El_\varepsilon. \tag{4.13}$$

Proof. We introduce projection operators

$$P_+ : V \to V_+, \qquad P_- : V \to V_- \tag{4.14}$$

for the spaces defined in (4.4) by the formulas

$$P_+ v = \sum_{j=1}^{K} (v, d_j)_{V^0(G)} e_j, \qquad P_- v = v - P_+ v \tag{4.15}$$

where basises $\{e_j\}, \{d_j\}$ are defined correspondingly after and before formulation of Lemma 3.5. Introduce also the following notations:

$$Qv(x) = v(x) + w(x), \quad \text{where} \quad w = F(v) \in V_{00}^1(\omega), \tag{4.16}$$

and F is the operator we are looking for. By (4.14) and definition (4.6) of invariant manifold $\mathcal{W}_-$ the desired inclusion $Qv \in \mathcal{W}_-$ is equivalent to the following equality:

$$P_+ Qv = f(P_- Qv) \tag{4.17}$$

where f is operator (4.7). Besides, we have to ensure that the equality

$$(Qv)(x) \equiv v(x), \quad x \in G \setminus \omega \tag{4.18}$$

is true. By (4.15) basis $\{e_j(x)\}$ generates V_+ and therefore the map $f(u)$ can be written in the form

$$f(u) = \sum_{j=1}^{K} e_j f_j(u) \qquad \text{where} \quad f_j(u) = (f, d_j)_{V^0(G)}$$

and equality (4.17) is equivalent to the following one:

$$\int_G Qv(x) d_j(x)\, dx = f_j(P_- Qz), \quad j = 1, \ldots, K. \tag{4.19}$$

Similarly to (3.42) we look for the vector field $w(x)$ from (4.16) in the form

$$w = -(-\hat{\pi}\Delta)_{\omega_1}^{-1} \sum_{j=1}^{K} p_j d_j \tag{4.20}$$

To find coefficients $(p_1, \ldots, p_K) \equiv \vec{p}$ we substitute (4.20) into (4.16) taking into account (4.19). As a result we get

$$\vec{v} - M\vec{p} = \vec{f}\big(v - (\vec{p}, (-\hat{\pi}\Delta)_{\omega_1}^{-1}\vec{d}) - (\vec{e}, \vec{v} - M\vec{p})\big), \tag{4.21}$$

where $\vec{v} = (v_1, \ldots, v_K)$, $M = \|m_{jk}\|$ and

$$v_j = \int_G (v(x), d_j(x))\, dx, \quad m_{jk} = \int_G ((-\hat{\pi}\Delta)_{\omega_1}^{-1} d_k(x), d_j(x))\, dx,$$

$$\vec{f}(u) = (f_1(u), \ldots, f_K(u)), \quad \vec{e} = (e_1(x), \ldots, e_K(x)), \quad \vec{d} = (d_1(x), \ldots, d_K(x)),$$

$$(\vec{c}, \vec{d}) = \sum_{j=1}^{K} c_j d_j.$$

In order right side of (4.21) to be well-defined we have to assume in virtue of Theorem 4.2 that

$$v - (\vec{p}, (-\hat{\pi}\Delta)^{-1}_{\omega_1}\vec{d}) \in El_\rho \tag{4.22}$$

where the magnitude ρ is defined in the mentioned Theorem. Taking into account invertibility of matrix $M = \|m_{jk}\|$ ascertained in Lemma's 3.7 proof one can apply to relations (4.21),(4.22) contraction mapping principle (see for details e.g. [11]). As a result we obtain that if $\|\vec{v}\|$ is sufficiently small, equation (4.21) possesses unique solution $\vec{p}$. The last assumption is fulfilled because ε in (4.12) is small enough.

□

Now in virtue of Theorem 4.3 for stabilization of problem (3.2),(3.3) one has to take $u = F(y_0)$.

5. **Stabilization by impulse control.** In this section we construct feedback impulse control for stabilization of Oseen equations. We begin from the motivations.

5.1. **Motivations and setting of the problem.** Stabilization construction by start control described above was implicitly used in [7]-[11] (in the case $\mathcal{O} = B\rho(V_0^1(\Omega))$) for stabilization by control on boundary. That method was used in [22] for numerical stabilization of Taylor curls in Cuette problem. Since start control is supported in small subdomain, it generated big gradients, almost singularities, in boundary control near $t = 0$. It was clear that this singularity should prevent to use the stabilization technique for bigger Reynolds numbers. To overcome this difficulty it is possible to impose restriction on magnitude of the start control norm using such restricted start control in several time moments. Below we show that this algorithm leads to stabilization by impulse control. The corresponding control problem can be written as follows:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = h(x) + \sum_{j=0}^{N} \delta(t-t_j)u_j(x), \quad \operatorname{div} v(t,x) = 0 \tag{5.1}$$

$$v(t,x)|_{t=0} = v_0(x). \tag{5.2}$$

where $v(t,x)$ additionally satisfies boundary condition (2.2), $\delta(t - t_j)$ is Dirac δ-function supported in t_j, $0 = t_0 < t_1 < \cdots < t_j < \cdots < t_N$ with $N = N(v_0)$, $u_j(x) = (u_{j1}, \ldots, u_{jd})$ is a solenoidal vector field supported in a given fixed subdomain $\omega \subset G$. The following restriction is imposed on u_j:

$$\|u_j\|_{V_0^0(G)} \leqslant \gamma, \quad \forall j \tag{5.3}$$

where $\gamma > 0$ is a given fixed magnitude.

There is a close connection between impulse control and start control. indeed, the following assertion is true:

Lemma 5.1. *Stabilization problem (2.1),(2.2),(2.3)with a start control is equivalent to stabilization problem (5.1),(5.2),(2.2) with impulse control satisfying condition*

$$u_j(x) \equiv 0 \quad \forall\, j \geqslant 1,\ \gamma = \infty.[7] \tag{5.4}$$

[7] I.e. when we have only one impulse at $t = 0$, and there is no any restriction on control u_0.

Proof. Recall, that generalized solution of problem (2.1),(2.2),(2.3) with given initial condition $v_0 + u \in V_0^1(G)$ and right side $h \in L_2(G)^d$ is the vector field $v \in L_\infty(0,T;V_0^0(G)) \cap L_2(0,T;V_0^1(G))$ satisfying

$$\begin{aligned} - \quad & \int_0^T \int_G [(v(t,x), \partial_t \varphi(t,x)) + \textstyle\sum_{j=1}^d (\partial_{x_j} v, \partial_{x_j}\varphi) + (v, \sum_{j=1}^d v_j \partial_{x_j}\varphi)]\, dxdt \\ + \quad & \int_G (v(T,x), \varphi(T,x))\, dx = \int_0^T \int_G (h(x), \varphi(t,x))\, dxdt + \int_G (v_0 + u, \varphi(0,\cdot))\, dx \\ & \forall \varphi(t,x) \in C^1(0,T;\mathcal{V}(G)) \end{aligned} \tag{5.5}$$

To get (5.5) we multiply scalarly (2.1) on φ and integrate by parts using (2.3).

If we multiply scalarly on φ equation (5.1) that satisfies (5.4) and integrate by parts using (5.2) we obtain (5.5) as well. Therefore vector field $v(t,x) \in L_\infty(0,T;V_0^0(G)) \cap L_2(0,T;V_0^1(G))$ that satisfies equality (5.5) is generalized solution of boundary value problem (2.1),(2.2),(2.3) with given $v_0 + u \in V_0^1(G)$, and $h \in L_2(G)^d$ as well as of boundary value problem (5.1)-(5.2),(5.4) with given $v_0, u \in V_0^1(G)$, and $h \in L_2(G)^d$. Note that in virtue of (2.9) our generalized solution satisfies inclusion $v \in L_\infty(0,T;V_0^1(G))$. As well-known (see [23],[28]) the last inclusion guarantees its uniqueness. That is why the boundary value problems (2.1),(2.2),(2.3) and (5.1)-(5.2),(5.4) with given $v_0, u \in V_0^1(G)$, and $h \in L_2(G)^d$ are equivalent. Thus, corresponding control problems are equivalent as well. $\square$

For briefness we study below stabilization by impulse control only for Oseen equations although considering the case of Navier-Stokes equations is also possible. The point is that one can overcome difficulties from [22] mentioned above with help of distributed control as well, and we realize this kind of control in nonlinear case of Navier-Stokes equations in the next section. Actually, impulse control is intermediate between start and distributed controls. Note also that the stabilization of 2D Navier-Stokes equations by impulse control will be considered below in Section 7 in the case of Cauchy problem (i.e. in the case of unbounded domain $G = \mathbb{R}^2$).

5.2. **The case of Oseen equations.** Let consider the following stabilization problem:

$$\frac{dy(t,\cdot)}{dt} + Ay(t,\cdot) = \sum_{j=0}^{N} \delta(t - t_j) u_j(x), \quad y|_{t=0} = y_0. \tag{5.6}$$

$$u_j(x) \in V_{00}^1(\omega), \qquad \|u_j\|_{V_0^1(G)} \leqslant \gamma \qquad \forall j \tag{5.7}$$

where A is the operator (3.8), $y_0 \in V_0^1(G)$ is the given initial datum, $\omega \subset G$, and the constant $\gamma > 0$ is prescribed restriction.

We have to look for instants $\{t_j\}$ and controls u_j such that solution y of (5.6) satisfies inequality

$$\|y(t,\cdot)\|_{V_0^1(G)} \leqslant (c_1 \|y_0\|_{V_0^1(G)} + c_2) \min(1, e^{-\sigma(t-\widehat{t})}) \tag{5.8}$$

where $\widehat{t}$ is the instant when the solution $y(t,x)$ reaches the subspace V_-, and constants c_1, c_2 do not depend on $y_0 \in V_0^1(G)$.

Moreover, we look for feedback control that is defined as follows.

Definition 5.2. Control $(t_j, u_j, j = 1, 2, \dots)$ is called feedback if for every j

$$u_j = Ey(t_j, \cdot) \qquad \text{where} \quad E : V_0^1(G) \to V_{00}^1(\omega) \tag{5.9}$$

is a linear bounded operator that should be constructed in a such way that after substitution (5.9) into (5.6) solution y of (5.6) will satisfy inequality (5.8).

Taking into account close connection between start and impulse controls (see Lemma 5.1) we can take as E from (5.9) the feedback operator constructed in subsection 3.4 for the case of start control. Thus, the main problem now is to choose properly instants $\{t_j\}$. By reasons cited above and connected with big gradients it is useful to take distances $|t_{j+1}-t_j|$ as big as possible. But, obviously, it is impossible to make this distances too big. indeed, the aim of stabilization is to push the solution y to the subspace V_-. In virtue of restriction (5.7) control u_j usually can only approach solution to V_- and in order not to give solution to move off from V_- far away the distance $|t_{j+1}-t_j|$ should not be too big. Let define $u_0 = Ey_0$ where E is operator (5.9) and y_0 is initial condition from (5.6). Our goal is to define by induction u_j, t_j, with proper distance $|t_j - t_{j-1}|$.

Denote

$$\|\cdot\| := \|\cdot\|_{V_0^1(G)} \tag{5.10}$$

where A is operator (3.8). Recall that we assume that the Condition 3.1 holds. Assume also that

$$\|Ey_0\| > \gamma \tag{5.11}$$

where y_0, γ are defined in (5.6),(5.7): otherwise to solve stabilization problem (5.6),(5.7) it is enough to use one impulse at $t=0$, i.e. to use start control.

To solve stabilization problem (5.6),(5.7) we have to push solution y to subspace V_-. We will do it with help of impulse control $\sum u_k\delta(t-t_k)$, and in virtue of restriction (5.7) we are forced to use the following modification of formula (3.41):

$$u_{k-1} = \frac{\gamma}{\|Ey_{k-1}\|} E y_{k-1} \tag{5.12}$$

where $y_{k-1} = y(t_{k-1}, \cdot)$ that only bring our solution nearer to V_-. Up to next impulse our solution is defined by the formula

$$y(t,\cdot) = e^{-A(t-t_{k-1})}\left(y_{k-1} + \frac{\gamma E y_{k-1}}{\|Ey_{k-1}\|}\}\right). \tag{5.13}$$

Let $t_0 = 0, y_0$ is initial condition from (5.6). Define by induction the sequence $(y_k, t_k), k = 1, 2, \dots$ with help of the formulas

$$y_k = e^{-A(t_k-t_{k-1})}\left(y_{k-1} + \frac{\gamma E y_{k-1}}{\|Ey_{k-1}\|}\}\right) \quad \text{if} \quad \|Ey_{k-1}\| > \gamma \tag{5.14}$$

where $t_k > t_{k-1}$ is minimal from magnitudes satisfying

$$\left(1 - \frac{\gamma}{\|EP_+y_{k-1}\|}\right) \|e^{-A(t_k-t_{k-1})}P_+y_{k-1}\| = \left(\|P_+y_{k-1}\| - \frac{\gamma}{2\|E\|}\right). \tag{5.15}$$

We stop this process when for some k

$$\|Ey_{k-1}\| \leqslant \gamma. \tag{5.16}$$

After that for stabilization of Oseen equation we use start control at instant $t = t_{k-1}$ constructed in Theorem 3.8.

Lemma 5.3. *Let (y_{k-1}, t_{k-1}) with $k > 1$ that does not satisfy (5.16) is constructed. Then there exists unique pair (y_k, t_k) that satisfies (5.14),(5.15). Moreover,*

$$\|P_+y_k\| = \|P_+y_{k-1}\| - \frac{\gamma}{2\|E\|}. \tag{5.17}$$

Therefore after several steps $k-1 \to k$ parameter k will satisfying (5.16).

Proof. In virtue of Condition 3.1 the set of values for the function

$$f(t) = \|e^{-A(t-t_{k-1})}P_+ y_{k-1}\|, \ t > t_{k-1}$$

contains the set $[\|P_+ y_{k-1}\|, \infty)$. Besides, evidently, the inequality

$$\left(1 - \frac{\gamma}{\|EP_+ y_{k-1}\|}\right)\|P_+ y_{k-1}\| < \left(\|P_+ y_{k-1}\| - \frac{\gamma}{2\|E\|}\right)$$

is true. Hence, there exists $t = t_k > t_{k-1}$ that satisfies equality (5.15). The minimal from all such solutions we denote by t_k.

Let prove that if y_k is defined by formula (5.14) then equality (5.17) holds. By definition (3.42) of operator E and by Theorem 3.6 we get: $\operatorname{Ker} E = V_-$, $y + Ey \in V_-$. Therefore by definition (4.15) of P_+ we obtain from (5.14):

$$\begin{aligned} P_+ y_k &= e^{-A(t_k - t_{k-1})}\left(P_+ y_{k-1} + \frac{\gamma}{\|Ey_{k-1}\|}P_+(Ey_{k-1} + y_{k-1}) - \frac{\gamma}{\|Ey_{k-1}\|}P_+ y_{k-1}\right) \\ &= \left(1 - \frac{\gamma}{\|EP_+ y_{k-1}\|}\right)e^{-A(t_k - t_{k-1})}P_+ y_{k-1} \end{aligned}$$

This relation and (5.15) imply the desired equality. □

Using (5.17) we can calculate the number m of impulses required for phase variable $y(t,\cdot)$ to reach subspace V_-. It can be defined from relations

$$\frac{\gamma m}{2\|E\|} < \|P_+ y_0\| \leqslant \frac{\gamma(m+1)}{2\|E\|}$$

that are equivalent to

$$\frac{2\|E\|\|P_+ y_0\|}{\gamma} - 1 \leqslant m < \frac{2\|E\|\|P_+ y_0\|}{\gamma} \tag{5.18}$$

In other words

$$m = \begin{cases} \left[\frac{2\|E\|\|P_+ y_0\|}{\gamma}\right], & \frac{2\|E\|\|P_+ y_0\|}{\gamma} \ \text{is not integer} \\ \left[\frac{2\|E\|\|P_+ y_0\|}{\gamma}\right] - 1, & \frac{2\|E\|\|P_+ y_0\|}{\gamma} \ \text{is integer} \end{cases} \tag{5.19}$$

where $[a]$ denotes integer part of a.

Let

$$\tau = \tau(y_0) := \min_{k=1,\dots,m}(t_k - t_{k-1}) \tag{5.20}$$

Lemma 5.4. *The following estimate holds:*

$$\|P_- y_k\| \leqslant ce^{-\sigma k\tau}\|P_- y_0\| + \frac{\gamma c\|P_-\|e^{-\sigma\tau}}{1 - e^{-\sigma\tau}} \tag{5.21}$$

Proof. We get from (5.14) that

$$\begin{aligned} y_k &= e^{-A(t_k - t_{k-1})}\left(e^{-A(t_{k-1} - t_{k-2})}\left(y_{k-2} + \frac{\gamma Ey_{k-2}}{\|Ey_{k-2}\|}\right) + \frac{\gamma Ey_{k-1}}{\|Ey_{k-1}\|}\right) \\ &= e^{-A(t_k - t_{k-2})}y_{k-2} + e^{-A(t_k - t_{k-1})}\frac{\gamma Ey_{k-1}}{\|Ey_{k-1}\|} + e^{-A(t_k - t_{k-2})}\frac{\gamma Ey_{k-2}}{\|Ey_{k-2}\|} = \dots \\ &= e^{-At_k}y_0 + \gamma\sum_{j=1}^{k} e^{-A(t_k - t_{k-j})}\frac{\gamma Ey_{k-j}}{\|Ey_{k-j}\|} \end{aligned}$$

This equalities imply the estimates

$$\begin{aligned} &\|P_- y_k\| \leqslant c e^{-\sigma t_k}\|P_- y_0\| + \gamma c\|P_-\| \sum_{j=1}^k e^{-\sigma(t_k - t_{k-j})} \\ &\leqslant c e^{-\sigma k\tau}\|P_- y_0\| + \gamma c\|P_-\| \tfrac{e^{\sigma\tau}}{1-e^{\sigma\tau}} \end{aligned} \tag{5.22}$$

□

Now we are ready to obtain the estimate for the solution stabilized by impulse control.

Theorem 5.5. *Let $y(t,\cdot)$ be the solution of the stabilization problem (5.6),(5.7) by the method descibed above. Then*

$$\|y(t,\cdot)\| \leqslant \begin{cases} \|P_+ y_0\| - \frac{k\gamma}{2\|E\|} + c e^{-\sigma(k-1)\tau}\|P_- y_0\| + \frac{c\gamma\|P_-\|e^{-\sigma\tau}}{1-e^{-\sigma\tau}}, \; t \in (t_{k-1}, t_k), \; k \leqslant m \\ c\left(e^{-\sigma m\tau}\|P_- y_0\| + \frac{c\gamma\|P_-\|e^{-\sigma\tau}}{1-e^{-\sigma\tau}}\right) e^{\sigma(t-t_m)}, \quad t > t_m \end{cases} \tag{5.23}$$

Proof. Evidently, $\|y(t,\cdot)\| \leqslant \|P_+ y(t)\| + \|P_- y(t)\|$. By definition (5.15) of t_k we get that $\|P_+ y(t)\| \leqslant \|P_+ y_{k-1}\| - \gamma/(2\|E\|)$ for $t \in (t_{k-1}, t_k)$. Besides, similarly to (5.22) we get for this t that

$$\|P_- y(t)\| \leqslant c e^{-\sigma(k-1)\tau}\|P_- y_0\| + \frac{c\gamma\|P_-\|e^{-\sigma\tau}}{1-e^{-\sigma\tau}}.$$

This proves (5.23) for $t \in (t_{k-1}, t_k)$. Inequality (5.23) for $t > t_m$ follows from Theorem 3.8 and estimate (5.22) with $k = m$. □

6. Stabilization by distributed control in right-hand-side supported in subdomain.

6.1. The case of controlled Oseen equations.

Let us consider the boundary value problem

$$\frac{v(t,\cdot)}{dt} + Av(t,\cdot) = u(t,\cdot), \quad v|_{t=0} = v_0 \tag{6.1}$$

with $v_0 \in V_0^1(G)$, operator A from (3.8), control $u(t,x) \in L_2(\mathbb{R}_+; V_{00}^1(\omega))$ we are looking for. This control should satisfy the following conditions:

i) The solution of problem (6.1) satisfies the estimate

$$\|v(t,\cdot)\|_{V_0^1(G)} \leqslant C\|v_0\|_{V_0^1(G)} e^{-\sigma t} \tag{6.2}$$

where constant $C = C_\sigma$ does not depend on $\|v_0\|_{V_0^1(G)}$.
Moreover we are looking for the feedback control. By the definition it means that

ii) There exists a linear bounded operator: $\widehat{E} : V_0^1(G) \to V_{00}^1(\omega)$ such that control $u(t,\cdot)$ is expressed by phase function $y(t,\cdot)$ with help of the formula

$$u(t,\cdot) = \widehat{E} v(t,\cdot) \tag{6.3}$$

Theorem 6.1. *There exists a control $u(t,x) \in L_2(\mathbb{R}_+; V_{00}^1(\omega))$ that satisfies conditions $i), ii)$ written above. Moreover, operator $\widehat{E}$ from (6.3) is defined by the formula*

$$\widehat{E} = -\Lambda E \tag{6.4}$$

where $\Lambda > 0$ is a sufficiently large magnitude, and $E : V_0^1(G) \to V_{00}^1(\omega)$ is feedback operator defined in the proof of of theorem 3.6 (see (3.42), (3.43)).

Proof. Recall that phase space $V = V_0^1(G)$ admits decomposition (4.4), in $V_+ = X_\sigma^+(A)$ one can choose a basis $(e_1(x), \dots, e_k(x))$ constructed from eigenfunctions and generalized eigenfunctions of operator A corresponding to eigenvalues λ_j with $\text{Re}\,\lambda_j < \sigma$, (see [7],[8]). Besides, in $X_\sigma^+(A^*)$ one can choose a basis $(d_1(x), \dots, d_k(x))$ constructed from eigenfunctions and generalized eigenfunctions of operator A^* corresponding to eigenvalues μ_j with $\text{Re}\,\mu_j < \sigma$, ([7], [8]). These basises are biorthogonal, i.e. they satisfy: $(e_j, d_m)_{L_2(G)} = \delta_{jm}$, where δ_{jm} is Kronecker symbol. Therefore $v \in V_+$ if and only if

$$v = \sum_{j=1}^{k} v_j e_j(x), \quad \text{where} \quad v_j = (v, d_j)_{L_2(\Omega)} \tag{6.5}$$

We define desired operator E by formulas (3.42), (3.43) explained in the proof of the Theorem 3.6. Comparing (3.43), (6.5) we see that in fact

$$u(t, \cdot) = \Lambda E v(t, \cdot) = -\Lambda E P_+ v(t, \cdot) \tag{6.6}$$

where P_+ is the projector defined in (4.15). After substitution (6.6) into (6.1) and applying to obtained equation projectors P_+, P_- we get using notation

$$v_+(t, \cdot) = P_+ v(t, \cdot), \quad v_-(t, \cdot) = P_- v(t, \cdot)$$

that problem (6.1), (6.6) is equivalent to the following one:

$$\frac{dv_+(t, \cdot)}{dt} + Av_+ = -\Lambda P_+ E v_+(t, \cdot), \quad v_+|_{t=0} = v_{0+} \equiv P_+ v_0 \tag{6.7}$$

$$\frac{dv_-(t, \cdot)}{dt} + Av_- = -\Lambda P_- E v_+(t, \cdot), \quad v_-|_{t=0} = v_{0-} \equiv P_- v_0 \tag{6.8}$$

Using the notations: $\bar{c} = (c_1, \dots, c_k)$, $\bar{v} = (v_1, \dots, v_k)$,

$$m_{kj} = \int\limits_G d_k(x)(-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x)dx = \int\limits_{\omega_1} \nabla(-\hat{\pi}\Delta)_{\omega_1}^{-1} d_k(x) \cdot \nabla(-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x)dx,$$

$$M = (m_{kj})_{k,j=1}^K$$

we can rewrite (3.43) in the form $m\bar{c} = -\bar{v}$, and (3.42) as follows:

$$Ev(x) = -\sum_{j=1}^{K} (M^{-1}\bar{v})_j (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x), \quad \text{where } (M^{-1}\bar{v})_j = c_j \tag{6.9}$$

Applying to (6.9) operator P_+ we get

$$P_+ Ev(x) = -\sum_{j=1}^{K} (M^{-1}\bar{v})_j m_{kj} e_k(x) = -\sum_{k=1}^{K} (MM^{-1}\bar{v})_k e_k(x) = -P_+ v(x)$$

Therefore (6.7) is equivalent to the problem:

$$\frac{dv_+(t, \cdot)}{dt} + (A|_{V_+} + \Lambda I)v_+(t, \cdot) = 0, \quad v_+|_{t=0} = v_{0+} \tag{6.10}$$

where $A|_{V_+}$ is restriction of operator A on V_+ (recall that V_+ is invariant with respect of A), and I is identity operator. We choose now $\Lambda > 0$ such that

$$\text{Re}\,\lambda_j + \Lambda > \sigma + \varepsilon \tag{6.11}$$

for each eigenvalue λ_j of operator $A|_{V_+}$ where $\varepsilon > 0$ is fixed. Then (6.10), (6.11) implies:

$$\|v_+(t, \cdot)\|_{V_+} \leqslant C\|P_+ v_0\|_{v_+} e^{-(\sigma+\varepsilon)t} \tag{6.12}$$

where $C = C_{\sigma+\varepsilon}$ does not depend on v_0, and solution v_- is defined by the formula:

$$v_-(t,\cdot) = e^{-At}P_-v_0 + \int_0^t e^{-A(t-\tau)}(P_-Ev_+(\tau,\cdot))d\tau \tag{6.13}$$

where e^{-At} is operator (3.19). In [7, 8] the following estimate for operator e^{-At} had been proved

$$\|e^{-At}P_-v_0\|_{V_-} \leqslant Ce^{-\sigma t}\|P_-v_0\|_{V_-} \tag{6.14}$$

with constant $C = C_\sigma$ independent of $\|P_-v_0\|_{V_-}$.

Applying (6.12), (6.14) to (6.13) we obtain

$$\begin{aligned} &\|v_-(t,\cdot)\|_{V_-} \leqslant C_1e^{-\sigma t}\|P_-v_0\|_{V_-} + C_2\textstyle\int_0^t e^{-\sigma(t-\tau)}e^{-(\sigma+\varepsilon)\tau}d\tau\|P_+v_0\|_{V_+} \\ &\leqslant e^{-\sigma t}(C_1\|P_-v_0\|_{V_-} + C_2\tfrac{1-e^{-\varepsilon t}}{\varepsilon}\|P_+v_0\|_{V_+}) \leqslant C\|v_0\|_V e^{-\sigma t} \end{aligned} \tag{6.15}$$

Bounds (6.12), (6.15) imply (6.2).

□

6.2. **The case of controlled Navier-Stokes equations.** Let consider controlled Navier-Stokes equation written in abstract form

$$\frac{dv(t,\cdot)}{dt} + Av + B(v) = u(t,\cdot), \qquad v|_{t=0} = v_0 \tag{6.16}$$

where A is operator (3.8),

$$B(v) = \pi[(v,\nabla)v], \tag{6.17}$$

π is projector (3.7), $v_0 \in V_0^1(G)$ is given, $u(t,x) \in L_2(\mathbb{R}_+; V_{00}^1(\omega))$ is a control. As in the case of Oseen equation we look for control u in the form (6.3)

After substitution (6.3) into (6.16) we get

$$\frac{dv(t,\cdot)}{dt} + \mathbb{A}v + B(v) = 0, \qquad v|_{t=0} = v_0 \tag{6.18}$$

where

$$\mathbb{A} = A + \Lambda E \tag{6.19}$$

We will use the spaces (3.12) for problem (6.18),(6.19). Introduce also the spaces of vector fields defined on cylinder $Q = \mathbb{R}_+ \times G$

$$V^{1,2(s)}(Q) = L_2(\mathbb{R}_+; V^{2+s}) \cap H^1(\mathbb{R}_+; V^s). \tag{6.20}$$

(They are analog of spaces (3.46)) Recall that γ_0 is operator of restriction at $t = 0:\ \gamma_0 v = v|_{t=0}$.

Lemma 6.2. *Let $\mathbb{A}$ be operator (6.19) with operators A, E from (3.8),(3.42),(3.43), and magnitude $\Lambda > 0$ satisfying (6.11). Then*

i) Each eigenvalue $\tilde\lambda_j$ of operator $\mathbb{A}$ satisfies condition

$$\tilde\lambda_j > \sigma > 0. \tag{6.21}$$

ii) For every $s \in [-1, 0]$ operator

$$\left(\frac{d}{dt} + \mathbb{A}, \gamma_0\right):\ V^{1,2(s)}(Q) \longrightarrow L_2(\mathbb{R}_+; V^s) \times V^{s+1} \tag{6.22}$$

realizes isomorphism of the spaces.

Proof. i) It has been proved in Theorem 6.1 that for each $v_0 \in V_0^1(G)$ solution $v(t,x)$ of the problem

$$\frac{dv(t,\cdot)}{dt} + \mathbb{A}v = 0, \quad v|_{t=0} = v_0$$

satisfies estimate (6.2). This estimate implies (see e.g.[21] Ch.IX, Sect.4) that each eigenvalue $\tilde{\lambda}_j$ of $\mathbb{A}$ satisfies equality $\mathrm{Re}\lambda_j \geqslant \sigma$. Suppose that $\mathbb{A}$ possesss eigenvector $e(x)$ and eigenvalue λ with $\mathrm{Re}\lambda = \sigma$:

$$\mathbb{A}e = \lambda e. \tag{6.23}$$

Denote $e_+ = P_+ e,\ e_- = P_- e$ and recall (see proof of Theorem 6.1) that

$$\mathbb{A}e = (A + \Lambda E)(e_+ + e_-) = (A + \Lambda E)e_+ + Ae_-; \quad P_+(A + \Lambda E)e_+ = (A + \Lambda I)e_+. \tag{6.24}$$

That is why applying to both parts of (6.23) operators P_+, P_- we can rewrite (6.23) in the following equivalent form:

$$(A + \Lambda I)e_+ = \lambda e_+, \qquad Ae_- + P_-(A + \Lambda E)e_+ = \lambda e_-. \tag{6.25}$$

In virtue of (6.11) and equality $\mathrm{Re}\lambda = \sigma$ we get that the first equality in (6.25) holds only if $e_+ = P_+ e = 0$. Then the second equality in (6.25) has the form $Ae_- = \lambda e_-$. But since all eigenvalues λ_j of operator $A|_{V_-}$ satisfy $\mathrm{Re}\lambda_j > \sigma$, equality $Ae_- = \lambda e_-$ implies $e_- = 0$. Hence $e = e_+ + e_- = 0$ and equality $\mathrm{Re}\lambda = \sigma$ is impossible.

ii) Since $\mathbb{A} = A + \Lambda E$, E is finite dimensional operator (3.42),(3.43), and continuity of operator (6.22) with $\mathbb{A}$ changed on A is well-known (see, e.g. [14]), continuity of operator (6.22) is also true. To prove reversibility of operator (6.22) we have to solve the problem

$$\frac{dv(t,x)}{dt} + \mathbb{A}v = f(t,x), \quad v|_{t=0} = v_0 \tag{6.26}$$

for each $v \in V^{s+1},\ f \in L_2(\mathbb{R}_+; V^s)$. It is enough to solve (6.26) with $f = 0$ and arbitrary $v_0 \in V^{s+1}$ and after that with $v_0 = 0$ and arbitrary $f \in L_2(\mathbb{R}_+; V^s)$.

Let $v(t,x)$ be solution of (6.26) with $f = 0$. Then

$$v = v_+ + v_-, \quad \text{where} \quad v_\pm = P_\pm v, \qquad v_+(t,\cdot) = e^{-(A+\Lambda I)t} P_+ v_0 \tag{6.27}$$

and $v_-(t,x)$ is defined by (6.13). It is well-known that (6.13) implies

$$\|v_-\|^2_{V^{1,2(s)}(Q)} \leqslant c(\|P_- v_0\|^2_{V^{1+s}} + \|P_- E v_+\|^2_{L_2(\mathbb{R}_+; V^s)}) \tag{6.28}$$

Since for each $t > 0$ function $v_+(t,\cdot)$ belongs to finite dimensional space $V^s_+ = P_+ V^s$, we get using (6.27),(6.11) that

$$\|P_- E v_+\|^2_{L_2(\mathbb{R}_+; V^s)} \leqslant c\|v_+\|^2_{L_2(\mathbb{R}_+; V^s)} \leqslant c\int_0^\infty e^{-2(\sigma+\varepsilon)} dt \|P_+ v_0\|_{V^s_+} \tag{6.29}$$

Bounds (6.28),(6.29) imply

$$\|v_-\|^2_{V^{1,2(s)}(Q)} \leqslant c\|v_0\|_{V^{s+1}}. \tag{6.30}$$

Let v be solution of (6.26) with $v_0 = 0$. Using notations $v_\pm$ from (6.27) and $f_\pm = P_\pm f$ we get

$$\frac{dv_+}{dt} + (A + \Lambda I)v_+ = f_+, \quad \frac{dv_-}{dt} + Av_- = f_- - AP_- E v_+, \quad v|_{t=0} = v_0 \tag{6.31}$$

These equations imply

$$\|v_+\|^2_{V^{1,2(s)}(Q)} \leqslant c\|f_+\|_{L_2(\mathbb{R}_+; V^s)}, \quad \|v_-\|^2_{V^{1,2(s)}(Q)} \leqslant c\|f_- - AP_- E v_+\|_{L_2(\mathbb{R}_+; V^s)} \tag{6.32}$$

Since for each $t > 0$ function $v_+(t,\cdot)$ belongs to finite dimensional space V^s_+ where all norms are equivalent, we get using the first inequality in (6.32)

$$\|AP_-Ev_+\|_{L_2(\mathbb{R}_+;V^s)} \leqslant c\|v_+\|_{L_2(\mathbb{R}_+;V^{s+2})} \leqslant c\|f_+\|_{L_2(\mathbb{R}_+;V^s)} \tag{6.33}$$

Inequalities (6.32),(6.33) imply

$$\|v\|^2_{V^{1,2(s)}(Q)} \leqslant c\|f\|_{L_2(\mathbb{R}_+;V^s)}$$

□

Let consider the set

$$El_\rho = \{v = \sum_{j=1}^{\infty} v_j\hat{e}_j(x) \in V^1 : \sum_{j=1}^{\infty} \hat{\lambda}_j^{1/2} v_j^2 < \rho\} \tag{6.34}$$

where $\rho > 0$. Recall that this set is unbounded ellipsoid in V^1 (see (4.10) and explaination below this formula).

Now we are in position to consider nonlinear stabilization problem (6.16) with feedback law (6.3).

Theorem 6.3. *Let feedback law (6.3) of nonlinear stabilization problem (6.16) satisfies conditions of Lemma 6.2. Then there exists $\rho > 0$ such that solution $v(t,x)$ of this stabilization problem with arbitrary initial condition $v_0 \in El_\rho$ exists, is unique, and satisfies the estimate:*

$$\|v(t,\cdot)\|_{V^1} \leqslant c\|v_0\|_{V^1} e^{-\sigma t} \qquad \forall t > 0 \tag{6.35}$$

where constant c does not depend on $v_0 \in El_\rho$.

Proof. The main idea of the proof is very simple and is as follows: It is well-known (see e.g. [14]) that operator B defined in (6.17) is continuous in the following spaces:

$$B:\ V^{1,2(s)}(Q) \longrightarrow L_2(\mathbb{R}_+;V^s), \qquad s \geqslant -1/2 \tag{6.36}$$

Since feedback stabilization problem (6.16),(6.3) is equivalent to boundary value problem (6.18), let consider problem (6.18) with $v_0 \in V^{1/2}$. Taking into account that by Lemma 6.2 linear part from left side of (6.18) realized isomorphism of spaces in (6.22) with $s = -1/2$, and nonlinear part is continuous in spaces (6.36) with $s = -1/2$ we get by Theorem on Inverse Operator that for $v_0 \in B^{1/2}_\rho := \{\|v_0\|_{V^{1/2}} < \rho\}$ with small enough ρ there exists unique solution $v \in V^{1,2(-1/2)}(Q)$ of problem (6.18). Moreover, simple reasonings yield the estimate

$$\|v(t,\cdot)\|_{V^{1/2}} \leqslant c\|v_0\|_{V^{1/2}} e^{-\sigma t} \tag{6.37}$$

where constant c does not depend on $v_0 \in B^{1/2}_\rho$. Note that the ball $B^{1/2}_\rho$ of space $V^{1/2}$ after intersection with V^1 becomes ellipsoid $El_\rho \subset V^1$ and the bound (6.37) can be transformed to bound (6.35). Detailed proof of this theorem can be obtained by almost word for word repeating the proof of Theorem 2.2 from [15]. □

7. **Impulse stabilization of 2D Navier-Stokes system in $\mathbb{R}^2$.** In this section we consider the case when the domain G filled with a liquid is unbounded, more exactly $G = \mathbb{R}^2$. In other words, in Subsections 7.2,7.3 below we study Cauchy problem for 2D Stokes and Navier-Stokes systems with impulse control in right-hand side. In the context of the stabilization problem the case of unbounded domains differs from bounded ones because as we show in Subsection 7.1 in this case only power-like stabilization can be realized by means of control unlike the case of bounded domains.

7.1. **Absence of exponential stabilizability in $\mathbb{R}_+$.** We give here an example of initial datum when the solution of the heat equation defined in half-line $\mathbb{R}_+$ can not tend to zero with the exponential rate $e^{-\sigma t}$.

Let consider the heat equation

$$\partial_t y(t,x) - \partial_{xx} y(t,x) = 0,\ (t,x) \in Q = (0,\infty)\times(0,\infty) \tag{7.1}$$

with a given initial datum

$$y(0,x) = y_0(x) \tag{7.2}$$

and the boundary condition

$$y(t,0) = u(t) \tag{7.3}$$

where $u(t)$ is the control function.

We suppose that $y_0(x) \in C^1(\mathbb{R}_+)$ with support on the segment $[0,1]$.

We will look for solutions $y(t,x)$ of (7.1) in anisotropic Sobolev space $H^{1,2}(Q)$ of functions $y(t,x)$ that are square-integrable on Q together with $\partial_t y(t,x)$, $\partial_x y(t,x)$, $\partial_{xx} y(t,x)$. Then control function $u(t) = y(t,0)$ will belong to the Sobolev space of traces $H^{\frac{3}{4}}(\mathbb{R}_+)$. The proof of this assertion as well as for more details about Sobolev spaces with fractional superscripts see [25].

Definition 7.1. The solution of (7.1) supplied with initial condition (7.2) does not possess the property of exponential stabilization if for no $\sigma > 0$ does there exist a control function $u(t) \in H^{\frac{3}{4}}(\mathbb{R}_+)$ such that the solution $y(t,x) \in H^{1,2}(Q)$ of boundary-value problem (7.1)- (7.3) satisfies for all $x \in \mathbb{R}_+$ the stabilization condition

$$\int_0^\infty |e^{\sigma t} y(t,x)|^2 dt < \infty. \tag{7.4}$$

Define the following initial function

$$y_0(x) = \begin{cases} (x-1)^2, & x \in (0,1) \\ 0, & x \geq 1 \end{cases} \tag{7.5}$$

Proposition 1. *Problem (7.1)-(7.3) with initial function (7.5) does not possess the property of exponential stabilization.*

Proof. Fix $x \in \mathbb{R}_+$. By means of Laplace transform $y(\tau,x) = \int_0^\infty e^{-\tau t} y(t,x)dt$ we reduce our problem to the following elliptic one with parameter τ:

$$\partial^2_{xx}\hat{y}(\tau,x) - \tau\hat{y}(\tau,x) = -y_0(x),\ \hat{y}(\tau,0) = \hat{u}(\tau) \tag{7.6}$$

The solution of (7.6) is given by the formula:

$$\begin{aligned} \hat{y}(\tau,x) = \hat{u}(\tau)e^{-\sqrt{\tau}x} - \frac{e^{-\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^\infty e^{-\sqrt{\tau}z} y_0(z)dz \\ + \frac{e^{-\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^x e^{\sqrt{\tau}z} y_0(z)dz + \frac{e^{\sqrt{\tau}x}}{2\sqrt{\tau}}\int_x^\infty e^{-\sqrt{\tau}z} y_0(z)dz. \end{aligned} \tag{7.7}$$

Function $\hat{y}(\tau,x)$ is analytical in $\mathbb{C}\backslash\mathbb{R}_-$. In view of (7.4) it must be analytical for $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > -\sigma$, and particularly for λ from the neighborhood of the origin. But $\sqrt{\tau}$ is the multi-valued function and after one circuit around origin $\sqrt{\tau}$ transforms into $-\sqrt{\tau}$. So, formula (7.7) must be invariant when one changes $\sqrt{\tau}$ to $-\sqrt{\tau}$:

$$\hat{u}(\tau)e^{-\sqrt{\tau}x} - \frac{e^{-\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^\infty e^{-\sqrt{\tau}z}y_0(z)dz + \frac{e^{-\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^x e^{\sqrt{\tau}z}y_0(z)dz$$
$$+\frac{e^{\sqrt{\tau}x}}{2\sqrt{\tau}}\int_x^\infty e^{-\sqrt{\tau}z}y_0(z)dz = \hat{u}(\tau)e^{\sqrt{\tau}x} + \frac{e^{\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^\infty e^{\sqrt{\tau}z}y_0(z)dz$$
$$-\frac{e^{\sqrt{\tau}x}}{2\sqrt{\tau}}\int_0^x e^{-\sqrt{\tau}z}y_0(z)dz - \frac{e^{-\sqrt{\tau}x}}{2\sqrt{\tau}}\int_x^\infty e^{\sqrt{\tau}z}y_0(z)dz.$$

Then we have the following formula for $\hat{u}(\tau)$:

$$\hat{u}(\tau) = \int_0^\infty \frac{e^{-\sqrt{\tau}z} - e^{\sqrt{\tau}z}}{2\sqrt{\tau}}y_0(z)dz.$$

Some elementary calculations imply the following representation of $\hat{y}(\tau, x)$

$$\hat{y}(\tau, x) = \begin{cases} \frac{1}{\tau}(x-1)^2 + \frac{2}{\tau^2} - \frac{1}{\tau^2}e^{-\sqrt{\tau}(1-x)} - \frac{1}{\tau^2}e^{\sqrt{\tau}(1-x)}, & x \in (0,1) \\ 0, & x \geq 1 \end{cases}$$

We set $y(t,x) = 0$ when $t \leq 0$. Then $\hat{y}(i\xi, x)$, $\xi \in \mathbb{R}$ as the Fourier transform of $y(t,x)$ with respect to t must be square integrable by ξ. But one can easily see, that $y(\tau, x)$ grows exponentially. This contradiction completes the proof. □

This example shows, that under no-one boundary control the discontinuity of $\hat{y}(\tau, x)$ is nonremovable. But with appropriate choice of $u(t)$ this discontinuity can be done arbitrarily small which lead to power-like stabilization of solution (see [19]).

7.2. **Power-like stabilization for Stokes system defined in $\mathbb{R}^2$.** Let us pass on to stabilization problem for equations of viscous incompressible fluid. We consider 2D Stokes system defined in $\mathbb{R}^2$ with impulse control in the right-hand side

$$\partial_t v(t,x) - \Delta v + \nabla p = \sum_{k=0}^{N} \delta(t - t_k)u_k(x) \qquad (7.8)$$
$$\operatorname{div} v(t,x) = 0,$$

with a given initial datum

$$v(0,x) = v_0(x) \qquad (7.9)$$

Here $v(t,x) = (v_1(t,x), v_2(t,x))$ is a vector field, $t_k,\ 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_N$ are the time moments of control action, $\{u_k(\cdot)\}_{k=0}^N$ are the control functions with support in the ball $B_R = \{x \in \mathbb{R}^2 | |x| \leq R\}$, that satisfy the restriction

$$\|u_k\|_{H^1(\mathbb{R}^2)} \leq \gamma,\ \ \gamma > 0 \qquad (7.10)$$

We shall solve the following stabilization problem. Given $\sigma > 0$ find control functions $u_k(x)$, and instants $\{t_k\}_{k=0}^N$, such that the solution v of (7.8),(7.9) satisfies the following stability condition with some $C > 0$

$$\|v(t,\cdot)\|_{L_2(\mathbb{R}^2)} \leq \frac{C}{(1+t)^\sigma}, \qquad t > t_N \qquad (7.11)$$

In this section and hereafter we will use the function spaces

$$L_{2,m}(\mathbb{R}^2) = \{f \ : \ \|f\|^2_{L_{2,m}(\Omega)} = \int_{\mathbb{R}^2} |f(x)|^2(1+|x|^2)^m dx < \infty\}$$

$$L_\infty(\mathbb{R}_+; V^1(\mathbb{R}^2)) = \Big\{v(t,x) = (v_1, v_2) \in \Big(L_\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)\Big)^2, \ \text{div } v(t,x) = 0\Big\}$$

$$V^0_m(\mathbb{R}^2) = \{v^0(x) = (v^0_1, v^0_2) \in \Big(L_2(\mathbb{R}^2)\Big)^2, \ \text{rot } v^0 \in L_{2,m}(\mathbb{R}^2), \ \text{div } v^0(x) = 0\}$$

The main goal of this subsection is to prove

Theorem 7.2. *Fix $\sigma > 0$, $v_0 \in V^0_m(\mathbb{R}^2)$, $m > 2\sigma + 2$. Then there exist time moments $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_N$ and feedback controls $\{u_k(\cdot)\}_{k=0}^N \in C_0^\infty(B_R)$ such that the solution $(v(t,x), p(x)) \in L_\infty(\mathbb{R}_+; V^1(\mathbb{R}^2)) \times H^1(\mathbb{R}^2)$ of (7.8),(7.9), (7.10) satisfies inequality (7.11) for $t > t_N$.*

Let $w(t,x) = \text{rot } v(t,x) = \partial_{x_1} v_2 - \partial_{x_2} v_1$ will be the vorticity of vector field $v(t,x)$. Then our Stokes system can be reduced to one-dimensional heat equation

$$\partial_t w(t,x) - \Delta w = \sum_{k=0}^N \delta(t - t_k) z_k(x) \tag{7.12}$$

$$w(0,x) = w_0(x)$$

Here $w_0(x) = \text{rot } v_0(x)$, and $z_k(x) = \text{rot } u_k(x)$, $k = 0...N$ are the control functions for vorticity. Both $z_k(x), u_k(x)$ have compact support in B_R, so restriction (7.10) in vorticity form changes to

$$\|z_k\|_{L_2(\mathbb{R}^2)} \le \gamma' \tag{7.13}$$

with some $\gamma' > 0$.

Below we use the following notation for averaging in phase space:

$$\overline{f}(t) = \int_{\mathbb{R}^2} f(t,x) dx$$

For $x = (x_1, x_2) \in \mathbb{R}^2$ we use the notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2$.

Define the subspace

$$W_n^- = \{w(x) \in L_{2,m} : \int_{\mathbb{R}^2} w(x) x^\alpha dx = 0, \, |\alpha| \le n\},$$

Lemma 7.3. *Subspace W_n^- is invariant under the semigroup $e^{\Delta t}$.*

Proof. We prove this lemma by induction. By inductive step suppose, that $w \in W_{n-1}^-$. Then for $|\alpha| = n$ integrating by parts we will have the following relation:

$$\frac{d}{dt}\overline{x^\alpha w}(t) = \overline{x^\alpha \dot{w}}(t) = \overline{x^\alpha \Delta w}(t) = \overline{P_\alpha(x) w}(t) = 0, \tag{7.14}$$

where $P_\alpha(x)$ is some polynomial of degree $n - 2$. Integration by parts is well-founded because the solution of heat equation is smooth and exponentially decays when $|x| \to \infty$.

The basis cases $n = 0, 1$ of induction are evident in view of (7.14) with $P_\alpha = 0$. □

Corollary 2. *If $w \in W_n^-$, then $\overline{x^\alpha e^{\Delta t} w} = const$ for $|\alpha| = n + 2$, $t > 0$.*

Proof. This corollary immediately follows from the identity (7.14). □

Stabilization of the form (7.11) is based on the following result, which has been proved by Th. Gallay, C. E. Wayne[18]. We reformulate this theorem in the following form more convenient for our case.

Theorem 7.4. *Let n, m satisfy the relation $n+1 < m < n+2$, and $w_0 \in W_n^-$. Then for each $\varepsilon > 0$ there exists $C > 0$ such that the following inequality holds:*

$$\|e^{\Delta t} w_0\|_{L_2(\mathbb{R}^2)} \leq C t^{-\frac{1}{2}(m-1-\varepsilon)} \|w_0\|_{L_{2,m}} \tag{7.15}$$

Now we are ready to prove Theorem 7.2.

Proof. Fix n as in the Theorem 7.4. In view of (7.15) from now our main goal is to push our solution $w(t,x)$ into W_n^- by means of controls $z_k(x)$. First, we push the solution into W_0, and then we do the same sequentially with $W_1, ..., W_n$.

Denote

$$w_k(\cdot) = w(t_k, \cdot)$$

As we did it in the case of bounded domains, in accordance to lemma 5.1 we reduce our impulse control problem to the one with start control. So, at time $t_0 = 0$ we perturb initial data w_0 by control $z_0(\cdot)$

$$w(0,x) = w_0(x) + z_0(x)$$

Define

$$z_0(x) = c_0 \chi(x) \tag{7.16}$$

where $\chi(x)$ is some function from $C_0^\infty(B_R)$ satisfying $\chi(x) \geq 0$, $\|\chi(\cdot)\|_{L_2(B_R)} = 1$, and c_0 is an unknown constant.

Solution $w(t,x)$ for $t \in (t_0, t_1)$ satisfies to heat equation with zero right-hand side. Therefore by Lemma 7.3 its total mass $\overline{w} = \int_{\mathbb{R}^2} w(t,x)dx$ is preserved for all $t \in (t_0, t_1)$, and subspace W_0^- is invariant under semiflow $e^{\Delta t}$.

If we set $c_0 = -\overline{w}_0/\overline{\chi}$, then $\overline{e^{\Delta t} w(0,x)} = 0$. Moreover, if $|c_0| \leq \gamma'$ then solution $w(t,x)$ of (7.12) with one impulse $\delta(t) z_0(x)$ at $t = 0$ in right-hand side will belong to invariant subspace W_0^-.

If $|c_0| > \gamma'$ then we set $c_0 = -\gamma' \text{sign}(\overline{w}_0)$, and at the time $t = t_1$ the final state $w(t_1, x)$ will satisfy

$$|\overline{w}_1| = |\overline{w}_0 + c_0 \overline{\chi}| = |\overline{w}_0| - \gamma' |\overline{\chi}| = |\overline{w}_0| \Big(1 - \gamma' \frac{|\overline{\chi}|}{|\overline{w}_0|} \Big) < |\overline{w}_0|$$

Then at time t_1 we will generate impulse with control $z_1(x) = c_1 \chi(x)$, where $c_1 = -\overline{w}_1/\overline{\chi}$. If such impulse control satisfies (7.13), the solution of (7.12) with pair of two impulses (z_0, z_1) will belong to E_0. Otherwise, we correct c_1 in the same way $c_1 = -\gamma' \text{sign}(\overline{w}_1)$.

At the k-th iteration in time $t = t_k$ we will have the control impulse $z_k(x)$:

$$z_k(x) = \begin{cases} -\chi(x) \gamma' \text{sign}\big(\overline{w}_k\big), & |\overline{w}_k|/|\chi| > \gamma' \\ -\chi(x) |\overline{w}_k|/|\chi|, & |\overline{w}_k|/|\chi| \leq \gamma' \end{cases} \tag{7.17}$$

While $|\overline{w}_k|/|\chi| > \gamma'$ function $\overline{w}_k$ will tend to zero

$$\begin{aligned} |\overline{w}_{k+1}| = |\overline{w}_k + c_k \overline{\chi}| = |\overline{w}_k| - \gamma' |\overline{\chi}| \\ = |\overline{w}_k| \Big(1 - \gamma' \frac{|\overline{\chi}|}{|\overline{w}_k|} \Big) < |\overline{w}_0| \Big(1 - \gamma' \frac{|\overline{\chi}|}{|\overline{w}_0|} \Big)^{k+1} \end{aligned}$$

Since $|\overline{w}_k| \to 0$ when $k \to \infty$, there exists $K > 0$, such that $|\overline{w}_K| \le \gamma' |\overline{\chi}|$. Set $z_K = -\chi(x)\gamma' \mathrm{sign}\big(\overline{w}_K\big)$. Then by means of $z_0, z_1, ..., z_K$ the solution of (7.12) after time $t = t_K$ will belong to W_0^-.

By inductive step, suppose, that at an instant $t = t_k$ our vorticity function w_k lies in W_{n-1}^-, i.e.:

$$\overline{x^\alpha w_k} = \int_{\mathbb{R}^2} x^\alpha w(t_k, x) dx = 0 \ \forall \alpha : \ |\alpha| < n$$

Our goal is to push the solution $w(t, x)$ into W_n^- by impulse control. By (7.14) for multi-indices α with $|\alpha| = n$ the moments $\overline{x^\alpha w}(t)$ will stay constant during time $t > t_k$.

Let us make perturbation of w_k by adding control impulse z_k. Then the solution of (7.12) for $t = t_{k+1}$ will be given by the formula:

$$w_{k+1} = e^{\Delta(t_{k+1} - t_k)}(w_k + z_k)$$

For $t = t_k$ define the function

$$Ew_k = \chi(x) \sum_{|\alpha| \le n} c_\alpha^k x^\alpha$$

Then the condition $w_{k+1} \in W_n^-$ with $z_k(x) = Ew_k(x)$ is equivalent to the system

$$A\vec{c}_k = \vec{b}_k,$$

where

$$\begin{aligned} A &= \|a_{\alpha\beta}\|_{|\alpha| \le n, |\beta| \le n}, \qquad & a_{\alpha\beta} &= \int_{\mathbb{R}^2} \chi(x) x^\alpha x^\beta dx \\ \vec{b}_k &= \{b_\alpha^k\}_{|\alpha| \le n}, \qquad & b_\alpha^k &= -\overline{x^\alpha w_k} \\ \vec{c}_k &= \{c_\alpha^k\}_{|\alpha| \le n} \end{aligned}$$

It is worth to note, that since $w_k \in E_{n-1}$ then $b_\alpha^k = 0$ for $|\alpha| < n$. The dimension of vectors $\vec{b}_k$, $\vec{c}_k$ equal to number of combinations α_1, α_2 such that $\alpha_1 + \alpha_2 \le n$ which is exactly $(n + 1)(n + 2)/2$. Lets prove, that A is positive-definite, and so inverse matrix A^{-1} is well defined. Indeed

$$(A\eta, \eta) = \sum_{|\alpha| \le n, |\beta| \le n} a_{\alpha\beta} \eta_\alpha \eta_\beta = \int \Big| \sum_{|\alpha| \le n} \sqrt{\chi(x)} x^\alpha \eta_\alpha \Big|^2 dx > 0$$

Set

$$\vec{c}_k = A^{-1} \vec{b}_k$$

If $\|Ew_k(\cdot)\| \le \gamma'$, then $z_k(x) = Ew_k(x)$ is the last impulse in series $(z_1, z_2, ..., z_k)$ which realizes hit of $w(t, x)$ into W_n^-. Otherwise, if $\|Ew_k(\cdot)\| > \gamma'$ we make correction of $z_k(x)$ by formula

$$z_k(x) = \gamma' \frac{Ew_k(x)}{\|Ew_k(\cdot)\|_{L_2(\mathbb{R}^2)}} \tag{7.18}$$

Then for multi-indices $\beta = (\beta_1, \beta_2)$ we will have:

$$|\overline{x^\beta w_{k+1}}| = |\overline{x^\beta w_k} + \overline{x^\beta z_k}| = \Big| \overline{x^\beta w_k} + \gamma' \frac{\sum_{|\alpha| \le n} a_{\alpha\beta} c_\alpha^k}{\|Ew_k(\cdot)\|_{L_2(\mathbb{R}^2)}} \Big|$$

$$= \Big| \overline{x^\beta w_k} + \frac{\gamma' \cdot b_\beta^k}{\|Ew_k(\cdot)\|_{L_2(\mathbb{R}^2)}} \Big| = \Big| \overline{x^\beta w_k} \Big(1 - \frac{\gamma'}{\|Ew_k(\cdot)\|_{L_2(\mathbb{R}^2)}} \Big) \Big| < |\overline{x^\beta w_{k+1}}|$$

For any $k \in \mathbb{Z}_+$ the following relations hold with some $C > 0$

$$\|Ew_k\|_{L_2}^2 = \|\chi(x) \sum_{|\alpha| \le n} c_\alpha^k x^\alpha\|^2 \le C\|\sqrt{\chi(x)} \sum_{|\alpha| \le n} c_\alpha^k x^\alpha\|^2$$
$$= C \sum_{|\alpha| \le n} \sum_{|\beta| \le n} a_{\alpha\beta} c_\alpha^k c_\beta^k = C(\vec{c}_k, \vec{b}_k) \le C\|\vec{c}_k\|\|\vec{b}_k\| \le C\|A^{-1}\|\|\vec{b}_k\|^2$$

After l iterations we will have the estimate

$$|\overline{x^\beta w_{k+l+1}}| = |\overline{x^\beta w_{k+l}} + \overline{x^\beta z_{k+l}}| = |\overline{x^\beta w_{k+l}}|\Big(1 - \frac{\gamma'}{\|Ew_{k+l}(\cdot)\|_{L_2(\mathbb{R}^2)}}\Big)$$
$$= |\overline{x^\beta w_k}| \prod_{i=0}^{l} \Big(1 - \frac{\gamma'}{\|Ew_{k+i}(\cdot)\|_{L_2(\mathbb{R}^2)}}\Big) \le |\overline{x^\beta w_k}| \prod_{i=0}^{l} \Big(1 - \frac{\gamma'}{\sqrt{C}\|A^{-1}\|^{\frac{1}{2}}\|\vec{b}_{k+i}\|}\Big)$$
$$\le |\overline{x^\beta w_k}| \Big(1 - \frac{\gamma'}{\sqrt{C}\|A^{-1}\|^{\frac{1}{2}}\|\vec{b}_k\|}\Big)^{l+1}$$

This inequality means, that the moments $b_\alpha^k = \overline{x^\alpha w_k}$ as well as c_α^k exponentially decay when $k \to \infty$. Therefore, after some time $t = t_N$ control $z_N(\cdot)$ will satisfy (7.13). Time t_N will be the last instant of control action on the system (7.12), after that $w(t, x)$ will stay on W_n^-.

Fix $\varepsilon > 0$. Applying (7.15) we get the bound with some $C_1, C_2 > 0$ depending on m and t_N :

$$\|w(t, \cdot)\|_{L_2(\mathbb{R}^2)} \le Ct^{-\frac{1}{2}(n-\varepsilon)}\|w(t_N, \cdot)\|_{L_{2,m}(\mathbb{R}^2)} \le t^{-\frac{1}{2}(n-\varepsilon)}(C_1 + C_2\|w_0(\cdot)\|_{L_{2,m}(\mathbb{R}^2)}),\ t > t_N$$

Since $\overline{w}(t_N) = 0$, this inequality implies the estimate on velocity field $v(t, x)$ with some $C_3, C_4 > 0$(see [18], Lemma 2.1, Prop. B.1):

$$\|v(t, \cdot)\|_{L_2(\mathbb{R}^2)} + \|\nabla v(t, \cdot)\|_{L_2(\mathbb{R}^2)} \le t^{-\frac{1}{2}(n-1-\varepsilon)}(C_3 + C_4\|w_0(\cdot)\|_{L_{2,m}(\mathbb{R}^2)}),\ t > t_N$$

The last inequality with $\sigma = \frac{1}{2}(n - 1 - \varepsilon)$ implies stabilization condition (7.11).

Finally, let show that the control is feedback one. For this we have to construct a bounded operator $\mathcal{F} : L_{2,m}(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ satisfying $z_k(\cdot) = \mathcal{F}(w_k)(\cdot)$.

Remind that

$$Ew_k = -\chi(x) \sum_{|\alpha| \le n, |\beta| \le n} a_{\alpha\beta}^{-1} \overline{x^\beta w_k} x^\alpha,$$

where $\{a_{\alpha\beta}^{-1}\}_{|\alpha| \le n, |\beta| \le n}$ denote elements of A^{-1}.

Then operator $\mathcal{F}$ is defined as follows:

$$\mathcal{F}w_k = \begin{cases} Ew_k,\ \|Ew_k\|_{L_2} \le \gamma' \\ \gamma' \frac{Ew_k}{\|Ew_k\|_{L_2}},\ \|Ew_k\|_{L_2} > \gamma' \end{cases}$$

Let show that $\mathcal{F}$ is bounded. Due to smoothness of polynomials x^α we get

$$\|\mathcal{F}(w_k)\|_{L_2} \le C\|\vec{b}_k\|_{\mathbb{R}^d},$$

where $d = (n + 1)(n + 2)/2$ is the dimension of $\vec{b}_k$.

Since $m > n+1$, we obtain the inequality with some $\varepsilon > 0$ that completes the proof of boundness of $\mathcal{F}$:

$$|b_\alpha^k| \le \int_{\mathbb{R}^2} |w(t_k,x)x^\alpha| dx \le C_1 \int_{\mathbb{R}^2} |w(t_k,x)|(1+|x|)^n dx =$$
$$C_1 \int_{\mathbb{R}^2} \frac{|w(t_k,x)|(1+|x|)^{n+1+\varepsilon}}{(1+|x|)^{1+\varepsilon}} dx \le C_1 \int_{\mathbb{R}^2} |w(t_k,x)|^2 (1+|x|)^{2n+2+2\varepsilon} dx \times$$
$$\int_{\mathbb{R}^2} \frac{dx}{(1+|x|)^{2+2\varepsilon}} = C_2 \int_{\mathbb{R}^2} |w(t_k,x)|^2(1+|x|)^{2n+2+2\varepsilon} dx \le C_2 \|w_k\|^2_{L_{2,m}}$$

□

7.3. **Power-like stabilization for Navier-Stokes system defined in $\mathbb{R}^2$.** As we show below, in the case of Navies-Stokes system only the first order moments $\overline{x_i w}(t)$ stay constant. And therefore in the statement of stabilization problem the rate of convergence will be restricted to $\sigma = 1-\varepsilon$, $\varepsilon > 0$.

Consider Navier-Stokes system

$$\begin{aligned} \partial_t v(t,x) - \Delta v + (v,\nabla)w + \nabla p &= \sum_{k=0}^{N} \delta(t-t_k) u_k(x) \\ \mathrm{div} v(t,x) &= 0 \\ v(0,x) &= v_0(x) \end{aligned} \tag{7.19}$$

with the same restriction on control as above:

$$\|u_k\|_{H^1(\mathbb{R}^2)} \le \gamma, \ \gamma > 0 \tag{7.20}$$

Theorem 7.5. *Fix $\varepsilon > 0$, $v_0 \in V_3^0(\mathbb{R}^2)$ with sufficiently small $\|\mathrm{rot}\ v_0\|_{L_{2,3}}$. Then there exist time moments $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_N$ and feedback controls $\{u_k(\cdot)\}_{k=0}^N \in C_0^\infty(B_R)$ such that the solution $(v(t,x), p(x)) \in L_\infty(\mathbb{R}_+; V^1(\mathbb{R}^2)) \times H^1(\mathbb{R}^2)$ of problem (7.19),(7.20) satisfies for $t > t_N$ the following stability condition:*

$$\|v(t,\cdot)\|_{L_2(\mathbb{R}^2)} \le \frac{C}{(1+t)^{1-\varepsilon}} \tag{7.21}$$

Proof. Vorticity $w(t,x) = \mathrm{rot}\ v(t,x)$ is the solution of the Cauchy problem

$$\begin{aligned} \partial_t w(t,x) - \Delta w + (v,\nabla)w &= \sum_{k=0}^{N} \delta(t-t_k) z_k(x) \\ w(0,x) &= w_0(x) \end{aligned} \tag{7.22}$$

with $z_k(x) = \mathrm{rot}\ u_k(x)$, $w_0(x) = \mathrm{rot}\ v_0(x)$.

As in the linear case of Stokes system, our proof is based on the result of Th. Gallay and E. Wayne[18] but for 2D Navier-Stokes system.

Suppose that Navier-Stokes system is free from control's action, i.e. $z_k = 0$, $k = 0,..,N$. Define Oseen vector field and its derivatives by

$$\begin{aligned} v^G(t,x) &= \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \left(1 - e^{-\frac{|x|^2}{4(1+t)}}\right) \\ v^{F_i}(x) &= \sqrt{1+t}\,\partial_{x_i} v^G(x),\ i = 1,2 \end{aligned}$$

Oseen vector field plays very important role in asymptotical analysis for Navier-Stokes system in $\mathbb{R}^2$. Namely, as it follows from the theorem below, it is the main term in asymptotic decomposition of solution $v(t,x)$ for $t \to \infty$.

Theorem 7.6. *(Th. Gallay, C. E. Wayne.[18]) Fix $\varepsilon > 0$. There exist $r > 0$ and $C > 0$ such that for all initial data $w_0 \in L_{2,3}$ with $\|w_0\|_{L_{2,3}} \le r$ the solution of (7.19) with $z_k = 0$, $k = 0,..,N$ satisfies*

$$\Big\| v(t,\cdot) - \frac{A}{1+t} v^G\big(\frac{x}{\sqrt{1+t}}\big) + \sum_{i=1}^{2} \frac{B_i}{(1+t)^{\frac{3}{2}}} v^{F_i}\big(\frac{x}{\sqrt{1+t}}\big)\Big\|_{L_2(\mathbb{R}^2)} \le \frac{C}{(1+t)^{1-\varepsilon}}$$

where $A = \overline{w}_0$, $B_i = \overline{x_i w_0}$, $i = 1, 2$.

This theorem implies that our stability condition (7.21) is equivalent to $w(t_N, x) \in W_1^-$.

It is well known, that total mass $\overline{w}(t)$ stays constant and equal to A for all $t > 0$. In the following lemma we prove that the similar statement is valid for first moments $\overline{x_i w}$. In other words, invariant spaces W_n^- for both Stokes and Navier-Stokes systems are identical when $n = 1$.

Lemma 7.7. *Let w be solution of problem (7.22) with $z_k = 0$, $k = 0,..,N$. Then, first moments $\overline{x_i w}$, $i = 1, 2$ stay constant and equal to B_i for all $t > 0$.*

Proof. Without loss of generality we prove this lemma for $i = 1$. Applying free-divergence condition to vector-field v we get

$$\begin{aligned} \frac{d}{dt}\int x_1 w(t,x)dx &= \int x_1 \Delta w dx - \int x_1 (v, \nabla) w dx \\ = \int \big(\partial_{x_1}(x_1 w_{x_1}) + \partial_{x_2}(x_1 w_{x_2}) - \partial_{x_1} w\big)dx &+ \int v_1 w dx \\ = \int v_1(\partial_{x_1} v_2 - \partial_{x_2} v_1)dx = -&\int (v_2 \partial_{x_1} v_1 + \frac{1}{2}\partial_{x_2} v_1^2)dx \\ &= \frac{1}{2}\int \partial_{x_2}(v_1^2 + v_2^2)dx = 0 \end{aligned}$$

□

Right now the last part of the proof is identical to the proof of theorem 7.2. First, using finite set of instants $t_0, t_1, ..., t_K$ with impulses z_k as in (7.17) we put our solution into E_0. After that we define operator Ew_k and new controls z_k as in (7.18) with n=1. Then moments $\overline{x_i w_k}$, $i = 1, 2$ will tend to zero when $k \to \infty$ and will vanish after some $k = N$. Hence, $w(t, x)$ will be in W_1^- for all $t > t_N$. Then stabilization condition (7.21) immediately follows from Theorem 7.6. □

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