# Analyticity of stable invariant manifolds of 1D-semilinear parabolic equations

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Abstract. This paper is devoted to the proof of analyticity of the stable invariant manifold in a neighborhood of the zero steady-state solution of a semilinear parabolic equation under the assumption that this steady-state solution is unstable. This investigation may have possible applications in stabilization theory for semilinear parabolic equation.

## Introduction

In this paper we prove analyticity of the stable invariant manifold $M_-$ near the zero steady-state solution of a semilinear parabolic equation. We need this result in order to develop a stabilization theory for semilinear parabolic PDE defined in a bounded domain $\Omega$ with feedback Dirichlet control given on the boundary $\partial\Omega$ or on its open part $\Gamma$.

This theory was built in [**F1**], [**F2**], [**F3**], [**F4**], [**F5**] for general quasilinear parabolic equations and for Navier-Stokes system. The main idea of the proposed method was to extend the stabilized boundary value problem (BVP) from $\Omega$ to a BVP on a longer domain $G$ doing extension through that part $\Gamma$ of $\partial\Omega$ where the control is defined and taking away $\Gamma$ with boundary condition on it that defines this control. Simultaneously, the initial condition $y_0(x)$ is extended to the initial condition $z_0(x)$ that has to belong to the stable invariant manifold $M_-$ of the extended BVP. If we stabilize the BVP near the origin, the manifold $M_-$ has to be defined in a neighborhood of origin. By the definition of the stable invariant manifold, the solution $z(t,x)$ of the extended BVP starting from $z_0 \in M_-$ tends to 0 as $t \to \infty$. Therefore the restriction of $z(t,x)$ on $\Omega$ and $\Gamma$ gives the desired solution of the initial stabilization problem.

In order to prove in [**F1**]-[**F5**] the existence of the extension operator acting on the stable invariant manifold $M_-$ it was enough to use the well-known existence theorems for stable invariant manifolds from [**H**], [**BV**].

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We have to emphasize that the main reason for developing a stabilization theory is to provide reliable stable algorithms for numerical stabilization. [1] To construct such algorithms it is desirable to have a simple description for the infinite-dimensional invariant manifold $M_-$ which would allow one to calculate it easily at an arbitrary point. A functional-analytic decomposition of $M_-$ gives such a description. In this paper we investigate the possibility of such a description for $M_-$ in the case of one-dimensional semilinear parabolic equation.

Using the classical description of $M_-$ by means of a map $F(y_-)$ we look for this map as a series

$$F(y_-) = \sum_{k=2}^{\infty} F_k(y_-)$$

where the maps $F_k(y_-)$ are homogeneous in $y_-$ with power $k$. We obtain recurrence relations for $F_k$ using special differential equation in variational derivatives for the map $F$. These recurrence relations allow us to prove convergence of the series for $F(y_-)$.

We note that when these recurrence relations for $F_k$ have been obtained, we passed them right away to specialists in numerical calculations. Up to now these relations have already been used for numerical calculations of stable invariant manifolds and for applications to numerical stabilization of semilinear parabolic equation. Moreover, the numerical results thus obtained seem to us quite satisfactory (see [**K**]).

In the case of ordinary differential equations the analyticity of the stable and unstable invariant manifolds in a neighborhood of a hyperbolic singular point are proved in the classical Hadamard-Perron Theorem (see [**IL**] Chapter 1, Theorem 1.2). On the other hand the Poincare theory of normal forms (see [**A**], Chapter 5) shows that not every invariant manifold is analytic because sometimes the so called resonance condition can appear.

In this paper we restrict ourselves to considering the case of stable invariant manifold near the hyperbolic singular point $\widehat{z} \equiv 0$. Note however that the analyticity of the unstable invariant manifold in a neighborhood of a hyperbolic singular point can be proved similarly. Besides, in the Remark 3.10 below we explain how to get examples of invariant manifolds which are not analytic because of the resonance condition. In Remark 3.11 below we discuss possible generalizations.

To establish analyticity in the case of semilinear parabolic equations considered here one has to overcome certain difficulties specific for PDE. To do this we were forced to develop essentially the technique of [**VF1**], [**VF2**].

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[1] Indeed, the existence theorems for exact controllability problem (see, for instance, [**FI**]) are stronger than existence rezultes for corresponding stabilization problem. Another point is that the exact controllability problems are ill-posed in the case of parabolic equations or Navier-Stokes system and therefore they can not be solved numerically by adequate way.

## 1. Stable invariant manifolds

In this section we recall the definition of a stable invariant manifold near a steady-state solution (fixed point) of a semilinear parabolic equation and derive the equation that determines the invariant manifold.

**1.1. Semilinear parabolic equation.** We consider the following boundary value problem for a semilinear parabolic equation:

$$\frac{\partial y(t,x)}{\partial t} - \frac{\partial^2 y(t,x)}{\partial^2 x} - \kappa y(t,x) + f(y(t,x)) = 0, \quad t>0, \quad x \in (0,\pi) \tag{1.1}$$

$$y(t,x)|_{x=0} = y(t,x)|_{x=\pi} = 0 \tag{1.2}$$

$$y(t,x)|_{t=0} = y_0(x) \tag{1.3}$$

where $\kappa > 0$ is a parameter, and $f(y)$ is an analytic function that admits the decomposition

$$f(y) = \sum_{k=2}^{\infty} f_k y^k \tag{1.4}$$

with coefficients $f_k$ satisfying

$$|f_k| \le \gamma \rho^k \tag{1.5}$$

with some $\gamma > 0, \rho > 0$. We assume that

$$\kappa > 1 \quad \text{and} \quad \sqrt{\kappa} \quad \text{is not integer.} \tag{1.6}$$

Note that

$$e_k(x) = \sqrt{2/\pi}\sin kx, \quad \lambda_k = k^2 - \kappa \tag{1.7}$$

are the eigenfunctions and the eigenvalues of the spectral problem

$$Ae \equiv -\frac{\partial^2 e(x)}{\partial x^2} - \kappa e(x) = \lambda e(x), \quad e(0) = e(\pi) = 0. \tag{1.8}$$

Therefore by virtue of (1.6), the solutions $e^{-\lambda_k t} e_k(x)$ of the linear equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} - \kappa y = 0, \quad y|_{x=0} = y|_{x=\pi} = 0 \tag{1.9}$$

tend to infinity as $t \to \infty$ for

$$k = 1, \dots, N, \quad N = [\sqrt{\kappa}] \tag{1.10}$$

where $[\sqrt{\kappa}]$ is the integer part of $\sqrt{\kappa}$, and tend to zero as $t \to \infty$ for $k > N$.

We take the Sobolev space

(1.11)

$$H = H_0^1(0,\pi) = \{y(x) \in L_2(0,\pi) : \|y\|^2_{H_0^1} = \int_0^\pi |\partial y/\partial x|^2 dx, \ y(0) = y(\pi) = 0\}$$

as the phase space of the dynamical system generated by boundary value problem (1.1)- (1.3). We introduce the subspaces

$$H_+ = [e_1, \dots, e_N], \quad H_- = [e_{N+1}, e_{N+2} \dots], \quad N = [\sqrt{\kappa}] \tag{1.12}$$

of unstable and stable modes for equation (1.9).

Let $B_r(H_0^1(0,\pi)) = \{y_0(x) \in H_0^1(0,\pi) : \ \|y_0\|_{H_0^1(0,\pi)} < r\}$. It is well-known, that there exists $r = r(\rho) < 1/\rho$ such that for each $y_0(x) \in B_r(H_0^1(0,\pi))$ there exists a unique solution $y(t,x) \in C(0,T; H_0^1(0,\pi))$ of problem (1.1)-(1.3), where $T$

depends on initial condition $y_0$ (see, for instance, Theorem 3.3.3 in [**Hen**]). We denote by $S(t, y_0)$ the solution operator of the boundary value problem (1.1)-(1.3):

$$S(t, y_0) = y(t, \cdot) \tag{1.13}$$

where $y(t, x)$ is the solution of (1.1)-(1.3).

**1.2. Stable invariant manifold.** In this subsection we recall some commonly used definitions (see Chapter V in [**BV**]) adopted for our case.

The origin of the phase space $H = H_0^1(0, \pi)$, i.e. the function $y(x) \equiv 0$, is, evidently, a steady-state solution of problem (1.1)-(1.3). The set $M_- \subset H$ defined in a neighborhood of the origin is called *the stable invariant manifold* if for each $y_0 \in M_-$ the solution $S(t, y_0)$ is well-defined and belongs to $M_-$ for each $t > 0$, and

$$\|S(t, y_0)\|_{H_0^1} \le ce^{-rt} \quad \text{as} \quad t \to \infty \tag{1.14}$$

where $0 < r < \lambda_{N+1}$.

The stable invariant manifold can be defined as a graph in the phase space $H = H_+ \oplus H_-$ by the formula

$$M_- = \{y_- + F(y_-),\ y_- \in \mathcal{O}(H_-)\} \tag{1.15}$$

where $\mathcal{O}(H_-)$ is a neighborhood of the origin in the subspace $H_-$, and

$$F : \mathcal{O}(H_-) \to H_+ \tag{1.16}$$

is a certain map satisfying

$$\|F(y_-)\|_{H_+} / \|y_-\|_{H_-} \to 0 \quad \text{as} \quad \|y_-\|_{H_-} \to 0. \tag{1.17}$$

So, in order to construct the invariant manifold $M_-$ we have to calculate the map (1.16), (1.17).

**1.3. Equation for $F$.** Here we recall the derivation of the well-known equation for the map (1.16) that determines the invariant manifold $M_-$.

First of all we introduce some notation. We rewrite equations (1.1),(1.2), using definition (1.8) of the operator $A$, as follows:

$$\partial_t y(t) + Ay(t) + f(y(t)) = 0 \tag{1.18}$$

We define the orthoprojectors

$$P_+ : H \to H_+, \quad P_- : H \to H_- \tag{1.19}$$

and introduce the notation

$$P_+ y = y_+, \quad P_- y = y_-, \quad P_+ S(t, y_0) = S_+(t, y_0), \quad P_- S(t, y_0) = S_-(t, y_0) \tag{1.20}$$

Taking into account that the spaces $H_+, H_-$ are invariant with respect to $e^{-At}$ and using notation (1.20) we can rewrite (1.18) as follows:

$$\begin{aligned} &\partial_t y_+(t) + Ay_+(t) + P_+ f(y_+(t) + y_-(t)) = 0 \\ &\partial_t y_-(t) + Ay_-(t) + P_- f(y_+(t) + y_-(t)) = 0 \end{aligned} \tag{1.21}$$

Let $y_0 \in M_-$. Then by (1.15) it has the form $y_0 = y_- + F(y_-)$. By definition of an invariant manifold for each $t \in \mathbb{R}_+$ $S(t, y_0) \in M_-$ or, what is equivalent,

$$S_+(t, y_- + F(y_-)) = F(S_-(t, y_- + F(y_-)))$$

We differentiate this equation with respect to $t$ and express the $t$-derivatives with the help of equations (1.21). As a result we get:

$$\begin{aligned} &AS_+(t, y_- + F(y_-)) + P_+f(S(t, y_- + F(y_-))) \\ =&\langle F'(S_-(t, y_- + F(y_-))), AS_-(t, y_- + F(y_-)) \\ &+P_-f(S_+(t, y_- + F(y_-)) + S_-(t, y_- + F(y_-)))\rangle \end{aligned} \tag{1.22}$$

where by $\langle F'(z), h\rangle$ we denote the value of derivative $F'(z)$ on vector $h$. Passing to the limit in (1.22) as $t \to 0$ we get the desired equation for $F$:

$$AF(y_-) + P_+f(y_- + F(y_-)) = \langle F'(y_-), Ay_- + P_-f(y_- + F(y_-))\rangle \tag{1.23}$$

**1.4. Equation for $F$ in the basis $\{e_k\}$.** First of all we write equation (1.18) in the basis $\{e_k = \sqrt{2/\pi}\sin kx\}$. Let

$$y(t, x) = \sum_{k=1}^{\infty} \widehat{y}(t, k)e_k(x) \tag{1.24}$$

be the Fourier decomposition of a solution $y(t, x)$. After the substitution of (1.24) into (1.18) and using the orthogonality of $e_k, e_j$ for $k \neq j$ in $H^1_0(0, \pi)$ we get:
(1.25)

$$\partial_t\widehat{y}(t, \xi) + \lambda_k\widehat{y}(t, \xi) + \sum_{k=2}^{\infty} f_k \sum_{\eta_1=1}^{\infty} \cdots \sum_{\eta_k=1}^{\infty} b_k(\xi; \eta_1, \dots, \eta_k)\widehat{y}(t, \eta_1)\dots\widehat{y}(t, \eta_k) = 0$$

where

$$b_k(\xi; \eta_1, \dots, \eta_k) = (2/\pi)^{\frac{k+1}{2}} \int_0^{\pi} \sin\xi x\ \sin\eta_1 x \dots\ \sin\eta_k x\ dx. \tag{1.26}$$

Below it will be important for us to distinguish the coordinates of vectors belonging to $H_+$ and to $H_-$. To this end we denote the coordinates of vectors from $H_+$ by first letters of the Greek alphabet i.e. by $\alpha, \beta, \dots$. The coordinates of vectors from $H_-$ we denote by letters $\xi, \eta, \zeta, \dots$ that belong to the last part of Greek alphabet. So

$$H_- \ni y_- = \sum_{\eta=N+1}^{\infty} \widehat{y}(\eta)e_\eta(x), \quad H_+ \ni F(y_-) = \sum_{\alpha=1}^{N} F^\alpha(y_-)e_\alpha(x) \tag{1.27}$$

Besides, we use the notation:

$$(\eta_1, \dots, \eta_j) = \overline{\eta^j}; \quad \sum_{\overline{\eta^j}} = \sum_{\eta_1=N+1}^{\infty} \cdots \sum_{\eta_j=N+1}^{\infty}; \quad \sum_{\overline{\alpha^j}} = \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_j=1}^{N} \tag{1.28}$$

$$\widehat{y}(\eta_1)\widehat{y}(\eta_2)\dots\widehat{y}(\eta_j) = y(\overline{\eta^j}), \quad F^{\alpha_1} \cdot \dots \cdot F^{\alpha_j} = F^{\overline{\alpha^j}} \tag{1.29}$$

(To simplify notation we omit the sign "hat" over $y$ in the product $y(\overline{\eta^j})$). Using notation (1.26),(1.28),(1.29), and (1.4) we get

$$\begin{aligned} &\int_0^{\pi} \sin\alpha x f(y_- + F(y_-))dx = \sum_{k=2}^{\infty} f_k \sum_{j=0}^{k} C_k^j \int_0^{\pi} \sin\alpha x\ F^j(y_-)\ y_-^{k-j}\ dx \\ &= \sum_{k=2}^{\infty} f_k \sum_{j=0}^{k} C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}})F^{\overline{\beta^j}}(y_-)\ y(\overline{\eta^{k-j}}) \end{aligned} \tag{1.30}$$

where $C_k^j = \frac{k!}{j!(k-j)!}$ are the binomial coefficients. Equations (1.23), (1.27), (1.30) yield:

$$\lambda_\alpha F^\alpha(y_-) + \sum_{k=2}^{\infty} f_k \sum_{j=0}^{k} C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) F^{\overline{\beta^j}}(y_-) y(\overline{\eta^{k-j}}) \tag{1.31}$$

$$= \sum_{\xi=N+1}^{\infty} \frac{\partial F^\alpha(y_-)}{\partial \widehat{y}(\xi)} \left( \lambda_\xi \widehat{y}(\xi) + \sum_{k=2}^{\infty} f_k \sum_{j=0}^{k} C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\xi; \overline{\beta^j}, \overline{\eta^{k-j}}) F^{\overline{\beta^j}}(y_-) y(\overline{\eta^{k-j}}) \right)$$

where $\alpha = 1, \dots, N$. Note that the term with $j = 0$ in the sums $\sum_{j=0}^{k}$ from (1.31) does not contain summation over $\overline{\beta^j}$, and $F^{\overline{\beta^j}}(y_-) = 1$ there.

Equation (1.31) is the equation (1.23) written in the basis $\{e_j\}$.

## 2. Formal construction of the map $F$

In this section we look for the map $F$ that defines the stable invariant manifold $M_-$ in the class of analytical maps. We derive recurrence relations for the coefficients of the map $F$ in terms of power series.

**2.1. Analytic maps.** Let $H_i$ be Hilbert spaces with scalar products $(\cdot, \cdot)_i$ and norms $\| \cdot \|_i$ where $i = 1, 2$. We denote by $(H_1)^k = H_1 \times \cdots \times H_1$ ($k$ times) the direct product of $k$ copies of $H_1$ and by $F_k : (H_1)^k \to H_2$ a $k$-linear operator $F_k(h_1, \dots, h_k)$, i.e. the operator that is linear with respect to each variable $h_i, i = 1, \dots, k$. Then

$$\|F_k\| = \sup_{\|h_i\|_1 = 1, i=1,\dots,k} \|F_k(h_1, \dots, h_k)\|_2 \tag{2.1}$$

The restriction of $k$-linear operator $F_k(h_1, \dots, h_k)$ to the diagonal $h_1 = \cdots = h_k = h$ is called the power operator of order $k$:

$$F_k(h) = F_k(h, \dots, h) \tag{2.2}$$

A k-linear operator $F_k(h_1, \dots, h_k)$ is called symmetric if for each permutation $(j_1, \dots, j_k)$ of $(1, 2, \dots, k)$

$$F_k(h_{j_1}, \dots, h_{j_k}) = F_k(h_1, \dots, h_k)$$

Using derivatives one can restore the symmetric $k$-linear operator $F_k(h_1, \dots, h_k)$ from the power operator $F_k(h)$.

We denote by $\mathcal{O}(H_1)$ a neighborhood of the origin in the space $H_1$. The map

$$F : \mathcal{O}(H_1) \to H_2 \tag{2.3}$$

is called analytic if it can be decomposed in the series

$$F(h) = F_0 + \sum_{k=1}^{\infty} F_k(h) \tag{2.4}$$

where $F_0 \in H_2$ and $F_k(h)$ are power operators of order $k$. The series (2.4) converges if the numerical series $\|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k(h)\|_2$ converges.

Proposition 2.1. *Let norms (2.1) of the power operator $F_k(h)$ from (2.4) satisfy*

$$\|F_k\| \leq \gamma\rho^{-k} \tag{2.5}$$

*where $\gamma > 0, \rho > 0$ do not depend on $k$.* [2] *Then series (2.4) converges for each $h \in B_\rho(H_1) = \{h \in H_1 : \|h\|_1 < \rho\}$.*

Proof. There exists $\varepsilon > 0$ such that $\|h\|_1 \leq \rho - \varepsilon$. Then using (2.1), (2.5) we get

$$\|F(h)\|_2 \leq \|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k\| \|h\|_1^k \leq \gamma \sum_{k=1}^{\infty} (\frac{\rho-\varepsilon}{\rho})^k < \infty$$

□

Let us consider the special case when $H_1 = H_-, H_2 = H_+$ with Hilbert spaces $H_-, H_+$ defined in (1.12). In this case the analytic map (2.3), (2.4) can be rewritten as follows:

$$F : \mathcal{O}(H_-) \to H_+, \quad F(h_-) = \sum_{k=2}^{\infty} F_k(h_-) \tag{2.6}$$

It is convenient to restrict ourselves to the case where $F_0 = 0,\ F_1 = 0$ because by (1.17) the map $F$ defining stable invariant manifold $M_-$ has precisely this form.

We decompose the vectors in $H_+$ in the basis $[e_1, \dots e_N]$ and the vectors in $H_-$ in the basis $[e_{N+1}, e_{N+2}, \dots]$ using notation (1.28), (1.29). We have

$$F(h_-) = \sum_{\alpha=1}^{N} F^\alpha(h_-)e_\alpha = \sum_{\alpha=1}^{N}\sum_{k=2}^{\infty} F_k^\alpha(h_-)e_\alpha \tag{2.7}$$

Using the decomposition

$$h_- = \sum_{\eta=N+1}^{\infty} h(\eta)e_\eta$$

we get

$$\begin{aligned} F_k^\alpha(h_-) &= F_k^\alpha(h_-, \dots, h_-) \\ &= \sum_{\eta_1=N+1}^{\infty} \cdots \sum_{\eta_k=N+1}^{\infty} F_k^\alpha(e_{\eta_1}, \dots, e_{\eta_k})h(\eta_1)\dots h(\eta_k) \end{aligned} \tag{2.8}$$

Using the notation

$$F_k^\alpha(\eta_1, \dots \eta_k) = F_k^\alpha(e_{\eta_1}, \dots, e_{\eta_k}) \tag{2.9}$$

as well as the notation (1.28), (1.29) we can rewrite (2.8) as follows:

$$F_k^\alpha(h_-) = \sum_{\overline{\eta^k}} F_k^\alpha(\overline{\eta^k})h(\overline{\eta^k}) \tag{2.10}$$

and the serie $F^\alpha(h_-)$ from (2.7) in the following way:

$$F^\alpha(h_-) = \sum_{k=2}^{\infty}\sum_{\overline{\eta^k}} F_k^\alpha(\overline{\eta^k})h(\overline{\eta^k}) \tag{2.11}$$

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[2] For brevity we use for the power operator $F_k(h)$ the norm (2.1) although an alternative definition is possible (see details in Chapter 1 of [**VF2**])

Let $\mathbb{Z}_{N+1} = \{k \in \mathbb{Z} : k \geq N+1\}$, $\mathbb{Z}^r_{N+1} = \mathbb{Z}_{N+1} \times \cdots \times \mathbb{Z}_{N+1}$ ($r$ times). For each function $K(\eta_1, \dots, \eta_r)$ defined on $\mathbb{Z}^r_{N+1}$ we define the function $\sigma_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r)$ which is symmetric with respect to an arbitrary permutation $(\eta_{j_1}, \dots, \eta_{j_r})$ of variables $(\eta_1, \dots, \eta_r)$ by the formula:

$$\sigma_{\overline{\eta^r}} K(\eta_1, \dots \eta_r) = \frac{1}{r!} \sum_{(j_1, \dots, j_r)} K(\eta_{j_1}, \dots, \eta_{j_r}) \tag{2.12}$$

where the sum on the r.h.s. of (2.12) is over all permutations $(j_1, \dots, j_r)$ of the set $(1, \dots, r)$.

Lemma 2.2. *Let $K(\eta_1, \dots, \eta_r)$ be defined on $\mathbb{Z}^r_{N+1}$. Then*

(a) *The following equality is true:*

$$\sum_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r) h(\eta_1) \dots h(\eta_r) = \sigma_{\overline{\eta^r}} \sum_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r) h(\eta_1) \dots h(\eta_r)$$

*for any $h(\eta_r)$ such that the series on the l.h.s. converges,*

(b) *For any function $G(\eta_1, \dots, \eta_r)$ which is symmetric in its arguments*

$$G(\overline{\eta^r}) \sigma_{\overline{\eta^r}} K(\overline{\eta^r}) = \sigma_{\overline{\eta^r}} [G(\overline{\eta^r}) K(\overline{\eta^r})] \tag{2.13}$$

*Furthermore*

$$\sup_{\overline{\eta^r}} |\sigma_{\overline{\eta^r}} K(\overline{\eta^r})| \leq \sup_{\overline{\eta^r}} |K(\overline{\eta^r})| \tag{2.14}$$

(c) *If all functions $F^\alpha_k(\overline{\eta^k})$ from (2.11) are symmetric in their arguments then these functions are defined uniquely by values of the analytic functions $F^\alpha(h_-)$ from (2.11).*

The proof of this Lemma is evident.

**2.2. Calculation of $F^\alpha_2(\overline{\eta^2})$.** We look for a solution $F^\alpha(y_-), \alpha = 1, \dots, N$ of system (1.31) in the form (2.11). The aim of this section is to find recurrence relations for the coefficients $F^\alpha_k(\overline{\eta^k})$ from (2.11). We rewrite (1.31) as follows:

$$\begin{aligned}
&\lambda_\alpha F^\alpha(y_-) + \sum_{k=2}^\infty f_k \sum_{\overline{\eta^k}} b_k(\alpha; \overline{\eta^k}) y(\overline{\eta^k}) \\
&+ \sum_{k=2}^\infty f_k \sum_{j=1}^k C^j_k \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) F^{\overline{\beta^j}}(y_-) y(\overline{\eta^{k-j}}) \\
&= \sum_{\xi=N+1}^\infty \frac{\partial F^\alpha(y_-)}{\partial \widehat{y}(\xi)} \left( \lambda_\xi \widehat{y}(\xi) + \sum_{k=2}^\infty f_k \sum_{\overline{\eta^k}} b_k(\xi; \overline{\eta^k}) y(\overline{\eta^k}) \right. \\
&\left. + \sum_{k=2}^\infty f_k \sum_{j=1}^k C^j_k \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\xi; \overline{\beta^j}, \overline{\eta^{k-j}}) F^{\overline{\beta^j}}(y_-) y(\overline{\eta^{k-j}}) \right)
\end{aligned} \tag{2.15}$$

and substitute (2.11) into (2.15). Then we get

$$
\begin{aligned}
&\lambda_\alpha \sum_{p=2}^{\infty} \sum_{\overline{\eta^p}} F_p^\alpha(\overline{\eta^p}) y(\overline{\eta^p}) + \sum_{k=2}^{\infty} f_k \sum_{\overline{\eta^k}} b_k(\alpha; \overline{\eta^k}) y(\overline{\eta^k}) \\
&+ \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k} C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) y(\overline{\eta^{k-j}}) \\
&\times \sum_{m_1=2}^{\infty} \sum_{\overline{\zeta^{m_1}}} F_{m_1}^{\beta_1}(\overline{\zeta^{m_1}}) y(\overline{\zeta^{m_1}}) \cdots \sum_{m_j=2}^{\infty} \sum_{\overline{\zeta^{m_1}}} F_{m_j}^{\beta_j}(\overline{\zeta^{m_j}}) y(\overline{\zeta^{m_j}}) \\
&= \sum_{\xi=N+1}^{\infty} \left( \frac{\partial}{\partial \widehat{y}(\xi)} \sum_{p=2}^{\infty} \sum_{\overline{\eta^p}} F_p^\alpha(\overline{\eta^p}) y(\overline{\eta^p}) \right) \Bigg[ \lambda_\xi y(\xi) + \sum_{k=2}^{\infty} f_k \sum_{\overline{\eta^k}} b_k(\xi; \overline{\eta^k}) y(\overline{\eta^k}) \\
&+ \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k} C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\xi; \overline{\beta^j}, \overline{\eta^{k-j}}) y(\overline{\eta^{k-j}}) \\
&\times \sum_{m_1=2}^{\infty} \sum_{\overline{\zeta^{m_1}}} F_{m_1}^{\beta_1}(\overline{\zeta^{m_1}}) y(\overline{\zeta^{m_1}}) \cdots \sum_{m_j=2}^{\infty} \sum_{\overline{\zeta^{m_1}}} F_{m_j}^{\beta_j}(\overline{\zeta^{m_j}}) y(\overline{\zeta^{m_j}}) \Bigg]
\end{aligned} \tag{2.16}
$$

In order to get recurrence relations for $F_p^\alpha(\overline{\eta^p})$ we have to equate the coefficients of the monomials $y(\overline{\eta^k})$ on left and right sides of (2.16). To do this in the general case we first have to make essential transformations of (2.16). But in order to find $F_2^\alpha(\overline{\eta^2})$ we do not need any serious preparation . Let us find it.

Note that the terms of the second order in $y$ are contained only in the first and second summands on the left side and in the first summand on the right side of (2.16). They are as follows:

$$
\begin{aligned}
&\lambda_\alpha \sum_{\overline{\eta^2}} F_2^\alpha(\overline{\eta^2}) y(\overline{\eta^2}) + f_2 \sum_{\overline{\eta^2}} b_2(\alpha; \overline{\eta^2}) y(\overline{\eta^2}) \\
&= \sum_{\xi=N+1}^{\infty} \left( \frac{\partial}{\partial \widehat{y}(\xi)} \sum_{\overline{\eta^2}} F_2^\alpha(\overline{\eta^2}) y(\overline{\eta^2}) \right) \lambda_\xi \widehat{y}(\xi)
\end{aligned} \tag{2.17}
$$

Denote by $I$ the right side of (2.17). Then

$$
\begin{aligned}
I &= \sum_{\xi, \eta_2} F_2^\alpha(\xi, \eta_2) \lambda_\xi y(\xi) y(\eta_2) + \sum_{\eta_1, \xi} F_2^\alpha(\eta_1, \xi) \lambda_\xi y(\eta_2) y(\xi) \\
&= \sum_{\overline{\eta^2}} F_2^\alpha(\overline{\eta^2}) (\lambda_{\eta_1} + \lambda_{\eta_2}) y(\eta_1) y(\eta_2)
\end{aligned} \tag{2.18}
$$

Equalities (2.17), (2.18) imply:

$$
f_2 \sum_{\overline{\eta^2}} b_2(\alpha; \overline{\eta^2}) y(\overline{\eta^2}) = \sum_{\overline{\eta^2}} F_2^\alpha(\overline{\eta^2}) (\lambda_{\eta_1} + \lambda_{\eta_2} - \lambda_\alpha) y(\eta_1) y(\eta_2) \tag{2.19}
$$

By definition (1.26), $b_2(\alpha, \eta_1, \eta_2)$ is symmetric with respect to $\eta_1, \eta_2$. Since by definition $F_2^\alpha(\eta_1, \eta_2)$ is symmetric in $\eta_1, \eta_2$, the function $F_2^\alpha(\eta_1, \eta_2)(\lambda_{\eta_1} + \lambda_{\eta_2} - \lambda_\alpha)$

is symmetric as well. Therefore by Lemma 2.2 equation (2.19) implies

$$F_2^\alpha(\overline{\eta^2}) = \frac{f_2 b_2(\alpha, \overline{\eta^2})}{\lambda_{\eta_1} + \lambda_{\eta_2} - \lambda_\alpha} \tag{2.20}$$

Note that since $\eta_j \geq N+1$ and $\alpha \leq N$, we have $\lambda_{\eta_j} > 0, \lambda_\alpha < 0$ and therefore

$$\lambda_{\eta_1} + \lambda_{\eta_2} - \lambda_\alpha = |\lambda_{\eta_1}| + |\lambda_{\eta_2}| + |\lambda_\alpha| > 0 \tag{2.21}$$

**2.3. Recurrence relation for $F_q^\alpha(\overline{\eta^q})$.** First of all analogously to (2.17)-(2.19) we transform the first terms of the right and left sides of (2.16):

$$\begin{aligned} &\sum_\xi \left( \frac{\partial}{\partial y(\xi)} \sum_{p=2}^\infty \sum_{\overline{\eta^p}} F_p^\alpha(\overline{\eta^p}) y(\overline{\eta^p}) \right) \lambda_\xi \widehat{y}(\xi) - \lambda_\alpha \sum_{p=2}^\infty \sum_{\overline{\eta^p}} F_p^\alpha(\overline{\eta^p}) y(\overline{\eta^p}) \\ =&\sum_{p=2}^\infty \sum_{\overline{\eta^p}} F_p^\alpha(\eta_1, \dots, \eta_p)(\lambda_{\eta_1} + \dots + \lambda_{\eta_p} - \lambda_\alpha) y(\overline{\eta^p}) \end{aligned} \tag{2.22}$$

Let us now transform the third term on the left side of (2.16)

$$\begin{aligned} I_3 \equiv &\sum_{k=2}^\infty f_k \sum_{j=1}^k C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) y(\overline{\eta^{k-j}}) \\ &\times \sum_{m_1=2}^\infty \sum_{\overline{\zeta^{m_1}}} F_{m_1}^{\beta_1}(\overline{\zeta^{m_1}}) y(\overline{\zeta^{m_1}}) \cdots \sum_{m_j=2}^\infty \sum_{\overline{\zeta^{m_1}}} F_{m_j}^{\beta_j}(\overline{\zeta^{m_j}}) y(\overline{\zeta^{m_j}}) \\ &= \sum_{k=2}^\infty f_k \sum_{j=1}^k C_k^j \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{k-j}}} b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) y(\overline{\eta^{k-j}}) \\ &\times \sum_{p=2j}^\infty \sum_{\overline{\zeta^p}} \sum_{\substack{m_1+\dots+m_j=p, \\ m_l \geq 2}} (F_{m_1}^{\beta_1} \dots F_{m_j}^{\beta_j})(\overline{\zeta^p}) y(\overline{\zeta^p}) \end{aligned} \tag{2.23}$$

We make on the right side of (2.23) the change of variables $(k,j,p) \to (q,j,p)$ with $q = k - j + p$ ($q$ is a full power of $y(\overline{\eta^{k-j}})y(\overline{\zeta^p})$). To do this we introduce the set $Q_q$ of pairs $(j,p)$ writing it in several forms:

$$\begin{aligned} Q_q &= \{(j,p) \in \mathbb{Z}_+^2 : q + j - p \geq 2, 1 \leq j \leq q + j - p, p \geq 2j\} \\ &= \{(j,p) \in \mathbb{Z}_+^2 : 2 \leq 2j \leq p \leq q + j - 2, p \leq q\} \\ &= \{(j,p) \in \mathbb{Z}_+^2 : 1 \leq j \leq q - 2, 2j \leq p \leq \min(q, q + j - 2)\}. \end{aligned} \tag{2.24}$$

Now after applying aforementioned change of variables the r.s. of (2.23) can be rewritten as follows:

$$
\begin{aligned}
I_3 &\equiv \sum_{q=3}^{\infty} \sum_{(j,p)\in Q_q} C^j_{q+j-p} f_{q+j-p} \sum_{\overline{\beta^j}} \sum_{\overline{\eta^{q-p}},\overline{\zeta^p}} b_{q+j-p}(\alpha; \overline{\beta^j}, \overline{\eta^{q-p}}) y(\overline{\eta^{q-p}}) \\
&\times \sum_{\substack{m_1+\dots+m_j=p,\\ m_i\geq 2}} F^{\beta_1}_{m_1} \dots F^{\beta_j}_{m_j}(\overline{\zeta^p}) y(\overline{\zeta^p}) \\
&= \sum_{q=3}^{\infty} \sum_{\overline{\eta^q}} \sigma_{\overline{\eta^q}} \Bigg( \sum_{(j,p)\in Q_q} f_{q+j-p} C^j_{q+j-p} \\
&\times \sum_{\overline{\beta^j}} \sum_{\substack{m_1+\dots+m_j=p,\\ m_i\geq 2}} (b_{q+j-p}(\alpha; \overline{\beta^j}, \cdot) F^{\beta_1}_{m_1} \dots F^{\beta_j}_{m_j})(\overline{\eta^q})) y(\overline{\eta^q}) \Bigg)
\end{aligned} \tag{2.25}
$$

We now transform the second summand on r.s. of (2.16) analogously to (2.22):

$$
\begin{aligned}
J_2 &\equiv \sum_{\xi} \left( \frac{\partial}{\partial y(\xi)} \sum_{p=2}^{\infty} \sum_{\overline{\eta^p}} F^\alpha_p(\overline{\eta^p}) y(\overline{\eta^p}) \right) \sum_{k=2}^{\infty} f_k \sum_{\overline{\zeta^k}} b_k(\xi; \overline{\zeta^k}) y(\overline{\zeta^k}) \\
&= \sum_{p=2}^{\infty} \sum_{k=2}^{\infty} f_k \sum_{\overline{\eta^{p-1}},\overline{\zeta^k}} \sum_{l=1}^{p} \sum_{\xi} F^\alpha_p(\eta_1, \dots, \eta_{l-1}, \xi, \eta_l, \dots, \eta_{p-1}) \\
&\times b_k(\xi; \overline{\zeta^k}) y(\overline{\eta^{p-1}}) y(\overline{\zeta^k}) \\
&= \sum_{q=3}^{\infty} \sum_{\substack{p+k=q+1,\\ p,k\geq 2}} f_k \sum_{\overline{\eta^q},\xi} \sum_{l=1}^{p} F^\alpha_p(\eta_1, \dots, \eta_{l-1}, \xi, \eta_{l+k}, \dots, \eta_q) \\
&\times b_k(\xi; \eta_l, \eta_{l+1}, \dots, \eta_{l+k-1}) y(\overline{\eta^q})
\end{aligned} \tag{2.26}
$$

Finally we transform the last summand on r.s. of (2.16):

$$
\begin{aligned}
J_3 \equiv & \sum_{\xi}\left(\frac{\partial}{\partial y(\xi)}\sum_{m=2}^{\infty}\sum_{\overline{\eta^m}} F_m^{\alpha}(\overline{\eta^m})y(\overline{\eta^m})\right)\\
& \times \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k} C_k^j \sum_{\overline{\beta^j}}\sum_{\overline{\eta^{k-j}}} b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})y(\overline{\eta^{k-j}})\\
& \times \sum_{m_1=2}^{\infty}\sum_{\overline{\zeta^{m_1}}} F_{m_1}^{\beta_1}(\overline{\zeta^{m_1}})y(\overline{\zeta^{m_1}})\cdots \sum_{m_j=2}^{\infty}\sum_{\overline{\zeta^{m_1}}} F_{m_j}^{\beta_j}(\overline{\zeta^{m_j}})y(\overline{\zeta^{m_j}})\\
& = \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k}\frac{C_k^j}{j+1}\sum_{\overline{\beta^j},\xi}\frac{\partial}{\partial y(\xi)}\left(\sum_{p=2(j+1)}^{\infty}\sum_{\overline{\zeta^p}} y(\overline{\zeta^p})\right.\\
& \times \left.\sum_{m_1+\cdots+m_{j+1}=p}(F_{m_1}^{\beta_1}\dots F_{m_j}^{\beta_j}F_{m_{j+1}}^{\alpha})(\overline{\zeta^p})\right)\sum_{\overline{\eta^{k-j}}} b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})y(\overline{\eta^{k-j}})\\
& = \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k}\frac{C_k^j}{j+1}\sum_{\overline{\beta^j}}\sum_{p=2(j+1)}^{\infty}\sum_{\overline{\zeta^{p-1}},\overline{\eta^{k-j}}}\sum_{m_1+\cdots+m_{j+1}=p} y(\overline{\eta^{k-j}})y(\overline{\zeta^{p-1}})\\
& \times \sum_{l=1}^{p}\sum_{\xi}(F_{m_1}^{\beta_1}\dots F_{m_{j+1}}^{\alpha})(\zeta_1,\dots,\zeta_{l-1},\xi,\zeta_l,\dots,\zeta_{p-1})b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})\\
& = \sum_{k=2}^{\infty} f_k \sum_{j=1}^{k}\frac{C_k^j}{j+1}\sum_{\overline{\beta^j}}\sum_{p=2(j+1)}^{\infty}\sum_{\overline{\zeta^{p+k-j-1}}}\sum_{m_1+\cdots+m_{j+1}=p}\\
& \times \sum_{l=1}^{p}\sum_{\xi}(F_{m_1}^{\beta_1}\dots F_{m_{j+1}}^{\alpha})(\zeta_1,\dots,\zeta_{l-1},\xi,\zeta_{l+k-j},\dots,\zeta_{p+k-j-1})\\
& \times b_k(\xi;\overline{\beta^j},\zeta_l,\dots,\zeta_{l+k-j-1})y(\overline{\zeta^{p+k-j-1}})
\end{aligned}
\tag{2.27}
$$

We now make the change of variables $(k,j,p)\to(q,j,p)$ where $q=k-j+p-1$, i.e. $q$ is the number of variables $y: y(\overline{\zeta^{p+k-j-1}})=y(\overline{\zeta^q})$. Then

$$
\sum_{k=2}^{\infty}\sum_{j=1}^{k}\sum_{p=2(j+1)}^{\infty}=\sum_{q=4}^{\infty}\sum_{(j,p)\in\widehat{Q}^q}
$$

where

$$
\begin{aligned}
\widehat{Q}^q &= \{(j,p)\in\mathbb{Z}_+^2:\ 1\le j\le q+j-p+1, q+j-p+1\ge 2,\ p\ge 2(j+1)\}\\
&= \{(j,p)\in\mathbb{Z}_+^2:\ 4\le 2(j+1)\le p\le \min(q+j-1,q+1)\}\\
&= \{(j,p)\in\mathbb{Z}_+^2:\ 1\le j\le q-3,\ 2(j+1)\le p\le \min(q+1,q+j-3)\}.
\end{aligned}
\tag{2.28}
$$

Therefore

$$J_3 \equiv \sum_{q=4}^{\infty} \sum_{(j,p)\in\widehat{Q}^q} \sum_{\overline{\eta^q},\xi,\overline{\beta^j}} \sum_{m_1+\cdots+m_{j+1}=p} f_{q+j-p+1} \frac{C^j_{q+j-p+1}}{j+1}$$
$$\times \sum_{l=1}^{p} (F^{\beta_1}_{m_1} \dots F^{\beta_j}_{m_j} F^{\alpha}_{m_{j+1}})(\eta_1, \dots, \eta_{l-1}, \xi, \eta_{l+q-p+1}, \dots, \eta_q)$$
$$\times\, b_{q+j-p+1}(\xi; \overline{\beta^j}, \eta_l, \dots, \eta_{l+q-p}) y(\overline{\eta^q}) \tag{2.29}$$

We substitute (2.22), (2.25), (2.26), (2.29) into (2.16). Then using the notation

$$\lambda_{\eta_1} + \dots + \lambda_{\eta_q} = \lambda_{\overline{\eta^q}} \tag{2.30}$$

we obtain:

$$\sum_{q=2}^{\infty} \sum_{\overline{\eta^q}} F^{\alpha}_q(\overline{\eta^q})(\lambda_{\overline{\eta^q}} - \lambda_\alpha) y(\overline{\eta^q}) = I_3 - J_2 - J_3 + \sum_{q=2}^{\infty} f_q \sum_{\overline{\eta^q}} b_q(\alpha; \overline{\eta^q}) y(\overline{\eta^q}) \tag{2.31}$$

where $I_3, J_2, J_3$ are equal to the right hand sides of (2.25), (2.26), (2.29) respectively. By Lemma 2.2 setting equal in (2.31) the terms of the same order with respect to $y(\overline{\eta^q})$, we get the recurrence relations for the coefficients $F^{\alpha}_q(\overline{\eta^q})$:

$$F^{\alpha}_3(\overline{\eta^3}) = (\lambda_{\overline{\eta^3}} - \lambda_\alpha)^{-1}\{f_3 b_3(\alpha; \overline{\eta^3}) + f_2 \sigma_{\overline{\eta^3}}[2\sum_{\beta} b_2(\alpha; \beta, \eta_1) F^{\beta}_2(\eta_2, \eta_3)$$
$$- \sum_{\xi} (F^{\alpha}_2(\xi, \eta_3) b_2(\xi; \eta_1, \eta_2) + F^{\alpha}_2(\eta_1, \xi) b_2(\xi; \eta_2, \eta_3))]\} \tag{2.32}$$

and for $q \geq 4$:

$$F^{\alpha}_q(\overline{\eta^q}) = A^{\alpha}_q(\overline{\eta^q}) + B^{\alpha}_q(\overline{\eta^q}) + C^{\alpha}_q(\overline{\eta^q}) + D^{\alpha}_q(\overline{\eta^q}) \tag{2.33}$$

where (using the r.s. of (2.31))

$$A^{\alpha}_q(\overline{\eta^q}) = (\lambda_{\overline{\eta^q}} - \lambda_\alpha)^{-1} f_q b_q(\alpha; \overline{\eta^q}), \tag{2.34}$$

(using the r.s. of (2.31) and (2.25))

$$B^{\alpha}_q(\overline{\eta^q}) = (\lambda_{\overline{\eta^q}} - \lambda_\alpha)^{-1} \sigma_{\overline{\eta^q}}[\sum_{(j,p)\in Q^q} f_{q+j-p} C^j_{q+j-p}$$
$$\times \sum_{\overline{\beta^j}} \sum_{\substack{m_1+\cdots+m_j=p,\\ m_l\geq 2}} b_{q+j-p}(\alpha; \overline{\beta^j}, \overline{\eta^{q-p}})(F^{\beta_1}_{m_1} \dots F^{\beta_j}_{m_j})(\overline{\eta^p})] \tag{2.35}$$

where the set $Q^q$ is defined in (2.24). Using the r.s. of (2.31) and (2.26) we get:

$$C^{\alpha}_q(\overline{\eta^q}) = \sigma_{\overline{\eta^q}} \left[ \sum_{\substack{p+k=q+1,\\ p,k\geq 2}} \sum_{l=1}^{p} \sum_{\xi} F^{\alpha}_p(\eta_1, \dots \eta_{l-1}, \xi, \eta_{l+k}, \dots, \eta_q) \right.$$
$$\left. \times \frac{f_k b_k(\xi; \eta_l, \dots, \eta_{l+k-1})}{\lambda_{\overline{\eta^q}} - \lambda_\alpha} \right] \tag{2.36}$$

At last, using the r.s. of (2.31) and (2.29) we obtain:

$$
\begin{aligned}
D_q^\alpha(\overline{\eta^q}) = &- \sigma_{\overline{\eta^q}} \Bigg[ \sum_{(j,p)\in \widehat{Q}^q} \sum_{\xi,\overline{\beta^j}} \sum_{m_1+\dots+m_{j+1}=p} \frac{f_{q+j-p+1} C^j_{q+j-p+1}}{(\lambda_{\overline{\eta^q}}-\lambda_\alpha)(j+1)} \\
&\times \sum_{l=1}^{p} (F^{\beta_1}_{m_1}\dots F^{\beta_j}_{m_j} F^{\alpha}_{m_{j+1}})(\eta_1,\dots \eta_{l-1},\xi,\eta_{l+q-p+1},\dots,\eta_q) \\
&\times b_{q+j-p+1}(\xi;\overline{\beta^j},\eta_l,\dots,\eta_{l+q-p}) \Bigg]
\end{aligned} \tag{2.37}
$$

where the set $\widehat{Q}^q$ is defined in (2.28).

## 3. Analyticity of the map $F$

In this section we prove the convergence of the series (2.11) that defines the $\alpha$-coordinate of the map $F(y_-) = (F^1,\dots,F^N)$.

**3.1. Norms for series.** Here we define some norms which are used to prove the convergence of series (2.11). Although they are not directly connected with the Sobolev space $H^1_0(0,\pi)$, they will help us prove the convergence of (2.11) in $H^1_0(0,\pi)$ as well. For $F_k = (F^1_k(\overline{\eta^1}),\dots,F^N_k(\overline{\eta^1}))$ we set

$$
\|F_k(\overline{\eta^k})\| = \sum_{j=1}^{N} |F^\alpha_k(\overline{\eta^k})| \tag{3.1}
$$

and

$$
\|F_k\| = \sup_{\overline{\eta^k}\in \mathbb{Z}^k_{N+1}} \|F_k(\overline{\eta^k})\| \tag{3.2}
$$

where

$$
\mathbb{Z}_{N+1} = \{j\in\mathbb{Z}_+ : j\ge N+1\}, \qquad \mathbb{Z}^k_{N+1} = \mathbb{Z}_{N+1}\times\dots\times\mathbb{Z}_{N+1} (k \text{ times}). \tag{3.3}
$$

The norms for functions $b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})$, $b_k(\alpha;\overline{\beta^j},\overline{\eta^{k-j}})$ defined in (1.26) are introduced similarly:

$$
\|b_k\|_{(j)} = \sup_{\overline{\beta^j}\in\mathbb{Z}^j_{[1,N]}} \sup_{\overline{\eta^{k-j}}\in\mathbb{Z}^{k-j}_{N+1}} \sum_{\xi=N+1}^{\infty} |b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})|, \quad \|b_k\| = \max_{1\le j\le k}\|b_k\|_{(j)} \tag{3.4}
$$

$$
|||b_k|||_{(j)} = \sup_{\overline{\beta^j}\in\mathbb{Z}^j_{[1,N]}} \sup_{\overline{\eta^{k-j}}\in\mathbb{Z}^{k-j}_{N+1}} \sum_{\alpha=1}^{N} |b_k(\alpha;\overline{\beta^j},\overline{\eta^{k-j}})|, \quad |||b_k||| = \max_{1\le j\le k}|||b_k|||_{(j)} \tag{3.5}
$$

where

$$
\mathbb{Z}_{[1,N]} = \{1,2,\dots,N\}, \quad \mathbb{Z}^j_{[1,N]} = \mathbb{Z}_{[1,N]}\times\dots\times\mathbb{Z}_{[1,N]} \ (j \text{ times}) \tag{3.6}
$$

In the case $j=0$ the variable $\overline{\beta^j}$ is absent in (3.4), (3.5) as well as if $j=k$, then $\overline{\eta^{k-j}}$ is absent.

The goal of this section is to prove that the coefficients $F^\alpha_q(\overline{\eta^q})$ that are defined by the recurrence relations (2.20), (2.32)-(2.37) satisfy the inequality

$$
\|F_k\| \le \gamma_1 \rho_1^{-k} \tag{3.7}
$$

with some numbers $\gamma_1>0$, $\rho_1>0$ independent of $k$.

The following Proposition shows that condition (3.7) is sufficient for the convergence of the series

$$F(y_-) = \sum_{k=2}^{\infty} \sum_{\overline{\eta^k}} F_k(\overline{\eta^k}) y(\overline{\eta^k}) \tag{3.8}$$

in the ball

$$B_{\rho_1/r_N}(H_-) \equiv \{y_- \in H_- : \ \|y_-\|_{H_0^1(0,\pi)} < \rho_1/r_N\} \tag{3.9}$$

of the space $H_- \subset H_0^1(0,\pi)$ defined in (1.12). Here

$$r_N = \left( \sum_{\xi=N+1}^{\infty} \xi^{-2} \right)^{1/2} \tag{3.10}$$

Proposition 3.1. *Let the coefficients $F_k(\overline{\eta^k}) = F_k^1(\overline{\eta^k})e_1 + \cdots + F_k^N(\overline{\eta^k})e_N$ of series (3.8) satisfy (3.7) with $\gamma_1 > 0$, $\rho_1 > 0$ independent of $k$. Then series (3.8) converges in the ball $B_{\rho_1/r_N}$ where $r_N$ is the number that is defined in (3.10).*

Proof. By the Cauchy- Bunyakovskii inequality

$$\sum_{\xi=N+1}^{\infty} |y(\xi)| \le \left( \sum_{\xi=N+1}^{\infty} \frac{1}{\xi^2} \right)^{1/2} \left( \sum_{\xi=N+1}^{\infty} \xi^2 |y(\xi)|^2 \right)^{1/2} \tag{3.11}$$

Therefore if $y_- \in B_{\rho_1/r_N}(H_-)$ then

$$\sum_{\xi=N+1}^{\infty} |y(\xi)| \le \rho_1 - \varepsilon$$

with some $\varepsilon > 0$, and (3.7), (3.8) imply

$$\|F(y_-)\|_{H_+} \le c \sum_{k=2}^{\infty} \sum_{\overline{\eta^k}} \|F_k(\overline{\eta^k})\| |y(\eta_1)| \dots |y(\eta_k)|$$

$$\le c \sum_{k=2}^{\infty} \|F_k\| \left( \sum_{\xi=N+1}^{\infty} |y(\xi)| \right)^k \le c\gamma_1 \sum_{k=2}^{\infty} \left( \frac{\rho_1 - \varepsilon}{\rho_1} \right)^k < \infty$$

□

**3.2. Estimates for functions (1.26).** Recall that

$$b_k(\alpha; \overline{\beta^j}, \overline{\eta^{k-j}}) = (2/\pi)^{\frac{k+1}{2}} \int_0^{\pi} \sin \alpha x \sin \beta_1 x \dots \sin \beta_j x \sin \eta_1 x \dots \sin \eta_{k-j} x \; dx \tag{3.12}$$

$$b_k(\xi; \overline{\beta^j}, \overline{\eta^{k-j}}) = (2/\pi)^{\frac{k+1}{2}} \int_0^{\pi} \sin \xi x \sin \beta_1 x \dots \sin \beta_j x \sin \eta_1 x \dots \sin \eta_{k-j} x \; dx \tag{3.13}$$

where $\alpha, \beta_1, \dots, \beta_j \in \mathbb{Z}_{[1,N]}$, $\xi, \eta_1, \dots, \eta_{k-j} \in \mathbb{Z}_{N+1}$ and $\mathbb{Z}_{[1,N]}$, $\mathbb{Z}_{N+1}$ are defined in (3.3), (3.6). In the case when $j = 0$ the $\sin \beta_1 x \dots \sin \beta_j x$ are absent in (3.12), (3.13). When $j = k$ the terms $\sin \eta_1 x \dots \sin \eta_{k-j} x$ are absent from these formulas.

Lemma 3.2. *Let $b_k(\alpha,\overline{\beta^j},\overline{\eta^{k-j}})$ be the function (3.12) and let $|||b_k|||$ be defined in (3.5). Then*

$$|||b_k||| \le (2/\pi)^{\frac{k+1}{2}}\pi N \tag{3.14}$$

Proof. By definitions (3.12), (3.5), taking into account that $\forall t\in\mathbb{R}\ |\sin t|\le 1$ we get

$$|||b_k||| =$$

$$= \max_{1\le j\le k}\sup_{\overline{\beta^j}}\sup_{\overline{\eta^{k-j}}}\sum_{\alpha=1}^{N}(2/\pi)^{\frac{k+1}{2}}\left|\int_0^\pi \sin\alpha x\sin\beta_1 x\dots\sin\beta_j x\sin\eta_1 x\dots\sin\eta_{k-j}x\ dx\right|$$

$$\le (2/\pi)^{\frac{k+1}{2}}\pi N$$

□

For the function (3.13) we define

$$\|b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})\| = \sum_{\xi=N+1}^{\infty}|b_k(\xi;\overline{\beta^j},\overline{\eta^{k-j}})| \tag{3.15}$$

Lemma 3.3. *For the function (3.15) the following inequality holds:*

$$\|b_k(\cdot;\overline{\beta^j},\overline{\eta^{k-j}})\| \le (2/\pi)^{\frac{k+1}{2}}\frac{\pi(k+1)}{2N}\left(\sum_{1\le l\le j}\beta_l^2+\sum_{m=1}^{k-j}\eta_m^2\right) \tag{3.16}$$

Proof. We rename $(\overline{\beta^j},\overline{\eta^{k-j}})$ as follows:

$$(\overline{\beta^j},\overline{\eta^{k-j}}) = (\beta_1,\dots,\beta_j,\eta_1,\dots,\eta_{k-j}) = (\theta_1,\dots,\theta_j,\theta_{j+1},\dots,\theta_k) = (\overline{\theta^k}) \tag{3.17}$$

Then by (3.13), (3.15), (3.17), after integrating by parts and simple estimates we have

$$\begin{aligned}
&\|b_k(\cdot;\overline{\beta^j},\overline{\eta^{k-j}})\| = \|b_k(\cdot;\overline{\theta^k})\|\\
&\le \sum_{\xi=N+1}^{\infty}(2/\pi)^{\frac{k+1}{2}}\left|\int_0^\pi \sin\xi x\ (\sin\theta_1 x\dots\sin\theta_k x)\ dx\right|\\
&= \sum_{\xi=N+1}^{\infty}(2/\pi)^{\frac{k+1}{2}}\frac{1}{\xi}\left|\int_0^\pi \cos\xi x\frac{d}{dx}(\sin\theta_1 x\dots\sin\theta_k x)\ dx\right|\\
&= \sum_{\xi=N+1}^{\infty}(2/\pi)^{\frac{k+1}{2}}\frac{1}{\xi^2}\left|\int_0^\pi \sin\xi x\frac{d^2}{dx^2}(\sin\theta_1 x\dots\sin\theta_k x)\ dx\right|\\
&\le \sum_{\xi=N+1}^{\infty}(2/\pi)^{\frac{k+1}{2}}\frac{\pi}{\xi^2}\sum_{i,j=1}^{k}\theta_i\theta_j \le \pi\,(2/\pi)^{\frac{k+1}{2}}\frac{k+1}{2N}\sum_{i=1}^{k}\theta_i^2
\end{aligned} \tag{3.18}$$

Note that we used here that $k\ge 2$. Now (3.16) follows from (3.18) (3.17). □

**3.3. Estimates of coefficients $F_q^\alpha(\overline{\eta^q})$.** To get the desired estimates we use the recurrence relations (2.33)-(2.37). Recall that by our notations $\xi, \eta_j \geq N+1$ and $1 \leq \alpha, \beta_j \leq N$ where $N$ is defined in (1.10). Therefore by the definition (1.7) of $\lambda_k$ we get:

$$\lambda_{\overline{\eta^j}} - \lambda_\alpha = \sum_{j=1}^{q} \lambda_{\eta_j} + |\lambda_\alpha| = \sum_{j=1}^{q} (\eta_j^2 - \kappa) + (\kappa - \alpha^2) \geq (q+1)\widehat{\lambda} \tag{3.19}$$

where $\widehat{\lambda} = \min((N+1)^2 - \kappa, \kappa - N^2) > 0$.

By virtue of (1.5) (3.19), and (3.14), the coefficients $A_q^\alpha(\overline{\eta^q})$ from (2.34) can be bounded as follows:

$$\begin{aligned} \|A_q\| &= \sup_{\overline{\eta^q}} \sum_{\alpha=1}^{N} |A_q^\alpha(\overline{\eta^q})| \leq ((q+1)\widehat{\lambda})^{-1} \gamma \rho^q \, (2/\pi)^{\frac{q+1}{2}} \, \pi N \\ &= \frac{\sqrt{2\pi}\gamma N}{(q+1)\widehat{\lambda}} \left(\rho\sqrt{2/\pi}\right)^q \end{aligned} \tag{3.20}$$

To estimate $B_q^\alpha(\overline{\eta^q})$ from (2.35) we use (3.19), (2.14), and Lemma 3.2:

$$\begin{aligned} \|B_q\| &\leq ((q+1)\widehat{\lambda})^{-1} \sup_{\overline{\eta^q}} \sigma_{\overline{\eta^q}} \sum_{(j,p)\in Q_q} f_{q+j-p} C_{q+j-p}^j \\ &\times \sum_{\substack{m_1+\dots+m_j=p, \\ m_l\geq 2}} \sup_{\overline{\beta^j}} \sum_{\alpha}^{N} |b_{q+j-p}(\alpha; \overline{\beta^j}, \overline{\eta^{q-p}})| \\ &\times \sum_{\beta_1} |F_{m_1}^{\beta_1}| \cdots \sum_{\beta_j} |F_{m_j}^{\beta_j}|(\eta_{q-p+1}, \dots, \eta_q) \\ &\leq \frac{\sqrt{2\pi}\gamma N}{((q+1)\widehat{\lambda})} \sum_{(j,p)\in Q_q} C_{q+j-p}^j \left(\rho\sqrt{2/\pi}\right)^{q+j-p} \\ &\times \sum_{\substack{m_1+\dots+m_j=p, \\ m_l\geq 2}} \|F_{m_1}\| \dots \|F_{m_j}\| \end{aligned} \tag{3.21}$$

Using (2.13) and (3.16) with $j=0$ we estimate $C_q^\alpha(\overline{\eta^q})$ from (2.36)

$$\begin{aligned} \|C_q\| &\leq \sup_{\overline{\eta^q}} \sigma_{\overline{\eta^q}} \sum_{\substack{p+k=q+1, \\ p,k\geq 2}} \gamma\rho^k \\ &\times \sum_{l=1}^{p} \frac{\sup_\xi \|F_p(\overline{\eta^{l-1}}, \xi, \eta_{l+k}, \dots, \eta_q)\| \|b_k(\cdot; \eta_l, \dots \eta_{l+k-1})\|}{(\lambda_{\overline{\eta^q}} - \lambda_\alpha)} \\ &\leq \sup_{\overline{\eta^q}} \sum_{\substack{p+k=q+1, \\ p,k\geq 2}} \gamma\rho^k \|F_p\| \frac{\pi(2/\pi)^{\frac{k+1}{2}}(k+1)}{2N(\lambda_{\overline{\eta^q}} - \lambda_\alpha)} \sum_{l=1}^{p} \sum_{m=0}^{k-1} |\eta_{l+m}| \end{aligned} \tag{3.22}$$

Recall that by (1.10) $N = [\sqrt{\kappa}]$ and set $\theta = \sqrt{\kappa} - N$. Note that $0 < \theta < 1$ because of (1.6). Then by virtue of (1.7)

$$\sup_{\eta > N} \frac{\eta}{\lambda_\eta} = \frac{(N+1)^2}{(N+1)^2 - (N+\theta)^2} = \frac{(N+1)^2}{(1-\theta)(2N+1+\theta))} \leq \frac{2(N+1)}{3(1-\theta)} \tag{3.23}$$

and therefore since $p+k=q+1$

$$\begin{aligned}(3.24)\qquad &\sup_{\overline{\eta^q}}\frac{\sum_{l=1}^{p}\sum_{m=0}^{k-1}|\eta_{l+m}|^2}{\lambda_{\overline{\eta^q}}-\lambda_\alpha}\le\sup_{\overline{\eta^q}}\frac{k\sum_{l=1}^{q}|\eta_l|^2}{\lambda_{\overline{\eta^q}}-\lambda_\alpha}\\ &\le\frac{2k(N+1)}{3(1-\theta)}\sup_{\overline{\eta^q}}\frac{\lambda_{\overline{\eta^q}}}{\lambda_{\overline{\eta^q}}+|\lambda_\alpha|}\le\frac{2k(N+1)}{3(1-\theta)}\end{aligned}$$

Estimates (3.22), (3.24) imply the bound

$$(3.25)\qquad \|C_q\|\le\frac{\sqrt{2\pi}(N+1)\gamma}{3N(1-\theta)}\sum_{\substack{p+k=q+1,\\ p,k\ge 2}}\|F_p\|k(k+1)(\rho\sqrt{2/\pi})^k$$

At last, let us estimate $D_q^\alpha(\overline{\eta^q})$ defined in (2.37). Using (2.13), (1.5) we obtain

$$\begin{aligned}(3.26)\qquad \|D_q\|\le&\sup_{\overline{\eta^q}}\sigma_{\overline{\eta^q}}\sum_{(j,p)\in\widehat{Q}^q}\gamma\rho^{q+j-p+1}C^j_{q+j-p+1}\\ &\times\sum_{m_1+\dots+m_{j+1}=p}\|F_{m_1}\|\dots\|F_{m_{j+1}}\|\\ &\times\sum_{l=1}^{p}\sup_{\overline{\beta^j}}\frac{\|b_{q+j-p+1}(\cdot;\overline{\beta^j},\eta_l,\dots\eta_{l+q-p})\|}{(j+1)(\lambda_{\overline{\eta^q}}-\lambda_\alpha)}\end{aligned}$$

Let us estimate the last factor of the r.s. in (3.26). Using (3.16), (3.24), (3.19) we get:

$$\begin{aligned}&\sup_{\overline{\eta^q}}\sigma_{\overline{\eta^q}}\sum_{l=1}^{p}\sup_{\overline{\beta^j}}\frac{\|b_{q+j-p+1}(\cdot;\overline{\beta^j},\eta_l,\dots\eta_{l+q-p})\|}{(j+1)(\lambda_{\overline{\eta^q}}-\lambda_\alpha)}\\ &\le\sqrt{\pi/2}(\sqrt{2/\pi})^{q+j-p+1}\frac{q+j-p+2}{N}\sup_{\overline{\eta^q}}\sigma_{\overline{\eta^q}}\frac{jN^2p+\sum_{l=1}^{p}\sum_{m=0}^{q-p}|\eta_{l+m}|}{(j+1)(\lambda_{\overline{\eta^q}}-\lambda_\alpha)}\\ (3.27)\qquad &\le\sqrt{2\pi}(N+1)(\sqrt{2/\pi})^{q+j-p+1}(q+j-p+1)\\ &\times\left(\frac{jp}{(j+1)(q+1)\widehat{\lambda}}+\frac{2(q-p+1)}{3(1-\theta)(j+1)}\right)\\ &\le C_1(\sqrt{2/\pi})^{q+j-p+1}(q+j-p+1)\left(1+\frac{q-p+1}{j+1}\right)\end{aligned}$$

where $C_1$ depends only on $\kappa$ (see (1.10)) i.e. on the data of the problem.

Equalities (3.26), (3.27) imply:

$$\begin{aligned}(3.28)\qquad \|D_q\|\le C_2&\sum_{(j,p)\in\widehat{Q}^q}(\rho\sqrt{2/\pi})^{q+j-p+1}C^j_{q+j-p+1}(q+j-p+1)\\ &\times\left(1+\frac{q-p+1}{j+1}\right)\sum_{m_1+\dots+m_{j+1}=p}\|F_{m_1}\|\dots\|F_{m_{j+1}}\|\end{aligned}$$

where $C_2$ depends on the data of the problem only.

As a result we have proved the following theorem.

THEOREM 3.4. *The coefficients $F_q^\alpha(\overline{\eta^q})$ defined by recurrence relations (2.33)-(2.37) satisfy the estimate:*

(3.29)

$$\|F_q\| \le \widehat{C} \Bigg( \sigma^q + \sum_{(j,p)\in Q_q} C^j_{q+j-p+1} \sigma^{q+j-p} \sum_{\substack{m_1+\cdots+m_{j+1}=p,\\ m_l\ge 2}} \|F_{m_1}\|\dots\|F_{m_{j+1}}\|$$
$$+ \sum_{\substack{p+k=q+1,\\ p,k\ge 2}} k(k+1)\sigma^k \|F_p\|$$
$$+ \sum_{(j,p)\in \widehat{Q}^q} (q+j-p+1)\left(1+\frac{q-p+1}{j+1}\right)\sigma^{q+j-p+1} C^j_{q+j-p+1}$$
$$\times \sum_{m_1+\cdots+m_{j+1}=p} \|F_{m_1}\|\dots\|F_{m_{j+1}}\| \Bigg)$$

*where $q\ge 4, \sigma=\rho\sqrt{2/\pi}$, the constant $\widehat{C}$ depends on $\kappa,\gamma$ only (see (1.5), (1.10)), and the sets $Q_q, \widehat{Q}^q$ are defined in (2.24), (2.28).*

Inequality (3.29) follows directly from (2.33), (3.20), (3.21), (3.25), (3.28).

REMARK 3.5. Inequality (3.29) should be complemented with the analogous bound for $q=2$ that follows immediately from (2.20), (1.5), (3.14), (3.19):

$$\|F_2\| \le \widehat{C}\sigma^2. \tag{3.30}$$

Moreover, formulae (2.32), (1.5), (3.14), (3.19) imply the estimate for $F_3^\alpha$:

$$\|F_3\| \le \widehat{C}(\sigma^3 + 9\sigma^2\|F_2\|). \tag{3.31}$$

where $\widehat{C}$ and $\sigma$ are the same as in (3.29).

**3.4. Convergence of serie (2.6).** We now are in a position to prove the convergence of the serie (2.6) for the map $F(y_-)$ that determines the stable invariant manifold. For this purpose we define the majorants $\varphi_q$ for the coefficients $F_q^\alpha(\overline{\eta^q})$ using recurrence relations (3.29)-(3.31):

$$\varphi_2 = \widehat{C}\sigma^2, \quad \varphi_3 = \widehat{C}(\sigma^3 + 9\sigma^2\varphi_2) \tag{3.32}$$

(3.33)

$$\varphi_q = \widehat{C} \Bigg( \sigma^q + \sum_{(j,p)\in Q_q} C^j_{q+j-p+1} \sigma^{q+j-p} \sum_{\substack{m_1+\cdots+m_{j+1}=p,\\ m_l\ge 2}} \varphi_{m_1}\cdots\varphi_{m_j}$$
$$+ \sum_{\substack{p+k=q+1,\\ p,k\ge 2}} k(k+1)\sigma^k \varphi_p$$
$$+ \sum_{(j,p)\in \widehat{Q}^q} (q+j-p+1)\left(1+\frac{q-p+1}{j+1}\right)\sigma^{q+j-p+1} C^j_{q+j-p+1}$$
$$\times \sum_{m_1+\cdots+m_{j+1}=p} \varphi_{m_1}\cdots\varphi_{m_{j+1}} \Bigg), \ q\ge 4$$

where the constants $\widehat{C}, \sigma$ and sets $Q_q, \widehat{Q}^q$ are as in (3.29).

Formulas (3.29)-(3.33) imply that

$$\|F_q\| \le \varphi_q \quad \forall q \ge 2. \tag{3.34}$$

Thus to prove convergence of the series (2.6) in a neighborhood of the origin, it is enough to prove the inequalities

$$\varphi_q \le \gamma_1 \rho_1^{-q} \quad \forall q \ge 2. \tag{3.35}$$

with some positive constants $\gamma_1, \rho_1$ independent of $q$.

THEOREM 3.6. *There exist constants $\gamma_1 > 0, \rho_1 > 0$ such that for each $q \ge 2$ inequality (3.35) holds.*

PROOF. As in Theorem 1.1 of Chapter I in [**VF2**] we introduce the formal series

$$\varphi(t) = \sum_{q=2}^{\infty} \varphi_q t^q \tag{3.36}$$

and prove that it defines an analytic function for $t$ belonging to a neighborhood of origin in $\mathbb{C}$. To prove this we plan to derive en equation for $\varphi(t)$ and after that apply to this equation the Implicit Function Theorem.

We multiply both parts of equations (3.32), (3.33) by $t^q$ and sum the obtained equalities over $q \ge 2$. Then we get

$$\begin{aligned} \sum_{q=2}^{\infty} \varphi_q t^q = \widehat{C} \Bigg( & \sum_{q=2}^{\infty} \sigma^q t^q + \sum_{q=3}^{\infty} t^q \sum_{(j,p)\in Q_q} C^j_{q+j-p+1} \sigma^{q+j-p} \\ & \times \sum_{m_1+\dots+m_{j+1}=p} \varphi_{m_1} \dots \varphi_{m_j} \\ & + \sum_{q=3}^{\infty} t^q \sum_{p=2}^{q-1} (q-p+1)(q-p+2)\sigma^{q-p+1}\varphi_p \\ & + \sum_{q=4}^{\infty} t^q \sum_{(j,p)\in \widehat{Q}^q} \frac{(q+j-p+1)(q+j-p+2)}{j+1} \\ & \times \sigma^{q+j-p+1} C^j_{q+j-p+1} \sum_{m_1+\dots+m_{j+1}=p} \varphi_{m_1}\dots\varphi_{m_{j+1}} \Bigg). \end{aligned} \tag{3.37}$$

Making the change of variables $(q, j, p) \to (k, j, p)$, $q = k - j + p$ and using the definition of $Q_q$ in (2.24), we do the following transformation with the first and second sums on the right side of (3.37):

$$
\begin{aligned}
&\sum_{q=2}^{\infty}\sigma^q t^q+\sum_{q=3}^{\infty}t^q\sum_{(j,p)\in Q_q}C^j_{q+j-p}\sigma^{q+j-p}\sum_{m_1+\cdots+m_{j+1}=p}\varphi_{m_1}\dots\varphi_{m_j}\\
&=\sum_{q=2}^{\infty}\sigma^k t^k+\sum_{k=2}^{\infty}t^k\sum_{j=1}^{k}C^j_k\sigma^k t^{-j}\sum_{p=2j+2}^{\infty}t^p\sum_{m_1+\cdots+m_{j+1}=p}\varphi_{m_1}\dots\varphi_{m_j}\\
&=\sum_{q=2}^{\infty}(\sigma t)^k+\sum_{k=2}^{\infty}(\sigma t)^k\sum_{j=1}^{k}C^j_k t^{-j}\left(\sum_{m=2}^{\infty}\varphi^m t^m\right)^j\\
&=\sum_{q=2}^{\infty}(\sigma t)^k+\sum_{k=2}^{\infty}(\sigma t)^k[(1+\varphi(t)/t)^k-1]\\
&=\sum_{k=2}^{\infty}(\sigma t)^k(1+\varphi(t)/t)^k=\frac{(\sigma t)^2(1+\varphi(t)/t)^2}{1-\sigma t(1+\varphi(t)/t)}
\end{aligned}
\tag{3.38}
$$

Changing the order of summation and after that changing variables $(q,p)\to(q,k)$ with $k=q-p+1$ we transform the third sum on the r.s. of (3.37) as follows:

$$
\begin{aligned}
&\sum_{q=3}^{\infty}t^q\sum_{p=2}^{q-1}(q-p+1)(q-p+2)\sigma^{q-p+1}\varphi_p\\
&=\sum_{p=2}^{\infty}\sum_{q=p+1}^{\infty}t^q(q-p+1)(q-p+2)\sigma^{q-p+1}\varphi_p\\
&=\sum_{p=2}^{\infty}\varphi_p\sum_{k=2}^{\infty}t^{k+p-1}k(k+1)\sigma^k=\frac{\varphi(t)}{t}\sum_{k=2}^{\infty}k(k+1)(\sigma t)^k
\end{aligned}
\tag{3.39}
$$

At last, using definition (2.28) of $\widehat{Q}^q$ we make transformation of the forth sum from r.s. in (3.37) by changing the variables $(q,j,p)\to(k,j,p)$, $q=k-j+p-1$:

$$
\begin{aligned}
&\sum_{q=4}^{\infty}t^q\sum_{(j,p)\in\widehat{Q}^q}\frac{(q+j-p+1)(q+j-p+2)}{j+1}C^j_{q+j-p+1}\sigma^{q+j-p+1}\\
&\times\sum_{m_1+\cdots+m_{j+1}=p}\varphi_{m_1}\dots\varphi_{m_{j+1}}\\
&=\sum_{k=2}^{\infty}\sum_{j=1}^{k}t^{k-j-1}\frac{k(k+1)}{j+1}\frac{k!}{j!(k-j)!}\sigma^k\\
&\times\sum_{p=2j+2}^{\infty}t^p\sum_{m_1+\cdots+m_{j+1}=p}\varphi_{m_1}\dots\varphi_{m_{j+1}}\\
&=\sum_{k=2}^{\infty}k(\sigma t)^k\sum_{j=1}^{k}C^{k+1}_{j+1}(\varphi(t)/t)^{j+1}\\
&=\sum_{k=2}^{\infty}k(\sigma t)^k[(1+(\varphi(t)/t))^{k+1}-(k+1)\varphi(t)/t-1]
\end{aligned}
\tag{3.40}
$$

Substitution of the right sides of (3.38), (3.39) (3.40) into (3.37) yields the equality

(3.41)

$$\varphi(t)=\widehat{C}\left(\frac{(\sigma t)^2(1+\varphi(t)/t)^2}{1-\sigma t(1+\varphi(t)/t)}+\sum_{k=2}^{\infty}k(\sigma t)^k(1+(\varphi(t)/t))^{k+1}-\sum_{k=2}^{\infty}k(\sigma t)^k\right)$$

Applying the equality

$$\sum_{k=2}^{\infty}k\alpha^k=\alpha\partial_\gamma\sum_{k=2}^{\infty}\alpha^k=\alpha\partial_\gamma\frac{\alpha^2}{1-\alpha}=\frac{\alpha^2(2-\alpha)}{(1-\alpha)^2}$$

to the second and third sums from right side of (3.41) we get

$$\begin{aligned}\varphi(t)=\widehat{C}\Bigg(&\frac{(\sigma t)^2(1+\varphi(t)/t)^2}{1-\sigma t(1+\varphi(t)/t)}\\&+(1+\varphi(t)/t)^3\,\frac{(\sigma t)^2(2-\sigma t(1+\varphi(t)/t))}{(1-\sigma t(1+\varphi(t)/t))^2}-\frac{(\sigma t)^2(2-\sigma t)}{(1-\sigma t)^2}\Bigg)\end{aligned}\tag{3.42}$$

Making the change of variable $\mu(t)=\varphi(t)/t$ in (3.42) we obtain the equality

$$G(\mu(t),t)=0\tag{3.43}$$

where

(3.44)

$$G(\mu,t)=\mu-\widehat{C}\sigma^2t\left(\frac{(1+\mu)^2}{1-\sigma t(1+\mu)}+\frac{(1+\mu)^3(2-\sigma t(1+\mu))}{(1-\sigma t(1+\mu))^2}-\frac{(2-\sigma t)}{(1-\sigma t)^2}\right)$$

Since $G(0,0)=0$, $\frac{\partial G(\mu,t)}{\partial\mu}|_{(\mu,t)=(0,0)}=1$ we can apply to (3.43) the Implicit Function Theorem and claim that there exists a unique solution $\mu(t)$ of equation (3.43), defined for small $t$. Moreover, since function (3.44) is analytic in a neighborhood of origin in $\mathbb{C}^2$, this solution $\mu(t)$ is analytic in a neighborhood of origin in $\mathbb{C}$. This implies analyticity of $\varphi(t)=t\mu(t)$ that proves (3.35). □

Now we are in a position to prove the main theorem of this section.

THEOREM 3.7. *Let the coefficient $\kappa$ and the function $f$ in problem (1.1)-(1.3) satisfy (1.6) and (1.4), (1.5) respectively. Then the map (2.6) that defines the stable invariant manifold (1.15) is analytic in a neighborhood $\mathcal{O}(H_-)$ of the origin in the space $H_-$.*

PROOF. We have to prove that the serie (2.6) converges for each

$$h_-\in B_{\rho_0}(H_-)=\{h_-\in H_-:\|h\|_{H^1(0,\pi)}<\rho_0\}\tag{3.45}$$

with certain $\rho_0>0$. By virtue of Proposition 3.1 this can be reduced to establishing inequalities (3.7) for norms (3.1), (3.2) of the coefficients $F_k(\overline{\eta^k})$ from (3.8). By (3.9) the number $\rho_1$ from (3.7) is related with $\rho_0$ from (3.45) by the equality $\rho_1=\rho_0r_N$ where $r_N=\sum_{\xi=N+1}^{\infty}\xi^{-2}$ and $N$ is defined in (1.10). By virtue of Theorem 3.4 these norms $\|F_k\|$ satisfy inequality (3.34) with $\varphi_k$ defined in (3.32), (3.33). Finaly, inequalities (3.35) proved in Theorem 3.6 and bounds (3.34) imply (3.7). □

REMARK 3.8. One of the essential conditions allowing to prove Theorem 3.7, i.e. to prove analyticity of invariant manifold (1.15)-(1.17) is the condition

$$\lambda_{\overline{\eta^q}}-\lambda_\alpha\neq0\tag{3.46}$$

called the absence of a resonance. Indeed each term on right sides of the recurrence relations (2.33)-(2.37) determining the coefficients $F_q(\overline{\eta^q})$ of series (2.11) contains the factor $(\lambda_{\overline{\eta^q}} - \lambda_\alpha)^{-1}$ . In the case considered in this paper condition (3.46) is true because $\lambda_\alpha < 0$ and each summand $\lambda_{\eta_j}$ of $\lambda_{\overline{\eta^q}} = \lambda_{\eta_1} + \cdots + \lambda_{\eta_q}$ is positive.

Remark 3.9. The unstable invariant manifold is defined by the following formulas analogous to (1.15)-(1.17):

$$M_+ = \{y_+ + G(y_+),\ y_+ \in \mathcal{O}(H_+)\} \tag{3.47}$$

where $\mathcal{O}(H_+)$ is a neighborhood of the origin in the subspace $H_+$, and

$$G : \mathcal{O}(H_+) \to H_- \tag{3.48}$$

is a certain map satisfying

$$\|G(y_+)\|_{H_-}/\|y_+\|_{H_+} \to 0 \quad \text{as} \quad \|y_+\|_{H_+} \to 0. \tag{3.49}$$

To prove the analyticity of $G$ we use a differential equation for $G$ analogous to (1.23) by means of this we derive recurrence relations for the coefficients $G_q(\overline{\alpha^q})$ of series

$$G(y_+) = \sum_{k=2}^{\infty} \sum_{\alpha^k} G_k(\overline{\alpha^k}) y(\overline{\alpha^k})$$

(Recall that we use notation (1.28), (1.29) here.) These recurrence relations contain the factor $(\lambda_{\overline{\alpha^q}} - \lambda_\eta)^{-1}$. Since $\lambda_{\overline{\alpha^q}} = \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_q}$ with $\lambda_{\alpha_j} < 0$ for $j = 1, \ldots, q$ and $\lambda_\eta > 0$, we have $\lambda_{\overline{\alpha^q}} - \lambda_\eta \neq 0$. In other words in this case the resonances are absent, and therefore using the technique of this paper one can prove the analyticity of $G(y_+)$.

Remark 3.10. Let change the definition (1.12) of the subspaces $H_+, H_-$ as follows:

$$H_+ = [e_1, \ldots, e_{N+k_0}], \quad H_- = [e_{N+k_0+1}, e_{N+k_0+2} \ldots], \quad N = [\sqrt{\kappa}] \tag{3.50}$$

Using (3.50) we define $M_-$ similarly to (1.15)-(1.17). Then the corresponding map $F$ will be analytic. Indeed, in this case condition (3.46) for the absence of resonance holds because each summand $\lambda_{\eta_j}$ of $\lambda_{\overline{\eta^q}} = \lambda_{\eta_1} + \cdots + \lambda_{\eta_q}$ is greater than $\lambda_\alpha$, and to prove the analyticity of $F$ one simply has to repeat arguments of this paper.

From the other hand it is impossible to guarantee that the invariant manifold (3.47)-(3.49) corresponding to (3.50) is analytic because in this case the resonance condition

$$\lambda_{\overline{\lambda^q}} - \lambda_\eta = 0 \tag{3.51}$$

can take place since summands $\lambda_{\alpha_j}$ of $\lambda_{\overline{\alpha^q}} = \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_q}$ can be positive as well as negative.

Remark 3.11. Note that methods of this paper essentially use special form of eigenfunctions of operator (1.8). Therefore results of this paper can be automatically generalized on the case when problem (1.1)-(1.3) is defined on rectangle $\{x \in (0, \pi)^n\}$ and $\partial^2 y/\partial x^2$ is changaed on $\Delta y$. Besides, straightforward generalization on the case of a BVP with periodic boundary conditions is possible. Moreover, this BVP can be more general, than the BVP considered in this paper.

In fact a natural generalization on the many-dimensional case of the problem considered here is when the problem (1.1)-(1.3) is defined on arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary. This generalization as well as generalization from singular point $\widehat{z} \equiv 0$ to the case of arbitrary singular point $\widehat{z}$ can be made but for this additional tools should be used. These generalizations will be made elsewhere.

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