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Feedback stabilization for Navier-Stokes equations: Theory and Calculations

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Abstract

Stabilization problem for Navier-Stokes equations defined in a bounded domain by feedback control is considered. The cases of control in right side (distributed and impulse) supported in a subdomain, of control in initial condition and on boundary are studied; intercommunications between different kinds of control are presented. Feedback property is discussed, and feedback map expressing control via state variable is constructed for initial and distributed control (the last one for Oseen equations only). Numerical algorithms for calculation of stable invariant manifolds and projection operators on these sets are discussed. Results of numerical stabilization a certain fluid flow are presented.

1.1 Introduction

The aim of this paper is to give a relatively short presentation of mathematical and numerical results concerning stabilization of Navier-Stokes equations by feedback control. Description of the mathematical stabilization construction will be accompanied with discussion how much some mathematical notions of stabilization theory are adapted to calculations.

The control theory for partial differential equations to where the topic of this article can be included was developed last decades very intensively and transforms now to very wide and reach field even if we exclude extremal theory for PDE. To have some idea of this field see recent books Coron (2007) and Tucsnk & Weiss (2009), as well as more early survey in Fursikov & Imanuilov (1999) together with references in these publications.

The stabilization problem for 2D Navier-Stokes system by feedback distributed control supported in the whole domain filled the fluid was studied in Barbu & Sritharan (1998). For 2D Euler equations with feedback boundary control the similar problem was investigated in Coron (1999).

The local stabilization theory for Navier-Stokes equations by feedback control supported on the boundary of domain filled with liquid was created in Fursikov (2001a) - Fursikov (2004). In particular, feedback theory was developed in Fursikov (2002b), Fursikov (2002c) and some probability aspects of this theory were worked in Duan & Fursikov (2005). In these works classical mathematical notion of feedback control as a function on phase variable was used but in implicit veiled form.

The construction of local stabilization theory for Navier-Stokes equations with use of mentioned mathematical definition of feedback obtained by classical Riccati-based approach was begun in Barbu (2003), Barbu & Triggiani (2004), Barbu, Lasiecka, & Triggiani (2006) and had been completed in Raymond (2006), Raymond (2007), Raymond & Thevenet (2010) (see also Ravindran (2007)). It is necessary to note that this approach leads to complicate Riccati-based construction of the map connecting feedback control with phase variable.

That is why it seams quite natural that creation of numerical feedback stabilization theory became based on the mathematical notion of stable invariant manifold which is the key notion in mathematical stabilization theory developed in Fursikov (2001a)-Fursikov (2004).

General theory of invariant manifolds was developed in classical works of H.Poincare, A.M.Liapunov, J.Hadamard, O.Perron (see, for example, Hartman (1964), Anosov (1967), Hirsch, Pugh, & Shub (1977)) and with applications to hydrodynamical equations was developped in Ladyzhenskaya & Solonnikov (1973), Marsden & McCracken (1976), Babin & Vishik (1992).

The problem on numerical construction of invariant manifolds is also well-known; in the case of small dimensional spaces it was studied by many authors (see, for example Shil'nikov, Shil'nikov, Turaev, & Chua (2004))

The fisrt numerical solution of stabilization problem with feedback boundary control based on approach from Fursikov (2001a)-Fursikov (2004) has been made in Chizhonkov (2003), Chizhonkov (2004) in the case of 1D Chafee-Infante equation. Similar stabilization problems for

Navier-Stokes Equations, including stabilization of classical Couette flow between two rotating cylinders in 2D formulation have been solved numerically in Chizhonkov & Ivanchikov (2004), Ivanchikov (2006).

Simultaneously in Kornev (2003) - Kornev (2006) developed numerical methods of invariant manifold construction and feedback stabilization by boundary, initial, and right sides feedback control for Lorenz system, Chafee-Infante, Burgers and Navier-Stokes equations as well as for barotropic vorticity equation on a rotating sphere.

It is necessary to emphasize that all aforementioned stabilization results are related with stabilization near steady-state or periodic solution. This setting admits natural generalization. Indeed, in kindred exact controllability theory the notion of local exact controllability introduced in Fursikov & Imanuvilov (1995) (see also Fursikov & Imanuvilov (1996), Fursikov (2000)) has deal with controllability not relatively to steady-state solution but with respect to general time-dependent solution because the last variant is incomparably more natural. Stabilization to an arbitrary bounded time-depended solution is not less (and maybe much more) natural than stabilization to steady-state solution. Creation of mathematical theory for feedback stabilization to time-dependent solution is in the very begining now (see Barbu, Rodrigues, & Shirikyan) (2010)) in contrast to local exact controllability theory. But such problem arises in applications and its numerical solution is developed already several years.

The key mathematical notion of proposed numerical method of solution for stabilization problem is local stable manifold $\mathcal{W}_-$ corresponding to a given time-dependent solution of dynamical system $\{S(t,\cdot)\}$. Corresponding numerical schemes are constructed in terms $\mathcal{W}_-$ and $S(t,\cdot)$ as well. This gives opportunity to apply them to the whole class of problems.

General theory of local stable manifolds was developed from aforementioned results in Anosov (1967), Pesin (1977), Daletskiy & Krein (1974), for hydrodynamical equations in Ladyzhenskaya & Solonnikov (1973) and Yudovich (1989).

Numerical schemes for construction of local stable manifolds and application of these results were obtained in Kornev (2003) - Kornev & Ozeritskii (2010), Kalinina (2006). Note that for working out numerical algorithms in the case of fixed point theoretical constructions of Ladyzhenskaya & Solonnikov (1973) were essentially used, and in the case of trajectory constructions from Pesin (1977) were very important. In Vazquez & Krstic (2008) proposed the algorithm of other type for solving boundary stabilization problem for Navier-Stokes and magnetohydrodynamic channel flows.

In the first part of this paper we give a survey of results from Fursikov (2001a)-Fursikov (2004) that is built as follows. In fact aforementioned stabilization results on boundary feedback control contains implicitly feedback stabilization by initial and impulse controls. Here we explain why it is so: In section 1.2 we give settings of stabilization problems for Navier-Stokes equations with control (distributed and impulse) in right side supported in spatial subdomain, with initial control, and with control supported on a boundary. We investigate carefully correlation between different types of control, and in particular explain how stabilization by boundary control can be obtained by initial control. In sections 1.3,1.4 we give construction of stabilization by initial control based on projection of initial condition on stable invariant manifold. This projection is realized by classical feedback relation of the form $u = F(y_0)$ where u is the control, y_0 is the given initial condition, and the map F is constructed by some simple tools (without any Riccati equations as in Barbu et al. (2006), Raymond (2007)). Section 1.5 is devoted to discussion of feedback property and its realization for different types of control. Feedback state-control relation in the case for distributed control is a new result obtained in this paper although it is very close to relation for initial control.

The second part of the paper is devoted to numerical solution of stabilization problem. The original problem is reduced to projecting on the stable manifold of the resolving operator of the given (semi)dynamical systems. This approach makes it possible to apply the results to a wide class of dynamical systems including Navier-Stokes equations. Corresponding numerical algorithms are presented in section 1.6 in the case of stable invariant manifolds for fixed point as well as for stable manifolds corresponding time-dependent solution. In section 1.7 results of numerical solution of the quasi-two-dimensional Navier-Stokes equations in the initial data or in the boundary conditions or in the right-hand side is given.

1.2 Setting of the stabilization problem

In this section we recall formulation of the stabilization problem for three kinds of the control: for initial control, boundary control, and for distributed and impulse control in right side.

1.2.1 Formulation of the stabilization problem with a control in right-hand side.

Let $G \subset \mathbb{R}^d$, $d = 2,3$, $\partial G \in C^\infty$, $Q = \mathbb{R}_+ \times G$. We consider the Navier-Stokes equations with control in the right-hand side:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = h(x) + u(t,x), \quad \operatorname{div} v(t,x) = 0 \quad (1.1)$$

with boundary condition

$$v(t,x)|_{x\in\partial G} = 0 \quad (1.2)$$

and initial condition

$$v(t,x)|_{t=0} = v_0(x). \quad (1.3)$$

Here $(t,x) = (t,x_1,\dots,x_d) \in Q$, $v(t,x) = (v_1,\dots,v_d)$ is a velocity of fluid flow, $p(t,x)$ is a pressure, $h(x) = (h_1,\dots,h_d)$ is a given right side, and $u(t,x)$ is a control supported at each $t \in \mathbb{R}_+$ in a given fixed subdomain $\omega \subset \Omega$.

Denote, as usually, by $H^k(G)$, $k \in \mathbb{N}$ the Sobolev space of scalar functions, defined and square integrable on G together with all its derivatives up to order k and by $(H^k(G))^d$ the analogous space of vector fields. Besides, $H_0^1(G) = \{f(x) \in H^1(G) : f(x)|_{x\in\partial G} = 0\}$. We will use also the following spaces of solenoidal vector fields:

$$V^k(G) = \left\{v(x) = (v_1,\dots,v_d) \in (H^k(G))^d : \quad \operatorname{div} v = 0\right\}, k = 0,1,2,\dots$$

$$V_0^1(G) = V^1(G) \cap H_0^1(G)^d, \quad V_0^0(G) = \text{closure } \mathcal{V}(G) \text{ in } L_2(G)^d$$

where $\mathcal{V}(G) = \{v(x) \in C_0^\infty(G)^d : \operatorname{div} v = 0\}$. Evidently,

$$\|v\|_{V^0(G)} = \|v\|_{V_0^0(G)} := \|v\|_{(L_2(G))^d}; \quad \|v\|_{V_0^1(G)} := \|\nabla v\|_{(L_2(G))^{d^2}}.$$

We consider in (1.1) a control $u(t,x)$ of two kinds:

i) Impulse control, i.e. control of the form of kick forces:

$$u(t,x) = \sum_j \delta(t-t_j)u_j(x), \quad (1.4)$$

where $\delta(t-t_j)$ is Dirac δ-function supported in t_j, $0 = t_0 < t_1 < \dots < t_j < \dots$, $u_j(x) = (u_{j1},\dots,u_{jd})$ is a solenoidal vector field supported in a given fixed subdomain $\omega \subset G$.

ii) Distributed control in subdomain:

$$u(t,x) \in L_2(0,T;V_0^0(G)) \quad \forall T > 0, \quad \operatorname{supp} u(t,\cdot) \subset \omega \quad \forall t > 0 \quad (1.5)$$

The setting of stabilization problem for each kind of a control written above is as follows:

Let $h \in L_2(G)^d,\ v_0 \in V_0^1(G)$. Suppose that $\sigma > 0$ and an unstable steady-state solution $(\widehat{v}(x), \widehat{p}(x)) \in (V^2(G) \cap V_0^1(G)) \times H^1(G)$ of Navier-Stokes equations

$$-\Delta\widehat{v}(x) + (\widehat{v}, \nabla)\widehat{v} + \nabla\widehat{p}(x) = h(x), \quad \operatorname{div}\widehat{v}(x) = 0 \tag{1.6}$$

are given. The problem of stabilization a solution of (1.1)–(1.3) with rate σ is to construct a control $u(t,x)$ of kind (1.4) or of kind (1.5) such that the solution $v(t,x)$ of boundary value problem (1.1)–(1.3) satisfies:

$$\|v(t,\cdot) - \widehat{v}\|^2_{V_0^1(G)} \leqslant ce^{-\sigma t} \tag{1.7}$$

with a certain constant $c > 0$ depending on σ, $\|v_0\|_{V_0^1(G)}$, and u.

1.2.2 The case of a control in initial conditions.

Instead of problem (1.1)-(1.3) let consider the following control problem:

$$\partial_t v(t,x) - \Delta v + (v, \nabla)v + \nabla p(t,x) = h(x), \quad \operatorname{div} v(t,x) = 0 \tag{1.8}$$

$$v(t,x)|_{t=0} = v_0(x) + u(x). \tag{1.9}$$

supplied with boundary condition (1.2). In this problem control $u(x)$ is in initial condition, and we assume that $\operatorname{supp} u \subset \omega$ where ω is a given subdomain of G.

The stabilization problem is formulated in this case as follows:

Given $\sigma > 0,\ h \in L_2(G)^d,\ v_0 \in V_0^1(G)$ and steady-state solution $(\widehat{v}(x), \widehat{p}(x)) \in (V^2(G) \cap V_0^1(G)) \times H^1(G)$ of system (1.6). Find a control $u(x) \in V_0^1(G)$ with supp $u \subset \omega$ such that the solution (v,p) of problem (1.8), (1.2), (1.9) satify estimate (1.7).

The following assertion is true:

Lemma 1.2.1 *Stabilization problem (1.8),(1.2),(1.9)with a control in initial condition is equivalent to stabilization problem (1.1)-(1.3) with impulse control satisfying condition*

$$u_j(x) \equiv 0 \quad \forall\, j \geqslant 1.^{1} \tag{1.10}$$

Proof Recall, that generalized solution of problem (1.8),(1.2),(1.9) with

[1] I.e. when we have only one impulse at t=0.

given initial condition $v_0 + u \in V_0^1(G)$ and right side $h \in L_2(G)^d$ is the vector field $v \in L_\infty(0,T; V_0^0(G)) \cap L_2(0,T; V_0^1(G))$ satisfying

$$
\begin{gathered}
-\int_0^T \int_G [(v(t,x), \partial_t \varphi(t,x)) + \sum_{j=1}^d (\partial_{x_j} v, \partial_{x_j} \varphi) \\
+(v, \sum_{j=1}^d v_j \partial_{x_j} \varphi)]\, dxdt + \int_G (v(T,x), \varphi(T,x))\, dx \\
= \int_0^T \int_G (h(x), \varphi(t,x))\, dxdt + \int_G (v_0(x) + u(x), \varphi(0,x))\, dx, \\
\forall \varphi(t,x) \in C^1(0,T; \mathcal{V}(G))
\end{gathered}
\tag{1.11}
$$

To get (1.11) we multiply scalarly (1.8) on φ and integrate by parts using (1.9).

If we multiply scalarly on φ equation (1.1),(1.4) that satisfies (1.10) and integrate by parts using (1.3) we obtain (1.11) as well. Therefore vector field $v(t,x) \in L_\infty(0,T; V_0^0(G)) \cap L_2(0,T; V_0^1(G))$ that satisfies equality (1.11) is generalized solution of boundary value problem (1.8),(1.2),(1.9) with given $v_0 + u \in V_0^1(G)$, and $f \in L_2(G)^d$ as well as of boundary value problem (1.1)-(1.3),(1.10) with given $v_0, u \in V_0^1(G)$, and $h \in L_2(G)^d$. Note that in virtue of (1.7) our generalized solution satisfies inclusion $v \in L_\infty(0,T; V_0^1(G))$ that, as well-known (see Ladyzhenskaya (1963), Temam (1984)), guarantees its uniqueness. That is why boundary value problems (1.8),(1.2),(1.9) and (1.1)-(1.3),(1.10) with given $v_0, u \in V_0^1(G)$, and $h \in L_2(G)^d$ are equivalent. Thus, corresponding control problems are equivalent as well. $\square$

As we will see later the main mathematical tools for stabilization construction will be worked out to the case of control in initial conditions. The case of impulse control will be important for construction theory of feedback control based on the notion of real process.

1.2.3 The case of a control supported on a part of boundary.

Set now the stabilization problem with control in a boundary conditions. This case, perhaps, is the most interesting for applications.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3.$ be a bounded domain with the boundary $\partial\Omega$ of class C^∞. Consider now Navier-Stokes system with initial conditon

$$\partial_t v(t,x) - \Delta v + (v, \nabla)v + \nabla p(t,x) = g(x), \quad \operatorname{div} v(t,x) = 0 \tag{1.12}$$

$$v(t,x)|_{t=0} = v_0(x) \tag{1.13}$$

and introduce boundary condition by the following way.

Let $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_0, \ \Gamma \neq \emptyset$ where Γ, Γ_0 be open sets (in topology of $\partial\Omega$). Here, as usual, the over line means the closure of a set. We define $\Sigma = \mathbb{R}_+ \times \Gamma, \Sigma_0 = \mathbb{R}_+ \times \Gamma_0$, and set:

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u \tag{1.14}$$

where u is a control, supported on Σ.

Stabilization problem is formulated now as follows:

Given $\sigma > 0, \ g \in L_2(\Omega)^d, \ v_0 \in V^1(\Omega), \ v_0|_{\Gamma_0} = 0$, and steady-state solution $(\widehat{v}(x), \widehat{p}(x)) \in V^2(\Omega) \times H^1(\Omega)$ of the problem

$$-\Delta\widehat{v}(x) + (\widehat{v}, \nabla)\widehat{v} + \nabla\widehat{p} = g(x), \quad \operatorname{div}\widehat{v} = 0, \quad \widehat{v}|_{\Gamma_0} = 0 \tag{1.15}$$

Find a control $u(t, x)$ supported on Σ such that the solution (v, p) of problem (1.12), (1.14), (1.13) satisfies the estimate:

$$\|v(t, \cdot) - \widehat{v}\|^2_{v^1(\Omega)} \leqslant c\|v_0 - \widehat{v}\|_{V^1(\Omega)} e^{-\sigma t} \tag{1.16}$$

with a certain constant $c = c(\sigma, \Sigma) > 0$.

1.2.4 Reduction of the case with control on a boundary to the case with a control in initial condition.

Let $\omega \subset \mathbb{R}^d$ be a bounded domain such that $\Omega \cap \omega = \emptyset, \quad \overline{\Omega} \cap \overline{\omega} = \overline{\Gamma}$. We set

$$G = \operatorname{Int}(\overline{\Omega} \cup \overline{\omega}) \tag{1.17}$$

(the notation $\operatorname{Int} A$ means, as always, the interior of the set A).

We suppose that $\partial G \in C^\infty$. We extend problem (1.12)–(1.13) from Ω to G via Σ forgetting about the second condition in (1.14). For this we extend first the steady-state solution $(\widehat{v}, \widehat{p})$ of (1.6) satisfying $\widehat{v}|_{\Gamma_0} = 0$ from Ω in the pair $(a(x), q(x))$ defined on G satisfying:

$$a(x) \in V^2(G) \cap \left(H^1_0(G)\right)^d, \quad q(x) \in H^2(G), \tag{1.18}$$

After substitution (a, q) into the left part of equation (1.6) considered on G we obtain

$$-\Delta a(x) + (a, \nabla)a + \nabla q(x) = h(x), \quad \operatorname{div} a(x) = 0, \quad a|_{\partial G} = 0 \tag{1.19}$$

where, evidently, $h(x) \in L_2(G)$ is the extension of right side $g(x)$ from (1.15): $h|_\Omega = g$.

The extension of (1.12)–(1.14) from Ω to G can be written as follows:

$$\partial_t w(t, x) - \Delta w + (w(x), \nabla)w + \nabla p(t, x) = h(x), \quad \operatorname{div} w(t, x) = 0, \tag{1.20}$$

$$w(t,x)|_{t=0} = w_0(x) + u(x), \qquad w|_S = 0, \tag{1.21}$$

where $w_0(x) \in V_0^1(G)$ is some extension of initial condition v_0 from (1.13), $u(x) \in V_0^1(G)$, $\operatorname{supp} u \subset \omega_\varepsilon := \{x \in \omega : \operatorname{dist}(x, \partial\omega) > \varepsilon\}$ with small enough ε, and $S = \mathbb{R}_+ \times \partial G$.

Suppose, that we find a control u in (1.20),(1.21) such that the solution (w,p) of this boundary value problem satisfies the inequality

$$\|w(t,\cdot) - a\|_{V_0^1(G)} \leqslant ce^{-\sigma t}\|w_0 + u - a\|_{V_0^1(G)} \quad \text{for } t \geqslant 0 \tag{1.22}$$

For vector fields defined on G we denote by γ_Ω the operator of restriction on Ω and by γ_Γ we denote the operator of restriction on Γ:

$$\gamma_\Omega : V^k(G) \longrightarrow V^k(\Omega), \quad \gamma_\Gamma : V^k(G) \longrightarrow V^{k-1/2}(\Gamma),\ k \geqslant 0 \tag{1.23}$$

Evidently, these operators are well-defined and bounded (see Temam (1984)).

Introduce the functions:

$$v(t,\cdot) = \gamma_\Omega w(t,\cdot), \quad u(t,\cdot) = \gamma_\Gamma w(t,\cdot) \qquad \forall t \geqslant 0 \tag{1.24}$$

It is clear that if $w(t,\cdot)$ is the solution of boundary value problem (1.20)-(1.21) then $(v(t,\cdot), u(t,\cdot))$ is the solution of stabilization problem (1.12)–(1.16).

Evidently, if the solution w of (1.20)-(1.21) satisfies (1.22), the pair (v,u) defined in (1.24) satisfies (1.16). Hence (v,u) forms a solution of the initial stabilization problem (1.12)-(1.16).

1.3 Construction of stabilization for Oseen equations

In this and next sections we describe stabilization construction in the case of control belonging to initial condition.

1.3.1 Reduction to linear case

Let $G \subset \mathbb{R}^d$, $d = 2,3$ be a bounded domain with a boundary ∂G of class C^∞, $\omega \Subset G$ be a subdomain of G. We make change of unknown functions

$$w(t,x) = y(t,x) + a(t,x), \quad p(t,x) = s(t,x) + q(t,x) \tag{1.25}$$

in (1.20) where (a,q) is solution of (1.19). As a result we get

$$\begin{gathered} \partial_t y(t,x) - \Delta y + (a(x), \nabla)y + (y,\nabla)a \\ +(y,\nabla)y + \nabla s(t,x) = 0, \\ \operatorname{div} y = 0, \end{gathered} \tag{1.26}$$

$$y(t,x)|_{t=0} = y_0(x) + u(x) \tag{1.27}$$

where $y_0 = w_0 - a$. We omit in (1.26) nonlinear term $(y,\nabla)y$ and, changing notation s for pressure on p, we obtain:

$$\begin{gathered}\partial_t y(t,x) - \Delta y + (a(x),\nabla)y + (y,\nabla)a + \nabla p(t,x) = 0,\\ \operatorname{div} y = 0, \quad y|_{\partial G} = 0\end{gathered} \tag{1.28}$$

Set initial condition

$$y(t,x)|_{t=0} = y_0(x) \tag{1.29}$$

Our aim now is to describe the set of initial conditions $\{y_0\}$ such that solutions $y(t,x)$ of (1.28)-(1.29) satisfy estimate

$$\|y(t,\cdot)\|_{V_0^1(G)} \leqslant c\|y_0\|_{V_0^1(G)} e^{-\sigma t} \quad \text{for } t \geqslant 0 \tag{1.30}$$

1.3.2 Description of "correct" initial conditions

Denote by

$$\hat{\pi} : (L_2(G))^2 \longrightarrow V_0^0(G) \tag{1.31}$$

the operator of orthogonal projection. We consider the Oseen steady-state operator

$$Av := -\hat{\pi}\Delta v + \hat{\pi}[(a(x),\nabla)v + (v,\nabla)a] : V_0^0(G) \longrightarrow V_0^0(G) \tag{1.32}$$

and its adjoint operator A^*. These operators are closed and have the domain $\mathcal{D}(A) = V^2(G) \cap (H_0^1(G))^2$. Emphasize that $\mathcal{D}(A)$ consists of vector fields equal to zero on ∂G. The spectrums $\Sigma(A), \Sigma(A^*)$ of operators A and A^* are discrete subsets of a complex plane $\mathbb{C}$ which belong to a sector symmetric with respect to $\mathbb{R}$ and containing $\mathbb{R}_+$. In other words, A is a sectorial operator. So spectrums $\Sigma(A), \Sigma(A^*)$ contain only eigenvalues of A, A^*, respectively. Since $a(x)$ is real-valued vector field satisfying (1.18), they are symmetric with respect to $\mathbb{R}$, and moreover $\Sigma(A) = \Sigma(A^*)$.

We rewrite the boundary value problem (1.28)-(1.29) for Oseen equations in the following form

$$\frac{dy(t,\cdot)}{dt} + Ay(t,\cdot) = 0, \quad y|_{t=0} = y_0. \tag{1.33}$$

where A is the operator (1.32). Then for each $y_0 \in V_0^0(G)$ the solution $y(t,\cdot)$ of (1.33) is defined by $y(t,\cdot) = e^{-At}y_0$ where e^{-At} is the resolving semigroup of problem (1.33).

Let $\sigma > 0$ satisfy:

$$\Sigma(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = \sigma\} = \emptyset \tag{1.34}$$

The case when there are certain points of $\Sigma(A)$ which are in the left of the line $\{\mathrm{Re}\lambda = \sigma\}$ will be interesting for us.

Denote by $X_\sigma^+(A)$ the subspace of $V_0^0(G)$ generated by all eigenfunctions and associated functions of operator A corresponding to all eigenvalues of A placed in the set $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < \sigma\}$. By $X_\sigma^+(A^*)$ we denote analogous subspace corresponding to adjoint operator A^*. We denote the orthogonal complement to $X_\sigma^+(A^*)$ in $V_0^0(G)$ by $X_\sigma(A) \equiv X_\sigma$:

$$X_\sigma = V_0^0(G) \ominus X_\sigma^+(A^*) \tag{1.35}$$

One can show that subspaces $X_\sigma^+(A)$, X_σ are invariant with respect to the action of semigroup e^{-At}, and $X_\sigma + X_\sigma^+(A) = V_0^0(G)$.

Theorem 1.3.1 *Suppose that A is operator* (1.32) *and $\sigma > 0$ satisfies* (1.34). *Then for each $y_0 \in X_\sigma$ the inequality (1.30) holds. Besides, the solution of problem (1.33) with such initial conditions are defined by the formula*

$$y(t,\cdot) = e^{-At}y_0 = (2\pi i)^{-1}\int_\gamma (A - \lambda I)^{-1}e^{-\lambda t}y_0 d\lambda. \tag{1.36}$$

Here γ is a contour belonging to $\rho(A) := \mathbb{C} \setminus \Sigma(A)$ such that $\arg\lambda = \pm\theta$ for $\lambda \in \gamma, |\lambda| \geqslant N$ for certain $\theta \in (0, \pi/2)$ and for sufficiently large N. Moreover, γ encloses from the left the part of the spectrum $\Sigma(A)$ placed right of the line $\{\mathrm{Re}\lambda = \sigma\}$. The complementary part of the spectrum $\Sigma(A)$ is placed left of the contour γ.

Proof: See Fursikov (2001a), Fursikov (2001b).

1.3.3 Theorem on stabilization of Oseen equations

Recall that stabilization problem is to find a control

$$u \in V_{00}^1(\omega) := \{w \in V_0^1(G) : w(x) = 0 \ \forall x \in G \setminus \omega)\} \tag{1.37}$$

such that the solution y of (1.28),(1.27) satisfies (1.30) with y_0 changed on $y_0 + u$. To complete the construction of stabilization for Oseen equations (1.28), (1.27) we have to construct the operator $E : V_0^1(G) \to V_{00}^1(\omega)$ that transforms arbitrary initial condition y_0 from (1.27) to control u such that $y_0 + u \in X_\sigma$. We consider here analog of construction from Fursikov (2004).

It is known that in the space $X_\sigma^+(A^*)$ one can choose a basis $(d_1(x),\dots, d_K(x))$ such that restriction $(d_1(x)|_\omega,\dots,d_K(x)|_\omega)$ on an arbitrary subdomain $\omega \Subset G$ forms a linear independent set of vector fields. This property had been proved in Fursikov (2002a), Fursikov (2004) with help of Carleman estimates and one abstract result from Fursikov (2001a). We can define space (1.35) by the following equivalent form:

$$X_\sigma = \{v(x) \in V_0^0(G) : \int_G v(x)\cdot d_j(x)\,dx = 0, \quad j = 1,\dots,K\}. \tag{1.38}$$

Theorem 1.3.2 *(Fursikov (2002a), Fursikov (2004)) There exists a linear bounded operator*

$$E : V_0^1(G) \to V_{00}^1(\omega) \quad \text{such that} \quad y_0 + Ey_0 \in X_\sigma. \tag{1.39}$$

Proof Let subset $\omega_1 \subset \omega$ be a domain with C^∞- boundary $\partial\omega_1$. In this set we consider the Stokes problem:

$$-\Delta w(x) + \nabla p(x) = v(x), \quad \operatorname{div} w(x) = 0, \quad x \in \omega_1; \quad w|_{\partial\omega_1} = 0$$

As is well known, for each $v \in V^0(\omega_1)$ there exists a unique solution $w \in V_0^1(\omega_1)\cap V^2(\omega_1)$ of this problem. The resolving operator to this problem we denote as follows: $(-\hat\pi\Delta)_{\omega_1}^{-1}v = w$. Extension of $(-\hat\pi\Delta)_{\omega_1}^{-1}v$ from ω_1 in G by zero we also denote as $(-\hat\pi\Delta)_{\omega_1}^{-1}v$. Evidently, $(-\hat\pi\Delta)_{\omega_1}^{-1}v \in V_{00}^1(\omega_1)$.

We look for the desired operator E in the form

$$Ev(x) = \left[\sum_{j=1}^K c_j(-\hat\pi\Delta)_{\omega_1}^{-1}d_j\right](x), \tag{1.40}$$

where $c_j = c_j(v)$ are constants which should be determined. Evidently, $Ev \in V_0^1(G)$, $\operatorname{supp} Ev \subset \overline{\omega}_1$. To define constants c_j we note that by (1.38) $v + Ev \in X_\sigma$ if

$$\int_G d_k(x)\left[\sum_{j=1}^K c_j(-\hat\pi\Delta)_{\omega_1}^{-1}d_j(x)\right]dx = -\int_G d_k(x)v(x))\,dx \tag{1.41}$$

for $k = 1,\dots,K$. As in Fursikov (2002a), Fursikov (2004) one can prove that this system of linear equations has a unique solution. □

Thus, in virue of this theorem in order to stabilize problem (1.28),(1.27) one has to take $u = Ey_0$.

1.4 Stabilization for Navier-Stokes equations

In this section we give a construction for stabilization of problem (1.26), (1.27) obtained from Navier-Stokes system (1.20) by change of unknown function.

1.4.1 Definition of stable invariant manifold

Natural space for solution of problem (1.26), (1.29) is

$$V^{1,2(0)}(Q_T) = L_2(0,T;V^2(G)\cap V_0^1(G))\cap H^1(0,T;V_0^0(G)),$$

and in virtue of inclusion $C(0,T;V_0^1(G)) \subset V^{1,2}(Q_T)$ natural phase space V for corresponding dynamical system is $V_0^1(G)$. Definition of spaces given around (1.35) and relations for them imply:

$$\begin{array}{c} V = V_+ + V_- \quad \text{where} \quad V = V_0^1(G), \\ V_+ = X_\sigma^+(A),\ V_- = X_\sigma \cap V_0^1(G) \end{array} \tag{1.42}$$

It is well-known (see Ladyzhenskaya (1963), Temam (1984)), that for each $y_0 \in V$ there exists a unique solution $y(t,x) \in V^{1,2(0)}(Q_{T_{\|v_0\|}})$ of problem (1.26),(1.29), where $0 < T_{\|v_0\|} \to \infty$ as $\|v_0\| := \|v_0\|_V \to 0$. Denote by $S(t,y_0)$ the solution operator of the boundary value problem (1.26),(1.29):

$$S(t,y_0) = y(t,\cdot) \tag{1.43}$$

where $y(t,x)$ is the solution of (1.26),(1.29). Then for S the following semigroup property holds:

$$S(t_2,S(t_1,y_0)) = S(t_1+t_2,y_0), \quad \forall\, t_1,t_2 \geqslant 0.$$

The triple $\{V,S(t,\cdot),t\geqslant 0\}$ is called (semi)dynamical system with the space of states (phase space) V, resolving operator $S(t,\cdot)$, and continuous time $t \geqslant 0$.

Recall now some commonly used concept of *the stable invariant manifold* $\mathcal{W}_- = \mathcal{W}_-(\mathcal{O})$ defined in a neighborhood $\mathcal{O}$ of the origin. By the definition

$$\begin{array}{c} \mathcal{W}_-(\mathcal{O}) = \{y_0 \in \mathcal{O} : \\ S(t,y_0) \subset \mathcal{O}, \|S(t,y_0)\|_V \leqslant c\|y_0\|_V e^{-\sigma t}, \quad t \geqslant 0\} \end{array} \tag{1.44}$$

where quantities $c > 0, \sigma > 0$ does not depend on y_0. Manifold $\mathcal{W}_-(\mathcal{O})$ contains all points y_0 of the neighborhood $\mathcal{O}$, such that their trajectories $S(t,y_0)$ tends to zero with asymptotic rate not less than $e^{-\sigma t}$. Using this property one can reduce solution of stabilization problem (1.7), (1.24) to projection on $\mathcal{W}_-(\mathcal{O})$.

Stable invariant manifold satisfies the following invariantness condition: $S(t, \mathcal{W}_-(\mathcal{O})) \subset \mathcal{W}_-(\mathcal{O})$. Moreover, in the small neighbourhood $\mathcal{O}$ the stable invariant manifold can be defined as a graph in the phase space $V = V_+ + V_-$ by the formula

$$\mathcal{W}_-(\mathcal{O}) = \mathcal{W}_-(\mathcal{O}, f) := \{y \in V : y = y_- + f(y_-),\ y_- \in \mathcal{O}(V_-)\} \quad (1.45)$$

where $\mathcal{O}(V_-)$ is a neighborhood of the origin in the subspace V_-, and

$$f : \mathcal{O}(V_-) \to V_+ \quad (1.46)$$

is a certain map satisfying

$$\|f(y_-)\|_+ / \|y_-\|_{V_-} \to 0 \quad \text{as} \quad \|y_-\|_{V_-} \to 0. \quad (1.47)$$

Since $\mathcal{W}_-$ is defined by the map $y_+ = f(y_-)$, using the term manifold is quite natural in this case.

The following existence theorem for invariant manifold $\mathcal{W}_-$ holds:

Theorem 1.4.1 *There exists unique map (1.46) such that the set $\mathcal{W}_-$ defined by formula (1.45) is stable invariant manifold for family of maps $S(t, \cdot)$ defined in (1.43). Moreover,*

$$\|S(t, y_0)\|_V \leqslant ce^{-\sigma t}\|y_0\|_V \qquad \text{as} \quad t \to \infty \quad (1.48)$$

where constants $c > 0, \sigma > 0$ do not depend on $y_0 \in \mathcal{W}_-$

This theorem as well as method of its proof is well-known (see Ladyzhenskaya & Solonnikov (1973), Marsden & McCracken (1976), Henry (1981), Babin & Vishik (1992) and references there in). As it had been established in Fursikov (2010), the domain $\mathcal{O}(V_-)$ of operator (1.46) is unbounded with respect to the norm of the space V_-.

1.4.2 Feedback operator and stabilization.

Here we construct feedback operator for Navier—Stokes equations. This operator is nonlinear analog of feedback operator (1.39) constructed for Oseen equations.

As in Theorem 1.3.2 we use the domain $\omega \Subset G$ and the space $V^1_{00}(\omega)$ defined in (1.37). Denote $\mathcal{O}_\varepsilon = \{v \in V : \|v\|_V < \varepsilon\}$.

Theorem 1.4.2 *Suppose that $\mathcal{W}_-$ is the invariant manifold constructed in a neighborhood of origin in $V = V_0^1(G)$ in Theorem 1.4.1. Then for sufficiently small ε there exists a continuous operator*

$$F : \mathcal{O}_\varepsilon \to V_{00}^1(\omega), \tag{1.49}$$

such that

$$v + F(v) \in \mathcal{W}_- \quad \forall v \in \mathcal{O}_\varepsilon. \tag{1.50}$$

Proof We introduce projection operators

$$P_+ : V \to V_+, \qquad P_- : V \to V_- \tag{1.51}$$

for the spaces defined in (1.42) and the following notations:

$$Qv(x) = v(x) + w(x), \quad \text{where} \quad w = F(v) \in V_{00}^1(\omega), \tag{1.52}$$

and F is the operator we are looking for. By (1.51) and definition (1.45) of invariant manifold $\mathcal{W}_-$ the desired inclusion $Qv \in \mathcal{W}_-$ is equivalent to the following equality:

$$P_+ Qv = f(P_- Qv) \tag{1.53}$$

where f is operator (1.46). Besides, we have to ensure that the equality

$$(Qv)(x) \equiv v(x), \quad x \in G \setminus \omega \tag{1.54}$$

is true. By (1.42),(1.38) $\{d_j(x)\}$ generates V_+ and therefore the map $f(u)$ can be written in the form

$$f(u) = \sum_{j=1}^{K} d_j f_j(u)$$

and equality (1.53) is equivalent to the following one:

$$\int_G Qv(x) d_j(x)\, dx = f_j(P_- Qz), \quad j = 1, \ldots, K. \tag{1.55}$$

Similarly to (1.40) we look for the vector field $w(x)$ from (1.52) in the form

$$w = -(-\hat{\pi}\Delta)_{\omega_1}^{-1} \sum_{j=1}^{K} p_j d_j \tag{1.56}$$

To find coefficients $(p_1, \ldots, p_K) \equiv \vec{p}$ we substitute (1.56) into (1.55) taking into account (1.52). As a result we get

$$\vec{v} - A\vec{p} = \vec{f}\big(v - (\vec{p}, (-\hat{\pi}\Delta)_{\omega_1}^{-1} \vec{d}) - (\vec{d}, \vec{v} - A\vec{p})\big), \tag{1.57}$$

where $\vec{v} = (v_1, \dots, v_K)$, $A = \|a_{jk}\|$ and

$$v_j = \int\limits_G (v(x), d_j(x))\, dx, \quad a_{jk} = \int\limits_G ((-\hat{\pi}\Delta)^{-1}_{\omega_1} d_k(x), d_j(x))\, dx,$$

$$\vec{f}(u) = (f_1(u), \dots, f_K(u)), \quad \vec{d} = (d_1(x), \dots, d_K(x)), \quad (\vec{c}, \vec{d}) = \sum_{j=1}^{K} c_j d_j.$$

Taking into account invertibility of matrix $A = \|a_{jk}\|$ ascertained in Theorem's 1.3.2 proof one can apply to equation (1.57) contraction mapping principle (see Fursikov (2004)) As a result we obtain that if $\|\vec{v}\|$ is sufficiently small, equation (1.57) possesses unique solution $\vec{p}$. The last assumption is fulfilled because ε in (1.49) is small enough.

□

Now in virtue of Theorem 1.4.2 for stabilization of problem (1.28), (1.37) one has to take $u = F(y_0)$.

1.5 Feedback property for a control

In this section we discuss feedback property that plays the central role in the stabilization problem. We consider here the cases of initial control and a control in right-hand side, both usual and impulse.

1.5.1 Definitions. The case of initial control

The important and distinctive property of the control used for stabilization of a solution to unstable dynamical system is feedback property. Just this property allows to stabilize a system in unstable situation and, in particular, to create numerical algorithm simulating original stabilization problem that can be realized in real time, i.e. simultaneously with functioning of original stabilization problem.

Let

$$v'(t) = f(v(t)) + B(u(t)), \ v|_{t=0} = v_0 \tag{1.58}$$

be a controlled dynamical system in phase space V $(v(t) \in V, \forall t > 0)$ with space of controls u $(u(t) \in U, \forall t > 0)$. Here $v(t)$ is the state variable, $u(t)$ is a controle, $B : U \to V$ is a continuous operator, $B(0) = 0$. Suppose also that $\hat{v} \in V$ is unstable steady-state solution of problem (1.58) without control: $f(\hat{v}) = 0$, and stabilization problem to $\hat{v}$ of dynamical system (1.58) by means of control u is considered.

In applied sciences the following not rigorous but very clear definition of feedback control is used very often.

Definition 1.5.1 *A control $u(t)$ stabilizing dynamical system* (1.58) *is called feedback if it can react on unpredictable fluctuations of state variable $v(t)$ dumping them.*

The most popular mathematical formalization of this notion is as follows:

Definition 1.5.2 *The control $u(t)$ is called feedback if there exists a continuous operator $F: V \to U$ such that $u(t) = F(v(t))$ for each $t \geqslant 0$ and the dynamical system*

$$v'(t) = f(v(t)) + B(F(v(t))), \; v|_{t=0} = v_0$$

is stable in a neighborhood of $\hat{v}$ with respect to fluctuations of initial condition v_0.

Let pass from general definitions to the concrete stabilization problem (1.26), (1.27), with a control $u \in V^1_{00}(\omega)$ in initial condition. In this problem no unpredictable fluctuations were introduced, and therefore Definition 1.5.1 can not be applied to this case. From to other side Definition 1.5.2 also cannot be applied because control u depends on t there. But if we consider analog of problem (1.58) with control u independent from time, the (1.26), (1.27) will be a particular case of such problem, and the control constructed in Theorem 1.4.2 for problem (1.26), (1.27) will satisfy feedback property in the meaning of Definition 1.5.2 where control does not depend on t and steady-state solution $\hat{v} \equiv 0$. Applying to (1.26) change of functions (1.25) we get that stabilization problem with initial control (1.20), (1.21) possesses solution with feedback control as well. In subsection 1.5.3 we introduce the notion of unpredictable fluctuation of state variable and show that stabilization problem (1.1)-(1.3) with impulse control (1.4) possesses solution with feedback control in the meaning of Definition 1.5.1, but first we construct in subsection 1.5.2 feedback control for stabilization problem with usual (not impulse) control in right-hand side.

1.5.2 The case of distributed control supported in subdomain

For simplicity we consider here only the case of Ozeen equation. Generalization on Navier-Stokes system can be made similarly as in subsection

1.4.2 above (see details in Fursikov (2001a), Fursikov (2001b), Fursikov (2004)).

So let consider the boundary value problem

$$\frac{v(t,\cdot)}{dt} + Av(t,\cdot) = u(t,\cdot), \quad v|_{t=0} = v_0 \tag{1.59}$$

with $v_0 \in V_0^1(G)$ be given, A be operator (1.32), $u(t,x) \in L_2(\mathbb{R}_+; V_{00}^1(\omega))$ be a control that we are looking for. This control has to satisfy the following conditions:

i) The solution of problem (1.59) satisfies the estimate

$$\|v(t,\cdot)\|_{V_0^1(G)} \leqslant C\|v_0\|_{V_0^1(G)} e^{-\sigma t} \tag{1.60}$$

where constant $C = C_\sigma$ does not depend on $\|v_0\|_{V_0^1(G)}$.
Moreover we are looking for the feedback control. By the definition this means that

ii) There exists a linear bounded operator: $E : V_0^1(G) \to V_{00}^1(\omega)$ such that control $u(t,\cdot)$ is expressed by phase function $y(t,\cdot)$ with help of the formula

$$u(t,\cdot) = \Lambda E v(t,\cdot) \tag{1.61}$$

where magnitude $\Lambda > 0$ will be chosen later.

Theorem 1.5.3 *There exists a control $u(t,x) \in L_2(\mathbb{R}_+; V_{00}^1(\omega))$ that satisfies conditions $i), ii)$ written above.*

Proof Recall that phase space $V = V_0^1(G)$ admits decomposition (1.42), in $V_+ = X_\sigma^+(A)$ one can choose a basis $(e_1(x), \dots, e_K(x))$ constructed from eigen and associated functions of operator A corresponding to eigenvalues λ_j with $\operatorname{Re}\lambda_j < \sigma$, (see Fursikov (2001a), Fursikov (2001b)). Besided, in $X_\sigma^+(A^*)$ one can choose a basis $(d_1(x), \dots, d_K(x))$ constructed from eigen and associated functions of operator A^* corresponding to eigenvalues μ_j with $\operatorname{Re}\mu_j < \sigma$, (Fursikov (2001a), Fursikov (2001b)). These basises are biorthogonal, i.e. they satisfy: $(e_j, d_m)_{L_2(G)} = \delta_{jm}$, where δ_{jm} is Kronecker symbol. Therefore $v \in V_+$ if and only if

$$v = \sum_{j=1}^{K} v_j e_j(x), \quad \text{where} \quad v_j = (v, d_j)_{L_2(\Omega)} \tag{1.62}$$

We define desired operator E by formulas (1.40), (1.41) explained in the proof of the Theorem 1.3.2. Compeering (1.41), (1.62) we see that in fact

$$u(t,\cdot) = \Lambda E v(t,\cdot) = \Lambda E P_+ v(t,\cdot) \tag{1.63}$$

where P_+ is the projector defined in (1.51). After substitution (1.63) into (1.59) and applying to obtained equation projector P_+, P_- we get using notation

$$v_+(t,\cdot) = P_+ v(t,\cdot), \quad v_-(t,\cdot) = P_- v(t,\cdot)$$

that problem (1.59), (1.63) is equivalent to the following one:

$$\frac{dv_+(t,\cdot)}{dt} + Av_+ = \Lambda P_+ E v_+(t,\cdot), \quad v_+|_{t=0} = v_{0+} \equiv P_+ v_0 \tag{1.64}$$

$$\frac{dv_-(t,\cdot)}{dt} + Av_- = \Lambda P_- E v_+(t,\cdot), \quad v_-|_{t=0} = v_{0-} \equiv P_- v_0 \tag{1.65}$$

Using the notations: $\bar{c} = (c_1, \dots, c_K)$, $\bar{v} = (v_1, \dots, v_K)$,

$$F_{kj} = \int_G d_k(x)(-\hat{\pi}\Delta)^{-1}_{\omega_1} d_j(x)dx =$$
$$\int_{\omega_1} \nabla(-\hat{\pi}\Delta)^{-1}_{\omega_1} d_k(x) \cdot \nabla(-\hat{\pi}\Delta)^{-1}_{\omega_1} d_j(x)dx,$$
$$F = (F_{kj})^K_{k,j=1}$$

we can rewrite (1.41) in the form $F\bar{c} = -\bar{v}$, and (1.40) as follows:

$$Ev(x) = -\sum_{j=1}^K (F^{-1}\bar{v})_j(-\hat{\pi}\Delta)^{-1}_{\omega_1} d_j(x), \quad \text{where } (F^{-1}\bar{v})_j = c_j \tag{1.66}$$

Applying to (1.66) operator P_+ we get

$$P_+ Ev(x) = -\sum_{k,j=1}^K (F^{-1}\bar{v})_j F_{kj} e_k(x) = -\sum_{k=1}^K (FF^{-1}\bar{v})_k e_k(x) = -P_+ v(x)$$

Therefore (1.64) is equivalent to the problem:

$$\frac{dv_+(t,\cdot)}{dt} + (A|_{V_+} + \Lambda E)v_+(t,\cdot) = 0, \quad v_+|_{t=0} = v_{0+} \tag{1.67}$$

where $A|_{V_+}$ is restriction of operator A on V_+ (recall that V_+ is invariant with respect of A), and E is identity operator. We choose now $\Lambda > 0$ such that

$$\text{Re}\,\lambda_j + \Lambda > \sigma + \varepsilon \tag{1.68}$$

for each eigenvalue λ_j of operator $A|_{V_+}$ where $\varepsilon > 0$ is fixed. Then (1.67), (1.68) implies:

$$\|v_+(t,\cdot)\|_{V_+} \leqslant C\|P_+ v_0\|_{v_+} e^{-(\sigma+\varepsilon)t} \tag{1.69}$$

where $C = C_{\sigma+\varepsilon}$ does not depend on v_0, and solution v_- is defined by the formula:

$$v_-(t,\cdot) = e^{-At}P_-v_0 + \int_0^t e^{-A(t-\tau)}(P_-Ev_+(\tau,\cdot))d\tau \tag{1.70}$$

where e^{-At} is operator (1.36). In Fursikov (2001a), Fursikov (2001b) the following estimate for operator e^{-At} had been proved

$$\|e^{-At}P_-v_0\|_{V_-} \leqslant Ce^{-\sigma t}\|P_-v_0\|_{V_-} \tag{1.71}$$

with constant $C = C_\sigma$ independent from $\|P_-v_0\|_{V_-}$.

Applying (1.69), (1.71) to (1.70) we obtain

$$\begin{aligned} &\|v_-(t,\cdot)\|_{V_-} \leqslant C_1e^{-\sigma t}\|P_-v_0\|_{V_-} \\ &+C_2\textstyle\int_0^t e^{-\sigma(t-\tau)}e^{-(\sigma+\varepsilon)\tau}d\tau\|P_+v_0\|_{V_+} \\ &\leqslant e^{-\sigma t}(C_1\|P_-v_0\|_{V_-} + C_2\frac{1-e^{-\tau}}{\varepsilon}\|P_+v_0\|_{V_+}) \leqslant C\|v_0\|_V e^{-\sigma t} \end{aligned} \tag{1.72}$$

Bounds (1.69), (1.72) imply (1.60).

□

1.5.3 Real processes

General theory of real processes had been work out in Fursikov (2002b), Fursikov (2002c) for stabilization problems with boundary control. Here we recall the main ideas of this theory in the case of impulse control. Note that in this case the theory of real processes is more transparent than for boundary control. Let consider stabilization problem (1.26), (1.27) with control in initial condition. If initial condition y_0 from (1.27) satisfies the bound $\|y_0\|_{V_0^1(G)} < \varepsilon$ with small enough ε, then by Theorem 1.4.2 we can take in (1.27) feedback control $u = F(y_0)$ and obtain $y_0 + u = y_0 + F(y_0) \in \mathcal{W}_-$ where $\mathcal{W}_-$ is stable invariant manifold (1.45). This solves problem from pure mathematical point of view. Our aim now is to justify numerical solution of this problem. Suppose that we calculate problem (1.26), (1.27) with initial condition $y_0 + F(y_0)$ in discrete time instants $t_k = k\tau$ where $k = 0, 1, 2, \ldots$ and $\tau > 0$ is fixed. Denote $S(y_0) = S(\tau, y_0)$ where $S(t, y_0)$ is solution operator (1.43) of problem (1.26), (1.29). Let w^k be the result of our calculation at time instant t_k. Since numerical calculation can not be exact, we have

$$w^k = S(w^{k-1}) + \tau\varphi^k \tag{1.73}$$

where φ^k is an error of calculation which is unknown for us before time t_k (i.e. φ^k is unpredictable fluctuation; we introduce multiplier τ in (1.73) for convenience of normalization). The sequence $\{w^k\}$ defined in (1.73) is called uncontrolled real process. We suppose that we can estimate the error of our calculation a priori:

$$\|\varphi^k\|_{V_0^1(G)} \leqslant \tilde{\varepsilon} \ll \varepsilon, \quad \forall k > 0. \tag{1.74}$$

where $\varepsilon > 0$ is magnitude from Theorem 1.4.2. It follows from (1.73) that $w^k \notin \mathcal{W}_-$ begining from $k = 1$. Therefore in virtue of wellknown structure of phase flow in a neighborhood of steady-state solution, w^k moves away origin as $k \to \infty$ and our stabilization construction collapses.

To keep our stabilization construction we have to pass from control in initial condition to impulse feedback control. In other words we have to change recurrence relation (1.73) on the following one (we assume that calculation of initial condition $\tilde{w}_0$ is absolutely exact:

$$\tilde{w}^0 = y_0 + F(y_0), \tilde{w}^k = S(\tilde{w}^{k-1} + F(\tilde{w}^{k-1})) + \tau\varphi^k, \quad k = 1, 1, \dots \tag{1.75}$$

It follows from (1.75) that for each instant $t_k = \tau k$ real process $\tilde{w}^k$ does not belong to invariant manifolds $\mathcal{W}_-$ and that is why we apply at each $t_k = \tau k$ impulse feedback control $\tilde{w}^k \to \tilde{w}^k + F(\tilde{w}^k)$ to return on $\mathcal{W}_-$. The following theorem is true:

Theorem 1.5.4 *Let F be feedback map constructed in Theorem 1.4.2, unpredictable fluctuations φ^k satisfy (1.74), and $\|y_0\|_V < \varepsilon$ with ε defined in Theorem 1.4.2. Then real process $\tilde{w}^k$ constructed by recurrence relation (1.75) satisfies the following estimate:*

$$\|\tilde{w}^k\|_V \leqslant \hat{C}(e^{-\sigma k\tau}\|\tilde{w}^0\|_V + (1 + \sigma^{-1})\hat{\varepsilon}), \quad k \to \infty \tag{1.76}$$

where constant $\hat{C}$ is constructed by constants estimating operator F from (1.46), (1.50).

This Theorem is proved similarly to analogous assertion obtained in Fursikov (2002b). Note that in contrast to estimates (1.48), (1.60) proved for feedback initial control in Theorems 1.4.2, 1.4.1 and for feedback control in right-hand side in Theorem 1.5.3, the right side of bounds (1.76) does not tend to zero as $k \to \infty$. This is quite natural because by definition of real process unpredictable fluctuation arise at $t_k = k\tau$ for each $k \in N$.

It is necessary to mention that if we assume that inpredictable fluctuations $\{\varphi^i, i \in N\}$ in (1.75) is independently identically distributed sequence of random vector fields $\varphi^i \in V_0^1(G)$ then under additional natural assumptions on random sequence $\{\varphi^i\}$, the random dynamical system $\tilde{w}^k$ defined in (1.75) is ergodic (i.e. it has unique stationary measure $\tilde{\mu}$), and it possesses the property of exponential mixing (i.e. probability distribution of w^k tends exponentially to $\tilde{\mu}$ in some natural meaning). This fact has been established in Duan & Fursikov (2005) in the case of stabilization problem for Ozeen system.

We will not discuss here theory of real processes in the case of boundary control because this theory is exposed in details in Fursikov (2002b), Fursikov (2002c).

1.6 Description of numerical algorithms

The second part of the paper is devoted to description of numerical schemes of the stabilization construction. The most difficult part of this construction is connected with calculations of stable invariant manifolds (see (1.45)) and with calculation of projection operators on these sets. Since the axiomatic presentation is used below, explain that operator $S(\cdot)$ used there is abstract analog of the operator $S(T,\cdot)$ with big enough $T>0$ where $S(t,\cdot),\ t>0$ is the solution operator of the boundary value problem (1.26),(1.29). Besides, $S^{n+1}(\cdot)=S(S^n(\cdot)),\ n=1,2,\ldots$. Below we study stable invariant manifold (1.45) and local stable manifold as well. Emphasize that the local stable manifold studied below is much more general object than stable invariant manifold (1.45), because the last one is connected with a fixed point, and local stable manifolds are connected with time-dependent trajectory.

1.6.1 General definitions

We describe numerical algorithms construction in terms of (semi)dynamical systems with discrete time. Let V be a Banach space with the norm $\|\cdot\|$, $S(\cdot):V\to V$ be a smooth map that, evidently, satisfy the following semigroup property:

$$S^{i_1}(S^{i_2}(u))=S^{i_1+i_2}(u),\quad \forall i_1,i_2\in\mathbb{N},\forall u\in V.$$

Then $\{V,S^i(\cdot),i=0,1,2,\ldots\}$ is called (semi)dynamical system with the state space V, resolving operator S and discrete time $i=0,1,2,\ldots$.

The set $\Gamma_+(z_0)=\{z_i=S^i(z_0),i=0,1,\ldots,n,\ldots\}$ is called trajectory of the point $z_0\in V$. We suppose that the map S is smooth enough, so one can construct the linearization of S: $S(z_i+u)=S(z_i)+L(z_i)u+R(z_i)[u]$ in a neighborhood $\mathcal{O}_{z_i}$ of each point $z_i\in\Gamma_+(z_0)$. Moreover, for the bounded linear operator $L(z_i):V\to V$ and for continuous map

$R(z_i)[u] = S(z_i + u) - S(z_i) - L(z_i)u$ there exist projection operators $P_\pm(z_i)$ and magnitudes $\mu_-^{(i)}$, $\mu_+^{(i)}$, $r^{(i)}$, $C_\pm^{(i)} > 0$, such that the following hyperbolicity conditions (Anosov (1967), Pesin (1977)) take place in a neighborhood $\mathcal{O}_{z_i} = \{u : \|P_\pm(z_i)(z_i - u)\| \leqslant r^{(i)}\}$:

Conditions (A)

$$
\begin{array}{ll}
A_1) & P_+(z_i) + P_-(z_i) = I,\ \|P_\pm(z_i)\| \leqslant C_\pm^{(i)}; \\
A_2) & L(z_i)\big(P_+(z_i)V\big) = P_+(z_{i+1})V, \quad L(z_i)\big(P_-(z_i)V\big) \subset P_-(z_{i+1})V; \\
A_3) & \|L(z_i)w\| \leqslant \mu_-^{(i)}\|w\| \quad \forall w \in P_-(z_i)V, \quad \mu_-^{(i)} < \mu; \\
A_4) & \|L(z_i)v\| \geqslant \mu_+^{(i)}\|v\| \quad \forall v \in P_+(z_i)V, \quad \mu_+^{(i)} > \mu; \\
A_5) & \|P_\pm(z_{i+1})\{R(z_i)[u_1] - R(z_i)[u_2]\}\| \leqslant \\
 & \qquad \theta_\pm^{(i)}\Big(\max\{\|u_1\|, \|u_2\|\}\Big)\|u_1 - u_2\|, \quad \forall u_{1,2} : z_i + u_{1,2} \in \mathcal{O}_{z_i}
\end{array}
$$

where $\theta_\pm^{(i)}(\gamma)$ are continuous positive nondecreasing functions of $\gamma > 0$, $\theta_\pm^{(i)}(0) = 0$ and $\mu_\pm^{(i)}$, $r^{(i)}$, $C_\pm^{(i)}$ are certain parameters. We suppose also that $\mu \leqslant 1$.

These conditions (A) mean that in a neighborhood of each point z_i there exist subspaces $P_+(z_i)V$ and $P_-(z_i)V$ that are expanded and are contracted, correspondingly, by acting of the linear part $L(z_i)$ of map S. The phase space V is decomposed in their direct sum: $V = P_+(z_i)V + P_-(z_i)V$. Applying to this decomposition operator $L(z_i)$ we get analogous decomposition at point z_{i+1}. Assume that dimension of the subspace $P_+(z_i)V$ is finite, the stable subsapce $P_-(z_i)V$ has finite codimension. These properties are typical for problem of mathematical physics.

Generalized Hadamard-Perron theorem claims that if parameters and functions from condition (A) satisfy certain relations (i.e. for so called trajectories of hyperbolic type Anosov (1967), or for partially nonuniformly hyperbolic trajectories Pesin (1977)), then there exists a stable invariant manifold

$$\mathcal{W}_-(S, \mathcal{O}) = \{m : m \in \mathcal{O}_{z_i}, \|S^n(m) - S^n(z_i)\| \leqslant C\mu^n, n, i \geqslant 0\}$$

in the neighborhood $\mathcal{O} = \cup_{i=0}^\infty \mathcal{O}_{z_i}$. Moreover, in a neighborhood of each point z_i this manifold can be defined by a map $\mathbf{f}^{(i)}$:

$$\mathcal{W}_-(S, \mathcal{O})\big|_{\mathcal{O}_{z_i}} = \mathcal{W}_-(z_i, \mathbf{f}^{(i)})$$

(as in formula (1.44), (1.45),(1.47) above; here we take $\mu = e^{-\sigma T}$).

Our goal is to realize approximate construction of local stable manifold $\mathcal{W}_-(z_0, \mathbf{f}^{(0)})$, i.e. of the map $\mathbf{f}^{(0)}$ that determines this manifold. Note again that $\mathcal{W}_-(z_0, \mathbf{f}^{(0)})$ contains all points belonging to $\mathcal{O}_{z_0}$ whose trajectories tend to trajectory of z_0 with prescribed rate $C\mu^n$. Since numerical stabilization problem are considered on finite time interval, we assume below that conditions (A) are realized only for finite segment of trajectory $\Gamma_+^n(z_0) = \{z_i = S^i(z), i = 0, 1, \dots, n\}$. (Although schemes considered below admit uniform closure (at least formally) on the case of infinite n and can give constructive proof for existence of local stable manifold $\mathcal{W}_-(z_0, \mathbf{f}^{(0)})$.

Remark 1.6.1 *When numerical solving of stabilization problems on V, evolution operator $S_\tau(\cdot)$ with "slow" discrete time usually is given. In this situation one has to do a formal change $S(u) := S_\tau^{N_0}(u)$ with some "typical" $N_0 \in \mathbb{N}$. Here and below we write $S^i(u) = S(S^{i-1}(u))$, $i = 1, 2, \dots$. For operator $S(t, u)$ with continuous time $t \in [0, \infty[$ corresponding, for instance, to problem (1.26),(1.29) the passage to discrete teme is realized by the change $S(u) := S_\tau^{N_0}(u) := S(N_0\tau, u)$ with some not necessary small $\tau > 0$. This change helps to realize implementation of conditions $(A_{3,4})$ in the case nontrivial Jordan boxes and increases effectiveness of proposed schemes.*

1.6.2 Stable invariant manifold for a fixed point

Let consider the problem of approximate construction of a stable invariant manifold in a neighborhood of a fixed point $z_0 = S(z_0)$. Assume for simplicity that $z_0 = 0$. In this case all operators, functions, and constants from conditions (A) do not depend on index (i), and we will use the following notations: $S(u) = Lu + R(u)$, $P_\pm$, $\mu_\pm$, $r, C_\pm$. Note also that subspace P_+V is a union of root subspaces and that is why subspaces $P_\pm V$ can be constructed solving the problem of spectrum dichotomy for operator L by circumference of sertain radius (taking into account nontriviality of Jordan boxes).

In virtue of conditions (A) the operator $S(u) = Lu + R(u)$ with $u = v + w$, $v \in P_+\mathcal{O}$, $w \in P_-\mathcal{O}$, can be written in the form $S(u) = S_+(u) + S_-(u)$, where $S_\pm(u) = P_\pm S(u)$. Here

$$\begin{aligned}
&S_+(v + w) = L_+v + R_+(v + w);\\
&S_-(v + w) = L_-w + R_-(v + w);\\
&L_\pm u = P_\pm Lu, \quad R_\pm(u) = P_\pm R(u).
\end{aligned}$$

Let consider the class $B_\gamma(\mathcal{O})$ of all Holder maps $f(w) : P_-\mathcal{O} \to P_+\mathcal{O}$, where $\mathcal{O} = \{u : \|P_\pm(u)\| \leqslant r\}$, that satisfy conditions $f(0) = 0$, $\|f(w_1) - f(w_2)\| \leqslant \gamma\|w_1 - w_2\|$ with fixed Holder constant γ. For this class of elements $B_\gamma(\mathcal{O})$ we define the norm $|f| = \sup_{w \in P_-\mathcal{O}} |f(w)|$. We look for a manifold in the form

$$\mathcal{W}_-(\mathcal{O}) = \{w + f(w),\, w \in P_-[\mathcal{O}]\}$$

with a certain map $f(w) \in B_\gamma(\mathcal{O})$. The following invariance condition for manifold $\mathcal{W}_-$ with respect to the map S is true:

$$P_+S\Big(f(w) + w\Big) = f\Big(P_-S(f(w) + w)\Big),$$

that can be rewritten in a form

$$L_+(f(w) + w) + R_+(f(w) + w) = f\Big(L_-(f(w) + w) + R_-(f(w) + w)\Big).$$

In virtue of conditions (A) this equality is equivalent to the following one:

$$L_+f(w) + R_+(f(w) + w) = f\Big(L_-w + R_-(f(w) + w)\Big). \qquad (1.77)$$

Obtained equation (1.77) with respect to unknown map f is the basic for construction of numerical algorithms.

Note that numerical solution of this equation for a concrete map $S(\cdot)$ can be nontrivial problem. Moreover, even construction of projection operators $P_\pm$, for example, in the case of 3D Navier-Stokes equations needs high-performance computers and effective mathematical schemes. Nevertheless, we assume below that all necessary algorithms have been realized numerically.

The simplest approximate method for solution of equation (1.77) is

Zero-Approximation Method: $f(w) \approx 0$ for $w \in P_-[\mathcal{O}]$. In fact, taking in (1.77) $R(u) \equiv 0$ (i.e. assuming that $S = L$) we get $L_+f(w) = f(L_-w)$, and therefore:

$$f(w) = L_+^{-k} f(L_-^k w), \quad \|f(w)\| \leqslant \left(\tfrac{\mu_-}{\mu_+}\right)^k \gamma \|w\|.$$

Since this bound holds for all $k \geqslant 0$, the equality $f \equiv 0$ holds. Thus, in linear case $\mathcal{W}_-(V, f) = P_-V$, that follows also from definitions of manifold $\mathcal{W}_-$ and subspace P_-V. This method was used in Chizhonkov (2003), Chizhonkov (2004) for solution of stabilization to 1D Chafee-Infanta equation by boundary control. Later the stabilization problem for Couette flow had been solved in Chizhonkov & Ivanchikov (2004), Ivanchikov (2006) by this method and it was marked there that this method is applicable only for small Reynolds numbers, i.e. $f(w) \approx 0$ only on a small neighborhood of a fixed point.

For more proximate solution of nonlinear equation (1.77) relative to $f(w) \in B_\gamma(\mathcal{O})$ one can apply either

Linear Contraction Mapping Method:

$$L_+ f_{k+1}(w) + R_+(f_k(w) + w) = f_k\Big(L_- w + R_-(f_k(w) + w)\Big), \quad f_0(w) \equiv 0 \tag{1.78}$$

or

Nonlinear Contraction Mapping Method:

$$L_+ f_{k+1}(w) + R_+(f_{k+1}(w) + w) = f_k\big(L_- w + R_-(f_k(w) + w)\big), \quad f_0(w) \equiv 0. \tag{1.79}$$

These schemes require not only calculation of operators $P_\pm$, but inversion of operator L on subspace P_+V as well. Linear method in indicated form was proposed in Ladyzhenskaya & Solonnikov (1973) to prove existence theorem for stable manifold in a neighborhood of a fixed point in the case of magnetohydrodynamic equations. Nonlinear method (see Kornev (2004)) has more high calculation's complexity but it is more effective for applications, since it converges in a wider neighborhood.

Let consider the first step of nonlinear contraction mapping method for operator $S(u) := S_\tau^{N_0}(u)$ and increase N_0 sequentially. In this case we get

Method of Nonlinear Equation:

$$P_+[S_\tau^{N_0}\big(f_{1,N_0}(w) + w\big)] = 0, \quad N_0 = 0, 1, 2, \ldots. \tag{1.80}$$

For numerical simulations this method was proposed in Kornev (2005). This method is highly technological since it reduces equation (1.77) with respect of $f(\cdot)$ to the standard nonlinear equation relatively to $f_{1,N_0}(w) \in P_+\mathcal{O}$. Note that it is not easy to solve this equation with large N_0. That is why from our point of view the most effective applied method of solution to equation (1.77) is the nonlinear contraction mapping method with sufficiently large N_0. The following assertion holds.

Theorem 1.6.2 *Let $S(0) = 0$ and conditions (A) be true. Then in some neighborhood $\mathcal{O}$ iteration processes (1.78), (1.79), (1.80) are solvable on each step, and they converges in $B_\gamma(\mathcal{O})$ with rate Cq^n, $q < 1$, to a function f tangent to P_-V at zero that defines stable invariant manifold $\mathcal{W}_-(\mathcal{O}, f)$.*

In this case $q \sim \mu_-/\mu_+$, and the map $S(\cdot)$ is subordinated to L in the neighborhood $\mathcal{O}$ in the meaning of conditions $A_{3,4}$. Convergence of methods (1.78),(1.79) had been proved correspondingly in Ladyzhenskaya & Solonnikov (1973) and Kornev (2004), Kornev (2006). One can prove convergence of nonlinear equation method (1.80) and to generalize aforementioned results on the case of trajectory $\{S^i(z_0)\}$ (see Kornev (2005), Kornev (2006)) with help of the following

Inverse Iteration Method

$$S_+\big(f_{k+1}(w)+w\big) = f_k\big(S_-(f_{k+1}(w)+w)\big),\ k=0,1,2,\dots \tag{1.81}$$

with initial function $f_0 \in B_\gamma(\mathcal{O})$.

In this case recurrence relations of iteration process mean that points of calculated manifold

$$\mathcal{W}_-(\mathcal{O}, f_{k+1}) = \{w + f_{k+1}(w),\, w \in P_-[\mathcal{O}]\}$$

pass to points of manifold $\mathcal{W}_-(\mathcal{O}, f_k)$ under acting of the map $S(\cdot)$. This implies that $S^n(\mathcal{W}_-(\mathcal{O}, f_n)) \subset \mathcal{W}_-(\mathcal{O}, f_0)$. Note that the method (1.81) is similar to the Graph Transformation Method, Anosov (1959)

1.6.3 Projection on stable invariant manifold

Since stable manifold $\mathcal{W}_-(\mathcal{O}, \mathbf{f})$ contains all points of the neighborhood $\mathcal{O}$ whose trajectories converge to zero, one can set the stabilization problem by initial data for trajectory $\Gamma_+(a_0) = \{S^i(a_0)\}$ as the problem to project initial conditions a_0 to $\mathcal{W}_-(\mathcal{O}, \mathbf{f})$ along a certain given subspace $\mathcal{L} = \{e_1, \dots, e_{i_0}\}$. Formally this means construction of $u = a_0 + l$, $l \in \mathcal{L}$ such that $u \in \mathcal{W}_-(\mathcal{O}, \mathbf{f})$. From here one can find the desired correction l. Using inclusion $S(a_0 + l) \in \mathcal{W}_-(\mathcal{O}, \mathbf{f})$, one can construct more effective method that is formally equivalent to original one. This condition approximately takes the following operator form

$$P_+\big[S(a_0+l)\big] = f_n\big(P_-\big[S(a_0+l)\big]\big) \tag{1.82}$$

for suffisiently large n. To solve obtained problem let consider the iteration process:

$$\begin{aligned} &P_+\big[L(b_0+l_{k+1}) + R(b_0+l_k)\big] = \\ &f_n\big(P_-\big[L^{(0)}(b_0+l_k) + R^{(0)}(b_0+l_k)\big]\big), \end{aligned} \tag{1.83}$$

where $l_k = \sum_{i=1}^{i_0} c_i^k e_i$, $b_0 = a_0 - z_0$.

Theorem 1.6.3 *Let $S(0) = 0$ and conditions (A) be true. Let a function $f_n \in B_\gamma(\mathcal{O})$ be tangent to subspace P_-V at zero, $\dim P_+V = i_0$, and system of vectors $\{P_+[e_i]\}_1^{i_0}$ form basis in P_+V. Then there exists $r > 0$*

such that for $b_0 \in \mathcal{O}$ the problem (1.82) possesses the unique solution. For an arbitrary initial approximation $u_0 = a_0 + l_0$, $u_0 \in \mathcal{O}$, the method (1.83) converges to $u_n \in \mathcal{W}_-(\mathcal{O}, f_n)$ with a geometric progression rate.

1.6.4 Stable manifold corresponding to a trajectory

Let consider the problem of approximate construction of stable manifold corresponding to a trajectory. Below we use the following notations for operators from conditions (A):

$$L(z_i) = L^{(i)}, \quad R(z_i)[u] = R^{(i)}(u), \quad P_\pm(z_i) = P_\pm^{(i)}$$

Assuming that $S(z_0) \neq z_0$ fix natural $n > 0$ and take a segment of trajectory

$$\Gamma_n^+(z_0) = \{z_i = S^i(z), i = 0, 1, \ldots, n\}.$$

For each $i = 0, \ldots, n$ we consider a class $B_{\gamma^{(i)}}\left(\mathcal{O}^{(i)}\right)$ of all continuous maps

$$f(w) : P_-^{(i)}\mathcal{O}^{(i)} \to P_+^{(i)}\mathcal{O}^{(i)}, \quad \text{with} \quad \mathcal{O}^{(i)} = \{u : \|P_\pm^{(i)}(u)\| \leqslant r^{(i)}\}$$

such that

$$f(0) = 0, \quad \|f(w_1) - f(w_2)\| \leqslant \gamma^{(i)}\|w_1 - w_2\|,\ 0 \leqslant \gamma^{(i)} \leqslant 1.$$

Define the norm

$$|f|_i = \sup_{w \in P_-^{(i)}\mathcal{O}^{(i)}} |f(w)|, \quad \forall f \in B_{\gamma^{(i)}}\left(\mathcal{O}^{(i)}\right).$$

Let $f^{(n)} \in B_{\gamma^{(n)}}\left(\mathcal{O}^{(n)}\right)$ be a map that defines in a neighborhood $\mathcal{O}_{z_n}$ of z_n a local manifold

$$\mathcal{W}_-(z_n, f^{(n)}) = \big\{m = z_n + v + w : m \in \mathcal{O}_{z_n},\\ w = P_-^{(n)}(m - z_n), v = f^{(n)}(w)\big\}.$$

Let consider the following

PROBLEM (ff): Given point z_0 find a map $f^{(0)} \in B_{\gamma^{(0)}}\left(\mathcal{O}^{(0)}\right)$ such that the set $\{z_0 + w + f^{(0)}(w),\ w \in P_-^{(0)}\mathcal{O}^{(0)}\}$ is transformed by the map $S^n(\cdot)$ into the manifold $\mathcal{W}_-(z_n, f^{(n)})$:

$$S^n(z_0 + w + f^{(0)}(w)) \subset \mathcal{W}_-(z_n, f^{(n)}). \tag{1.84}$$

Describe the method of solution the problem (1.84).

For given function $f^{(n)}$ we sequentially, in n steps, construct $f^{(0)}$ using conditions of enclosure

$$S\Big(\mathcal{W}_-(z_i, f^{(i)})\Big) \subset \mathcal{W}_-\big(z_{i+1}, f^{(i+1)}\big) \tag{1.85}$$

for $i = n-1, \dots, 0$. We wrote condition (1.85) in operator form as equation for function $f^{(i)}$:

$$\begin{aligned} &L_+^{(i)} f^{(i)}(w) + R_+^{(i)}(f^{(i)}(w) + w) = \\ &f^{(i+1)}\big(L_-^{(i)} w + R_-^{(i)}(f^{(i)}(w) + w)\big), \end{aligned} \tag{1.86}$$

where $L_\pm^{(i)} = P_\pm^{(i+1)} L^{(i)}$, $R_\pm^{(i)} = P_\pm^{(i+1)} R^{(i)}$. To solve problem (1.86) let consider the following iteration process:

$$\begin{aligned} &L_+^{(i)} f_{k+1}^{(i)}(w) + R_+^{(i)}(f_k^{(i)}(w) + w) = \\ &f^{(i+1)}\big(L_-^{(i)} w + R_-^{(i)}(f_k^{(i)}(w) + w)\big). \end{aligned} \tag{1.87}$$

Results of Kornev (2006) imply

Theorem 1.6.4 *Let conditions* (A) *take place. Then there exist* $\{r^{(i)}, \gamma^{(i)}, i = 0, 1, \dots, n\}$ *such that problems* (1.86) – (1.87) *possess unique solution for an arbitrary function* $f^{(n)} \in B_{\gamma^{(n)}}\big(\mathcal{O}^{(n)}\big)$. *This solution* $f^{(0)} \in B_{\gamma^{(0)}}\big(\mathcal{O}^{(0)}\big)$ *satisfies the following conditions*

$$\begin{aligned} &S^n\Big(\mathcal{W}_-(z_0, f^{(0)})\Big) \subset \mathcal{W}_-(z_n, f^{(n)}), \\ &\|S^n\big(z_0 + f^{(0)}(w) + w\big) - S^n\big(z_0\big)\| \leqslant Cp^n. \end{aligned}$$

In this case $p \sim \mu_-^{(i)}$, and the method (1.87) converges with the rate of geometric progression with denominator $q \sim \mu_-^{(i)}/\mu_+^{(i)}$. If theorem admits uniform closure as $n \to \infty$, then there exists a local stable manifold $\mathcal{W}_-(S, \mathcal{O})$, and the function $f^{(0)}$ determined by such a way approximates $\mathbf{f}^{(0)}$ in a neighborhood $\mathcal{O}_{z_0}$. For numerical simulations we take zero as initial function: $f^{(n)}(w) \equiv 0$. Since $\mathbf{f}^{(n)}$ is tangent for subspace $P_-^{(n)} V$, this approximation has an error $O\big((r^{(n)})^2\big)$.

1.6.5 Projection on stable manifold

Let consider

Problem (lf): Project initial conditions a_0 to manifold $\mathcal{W}_-(z_0, f^{(0)})$ along a given subspace $\mathcal{L} = \operatorname{span} < e_1, \dots, e_{i_0} >$.

By definition this means to construct $u = a_0 + l,\ l \in \mathcal{L}$ such that $u \in \mathcal{W}_-(z_0, f^{(0)})$. In other words we have to construct $u = a_0 + l$ that satisfies condition $S(u) \in \mathcal{W}_-(z_1, f^{(1)})$. The corresponding equation is written as follows:

$$P_+^{(1)}\big[S(a_0 + l) - S(z_0)\big] = f^{(1)}\big(P_-^{(1)}\big[S(a_0 + l) - S(z_0)\big]\big). \tag{1.88}$$

To solve problem (1.88) we consider the following iteration process:

$$\begin{aligned} &P_+^{(1)}\big[L^{(0)}(b_0+l^{k+1})+R^{(0)}(b_0+l^k)\big]= \\ &=f^{(1)}\big(P_-^{(1)}\big[L^{(0)}(b_0+l^k)+R^{(0)}(b_0+l^k)\big]\big), \end{aligned} \tag{1.89}$$

where $l^k=\sum_{i=1}^{i_0}c_i^k e_i$, $b_0=a_0-z_0$. Convergence of the obtained scheme had been proved in Kornev (2006).

It reasonable to choose starting approximation $u_0=a_0+l^0$ using condition $u_0\in P_-^{(0)}\mathcal{O}^{(0)}$. Science the function $f^{(1)}$ is tangent to subspace $P_-^{(1)}V$, and $f^{(0)}$ is tangent to $P_-^{(0)}V$, this approximation has an error $O\big((r^{(0)})^2\big)$.

The problem of approximate projection to a stable manifold along subspace $\mathcal{L}$ can be reduced to solution of the following equation:

$$P_+^{(n)}[S^n(a_0+l^n)-S^n(z_0)]=0, \quad l^n=\sum_{i=1}^{i_0}c_i^n e_i, \tag{1.90}$$

relatively unknown coefficients c_i^n. Note that this equation corresponds to equations for problems (ff) and (lf) with $f^{(n)}\equiv 0$.

The following theorem holds:

Theorem 1.6.5 *Let conditions of Theorem 1.6.4 be fulfilled, function* $f^{(1)}\in B_{\gamma^{(1)}}(\mathcal{O}^{(1)})$ *be tangent to subspace* $P_-^{(1)}V$ *at zero, vectors system* $\{P_+^{(0)}[e_i]\}_1^{i_0}$, $\dim P_+^{(0)}V=i_0$ *form basis in* $P_+^{(0)}V$. *Then there exists* $r^{(0)}>0$ *such that for* $b_0\in\mathcal{O}^{(0)}$ *problem (1.88) possesses unique solution* $u\in\mathcal{W}_-(z_0,f^{(0)})$. *For arbitrary starting approximation* $u_0=a_0+l_0$, $u_0\in\mathcal{O}_{z_0}$, *method (1.89) converges to* u *with a geometric progression rate. Moreover* $u=a_0+l$, $l=\sum_{i=1}^{i_0}c_i l_i$ *and the following bound from Theorem 1.6.4 holds:*

$$\|S^n\big(a_0+l\big)-S^n\big(z_0\big)\|\leqslant Cp^n.$$

Note that as it was mentioned above in subsection 1.2.2 control in initial conditions is equivalent to impulse control, i.e. it is "instantaneous control". That is why in problems from applied sciences this kind of control requires some modifications. Such modification from the point of view of Partial Differential Equations is stabilization method by boundary control in the form proposed in Fursikov (2001a)-Fursikov (2004) and used for calculations in Chizhonkov (2003), Chizhonkov (2004), Chizhonkov & Ivanchikov (2004), Ivanchikov (2006), Ivanchikov, Kornev, & Ozeritskii (2009). Connection between initial and boundary controls was explained above, in subsection 1.2.4

1.6.6 Calculations with control in right side

Some other kind of control that can be used in applications is stabilization by control in right-hand sides. Describe one method connected with stabilization by this kind of control that differs from one stated above in, subsection 1.5.2. We formulate stabilization problem by control in right side in the following form (see Kornev (2008)).

Let

$$S_F(z_i + u) = S(z_i) + L^{(i)}u + R^{(i)}(u) + \hat{S}^{(i)}(u, F)$$

where F is desired control function and operator $\hat{S}^{(i)}(\cdot, \cdot)$ prescribes the rule of applying this control. The case of zero function $F \equiv 0$ corresponds to resolving operator for problem without control, i.e. $S_0(\cdot) \equiv S(\cdot)$. Given initial conditions $z_0, a_0 \in V$ and $q_F \geqslant 0$, find $F \in \mathcal{F}$ such that

$$\begin{cases} \|P_+^{(n)}[S_F^n(a_0) - S^n(z_0)]\| \leqslant Q, \\ \|F\| \to \inf, \quad Q = q_F \|P_+^{(0)}[a_0 - z_0]\|, \\ F \in \mathcal{F}, \quad \|\cdot\| = (\cdot, \cdot)^{1/2} \end{cases} \tag{1.91}$$

In this case $\mathcal{F}$ gives the subspace of admissible right-hand sides, and magnitude $0 \leqslant q_F < 1$ defines stabilization rate along subspace $P_+^{(n)}V$. Note that since operator $S(\cdot)$ of initial problem is nonlinear, stabilization along subspace $P_+^{(n)}V$ does not guarantees stabilization on the whole space V. In the case $Q = 0$ equation (1.91) takes the form $\|P_+^{(n)}[S_F^n(a_0) - S^n(z_0)]\| = 0$ and considered method is the method of nonlinear equation with respect to F.

We construct approximate solution for problem (1.91) by the following way. Write linearization of relation (1.91), taking $R^{(i)}(u) \equiv 0$, $\hat{S}^{(i)}(u, F) \approx J^{(i)}F$. Then we get:

$$\begin{cases} \|P_+^{(n)}[L_a a_0 + L_F F]\| \leqslant q_F \|P_+^{(0)}[a_0 - z_0]\|, \\ \|F\| \to \inf, \ F \in \mathcal{F}, \\ L_a = L^{(n-1)}L^{(n-2)} \dots L^{(0)}, \\ L_F = L^{(n-1)}L^{(n-2)} \dots L^{(1)}J^{(0)} + \dots \\ +L^{(n-1)}L^{(n-2)}J^{(n-1)} + L^{(n-1)}J^{(n-2)} + J^{(n-1)}. \end{cases} \tag{1.92}$$

We denote solution of problem (1.92) with $F_{L^n,Q}$. If one knows finite bases in subspaces $P_+^{(n)}V$, $P_-^{(n),\perp}V$, $\mathcal{F}$ then (1.92) is reduced to generalized least squares problem and it can be solved by standard methods. Let one has found function $F_{L^n,Q}$. Then calculate $h = L_a a_0 + L_F F_{L^n,Q}$

and consider the following nonlinear equation with respect to F:

$$\begin{cases} P_+^{(n)}[S_F^n(a_0) - S^n(z_0)] = P_+^{(n)}[h], \\ h = L_a a_0 + L_F F_{L^n,Q}. \end{cases} \tag{1.93}$$

Apply the obtained solution $F_{S^n,Q}$ of (1.93) for stabilization of initial nonlinear problem. In this case control $F_{S^n,Q}$ provides stabilization of nonlinear problem (1.91) in subspace $P_+^{(n)}V$ by the same way as optimal control $F_{L^n,Q}$ stabilizes linear problem (1.92). Optimality condition $\|F_{S^n,Q}\| \to \inf$ takes place only approximately.

Process of stabilization is realized here for $i = 0, 1, \dots, n-1$ with constant on i function $F_{S^n,Q}$. If additional stabilization is needed for $i = n, n+1, \dots, 2n-1$, then control function is calculated over again with help of the same algorithm for next time segment $i = n, n+1, \dots, 2n-1$, and so on. To solve problem (1.93) the following method of the simple iteration type was applied:

$$P_+^{(n)}[L_F F_{k+1} + R_F(F_k)] = P_+^{(n)}[v], \quad F_0 = F_{L^n,Q}. \tag{1.94}$$

1.7 Results of numerical calculations

1.7.1 Physical Model and Mathematical Setting

Using Zero-Approximation Method E.V.Chizhonkov and A.A.Ivanchikov solved numerically (Chizhonkov & Ivanchikov (2004), Ivanchikov (2006)) the stabilization problem with boundary control for Couette flow. As far as we know this is the first successful attempt to stabilize by boundary control unstable solution of Navier-Stokes equations in variables "velocity-pressure" that describes real physical experiment.

In this paper we consider the problem of numerical stabilization for unstable flow of four-vortex structure. Experimental plant is rectangular horizontal container of small deepness filled with electrolyte (a water solution of $CuSO_4$). On both opposite inner sides of container cupric electrodes are placed and under container the system of direct magnets are rigged up. Electrical current going through water provokes deflecting Lorentz force. This leads to appearance of the flow consisting of four-vortex that becomes unstable for high force of current.

It is known (see Dolzhanskii, Dovzenko, & Krymov (1996) and references therein) that for the certain diapason of plant's parameters the movement of liquid is described with high level of accuracy by quasi-two-dimensional Navier-Stokes system. In dimensionless variables "stream function-vortisity" this system is written as follows:

$$\begin{aligned}&\frac{\partial \omega}{\partial t} = \bar{\nu}\Delta\omega - \omega - [\psi, \omega] + \Delta h + \Delta u,\\ &\Delta\psi = \omega, \quad [\psi, \omega] = \psi_x \omega_y - \psi_y \omega_x.\end{aligned} \tag{1.95}$$

Here $\psi(t, x, y)$ is unknown stream function, $h(x, y)$ describes magnetic field, $u(t, x, y)$ is additional controlling field or is equal to zero. The bottom influence is reduced in this case to damping of horizontal flows by linear law. We add the system with the following initial-boundary conditions:

$$\begin{aligned}&\psi|_{\partial\Omega} = 0, \quad \Delta\psi|_{\partial\Omega} = 0; \quad \psi|_{t=0} = \psi^0,\\ &\Omega = [0, l_1] \times [0, l_2].\end{aligned} \tag{1.96}$$

In this case Dirichlet conditions are prescribed on the boundary for functions ψ, ω. Let $\bar{\nu} \approx 2.83 \cdot 10^{-4}$, $l_1 = 1$, $l_2 = 0.5$,

$$\begin{aligned}&\Delta h(x, y) = \sum_{m,n} c_{mn} \sin(\tfrac{\pi m x}{l_1}) \sin(\tfrac{\pi n y}{l_2}),\\ &c_{22} \approx -37.75, c_{13} = c_{31} = 0.01 c_{22}, \quad \text{and } c_{mn} = 0 \text{ for other } \ m, n.\end{aligned}$$

These simplifications have an influence on fluid quantitative characteristics, but does not change qualitative picture (cf. Danilov et al. (1996), Kornev & Ozeritskii (2010)). In this case the harmonic $\{m = 2, n = 2\}$ of right side forms structure of flow that is close to experimental one and two other harmonics realize additional instability.

1.7.2 The structure of phase portrait

For numerical solution of considered system (1.95), (1.96) we apply Krank-Nikolson finite-difference scheme for approximation in time; operator Δ we approximate by Δ^h on "cross" five-points stencil, and for approximation of operator $[\cdot, \cdot]$ we use Arakawa scheme. Unknown grid functions $\psi_{ij}^n, \omega_{ij}^n$ approximate at nods $(n\tau, ih_x, jh_y)$ the desired functions $\psi(t, x, y), \omega(t, x, y)$ correspondingly. Let $\psi^{n+1} = S_\tau(\psi^n)$ with $\psi^n = \{\psi_{ij}^n\}$, i.e. $S_\tau(\cdot)$ be solving operator of constructed differences scheme. We define operator S for corresponding dynamical system as the difference scheme operator for N_0 steps, i.e. $S(\cdot) = S_\tau^{N_0}(\cdot)$. All calculation indicated below were done for $h_x, h_y \sim 0.015$, $\tau \sim 0.001$.

For chosen parameters the difference equation has unstable steady-state solution $\bar{z}_{ij} = S(\bar{z}_{ij})$ of four-vortex structure (similar Fig. 1), and in its neighborhood there are stable quasiperiodic oscillations. Taking into account relative smallness of c_{13}, c_{31} we have that $\bar{z}_{ij} \approx z_{ij}$ where z_{ij} satisfies the equation

$$(\bar{\nu}\Delta^h\Delta^h - \Delta^h)z_{ij} = -c_{22}\sin(2\pi i h_x)\sin(4\pi j h_y).$$

During oscillations there is periodical confluence of two vortex placed on diagonal (that possess identically directed rotation) to one vortex directed along diagonal (similar to Fig. 2). After that this vortex disintegrates on two vortexes of primary structure and the second pair of vortexes flows together to one vortex directed along the second diagonal of container. This process repeats with high precision of periodicity.

The trajectory with initial condition $\psi^0_{ij} = z_{ij}$ is close to stationary one on small time segment but later it also goes to indicated oscillating behavior.

Let $a := S^{n_0}_\tau(z)$, $n_0 = 6100$, i.e. trajectory of point $a = \{a_{ij}\}$ advances beyond the trajectory of point $\{z_{ij}\}$ on n_0 steps. Functions z_{ij}, a_{ij} are depicted on Fig. 1,2 correspondingly.

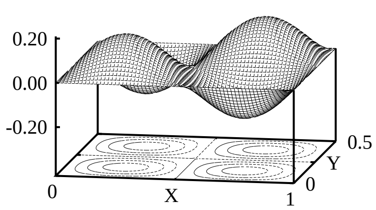

Fig. 1. Function z

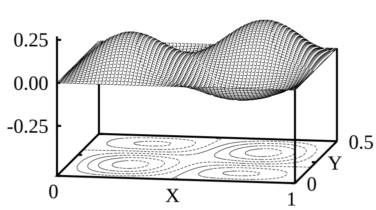

Fig. 2. Function a

Using described algorithms we solve numerically the problem of stabilization for trajectory of point a to trajectory of point z or to steady-state point $\bar{z}$. For this we have to construct linearization $L = L^{(n-1)} \dots L^{(0)}$ of operator S along trajectory of point z and then to calculate subspaces $P^{(i)}_+ V$. It is convenient to define unstable subspace $P^{(i)}_- V$ by its orthogonal complement $P^{(i),\perp}_- V$ that similarly can be constructed by means of operator L^*. In the considered parameters and $N_0 = 610, n = 1$ we have two-dimensional subspaces $P^{(0)}_+ V = \{\xi^{(0)}_1, \xi^{(0)}_2\}$, $P^{(0),\perp}_- V$.

If we suppose that the point z is fixed it is allows to construct $P^{(0)}_+ V$ using linearization $A(z)$ of right side for equation (1.95) only in z. The subspace $P^{(0),\perp}_- V$ can be constructed by means of operator $A^*(z)$.

1.7.3 Stabilization by the initial control

Describe result of stabilization by initial data with help of algorithm (1.88) (details of algorithms realization see in Kornev (2006), Kornev & Ozeritskii (2010)). Let the basis $\{e_1, e_2\}$ in the space of admissible displacements $\mathcal{L}$ have the form

$$e_k \equiv \xi_k^{(0)} \text{ for } (ih_x, jh_y) \in \bar{\Omega} := [0.5, 1] \times [0, 0.5];$$
$$e_k \equiv 0 \text{ for } (ih_x, jh_y) \notin \bar{\Omega}; k = 1, 2.$$

In this case initial function a changes only in subdomain $\bar{\Omega}$. Note that the choice of the space of admissible displacements is an important problem (see Chizhonkov (2004), Ivanchikov (2006)). In some sense the subspaces $P_+^{(0)}V$ and $P_-^{(0),\perp}V$ (or their ortogonal projections on the preassigned $\mathcal{L}$) are optimal. The function l for parameters of iteration process $n = 2$, $N_0 = 1220$ is depicted on Fig.3. The obtained trajectory $S_\tau^k(a+l)$ tends monotonously to trajectory $S_\tau^k(z)$ as k increasing to N, $N \approx 7400$. The accuracy function $\delta^N := S_\tau^N(z) - S_\tau^N(a+l)$ is depicted on Fig.4. Note that stabilization gives 17-times decreasing of initial error, i.e. $\delta^N/\delta^0 \sim 0.06$. For $k > N$ the trajectory of point a does not tend to the trajectory of point z.

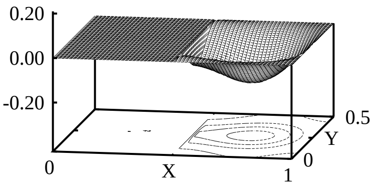

Fig. 3. Function l

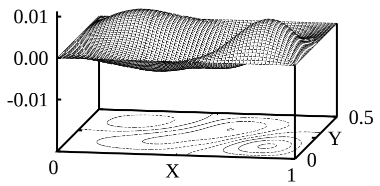

Fig. 4. Function δ^N

1.7.4 Stabilization by the control in right side

Describe solution of stabilization problem for trajectory $\{S_F^i(a), i = 0, 1, \ldots\}$ to the steady-state solution $\bar{z}_{ij}$ by right side F with help of algorithm (1.93) as stabilization to the segment of trajectory $\{S^i(z), i = 0, 1, \ldots, n\}$ for a small n. In our opinion the current problem is important from practical point of view. Details of algorithms realization see in Kornev (2008), Kornev & Ozeritskii (2010).

Let $N_0 = 1$, i.e $S(\cdot) = S_\tau(\cdot)$. The domain of control $\bar{\Omega}$ and form of control function we keep as above:

$$\mathcal{F} = \text{span} < e_k; k = 1, 2 >$$
$$\mathrm{e}_k \equiv \xi_k^{(0)} \text{ for } (ih_x, jh_y) \in \bar{\Omega};\ e_k \equiv 0 \text{ for } (x_i, y_j) \notin \bar{\Omega}.$$

However now the correction F we are looking for is added in a form $u^k(F) = c_1(k)e_1 + c_2(k)e_2$ to the right-hand side of (1.95) and it is a time-piecewise constant function on the each segment of stabilization: $nm \leqslant k < n(m+1)$, $m = 0, 1, \ldots$.

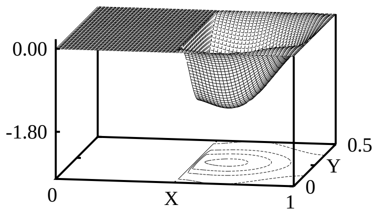

Fig. 5. Function u^0

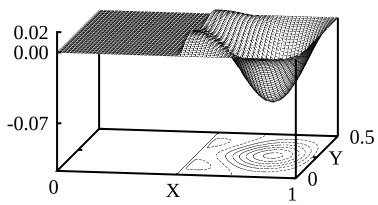

Fig. 6. Function u^N

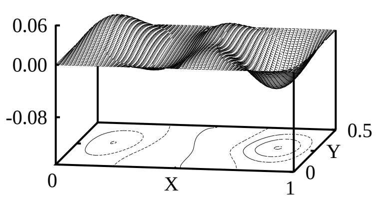

Fig. 7. Function δ^N

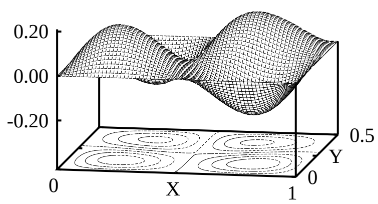

Fig. 8. Function $S_F^N(a)$

Let $q_F = 0.5$, $n = 61$. On Fig.5 the form of control function u^k of operator $S_F(\cdot)$ on the first segment $0 \leqslant k < n$ is depicted. On Fig. 6,7,8 are depicted the typical forms of control function u^k, function $S_F^k(a)$ and accuracy function δ^k for $k \geqslant N, N \approx 3000$. Note that in the numerical experiment accuracy function decrease from $\delta^0 \approx 0.04$ to 0.01 as $k \leqslant 6000$.

1.7.5 Stabilization by the boundary control

Show results of stabilization for trajectory $\{S^i(a), i = 0, 1, \ldots\}$ to the segment of trajectory $\{S^i(z), i = 0, 1, \ldots, N_0\}$ by boundary control mentioned in subsections 1.2.3, 1.2.4 (see details in Chizhonkov (2003), Ivanchikov (2006), Ivanchikov et al. (2009)). Assume that one can choose control functions for ω, ψ on the whole boundary $\partial\Omega$. On Fig.9, 10 the graphs of boundary control depending on time $t_i = i\tau$ are depicted when $G = [-0.5, 1.5] \times [-0.25, 0.75]$ is taken as extended domain.

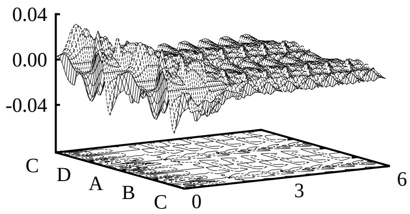

Fig. 9. Function $\psi^n|_{\partial\Omega}$

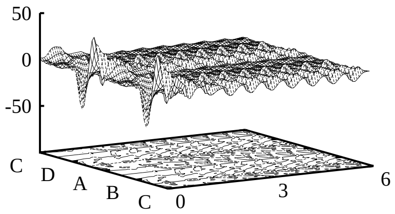

Fig. 10. Function $\omega^n|_{\partial\Omega}$

Here segments $[AB]$, $[BC]$, $[CD]$, $[DA]$ correspond to the sides of rectangle $\partial\Omega$. Note that in the considered parameters $\bar{\nu}$, G and $N_0 = 610, n = 1$ we have ten-dimensional subspaces $P_+^{(0)}V$, $P_-^{(0),\perp}V$. Functions $S^k_{\tau,\partial\Omega}(a)$ with boundary control for $k \geqslant 4500$ have "easily deformed" four-vortex stucture and accuracy function δ^N fluctuated near 0.0017.

1.7.6 Conclusions

In this paper we do not discuss the question on closeness of stable manifolds in initial differential problem and in chosen finite-difference scheme. We have to mark only that assertions of such kind can be proved in some situations by known methods. However if one assume that on considered times theorem on convergence of solutions for problem in finite differences to solution of differential equation is true, then it is possible to claim that controls obtained by such methods solves stabilization problem for initial system of differential equations. Note that to get approximation for the operators $L_+^{(i)}$, $P_+^{(n)}L_a$, $P_+^{(n)}L_F$ in numerical implementation of the considered algorithms one has to take numerical linearizations of operators S and S_F. This is especially important for complicate S and S_F.

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