ON THE NORMAL-TYPE PARABOLIC SYSTEM CORRESPONDING TO THE 3-DIMENSIONAL HELMHOLTZ SYSTEM

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Dedicated to the memory of Olga Alexandrovna Ladyzhenskaya

Abstract. A semilinear normal parabolic system is derived corresponding to the three-dimensional Helmholtz equations for the curl of the velicity vector field of viscous incompressible fluid. An explicit formula for the solution to the normal parabolic system is applied to examine the structure of the corresponding dynamic flow. The phase space of this dynamical system is shown to consist of the set of stability (on which the solution tends to zero with unboundedly increasing time $t \to \infty$), the set of explosions (on which the solutions explode at a finite time) and the set of increase (on which the solutions unboundedly increase as $t \to \infty$). A geometrical description of these sets is given.

1. Introduction

It is well known (see, for example, Hopf [1], Ladyzhenskaya [2], Temam [3]) that the existence of a weak solution of a three-dimensional Navier–Stokes system is based on the energy estimate, which, in turn, holds because the image $B(v)$ under the nonlinear operator generated by the nonlinear terms of the Navier–Stokes equations is L_2-orthogonal to the velocity vector v; that is, the vector $B(v)$ is tangent to the L_2-sphere of radius $\|v\|_{L_2}$ and center at the origin. However, insufficient smoothness does not allow one to prove the uniqueness of a weak solution. If the vectors $B(v)$ were tangent to the analogues sphere in the Sobolev space H^1, then similar methods would be capable of proving the existence of the strong solution of the Navier–Stokes system—in this case the solution is sufficiently smooth to guarantee its uniqueness. It appears that $B(v)$ fails to satisfy this property in the space H^1: in this case $B(v) = B_\tau(v) + B_n(v)$, where $B_\tau(v)$ is the component of the nonlinear operator which is tangent to the H^1-sphere, and $B_n(v)$ is the component normal to the sphere (collinear to the velocity vector v). The presence of the component $B_n(v)$ prevents the application of the methods based on the energy inequality to prove the existence of a strong solution.

Our aim in this paper is to implement the following plan. First, replacing the nonlinear operator $B(v)$ in the system under study by the normal component $B_n(v)$,

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we shall study the resulting system, to be referred to as the Normal Parabolic System (NPS). Then, using the information obtained, we shall try to better understand the problem of existence of a strong solution to the original system with the nonlinear operator $B(v) = B_\tau(v) + B_n(v)$.

This paper is devoted to realizing the first part of this plan: we examine the normal parabolic system constructed from the three-dimensional Helmholtz system describing the curl of the vector field of viscous incompressible fluid with periodic boundary conditions. Note that the normal parabolic equation constructed from the Burgers equation was studied earlier in [4], [5]. The study of the NPS corresponding a three-dimensional Helmholtz system was initiated in [6], in which the main results were only stated.

In the present paper, we give complete proofs of all these results. Moreover, we continue the study of the structure of the dynamic flow that was initiated in [4], [5], [6]. In particular, we introduce the parameter-independent set of stability, emphasizing that the parameter-dependent set of stability of [4], [5], [6] is employed only for technical purposes. Thanks to this, the use of the intermediate set from previous papers has proved unnecessary—instead there appears the set of increase.

We show that the phase space of the dynamical system corresponding to the NPS is the union of the set of stability, the set of increase, and the set of explosions—a fairly detailed geometrical description of these sets is given at the end of the paper. In Section 2, we derive the NPS corresponding to the Helmholtz system. The explicit formula for the solution of the NPS is obtained in Section 3. This is a key result providing quite sufficient insight into the NPS. The proofs of the unique solvability of the boundary-value problem for the NPS in corresponding function spaces and of the continuous dependence of the solution on the initial data are given in Section 4. Section 5 is concerned with the study of the functional entering the explicit formula for the solution of the NPS and governing many properties of the solutions. Finally, in Section 6 we examine the structure of the dynamic flow generated by the NPS.

2. Semilinear normal parabolic system (NPS)

2.1. **The Navier–Stokes system.** Consider the three-dimensional system of Navier–Stokes equations with periodic boundary conditions:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = 0, \quad \operatorname{div} v = 0, \tag{2.1}$$

$$v(t,\dots,x_i,\dots) = v(t,\dots,x_i+2\pi,\dots), \qquad i=1,2,3, \tag{2.2}$$

$$v(t,x)|_{t=0} = v_0(x); \tag{2.3}$$

here $t \in \mathbb{R}_+, x = (x_1,x_2,x_3) \in \mathbb{R}^3$, $v(t,x) = (v_1,v_2,v_3)$ is the fluid velocity vector field, ∇p is the pressure gradient, Δ is the Laplacian, and $(v,\nabla)v = \sum_{j=1}^3 v_j \partial_{x_j} v$. The periodic boundary conditions (2.2) means in particular that system (2.1) and the initial conditions (2.3) are defined on the torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.

For each $m \in \mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geqslant 0\}$ we define the space

$$V^m = V^m(\mathbb{T}^3) = \Big\{ v(x) \in (H^m(\mathbb{T}^3))^3 : \operatorname{div} v = 0,\ \int_{\mathbb{T}^3} v(x)\,dx = 0 \Big\}, \tag{2.4}$$

where $H^m(\mathbb{T}^3)$ is the Sobolev space.

It is well-known that the nonlinear term $(v,\nabla)v$ of problem (2.1)–(2.3) satisfies the relation

$$\int_{\mathbb{T}^3}(v(t,x),\nabla)v(t,x)\cdot v(t,x)\,dx=0.$$

Hence, multiplying scalarly the Navier–Stokes system (2.1) by v in $L_2(\mathbb{T}^3)$, integrating by parts (in x) and then integrating in t, we arrive at the well-known energy inequality

$$\int_{\mathbb{T}^3}|v(t,x)|^2dx+2\int_0^t\int_{\mathbb{T}^3}|\nabla_x v(\tau,x)|^2dx\,d\tau\leqslant\int_{\mathbb{T}^3}|v_0(x)|^2dx, \tag{2.5}$$

which enables one to prove the existence of a weak solution of problem (2.1)–(2.3). Note that in the case of a two-dimensional Navier–Stokes system (that is, for $x=(x_1,x_2)\in\mathbb{R}^2$, $v(t,x)=(v_1,v_2)$), Ladyzhenskaya [7], [2] employed the energy inequality (2.5) to prove the uniqueness and smoothness of a weak solution. However, in the three-dimensional setting, the smoothness of a weak solution is not enough to guarantee its uniqueness.

Unfortunately, the method of energy estimates is incapable of proving the existence of smooth solutions of system (2.1). Indeed, multiplying (2.1) scalarly by v in $V^1(\mathbb{T}^3)$, we will not obtain an analogue of the energy inequality in the phase space V^1, because the image of the quadratic operator $(v,\nabla)v$ from (2.1) is not V^1-orthogonal to v. Nevertheless, we shall try to better understand this situation by passing from the Navier–Stokes system to the Helmholtz system.

2.2. **The Helmholtz system.** We transform the boundary-value problem (2.1)–(2.3) with respect to the fluid velocity to the one with unknown curl of the velocity field:

$$y(t,x)=\operatorname{curl}v(t,x)=(\partial_{x_2}v_3-\partial_{x_3}v_2,\ \partial_{x_3}v_1-\partial_{x_1}v_3,\ \partial_{x_1}v_2-\partial_{x_2}v_1). \tag{2.6}$$

Recall that

$$(v,\nabla)v=y\times v+\nabla\frac{|v|^2}{2}, \tag{2.7}$$

$$\operatorname{curl}(y\times v)=(v,\nabla)y-(y,\nabla)v,\quad \text{provided that}\quad \operatorname{div}\,v=0,\quad \operatorname{div}\,y=0, \tag{2.8}$$

where $y\times v=(y_2v_3-y_3v_2,\ y_3v_1-y_1v_3,\ y_1v_2-y_2v_3)$ is the vector product of vector y, v, and $|v|^2=v_1^2+v_2^2+v_3^2$. We substitute (2.7) into the first of equations (2.1) and apply the curl operator to both sides of the resulting equality. Then, using (2.6), (2.8) and the formula $\operatorname{curl}\nabla F=0$, we obtain the system of Helmholtz equations:

$$\partial_t y(t,x)-\Delta y+(v,\nabla)y-(y,\nabla)v=0,\quad \operatorname{div}\,y=0. \tag{2.9}$$

We augment these equations with the initial conditions

$$y(t,x)|_{t=0}=y_0(x), \tag{2.10}$$

where $y_0=\operatorname{curl}v_0$.

Remark 2.1. Note that in the two-dimensional case $y(t,x)=\operatorname{curl}v(t,x)=\partial_{x_1}v_2-\partial_{x_2}v_1$ is a scalar function, and hence the first of equations (2.9) is a scalar equation without the term $(y,\nabla)v$. Hence, the solution of this equation satisfies the analogue of the energy inequality. Starting out from this, Yudovich [8] constructed the nonlocal existence and uniqueness theory for smooth solutions of the Euler equations governing the flow of an ideal incompressible fluid.

Taking into account that there is no analogue of the energy inequality in the three-dimensional setting for solving problem (2.9), (2.10), we shall proceed as follows.

2.3. **The normal parabolic system and its derivation.** Using the Fourier series expansion

$$v(x) = \sum_{k\in\mathbb{Z}^3} \widehat{v}(k)e^{ix\cdot k}, \quad \text{where} \quad \widehat{v}(k) = (2\pi)^{-3}\int_{\mathbb{T}^3} v(x)e^{-ix\cdot k}\,dx,$$

$x\cdot k = \sum_{j=1}^3 x_j k_j,\ k=(k_1,k_2,k_3)$, and applying the well-known formula

$$\operatorname{curl}\operatorname{curl} v = -\Delta v \quad \text{for} \quad \operatorname{div}\, v = 0,$$

one easily verifies that the inverse to the curl operator is well-defined on V^m and is given by

$$\operatorname{curl}^{-1} y(x) = i\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{k\times\widehat{y}(k)}{|k|^2}e^{ix\cdot k}. \tag{2.11}$$

Hence, the operator

$$\operatorname{curl}:\ V^1 \longrightarrow V^0$$

is an isomorphism of the spaces, and so the V^1-sphere for problem (2.1), (2.3) is equivalent to the V^0-sphere for problem (2.9), (2.10).

We let B denote the nonlinear term in the Helmholtz system,

$$B(y) = (v,\nabla)y - (y,\nabla)v \tag{2.12}$$

(the dependence of B on v is not indicated, since v is given in terms of y by (2.11)).

Multiplying scalarly equality (2.12) by $y=(y_1,y_2,y_3)$ in V^0 and integrating by parts, this gives

$$(B(y),y)_{V^0} = -\int_{\mathbb{T}^3}\sum_{j,k=1}^3 y_j\partial_j v_k y_k\ dx, \tag{2.13}$$

which, in general, is not zero. By this reason, the three-dimensional Helmholtz system does not satisfy the energy inequality. In other words, the operator B can be expanded as

$$B(y) = B_n(y) + B_\tau(y), \tag{2.14}$$

where the vector $B_n(y)$ is orthogonal to the sphere

$$\Sigma(\|y\|_{V^0}) = \{u\in V^0 : \|u\|_{V^0} = \|y\|_{V^0}\}$$

at the point y (that is, the vectors $B_n(y)$ and y are collinear), and the vector $B_\tau(y)$ touches the sphere $\Sigma(\|y\|_{V^0})$ at y. Note that, in general, both the operators B_n, B_τ in (2.14) are nonzero. The derivation of the energy inequality being hindered by the presence of the operator B_n, it is very likely that it is B_n that is responsible for possible singularities of the solution. It would therefore seem advisable to drop the component B_τ in the Helmholtz system and be concerned, in the early stage, with the analogue of system (2.9) in which the nonlinear operator $B(y)$ is replaced by its normal component $B_n(y)$. The so-obtained system will be referred to as a *normal-type system* or a *normal parabolic system* (*NPS*)[1].

Let us build the NPS for problem (2.9), (2.10) with respect to the V^0-sphere.

The term $(v,\nabla)y$ in (2.9) is a tangential operator; that is,

$$\int_{\mathbb{T}^3}(v,\nabla)y\cdot y\,dx=0.$$

Consequently, the normal part of the nonlinear operator in (2.9) is governed by the nonlinear term $(y,\nabla)v$. We shall seek it in the form $\Phi(y)y$, where Φ is an unknown functional determined from the equation

$$\int_{\mathbb{T}^3}\Phi(y)y(x)\cdot y(x)\,dx=\int_{\mathbb{T}^3}(y(x),\nabla)v(x)\cdot y(x)\,dx. \tag{2.15}$$

In view of (2.15),

$$\Phi(y)=\begin{cases}\int_{\mathbb{T}^3}(y(x),\nabla)\operatorname{curl}^{-1}y(x)\cdot y(x)\,dx\,/\int_{\mathbb{T}^3}|y(x)|^2dx, & y\neq 0,\\ 0, & y\equiv 0,\end{cases} \tag{2.16}$$

where $\operatorname{curl}^{-1}y(x)$ is defined in (2.11).

Thus, the normal parabolic system corresponding to the Helmholtz equations (2.9) is defined as follows:

$$\partial_t y(t,x)-\Delta y-\Phi(y)y=0,\qquad \operatorname{div}\,y=0; \tag{2.17}$$

here Φ is the functional from (2.16). This system, as equipped with the initial condition (2.10) and periodic boundary data, is the main object of study of the present paper.

3. Explicit formula for the solutions of NPS

Our aim in this section is to derive an explicit formula for the solutions of the NPS. This is a key result underlying many important properties thereof, of which some will be put forward in the subsequent sections.

Let $S(t,x;z_0)$ be the solution operator of the following Stokes system with periodic boundary conditions:

$$\partial_t z(t,x)-\Delta z(t,x)=0,\quad \operatorname{div}\,z=0,\quad z(t,x)|_{t=0}=z_0(x); \tag{3.1}$$

that is, $S(t,x;z_0)=z(t,x)$. (Of course, it is assumed that $\operatorname{div}z_0=0$.)

The following result holds.

Theorem 3.1. *The solution* $y(t,x):=y(t,x;y_0)$ *of problem* (2.17)*,* (2.10) *is given by*

$$y(t,x;y_0)=\frac{S(t,x;y_0)}{1-\int_0^t\Phi(S(\tau,\,\cdot\,;y_0))\,d\tau}. \tag{3.2}$$

Proof. The solution of problem (2.17), (2.10) will be sought in the form $y(t,x)=c(t)S(t,x;y_0)$, where $S(t,x;y_0)$ is the solution of problem (3.1) with $z_0=y_0$, and $c(t)$ is an unknown scalar function. Substituting this function $y(t,x)$ into (2.17),

[1]It has already been observed in the introduction that one needs to return in the sequel to the original Helmholtz system and then, using the available information about the NPS, examine the interaction of the operators $B_n(y)$ and $B_\tau(y)$. This question will be addressed in papers to follow.

(2.10) and taking into account that the functional $\Phi(v)$ is homogeneous in v, we see that

$$\partial_t y(t,x) - \Delta y - \Phi(y)y =$$
$$c(t)(\partial_t S(t,x;y_0) - \Delta S(t,x;y_0)) + (S(t,x;y_0)(dc(t)/dt - c^2(t)\Phi(S(t,\,\cdot\,;y_0)) = 0,$$

and now, in view of (3.1) and the straightforward relation $c(0)=1$,

$$1/c(t) = 1 - \int_0^t \Phi(S(\tau,\,\cdot\,;y_0))\,d\tau,$$

which proves (3.2). □

For brevity, we shall often conventionally write:

$$y(t;y_0) = y(t,\,\cdot\,;y_0), \qquad S(t;y_0) = S(t,\,\cdot\,;y_0). \tag{3.3}$$

4. Fundamental properties of the boundary value problem for NPS

We start with examining the most elementary properties of the solution of problem (2.17), (2.10). We shall first investigate for which classes of initial data the solution is well-defined by formula (3.2). Further, we prove the unique solvability of the problem in corresponding function spaces and the continuous dependence of the solution on the initial data.

4.1. **Properties of the functional $\Phi(u)$.** Let $s \in \mathbb{R}$. By definition, the Sobolev space $H^s(\mathbb{T}^3)$ is the space of periodic real distributions with finite norm

$$\|z\|^2_{H^s(\mathbb{T}^3)} \equiv \|z\|_s^2 = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^{2s}|\widehat{z}(k)|^2 < \infty \tag{4.1}$$

where $\widehat{z}(k)$ are the Fourier coefficients[2] of the function z.

We shall use the following generalization of the spaces (2.4) of solenoidal vector fields:

$$\begin{aligned} V^s &\equiv V^s(\mathbb{T}^3) \\ &= \Big\{v(x) \in (H^s(\mathbb{T}^3))^3 :\ \operatorname{div} v(x) = 0,\ \int_{\mathbb{T}^3} v(x)\,dx = 0\Big\}, \quad s\in\mathbb{R}. \end{aligned} \tag{4.2}$$

Lemma 4.1. *Let $\Phi(u)$ be the functional* (2.16). *Then there exists a constant $c>0$ such that, for every $u \in V^{3/2}$,*

$$|\Phi(u)| \leqslant c\|u\|_{3/2}. \tag{4.3}$$

Proof. The estimate

$$\begin{aligned} |\Phi(u)| &\leqslant \frac{\|u\|^2_{L_3(\mathbb{T}^3)}\|\nabla \operatorname{curl}^{-1} u\|_{L_3(\mathbb{T}^3)}}{\|u\|_0^2} \leqslant c\frac{\|u\|^2_{1/2}\|\nabla \operatorname{curl}^{-1} u\|_{1/2}}{\|u\|_0^2} \leqslant \\ &\leqslant c\frac{\|u\|^3_{1/2}}{\|u\|_0^2} \leqslant c\frac{\|u\|_0^2\|u\|_{3/2}}{\|u\|_0^2} = c\|u\|_{3/2} \end{aligned} \tag{4.4}$$

follows from definitions (2.16), (2.11), Sobolev's embedding theorem (according to which $H^{1/2}(\mathbb{T}^3) \subset L_3(\mathbb{T}^3)$) and the interpolation inequality $\|v\|^3_{1/2} \leqslant c\|v\|_0^2\|v\|_{3/2}$. □

[2]Strictly speaking, one needs to add $|\widehat{z}(0)|^2$ to the right-hand side of (4.1). However, we did not do this, since starting from (4.2) it will be assumed that this Fourier coefficient is zero.

Lemma 4.2. *Let Φ be the functional* (2.16)*. For any $\beta < 1/2$, there exists a constant $c_1 > 0$ such that*

$$\int_0^t |\Phi(S(\tau; y_0))| \, d\tau \leqslant c_1 \|y_0\|_{-\beta} \tag{4.5}$$

for any $y_0 \in V^{-\beta}(\mathbb{T}^3)$ and $t > 0$. Here, $S(t; y_0)$ is the solution operator of problem (3.1)*.*

Proof. Using (4.3) and the representation of the solution of problem (3.1) in terms of the Fourier series, we see that

$$\begin{aligned} &\int_0^t \left|\Phi(S(\tau; y_0))\right| d\tau \leqslant \\ &\qquad \leqslant c \int_0^t e^{-\tau/2} \left(\sum_{k \neq 0} (|\widehat{y}_0(k)|^2 |k|^{-2\beta}) |k|^{3+2\beta} e^{-(k^2-1)\tau} \right)^{1/2} d\tau. \end{aligned} \tag{4.6}$$

where $\widehat{y}_0(k)$ are the Fourier coefficients of the function y_0. The solution $\widehat{\rho} = \widehat{\rho}(t)$ of the extremal problem

$$f(t, \rho) = \rho^{3+2\beta} e^{-(\rho^2-1)t} \to \max, \quad \rho \geqslant 1,$$

is given by $\widehat{\rho}(t) = \sqrt{\frac{3+2\beta}{2t}}$. We also have

$$f(t, \widehat{\rho}(t)) = \begin{cases} \left(\frac{3+2\beta}{2t}\right)^{\frac{3+2\beta}{2}} e^{-(3+2\beta-2t)/2}, & t \leqslant \frac{3+2\beta}{2}, \\ 1, & t \geqslant \frac{3+2\beta}{2}. \end{cases} \tag{4.7}$$

Substituting (4.7) into (4.6), we arrive at (4.5), the result required. □

Remark 4.1. In view of Lemma 4.2, the functional on the left of (4.5) is well-defined for $y_0 \in V^{-\beta}(\mathbb{T}^3)$ with $\beta < 1/2$. In particular, Lemma 4.2 and (3.2) show that the solution of problem (2.17), (2.10) is well-defined for any initial data $y_0 \in V^0$ and is infinitely differentiable in these variables for each $t > 0$ and $x \in \mathbb{T}^3$.

In the following two subsections we will justify the choice of the space V^0 as the phase space of the corresponding dynamical system.

4.2. **Unique solvability of problem** (2.17)**,** (2.10)**.** We claim that problem (2.17), (2.10) has no other solutions than (3.2). Consider the space of solutions to this problem:

$$\begin{aligned} V^{1,2(-1)}(Q_T) = \Big\{ & y(t,x) \in L_2(0,T; V^1(\mathbb{T}^3)) : \\ & \qquad \frac{\partial y(t,x)}{\partial t} \in L_2(0,T; V^{-1}(\mathbb{T}^3)) \Big\}; \end{aligned} \tag{4.8}$$

here $Q_T = (0,T) \times \mathbb{T}^3$, $T > 0$ is some number or $T = \infty$. Assume that $y_0 \in V^0(\mathbb{T}^3)$, and impose on solution the following

Condition 4.1. *Solution $y(t, x; y_0) \in V^{1,2(-1)}(Q_T)$ of problem* (2.17)*,* (2.10) *with initial condition $y_0 \in V^0 \setminus \{0\}$ satisfies relation $\|y(t, \cdot, y_0)\|_0 \neq 0$ for any $t \in [0, T]$.*

Theorem 4.1. *For any $y_0 \in V^0$ there exist $T > 0$ such that problem* (2.17)*,* (2.10)*, has a unique solution $y(t, x; y_0)$ in the space $V^{1,2(-1)}(Q_T)$ satisfying Condition* 4.1.

Proof. By Lemma 4.2 the functional $\int_0^t \Phi(S(\tau;y_0))\,d\tau$ is well-defined for any $t>0$ and is continuous in t. Hence, the inequality

$$1-\int_0^t \Phi(S(\tau;y_0))\,d\tau \geqslant \varepsilon \quad \forall t\in[0,T] \tag{4.9}$$

holds in view of (4.5) for some $T>0$ and sufficiently small $\varepsilon>0$.

By Theorem 3.1, (4.9) and the well-known properties of the solution operator $S(t,x;y_0)$ (which in turn follow by expanding S into the Fourier series), the function $y(t,x;y_0)$, as defined by (3.2), lies in the space $V^{1,2(-1)}(Q_T)$. Moreover,

$$\|y(t,\,\cdot\,;y_0)\|_0 \geqslant \frac{1}{\varepsilon}\|S(t;y_0)\|_0 \geqslant \frac{1}{\varepsilon}\|S(T;y_0)\|_0 \quad \forall t\in[0,T]; \tag{4.10}$$

that is, Condition 4.1 is satisfied. This proves the solvability of problem (2.17), (2.10).

We next claim that this problem has a unique solution. Assume that $y(t,x)\in V^{1,2(-1)}(Q_T)$ is a solution of this problem satisfying Condition 4.1. It will be shown that there exists a function $\gamma(t)\in C[0,T]$ such that $\gamma(t)y(t,x)$ is the solution of the problem

$$\partial_t z(t,x)-\Delta z(t,x)=0, \quad z|_{t=0}=y_0(x). \tag{4.11}$$

Assuming that there exists a function $\gamma(t)$ and substituting $\gamma(t)y(t,x)$ into equation (4.11), we obtain

$$y\partial_t\gamma+\gamma(\partial_t y-\Delta y)=0.$$

Since $y(t,x)$ is the solution of equation (2.17), it follows from the previous equality that

$$y[\partial_t\gamma+\gamma\Phi(y)]=0,$$

and hence, since y satisfies Condition 4.1 and since $y(t,x)$ and $\gamma(t)y(t,x)$ agree for $t=0$,

$$\gamma(t)=e^{-\int_0^t \Phi(y(\tau,\,\cdot\,))\,d\tau}.$$

Thus, the function $y(t,x)e^{-\int_0^t \Phi(y(\tau,\,\cdot\,))\,d\tau}$ is the solution of problem (4.11), and hence, by the definition of the function $S(t,x;y_0)$ (see (3.1) and below),

$$S(t,x;y_0)=y(t,x)e^{-\int_0^t \Phi(y(\tau,\,\cdot\,))\,d\tau}. \tag{4.12}$$

Solving (4.12) for $y(t,x)$ and substituting the result into (2.17), this gives

$$e^{\int_0^t \Phi(y(\tau,\,\cdot\,))\,d\tau}\Big(\Phi(y(t,\,\cdot\,))S(t;y_0)+\partial_t S-\Delta S-e^{\int_0^t \Phi(y)\,d\tau}\Phi(S)S\Big)=0,$$

whence it follows that

$$\Phi(S(t;y_0))=\Phi(y(t,\,\cdot\,))e^{-\int_0^t \Phi(y)\,d\tau}=-\partial_t e^{-\int_0^t \Phi(y)\,d\tau}.$$

Consequently,

$$e^{-\int_0^t \Phi(y)\,d\tau}=1-\int_0^t \Phi(S(\tau;y_0))\,d\tau.$$

Substituting this into (4.12), we see that $y(t,x)$ is given by (3.2). This proves the uniqueness of the solution of problem (2.17), (2.10). $\square$

4.3. **Continuous dependence of the solution on the initial data.**

Lemma 4.3. *Assume that $y_m \to y_0$ as $m \to \infty$ in the space $V^0(\mathbb{T}^3)$. Then*

$$\int_0^t \Phi(S(\tau; y_m))\, d\tau \longrightarrow \int_0^t \Phi(S(\tau; y_0))\, d\tau \quad \text{as} \quad m \to \infty \quad \forall t \in (0, T]. \tag{4.13}$$

Proof. Since for $y_0 = 0$ the proof of this Lemma is evident, we assume that $y_0 \neq 0$. We let

$$A(y(x)) = (y(x), \nabla)\operatorname{curl}^{-1} y(x) \cdot y(x)$$

denote the integrand in the numerator of the fraction in (2.16). Formula (2.16) and the following well-known estimates for $S(t, x; y_0)$,

$$\gamma(T)\|y\|_0 \leqslant \|S(t; y)\|_0 \leqslant C\|y\|_0 \quad \forall t \in (0, T], \quad \|S(\,\cdot\,; y)\|_{L_2(0,T;V^1(\mathbb{T}^3))} \leqslant C\|y\|_0,$$

with some constants $\gamma(T) > 0, C > 0$ imply that

$$\begin{aligned} &\left|\int_0^t \Phi(S(\tau; y_0)) - \Phi(S(\tau; y_m))\, d\tau\right| \\ &\quad = \left|\int_0^t \frac{(A(S(\tau; y_0)) - A(S(\tau; y_m)))}{\|S(\tau; y_0)\|_0^2}\, d\tau + \right. \\ &\quad \left. + \int_0^t \frac{(\|S(\tau; y_m)\|_0^2 - \|S(\tau; y_0)\|_0^2) A(S(\tau; y_m))}{\|S(\tau; y_0)\|_0^2 \|S(\tau; y_m)\|_0^2}\, d\tau\right| \\ &\quad \leqslant c\int_0^t |A(S(\tau; y_0)) - A(S(\tau; y_m))| + \\ &\quad + \|S(\tau; y_0 - y_m)\|_0 \frac{|A(S(\tau; y_m))|}{\|S(\tau; y_m)\|_0^2}\, d\tau. \end{aligned} \tag{4.14}$$

By Lemma 4.2,

$$\begin{aligned} \int_0^t \|S(\tau; y_0 - y_m)\|_0 \frac{|A(S(\tau; y_m))|}{\|S(\tau; y_m)\|_0^2}\, d\tau &\leqslant c\|y_0 - y_m\|_0 \int_0^t \left|\Phi(S(\tau; y_m))\right| d\tau \\ &\leqslant c_1\|y_0 - y_m\|_0 \|y_m\|_0. \end{aligned} \tag{4.15}$$

Besides,

$$\begin{aligned} &\int_0^t |A(S(\tau; y_0)) - A(S(\tau; y_m))|\, d\tau \\ &\quad = \int_0^t |(S(\tau; y_0 - y_m), \nabla)\operatorname{curl}^{-1} S(\tau; y_0) \cdot S(\tau; y_0) + \\ &\quad + (S(\tau; y_m), \nabla)\operatorname{curl}^{-1} S(\tau; y_0 - y_m) \cdot S(\tau; y_0) + \\ &\quad + (S(\tau; y_m), \nabla)\operatorname{curl}^{-1} S(\tau; y_m) \cdot S(\tau; y_0 - y_m)|\, d\tau \\ &\quad \leqslant c\|y_0 - y_m\|_0 \int_0^t (\|S(\tau; y_0)\|_{3/4}^2 + \\ &\quad + \|S(\tau; y_m)\|_{3/4} \|S(\tau; y_0)\|_{3/4} + \|S(\tau; y_m)\|_{3/4}^2)\, d\tau \\ &\quad \leqslant c_1\|y_0 - y_m\|_0 (\|y_0\|_0^2 + \|y_m\|_0^2). \end{aligned} \tag{4.16}$$

Finally, (4.13) follows from (4.14), (4.15), (4.16). □

Let us prove the theorem on the continuous dependence of the solution to problem (2.17), (2.10) on the initial data. This result will be used for later purposes.

Theorem 4.2. *The solution* $y(t,x;y_0) \in V^{1,2(-1)}(Q_T)$ *of problem* (2.17), (2.10) *depends continuously on the initial data* $y_0 \in V^0(\mathbb{T}^3)$*; that is, for any* $t \in (0,T]$*, the mapping*

$$V^0(\mathbb{T}^3) \ni y_0 \longrightarrow y(t,\,\cdot\,;y_0) \in V^0(\mathbb{T}^3)$$

is continuous.

Proof. Assume that $y_m \to y_0$ as $m \to \infty$ in the space $V^0(\mathbb{T}^3)$. Then, using (3.2), employing the well-known properties of the operator function $S(t,x;y_0)$ (which readily follow on expanding into the Fourier series), and invoking (4.13), we obtain the assertion of the theorem:

$$\begin{aligned} \|y(t;y_0) - y(t;y_m)\|_0 \leqslant {} & \frac{\|S(t;y_0-y_m)\|_0}{1-\int_0^t \Phi(S(\tau;y_0))\,d\tau} + \\ & + \frac{\left|\int_0^t \Phi(S(\tau;y_m))\,d\tau - \int_0^t \Phi(S(\tau;y_0))\,d\tau\right| \|S(t;y_m)\|_0}{\left(1-\int_0^t \Phi(S(\tau;y_0))\,d\tau\right)\left(1-\int_0^t \Phi(S(\tau;y_m))\,d\tau\right)} \\ & \leqslant c_1\|y_0-y_m\|_0 + \\ & + c_2\left|\int_0^t \Phi(S(\tau;y_m))\,d\tau - \int_0^t \Phi(S(\tau;y_0))\,d\tau\right| \to 0 \quad \text{as} \quad m\to\infty. \end{aligned} \tag{4.17}$$

$\square$

5. Properties of the functional $\Phi(S(t;y_0))$

The explicit formula (3.2) implies that in order to study the dynamics generated by problem (2.17), (2.10) one first needs to examine in detail the properties of the functional $\Phi(S(t;y_0))$.

5.1. On the kernel of the functional $\Phi(S(t;u))$.

In view of the results obtained in the previous section, it is natural to take the space V^0 as the phase space of the dynamical system corresponding to problem (2.17), (2.10). The kernel of the functional $\Phi(S(t;u))$ is defined by

$$K\Phi = \{u \in V^0 : \ \Phi(S(t;u)) \equiv 0 \quad \forall t \in \mathbb{R}_+\}. \tag{5.1}$$

The functional $u \to \Phi(S(t;u))$ is first-order homogeneous (that is, $\Phi(S(t;\lambda u)) = \lambda\Phi(S(t;u))$ for all $\lambda \in \mathbb{R}$), and hence $K\Phi$ is a double-sided cone; that is, if $u \in K\Phi$, then $\lambda u \in K\Phi$ for all $\lambda \in \mathbb{R}$.

We claim that the cone $K\Phi$ is nonempty. To this end, we introduce the linear subspace

$$L = \{z \in V^0 : \ z(x) = \sum_{k\in\mathcal{U}_L} \hat{z}(k)e^{ik\cdot x}, \ \hat{z}(-k) = \overline{\hat{z}(k)}\}, \tag{5.2}$$

where

$$\mathcal{U}_L = \{k = (k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus \{0\} : \ k_1+k_2+k_3 - \text{is an odd number}\}. \tag{5.3}$$

The space L consists of vector fields $z(x) \in V^0$ whose Fourier coefficients $\hat{z}(k)$ are nonzero for $k \notin \mathcal{U}_L$. Clearly, L is an infinite-dimensional subspace of the space V^0.

Lemma 5.1. *The following inclusion holds:*

$$L \subset K\Phi; \tag{5.4}$$

here $L, K\Phi$ *are defined in* (5.2), (5.3), (5.1).

Proof. In view of (2.11),

$$\int_{\mathbb{T}^3} (z(x),\nabla)\operatorname{curl}^{-1} z(x)\cdot z(x)\,dx = -\sum_{k,m\in\mathcal{U}_L} \hat z(k)\cdot m\frac{m\times \hat z(m)}{|m|^2}\cdot\overline{\hat z(k+m)} = 0 \tag{5.5}$$

for any $z\in L$.

Indeed, $\overline{\hat z(k+m)}=0$, because $(k_1+m_1)+(k_2+m_2)+(k_3+m_3)$ is an even number for $k,m\in\mathcal{U}$. Therefore, by (5.5) and the definition of the operator $S(t,x;z)$,

$$\begin{aligned}\int_{\mathbb{T}^3} (S(t,x;z),\nabla)\operatorname{curl}^{-1} S(t,x;z)\cdot S(t,x;z)\,dx = \\ = -\sum_{k,m\in\mathcal{U}_L} e^{-2(k^2+m^2+k\cdot m)t}\hat z(k)\cdot m\frac{m\times\hat z(m)}{|m|^2}\cdot\overline{\hat z(k+m)}=0,\end{aligned} \tag{5.6}$$

for any $t\in\mathbb{R}_+$.

Finally, $\Phi(S(t;z))=0$ for any $t\in\mathbb{R}_+$ with $z\in L$ by (5.6) and the definition (2.16) of the functional Φ. ☐

5.2. Properties of the function $t\to\Phi(S(t;u))$.

Lemma 5.2. *For any $z\in V^0\setminus K\Phi$ there exists $t_0>0$, such that for $t>t_0$ the function $t\to\Phi(S(t;z))$ is sign-definite.*

Proof. From definition (2.16) of the functional Φ we have, similarly to (5.6),

$$\Phi(S(t,z)) = \frac{-\sum_{k,m\in\mathbb{Z}^3\setminus\{0\}} e^{-2(k^2+m^2+k\cdot m)t}\hat z(k)\cdot m\frac{m\times\hat z(m)}{|m|^2}\cdot\overline{\hat z(k+m)}}{\sum_{k\neq 0}|\hat z(k)|^2e^{-2k^2t}}. \tag{5.7}$$

Assume that $z(x)$ is such that $\hat z(k)=0$ for $|k|<\gamma$, where $\gamma\geqslant 1$ is some number, the first nonzero Fourier coefficients appearing for $|k|=\gamma$. The numerator in (5.7) can be written down as follows

$$\begin{aligned}-\sum_{k,m\in\mathbb{Z}^3\setminus\{0\}} e^{-2(k^2+m^2+k\cdot m)t}\hat z(k)\cdot m\frac{m\times\hat z(m)}{|m|^2}\cdot\overline{\hat z(k+m)} = \\ = e^{-(2\gamma^2+\alpha)t}\left(a_0+\sum_{m=1}^{\infty}a_me^{-mt}\right),\end{aligned}$$

where $\alpha\geqslant 2$ is an even number, $a_0\neq 0$, $a_m\in\mathbb{R}$. For the denominator, we have

$$\sum_{k\neq 0}|\hat z(k)|^2e^{-2k^2t} = e^{-2\gamma^2t}\left(b_0+\sum_{m=1}^{\infty}b_me^{-mt}\right),$$

where $b_0>0, b_m\geqslant 0$. Hence, in view of (5.7),

$$\begin{aligned}\Phi(S(t,z)) &= e^{-\alpha t}\frac{a_0+\sum_{m=1}^{\infty}a_me^{-mt}}{b_0+\sum_{m=1}^{\infty}b_me^{-mt}} \\ &= e^{-\alpha t}\left(\frac{a_0}{b_0}+\sum_{m=1}^{\infty}\frac{a_m}{b_0}e^{-mt}\right)\sum_{j=0}^{\infty}(-1)^j\left(\sum_{m=1}^{\infty}\frac{b_m}{b_0}e^{-mt}\right)^j \\ &= e^{-\alpha t}\left(\frac{a_0}{b_0}+\sum_{m=1}^{\infty}c_me^{-mt}\right),\end{aligned} \tag{5.8}$$

where $c_m \in \mathbb{R}$ are some constants, the series on the right of (5.8) converging absolutely for all sufficiently large t. Clearly, for large t, the sign of the bracketed expression in the last line of (5.8) agrees that of a_0/b_0. $\square$

Lemma 5.3. *For any $z \notin K\Phi$ and any $\alpha \in \mathbb{R}$, the equation*

$$\int_0^t \Phi(S(\tau; z))\, d\tau = \alpha \tag{5.9}$$

has at most finite number of solutions $0 \leqslant t_1 < t_2 < \cdots < t_N$, where either $t_N < \infty$ or $t_N = \infty$.

Proof. By Lemma 5.2 the function $t \to \int_0^t \Phi(S(\tau; z))\, d\tau$ is analytic for all $t \in \mathbb{R}_+$ and is strictly monotone for $t > t_0$ for some sufficiently large t_0. Hence, all the solutions of (5.9), with at most one exception, lie on the interval $[0, t_0]$. If the interval $[0, t_0]$ contains an infinite number of solutions, then one may choose a subsequence converging to $\hat{t} \in [0, t_0]$. But then all the derivatives of the function $\int_0^t \Phi(S(\tau; z))\, d\tau$ vanish at $t = \hat{t}$, and hence, being analytic, this function is identically constant. But this contradicts its strict monotonicity for $t \in [t_0, \infty]$. Hence, the number of solutions of (5.9) is at most finite. $\square$

5.3. **Some auxiliary results.** In this section, we perform some auxiliary calculations to be used later for the study of the behaviour of the function $t \to \int_0^t \Phi(S(\tau; z))\, d\tau$.

Consider the vectors:

$$e_1 = (1, 0, 0), \qquad e_2 = (0, 1, 0)$$

and specify the following Fourier coefficients

$$\begin{aligned} &\hat{z}(\pm n e_1) = (0, \alpha_2, \alpha_3), && \hat{z}(\pm n e_2) = (\beta_1, 0, \beta_3), \\ &\hat{z}(\pm n(e_1 + e_2)) = (\gamma, -\gamma, 0), && \hat{z}(\pm n(e_1 - e_2)) = (\delta, \delta, 0), \end{aligned} \tag{5.10}$$

where n is some fixed odd number and $\alpha_2, \alpha_3, \beta_1, \beta_3, \gamma, \delta \in \mathbb{R}$ are some real numbers.

Lemma 5.4. *Assume that all the Fourier coefficients of the vector field $z(x)$ are zero, except for those indicated in* (5.10). *Then*

$$\begin{aligned} &\int_{\mathbb{T}^3} (S(t, x; z), \nabla) \operatorname{curl}^{-1} S(t, x; z) \cdot S(t, x; z)\, dx \\ &\quad = -\sum_{k, m \in \mathbb{Z}^3 \setminus \{0\}} e^{-2(k^2 + m^2 + k \cdot m)t} \hat{z}(k) \cdot m \frac{m \times \hat{z}(m)}{|m|^2} \cdot \hat{z}(k + m) \\ &\qquad = (\alpha_2 \beta_3 + \alpha_3 \beta_1)(\gamma + \delta) e^{-4n^2 t}. \end{aligned} \tag{5.11}$$

Proof. We have

$$I(k, m) := \hat{z}(k) \cdot m \frac{m \times \hat{z}(m)}{|m|^2} \cdot \hat{z}(k + m) = \frac{\hat{z}(k) \cdot m}{|m|^2} \det(m, \hat{z}(m), \hat{z}(k + m))$$

and hence, using (5.10) it follows by simple calculations that

$$\begin{aligned} I(\pm n(e_1, e_2)) &= -I(\pm n(e_1, -e_1 - e_2)) = 2I(\pm n(e_1 + e_2, -e_1)) = \alpha_2 \beta_3 \gamma, \\ I(\pm n(e_1, -e_2)) &= -I(\pm n(e_1, -e_1 + e_2)) = 2I(\pm n(e_1 - e_2, -e_1)) = \alpha_2 \beta_3 \delta, \\ I(\pm n(e_2, e_1)) &= -I(\pm n(e_2, -e_2 - e_1)) = 2I(\pm n(e_2 + e_1, -e_2)) = \alpha_3 \beta_1 \gamma, \\ I(\pm n(e_2, e_1 - e_2)) &= -I(\pm n(e_2, -e_1)) = 2I(\pm n(e_2 - e_1, -e_2)) = \alpha_3 \beta_1 \delta, \end{aligned}$$

Substituting these formulae on the right of the first equality in (5.11), we obtain the right-hand side of the second equality. □

5.4. **Concrete examples of the function $t \to \Phi(S(t;z))$.** Below we will employ Lemma 5.4 to construct a series of concrete examples of the function $t \to \Phi(S(t;z))$ useful for later purposes.

Proposition 5.1. *We have*

$$K\Phi \setminus L \neq \varnothing, \tag{5.12}$$

where the sets $K\Phi$ and L are defined, respectively, in (5.1) *and* (5.2).

Proof. As an element of the nonempty set (5.12) one may take the vector field $z(x)$. For this field only the Fourier coefficients indicated in (5.10) are nonzero with $n = 1$, for which $\gamma + \delta = 0$. Now the inclusion $z \in K\Phi \setminus L$ readily follows from (5.11). □

Remark 5.1. In fact, the set $K\Phi \setminus L$ is much broader than that indicated in the proof of Proposition 5.1. Indeed, one easily sees that as an $z \in K\Phi \setminus L$ one may also take the vector field $z(x)$ whose nonzero Fourier coefficients are only those indicated in (5.10) with any odd $n > 0$, and for which $\gamma_n + \delta_n = 0$.

Proposition 5.2. *There exists $z \in V^0 \setminus K\Phi$ such that*

$$\int_0^t \Phi(S(\tau;z))\,d\tau \leqslant 1 \quad \forall t > 0 \quad \textit{and} \quad \max_{t>0} \int_0^t \Phi(S(\tau;z))\,d\tau = 1, \tag{5.13}$$

the maximum is attained only at $t = \infty$*. Moreover, for such z there exist constants $0 < c_1 < c_2$ such that*

$$c_1 e^{-2t} \leqslant 1 - \int_0^t \Phi(S(\tau;z))\,d\tau \leqslant c_2 e^{-2t} \tag{5.14}$$

for sufficiently large t.

Proof. Consider $z(x)$ whose nonzero Fourier coefficients are only those indicated in (5.10) with $n = 1$, where α_i, β_j, γ, and δ are such that

$$(\alpha_2\beta_3 + \alpha_3\beta_1)(\gamma + \delta) = A, \quad \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_3^2 = a_1, \quad 2\gamma^2 + 2\delta^2 = a_2 \tag{5.15}$$

with some $A > 0$, $a_1 > 0$, $a_2 > 0$. Hence, in view of (5.11),

$$\int_0^t \Phi(S(\tau;z))\,d\tau = \int_0^t \frac{Ae^{-4\tau}}{a_1 e^{-2\tau} + a_2 e^{-4\tau}}\,d\tau = \int_0^t \frac{Ae^{-2\tau}}{a_1 + a_2 e^{-2\tau}}\,d\tau$$

The integrand on the right is positive, and hence the corresponding integral increases to some limit as $t \to \infty$. Further, replacing in this equality z by λz with suitable $\lambda > 0$, we can achieve that this limit is 1. Finally, (5.14) follows, since, clearly,

$$\int_t^\infty \frac{Ae^{-2\tau}}{a_1 + a_2 e^{-2\tau}}\,d\tau \leqslant \frac{A}{a_1}\int_t^\infty e^{-2\tau}\,d\tau = \frac{A}{2a_1}e^{-2\tau},$$

and since, for sufficiently large t,

$$\int_t^\infty \frac{Ae^{-2\tau}}{a_1 + a_2 e^{-2\tau}}\,d\tau \geqslant \frac{A}{a_1(1+\varepsilon)}\int_t^\infty e^{-2\tau}\,d\tau = \frac{A}{2a_1(1+\varepsilon)}e^{-2\tau}.$$

□

Proposition 5.3. *There exists $z \in V^0 \setminus K\Phi$ such that*

$$\max_{t>0} \int_0^t \Phi(S(\tau; z))\, d\tau = 1 = \int_0^{t_0} \Phi(S(\tau; z))\, d\tau, \tag{5.16}$$

that is, this maximum is attained at finite $t = t_0$.

Proof. Consider $z(x)$ whose nonzero Fourier coefficients are only those indicated in (5.10) with $n = 1$ and $n = 3$, where α_i, β_j, γ, and δ corresponding to $n = 1$ satisfy (5.15) with $A < 0$, $a_1 > 0$, $a_2 > 0$, while α_i, β_j, γ, and δ corresponding to $n = 3$ are such that

$$(\alpha_2\beta_3 + \alpha_3\beta_1)(\gamma + \delta) = B, \quad \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_3^2 = b_1, \quad 2\gamma^2 + 2\delta^2 = b_2$$

with some $B > 0$, $b_1 > 0$, $b_2 > 0$. Hence, by (5.11),

$$\int_0^t \Phi(S(\tau; z))\, d\tau = \int_0^t \frac{Ae^{-4\tau} + Be^{-36\tau}}{a_1e^{-2\tau} + a_2e^{-4\tau} + b_1e^{-18\tau} + b_2e^{-36\tau}}\, d\tau =$$
$$= \int_0^t \frac{e^{-2\tau}(A + Be^{-32\tau})}{a_1 + a_2e^{-2\tau} + b_1e^{-16\tau} + b_2e^{-34\tau}}\, d\tau.$$

Let $A + B > 0$. Then, clearly, the integrand on the right of this equality is positive for small τ, it becomes negative for large τ, and it monotonically increases to zero as $\tau \to \infty$. It readily follows that this integrand assumes the maximum positive value at some finite $t = t_0$. In order that this maximum value be 1, it suffices to multiply z by a suitably chosen $\lambda > 0$. □

6. The structure of the dynamics generated by the NPS

The purpose of this section is to examine the structure of the dynamic flow generated by the NPS. To this end we partition the phase space of the dynamical system into three sets on each of which the behaviour of the dynamic flow is substantially different.

6.1. **Characteristic sets of the phase space of the NPS.** We first give the definition of three component subsets of the phase space, the NPS-generated dynamics being substantially different on these sets. Of course, these sets are invariant relative to the dynamical system under study. Recall that the space $V^0(\mathbb{T}^3) \equiv V^0$ (see (2.4)) was taken as the phase space of problem (2.17), (2.10).

Definition 6.1. A subset $M_- \subset V^0$ of initial data y_0 for which the solution $y(t, x; y_0)$ of problem (2.17), (2.10) exists, is unique, and satisfies the inequality

$$\|y(t, \cdot\,; y_0)\|_0 \leqslant \alpha\|y_0\|_0 e^{-t} \qquad \forall t > 0, \tag{6.1}$$

is called the *set of stability.* Here, $\alpha \geqslant 1$ is some fixed number depending, in general, on $\|y_0\|_0$.

Such a dependence of α on $\|y_0\|_0$ in (6.1) is not felt if y_0 lies in the subset

$$M_-(\alpha) = \{y_0 \in M_- :\ y(t, x; y_0) \quad \text{satisfies (6.1)}\} \tag{6.2}$$

of the set of stability. Here, $\alpha \geqslant 1$ is a fixed number and $y(t, x; y_0)$ is the solution of problem (2.17), (2.10). Clearly,

$$\cup_{\alpha\geqslant 1} M_-(\alpha) = M_-\,. \tag{6.3}$$

The sets $M_-(\alpha)$ prove useful in examining the set of stability M_-. The following result holds.

Lemma 6.1. *Suppose that $y_0 \in V^0$ satisfies the estimate*

$$\sup_{t\in\mathbb{R}_+}\int_0^t \Phi(S(\tau;y_0))\,d\tau \leqslant \frac{\alpha-1}{\alpha} \tag{6.4}$$

with $\alpha \geqslant 1$. Then $y_0 \in M_-(\alpha)$.

Proof. Indeed, in view of (3.2) and (6.4),

$$\|y(t,\,\cdot\,;y_0)\|_0 \leqslant \frac{\|S(t;y_0)\|_0}{1-\sup_{t\in\mathbb{R}_+}\int_0^t \Phi(S(\tau;y_0))\,d\tau} \leqslant \alpha\|y_0\|_0 e^{-t}.$$

□

Definition 6.2. A subset $M_+ \subset V^0$ of initial data y_0 of problem (2.17), (2.10) for which the solution $y(t,x;y_0)$ exists only on a finite time interval $t \in (0,t_0)$ with $t_0 > 0$ depending on y_0 and is exploded at $t = t_0$ [3] is called the *set of explosions.*

In view of (3.2),

$$M_+ = \Big\{y_0 \in V^0 :\ \exists\, 0 < t_0 < \infty,\ \int_0^{t_0} \Phi(S(\tau;y_0))\,d\tau = 1\Big\} \tag{6.5}$$

for the solution $y(t,x;y_0)$.

The smallest t_0 at which the equality in (6.5) is attained is called the *explosion moment* of the solution $y(t,x;y_0)$.

Definition 6.3. A subset $M_g \subset V^0$ of initial data y_0 of problem (2.17), (2.10) for which the solution $y(t,x;y_0)$ exists on an infinite time interval $t \in \mathbb{R}_+$ and $\|y(t,x;y_0)\|_0 \to \infty$ as $t \to \infty$ is called the *set of increase*.

Lemma 6.2. *Any of the sets M_-, M_+, M_g is nonempty.*

Proof. We have $K\Phi \subset M_-$ in view of (3.2) and (5.1), and moreover, $K\Phi \neq \varnothing$, by Lemma 5.1 and Proposition 5.1. Hence, $M_- \neq \varnothing$. Further, $M_+ \neq \varnothing$ in view of (3.2) and Proposition 5.3, and finally, $M_g \neq \varnothing$ by (3.2) and Proposition 5.2. □

Lemma 6.3. *The sets M_-, M_+, M_g are invariant under the operator $y_0 \to y(t,\,\cdot\,;y_0)$ of translation along solution of problem* (2.17), (2.10)*; that is,*

$$y(t;M_-) \subset M_-,\quad y(t;M_g) \subset M_g \quad \forall t > 0;\quad y(t;y_0) \in M_+,\quad \forall t \in (0,t_0), \tag{6.6}$$

where t_0 is the explosion moment of the initial data $y_0 \in M_+$.

Proof. Let $y_0 \in M_-$. Then estimate (6.4) with some $\alpha \geqslant 1$ is satisfied. We set $y_1 = y(t_1,\,\cdot\,;y_0)$. Hence, in view of (3.2), since the operator $S(t;y_0)$ is linear in y_0, further since the functional $\Phi(v)$ is homogeneous in v, and by (6.4),

$$\begin{aligned}\|y(t,\,\cdot\,;y_1)\|_0 &= \frac{\|S(t;y_1\|_0}{1-\int_0^t \Phi(S(\tau;\frac{S(t_1;y_0)}{1-\int_0^{t_1}\Phi(S(\tau_1;y_0)\,d\tau_1}))\,d\tau} = \\ &= \frac{(1-\int_0^{t_1}\Phi(S(\tau_1;y_0))\,d\tau_1)\|S(t;y_1\|_0}{1-\int_0^{t_1}\Phi(S(\tau_1;y_0))\,d\tau_1-\int_{t_1}^{t+t_1}\Phi(S(\tau;y_0))\,d\tau} \leqslant \alpha_1 e^{-t}\|y_1\|_0,\end{aligned}$$

where $\alpha_1 = \alpha(1-\int_0^{t_1}\Phi(S(\tau_1;y_0))\,d\tau_1) < \infty$, proving the first inclusion in (6.6).

[3] i.e. $\|y(t;y_0)\| \to \infty$ as $t \nearrow t_0$

Assume that $t_1 > 0$, $t_2 > 0$ and that the solution $y(t; y_0)$ is defined for $t = t_1$ and $t = t_1 + t_2$. If $y_1 = y(t_1; y_0)$, then, in view of (3.2), we have as before,

$$(6.7)\quad y(t_2; y_1) = \frac{S(t_2; \frac{S(t_1;y_0)}{1-\int_0^{t_1}\Phi(S(\tau;y_0))\,d\tau})}{1-\int_0^{t_2}\Phi(S(\tau_1; \frac{S(t_1;y_0)}{1-\int_0^{t_1}\Phi(S(\tau;y_0))\,d\tau}))\,d\tau_1} = \\ = \frac{S(t_1+t_2; y_0)}{1-\int_0^{t_1+t_2}\Phi(S(\tau;y_0))\,d\tau} = y(t_1+t_2, \cdot\,; y_0).$$

Now if $y_0 \in M_+$, t_0 is an explosion moment, $y_1 = y(t_1; y_0)$, $t_1 > 0$, $t_2 > 0$, $t_1 + t_2 < t_0$, then, in view of (6.7),

$$\|y(t_2; y_1)\|_0 = \|y(t_1 + t_2; y_0)\|_0 \to \infty \quad \text{for} \quad t_2 \to t_0 - t_1,$$

and hence, $y_1 \in M_+$, proving the third inclusion in (6.6).

Finally, if $y_0 \in M_g$, $y_1 = y(t_1; y_0)$, $t_1 > 0$, $t_2 > 0$, then a similar analysis shows that

$$\|y(t_2; y_1)\|_0 = \|y(t_1 + t_2; y_0)\|_0 \to \infty \quad \text{as} \quad t_2 \to \infty,$$

proving the second inclusion in (6.6). □

Below we shall show that $M_- \cup M_+ \cup M_g = V^0$ and study geometrical properties of these sets in V^0

6.2. **Some subsets of the unit sphere of the space V^0.** We let

$$(6.8)\qquad \Sigma = \{v \in V^0 : \quad \|v\|_0 = 1\}$$

denote the unit sphere of the phase space V^0.

In order to describe the structure of the phase flow corresponding to problem (2.17), (2.10), we need to define several subsets of the sphere Σ. Namely, we set

$$A_-(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau; v))\,d\tau \leqslant 0\},$$
$$A_0(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau; v))\,d\tau = 0\},$$

and define

$$(6.9)\qquad A_- = \cap_{t\geqslant 0} A_-(t), \quad A_0 = \cap_{t\geqslant 0} A_0(t).$$

All these sets are nonempty and closed. We have $A_0 = K\Phi \cap \Sigma$ by (5.1), (6.8), and hence $A_0 \neq \varnothing$ by Lemma 5.1 and Proposition 5.1. It is also clear that $A_0 \subset A_-$. The sets $A_-(t)$ and A_- have nonempty relative interior with respect to the induced topology on the sphere Σ from V^0. This follows from Lemma 4.3.

Consider also the set

$$(6.10)\qquad B_+ = \Sigma \setminus A_- \equiv \Big\{v \in \Sigma : \ \exists t_0 > 0 \int_0^{t_0} \Phi(S(\tau; v))\,d\tau > 0\Big\}.$$

Clearly, the set B_+ is open with respect to the induced topology on Σ. Moreover, the boundary ∂B_+ of B_+ is as follows:

$$\partial B_+ = \Big\{ v \in \Sigma : \ \forall t > 0 \int_0^t \Phi(S(\tau; v))\, d\tau \leqslant 0, \\ \exists t_0 \in \mathbb{R}_+ \cup \{\infty\} : \ \int_0^{t_0} \Phi(S(\tau; v))\, d\tau = 0 \Big\}.$$

Also, it is an easy matter to check that $A_0 \subset \partial B_+$ and $\partial B_+ \setminus A_0 \neq \varnothing$.

6.3. **On the structure of the phase space V^0.** We introduce the following function

$$B_+ \ni v \to b(v) = \max_{t \geqslant 0} \int_0^t \Phi(S(\tau; v))\, d\tau, \tag{6.11}$$

which is defined on the subset B_+ of the sphere Σ. Clearly, $b(v) > 0$ and $b(v) \to 0$ as $v \to \partial B_+$.

Next we define the mapping $\Gamma(v)$, which plays an important role in describing the phase flow generated by the boundary-value problem (2.17), (2.10):

$$B_+ \ni v \to \Gamma(v) = \frac{v}{b(v)} \in V^0; \tag{6.12}$$

here $b(v)$ is the function (6.11). Note that $\|\Gamma(v)\|_0 \to \infty$ as $v \to \partial B_+$.

It is easily seen that the image

$$\Gamma(B_+) = \{ y \in V^0 : y = \Gamma(v) \quad \text{for some} \quad v \in B_+ \} \tag{6.13}$$

under the mapping Γ splits the space V^0 into two parts:

$$V^0_- = \{ v \in V^0 : \ [0, v] \cap \Gamma(B_+) = \varnothing \}, \quad V^0_+ = \{ v \in V^0 : \ [0, v) \cap \Gamma(B_+) \neq \varnothing \}, \tag{6.14}$$

here $[0, v]$ is the closed interval in V^0 connecting a point $v \in V^0$ and the origin of the space V^0, and $[0, v)$ is the corresponding half-interval. One clearly has

$$V^0 = V^0_- \cup V^0_+ \cup \Gamma(B_+). \tag{6.15}$$

We partition the set B_+ into the following two subsets:

$$\begin{aligned} B_{+,f} &= \{ v \in B_+ : \quad \text{the maximum in (6.11) is attained for } t_0 < \infty \}, \\ B_{+,\infty} &= \{ v \in B_+ : \quad \text{the maximum in (6.11) is attained for } t_0 = \infty \}. \end{aligned} \tag{6.16}$$

It is readily seen that

$$\Gamma(B_+) = \Gamma(B_{+,f}) \cup \Gamma(B_{+,\infty}). \tag{6.17}$$

Theorem 6.1. *The following relations hold:*

$$M_- = V^0_-, \quad M_+ = V^0_+ \cup \Gamma(B_{+,f}), \quad M_g = \Gamma(B_{+,\infty}); \tag{6.18}$$

here V^0_-, V^0_+, $B_{+,f}$, $B_{+,\infty}$ are the sets defined in (6.14), (6.16), *Γ is the mapping* (6.12), *M_- is the set of stability, M_+ is the set of explosions, and M_g is the set of increase* (*see Definitions* 6.1, 6.2, 6.3).

Proof. Consider, first, the positivity cone of the functional Φ,

$$P\Phi = \{ v \in V^0 : \ \exists v_0 \in B_+, \lambda > 0, \text{ such that } v = \lambda v_0 \}, \tag{6.19}$$

where B_+ is defined in (6.10), and second, the negativity cone of the functional Φ,

$$N\Phi = \{ v \in V^0 : \ \exists v_0 \in A_-, \lambda > 0, \text{ such that } v = \lambda v_0 \}, \tag{6.20}$$

the set A_- is defined in (6.9). Clearly,

$$V^0 = P\Phi \cup N\Phi \cup \{0\}.$$

Let $\rho > 0$. Similarly to $\Gamma(v)$, we define the mapping $\Gamma_\rho(v)$:

$$B_+ \ni v \to \Gamma_\rho(v) = \frac{\rho v}{b(v)} \in V^0; \tag{6.21}$$

the function $b(v)$ is defined in (6.11). It is readily checked that

$$V_-^0 = N\Phi \cup \{\cup_{\rho\subset(0,1)}\Gamma_\rho(B_+)\} \cup \{0\}. \tag{6.22}$$

Using Lemma 6.1,

$$N\Phi \subset M_-(1). \tag{6.23}$$

Now assume that $y_0 \in \Gamma_\rho(B_+)$, where $\rho = (\alpha-1)/\alpha$ with some $\alpha > 1$. Then $y_0 = \Gamma_\rho(v)$ for some $v \in B_+$, and hence, in view of (6.21) and (6.11),

$$\sup_{t>0}\int_0^t \Phi(S(\tau; y_0))\,d\tau = \rho\,\frac{\sup_{t>0}\int_0^t \Phi(S(\tau;v))\,d\tau}{\sup_{t>0}\int_0^t \Phi(S(\tau;v))\,d\tau} = \frac{\alpha-1}{\alpha},$$

whence $\Gamma_\rho(B_+) \subset M_-(\alpha)$ for $\rho = (\alpha-1)/\alpha$ by Lemma 6.1. Hence, by (6.22), (6.23) and (6.3),

$$V_-^0 \subset M_-. \tag{6.24}$$

Let $v \in B_{+,f}$ and let $0 < t_0 < \infty$ be the least time at which the maximum in (6.11) is attained for this v. We set $y_0 = \frac{v}{b(v)} \in \Gamma(B_{+,f})$. Since the operator $S(t;v)$ is linear in v and since the functional $\Phi(v)$ is homogenous in v, it follows in view of (3.2) that

$$y(t,\,\cdot\,;y_0) = \frac{S(t;v/b(v))}{1-\int_0^t \Phi(S(\tau;v/b(v)))\,d\tau} = \frac{S(t;v)}{b(v)-\int_0^t \Phi(S(\tau;v))\,d\tau} \tag{6.25}$$

for any $t < t_0$.

Using (6.25), (6.11) it is seen that $\|y(t,\cdot;y_0)\|_0 \to \infty$ as $t \nearrow t_0$, and hence

$$\Gamma(B_{+,f}) \subset M_+\,. \tag{6.26}$$

Now let us consider the case $y_0 \in V_+^0$. From the definition of the set V_+^0 it follows that there exists $\rho > 1$ such that $y_0/\rho \in \Gamma(B_+)$ for some $v \in B_+$. But this means that $y_0 = \rho v/b(v)$. Hence, similarly to (6.25),

$$y(t,\,\cdot\,;y_0) = \frac{S(t;\rho v/b(v))}{1-\int_0^t \Phi(S(\tau;\rho v/b(v)))\,d\tau} = \frac{\rho S(t;v)}{b(v)-\rho\int_0^t \Phi(S(\tau;v))\,d\tau}. \tag{6.27}$$

We have $\rho > 1$, and hence there exists $0 < t_0 < \infty$ such that for $t < t_0$ the denominator on the right of the second fraction in (6.27) is positive and tends to zero as $t \to t_0$. As a result, $y_0 \in M_+$, and therefore,

$$V_+^0 \subset M_+. \tag{6.28}$$

Finally, assume that $y_0 = \frac{v}{b(v)} \in \Gamma(B_{+,\infty})$. Then the denominator of the second fraction in (6.25) is positive and tends to zero as $t \to \infty$. In view of (5.8), we have for sufficiently large t,

$$b(v) - \int_0^t \Phi(S(\tau;v))\,d\tau = \int_t^\infty \Phi(S(\tau;v))\,d\tau = e^{-\alpha t}\left(\frac{a_0}{b_0\alpha} + \sum_{m=1}^\infty \frac{c_m}{m+\alpha}e^{-mt}\right),$$

where $\alpha \geqslant 2$, $\frac{a_0}{b_0\alpha} > 0$, the series converging. Hence, using (6.25),

$$\|y(t,\cdot;y_0)\|_0 \asymp O(e^{-t}/e^{-\alpha t}) \to \infty \quad \text{as} \quad t \to \infty;$$

that is, $y_0 \in M_g$, and so

$$\Gamma(B_{+,\infty}) \subset M_g \tag{6.29}$$

Now (6.18) follows from the inclusions (6.24), (6.26), (6.28), (6.29), equalities (6.15), (6.17) and the clear relations $M_- \cap M_+ = M_- \cap M_g = M_+ \cap M_g = \varnothing$. □

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