Dedicated to Vsevolod Alekseevich Solonnikov

LOCAL EXISTENCE THEOREMS WITH UNBOUNDED SET OF INPUT DATA AND UNBOUNDEDNESS OF STABLE INVARIANT MANIFOLDS FOR 3D NAVIER-STOKES EQUATIONS

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Abstract. Local existence theorem of smooth solution $v(t,\cdot), t \in \mathbb{R}_+$ for 3D Navier-Stokes equations is proved, when initial data belongs to a certain unbounded ellipsoid of suitable function space. Unboundedness of stable invariant manifolds for 3D Navier-Stokes equations is proved as well.

Introduction

One of the main aim of this paper is to establish unboundedness of stable invariant manifold for dissipative evolution PDE. This property can help to weaken assumption of closeness to a steady-state mode $\widehat{v}(x)$ for a solution $v(t,x)$ when $v(t,x)$ is stabilized near $\widehat{v}$. Unboundedness of stable invariant manifold M_- was first observed by analyzing recurrence relations obtained in [F1], [F2] for coefficients of decomposition in a series of a map determining manifold M_-.

Subsequent investigations found out that real reasons of this phenomenon are not connected with analyticity property. Moreover, it turned out that smooth solutions of these equations determined for times $t \in \mathbb{R}_+$ as usual exist not only for initial data belonging to a ball of sufficiently small radius, but also for initial conditions belonging to a certain unbounded set of the corresponding function space. This existence theorem is true for a wide class of dissipative evolution PDE.

In this paper aforementioned fact is established for 3D evolution Navier-Stokes equations defined in a bounded domain Ω with a smooth boundary. In section 1 it is proved that in the space V^1 of initial data, that consists of divergence free vector fields square integrable together with their first derivatives, there exists such unbounded ellipsoid $El_\rho^{1/2}$ that for each initial datum $v_0 \in El_\rho^{1/2}$ smooth solution $v(t,x), t \in \mathbb{R}_+, x \in \Omega$ of Navier-Stokes system exists, and

$$\|v(t,\cdot)\|_{V^1} \le c\|v_0\|_{V^1} e^{-\sigma t}, \qquad t \in \mathbb{R}_+$$

where constants $\sigma > 0, c > 0$ do not depend on $v_0 \in El_\rho^{1/2}$.

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After some preliminaries given in section 2, unboundedness of stable invariant manifolds for Navier-stokes equations has been proved in Section 3, and these manifolds are defined with help of ellipsoid $El_\rho^{1/2}$. In sections 4, 5 some auxiliary assertions are proved, that are used in the main part of the paper.

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1. Unique Solvability of 3D Navier-Stokes System with Initial Data from an Unbounded Ellipsoid

In these section we prove existence and uniqueness of a smooth solution for the 3D Navier-Stokes boundary value problem with zero right side and initial datum belonging to certain unbounded ellipsoid. In addition we prove that this solution decays exponentially with increasing time.

1.1. **Setting of the problem.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary $\partial\Omega, Q = \mathbb{R}_+ \times \Omega, S = \mathbb{R}_+ \times \partial\Omega$. We consider in Q the following boundary value problem for 3D Navier-Stokes equations:

$$\partial_t v(t,x) - \Delta v(t,x) + (v, \nabla v) + \nabla p(t,x) = 0, \quad (t,x) \in Q, \tag{1.1}$$

$$\operatorname{div} v(t,x) = 0, \quad (t,x) \in Q \tag{1.2}$$

$$v(t,x) = 0, \quad (t,x) \in S \tag{1.3}$$

$$v(t,x)|_{t=0} = v^0(x), \quad x \in \Omega \tag{1.4}$$

Here $\partial_t v = \partial v/\partial t$, $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ is unknown velocity vector field, $\nabla p(t,x)$ is a pressure gradient , $(v, \nabla v) = \sum_{j=1}^3 v_j \partial v/\partial x_j, v^0(x)$ is a given initial datum for v.

Recall definition of function spaces where problem (1.1)-(1.4) is considered. We set

$$V^0(\Omega) = \{v(x) \in (L_2(\Omega))^3 : \operatorname{div} v = 0,\ v \cdot \nu|_{\partial\Omega} = 0\}; \quad \|v\|_{V^0(\Omega)} = \|v\|_{L_2(\Omega)} \tag{1.5}$$

where $\nu = \nu(x), x \in \partial\Omega$ is the vector field of outer normals to $\partial\Omega$; relations $\operatorname{div} v = 0,\ v \cdot \nu|_{\partial\Omega} = 0$ are understood in the sense of distributions theory (see [T] for details). Introduce also the spaces

$$V^1(\Omega) = \{v(x) \in (H^1(\Omega))^3 : \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0\}; \quad \|v\|_{V^1(\Omega)} = \|\nabla v\|_{L_2(\Omega)} \tag{1.6}$$

$$V^2(\Omega) = V^1(\Omega) \cap (H^2(\Omega))^3; \quad \|v\|_{V^2(\Omega)} = \|\Delta v\|_{L_2(\Omega)} \tag{1.7}$$

where $H^k(G)$ is the Sobolev space of functions belonging to $L_2(G)$ together with all their derivatives up to the order k, the norm in $H^k(\Omega)$ is defined as follows:

$$\|\varphi\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha \varphi(x)|^2 dx \right)^{1/2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_j are nonnegative integer, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_3^{\alpha_3}$. As well-known, $\|\cdot\|_{V^k(\Omega)}$ is equivalent on $V^k(\Omega)$ to $\|\cdot\|_{H^k(\Omega)}$ for $k = 0, 1, 2$.

Let $\{e_j(x), \lambda_j, j = 1, 2, \dots\}$ be eigenfunctions and eigenvalues of the following spectral problem for the Stokes operator:

$$-\Delta e(x) + \nabla p(x) = \lambda e(x), \quad \operatorname{div} e = 0, \quad x \in \Omega; \quad e|_{\partial\Omega} = 0 \tag{1.8}$$

As well-known, $0 < \lambda_1 \leq \lambda_2 \leq \ldots, \lambda_j \to \infty$ as $j \to \infty$, and $\{e_j(x)\}$ forms orthonormal basis in $V^0(\Omega)$:

$$\forall v \in V \quad v(x) = \sum_{j=1}^{\infty} v_j e_j(x), \quad \text{where } v_j = (v, e_j)_{V^0(\Omega)}, \quad \|v\|^2_{V^0(\Omega)} = \sum_{j=1}^{\infty} |v_j|^2$$

For each $s \in \mathbb{R}$ we introduce the space V^s by the formula

$$V^s = \left\{ v(x) = \sum_{j=1}^{\infty} v_j e_j(x),\ v_j \in \mathbb{R} : \|v\|_s^2 = \sum_{j=1}^{\infty} \lambda_j^s |v_j|^2 < \infty \right\} \tag{1.9}$$

It is well-known (see, for instance [F3], Ch.3,Sect.4) that

$$V^s = V^s(\Omega), \quad \text{and} \quad \|v\|_s = \|v\|_{V^s(\Omega)} \quad \text{for } s = 0, 1, 2 \tag{1.10}$$

where V^s is defined in (1.9), and $V^0(\Omega), V^1(\Omega), V^2(\Omega)$ are spaces (1.5), (1.6), (1.7).

We will look for solution $v(t,x)$ of problem (1.1)-(1.4) in certain spaces from the following family:

$$V^{1,2(s)} = \{v(t,\cdot) \in L_2(\mathbb{R}_+; V^{s+2}) :\ \partial_t v(t,\cdot) \in L_2(\mathbb{R}_+; V^s)\} \tag{1.11}$$

Note that the following inequality holds (see [F3]):

$$\|v\|_{L_\infty(\mathbb{R}_+; V^{1+s})} \leq 2\|v\|_{V^{1,2(s)}} \tag{1.12}$$

Assuming that initial condition (1.4) satisfies $v^0 \in V^1(\Omega)$, we look for component $v(t,x)$ of solution $(v, \nabla p)$ for (1.1)-(1.4) in he space $V^{1,2}(Q) = V^{1,2(0)}$. To get rid of component ∇p of solution $(v, \nabla p)$ we introduce the orthoprojector

$$\pi : (L_2(\Omega))^3 \longrightarrow V^0(\Omega) \tag{1.13}$$

Applying operator π to both parts of equation (1.1) one can reduce problem (1.1)-(1.4) to the following one:

$$\partial_t v(t,\cdot) + Av + B(v,v) = 0, \quad v|_{t=0} = v^0 \in V^1(\Omega) \tag{1.14}$$

where

$$A = -\pi\Delta, \quad B(v,w) = \pi\left(\sum_{j=1}^{3} v_j \frac{\partial w}{\partial x_j}\right) \tag{1.15}$$

Details of this reduction see, for instance, in [F3], Ch. 3, Sect, 4. Just there one can also find the proof of the following assertion:

Lemma 1.1. *Operator $B(v,w)$ defined in (1.15) is continuous in the following spaces:*

$$B : V^{s_1} \times V^{s_2+1} \longrightarrow V^{-s_3}, \quad B : V^{s_1} \times V^{s_3} \longrightarrow V^{-s_2-1} \tag{1.16}$$

where $s_j \geq 0, j = 1,2,3$ and

$$\text{either} \quad \sum_{j=1}^{3} s_j > \frac{3}{2}, \quad \text{or} \quad \sum_{j=1}^{3} s_j \geq \frac{3}{2}, \text{ and} \quad s_i + s_j > 0 \quad \forall i \neq j \tag{1.17}$$

It is well-known (see, for instance, [VF]) that if $v^0 \in B_\rho(V^1) = \{w \in V^1 : \|w\|_{V^1} < \rho\}$, and ρ is small enough then there exists unique solution $v(t,\cdot) \in V^{1,2(0)}$ of problem (1.14). Our goal is to show that similar result is true even if v^0 belongs to a certain unbounded set $El_\gamma(1/2) \subset V^1$.

1.2. **Formulation of the result.** Let us consider the set

$$El^{\alpha}_{\rho}=\{v=\sum_{j=1}^{\infty}v_je_j(x)\in V^1:\sum_{j=1}^{\infty}\lambda_j^{1-\alpha}v_j^2<\rho\} \tag{1.18}$$

where $\rho>0,\alpha\in[0,1]$. This set is unbounded ellipsoid in V^1. Indeed, by definition (1.9) of V^s, $\|v_je_j\|^2_{V^1}=\lambda_jv_j^2$, anf therefore (1.18) can be rewritten as follows

$$El^{\alpha}_{\rho}=\{v\in V^1:\sum_{j=1}^{\infty}\|v_je_j\|^2_{V^1}/(\lambda_j^{\alpha}\rho)<1\} \tag{1.19}$$

This formula means that El^{α}_{ρ} is ellipsoid in V^1 with axises of length $\sqrt{\lambda_j^{\alpha}\rho}$ directed along e_j. Since $\sqrt{\lambda^{\alpha}\rho}\to\infty$ as $j\to\infty$, El^{α}_{ρ} is an unbounded set.

The following theorem holds:

Theorem 1.1. *If ρ is sufficiently small, then for each $v^0\in El^{1/2}_{\rho}$ there exists unique solution $v(t,\cdot)\in V^{1,2(0)}$ of problem (1.14). Moreover,*

$$\|v(t,\cdot)\|_{V^1}\le c\|v^0\|_{V^1}e^{-\sigma t}\quad as\quad t\to\infty \tag{1.20}$$

with constants $\sigma\in(0,\lambda_1),c>0$ independent of time $t>0$ and datum $v^0\in El^{1/2}_{\rho}$. Here $\lambda_1>0$ is the first eigenvalue in spectral problem (1.8).

The rest of this section is devoted to the proof of this theorem.

1.3. **Transformation of equation and choice of the functions space.** In order to prove estimate (1.20) we make the following change in (1.14):

$$y(t,x)=e^{\sigma t}v(t,x)\quad\text{with}\quad\sigma\in(0,\lambda_1). \tag{1.21}$$

Note that in virtue of definition (1.15) of operator A

$$Ae_j=\lambda_je_j\quad\forall j\in\mathbb{N} \tag{1.22}$$

where $\{e_j(x),\lambda_j\}$ are eigenfunctions and eigenvalues of problem (1.8). Substitution of (1.21) into (1.14) yields

$$\partial_ty(t,\cdot)+A_1y(t,\cdot)+e^{-\sigma t}B(y(t,\cdot),y(t,\cdot))=0,\quad y|_{t=0}=v^0, \tag{1.23}$$

where $A_1=A-\sigma E$, and E is identity operator. In virtue of (1.22)

$$A_1e_j=(\lambda_j-\sigma)e_j\quad\forall j\in\mathbb{N}, \tag{1.24}$$

and since $\sigma\in(0,\lambda_1),\lambda_j-\sigma>0$. Thus A_1 is a positively defined operator as well as A.

Local existence theorems are proved usually with help of the Inverse Map Theorem (which is a corollary of Implicit Function Theorem). In order to apply the Inverse Map Theorem to solve problem (1.23), we have to choose function spaces for (1.23) by such a way that nonlinear operator $\widehat{B}(y)\equiv e^{-\sigma t}B(y,y)$ becomes subordinated to the linear part $(\partial_ty+A_1y,\gamma_0y)$ of (1.23) where $\gamma_0y=y|_{t=0}$.

As well-known, operator

$$(\partial_t+A_1,\gamma_0):V^{1,2(s)}\to L_2(\mathbb{R}_+;V^s)\times V^{1+s} \tag{1.25}$$

realizes isomorphism of the spaces for each $s\in\mathbb{R}$ (see [S], [VF], [F3]). From the other side, operator

$$\widehat{B}(\cdot):V^{1,2(s)}\to L_2(\mathbb{R}_+;V^s)\times V^{1+s}$$

is continuous for each $s \geq -1/2$ (see [F4], [F3]). The last fact follows easily from Lemma 1.1. So to apply Inverse Map Theorem for solution of problem (1.23) we should consider operator generated by (1.23) in the following spaces:

$$(\partial_t + A_1 + \widehat{B}, \gamma_0) : V^{1,2(s)} \to L_2(\mathbb{R}_+; V^s) \times V^{1+s}, \quad s \geq -1/2 \tag{1.26}$$

1.4. **Application of Inverse Map Theorem.** Here and everywhere below for Banach space X and quantity $\rho > 0$ we denote by $\Theta_\rho(X) = \{x \in X : \|x\|_X < \rho\}$ the ball in X of radius ρ with center in origin.

Let Y, Z be Banach spaces, $\Psi : Y \to Z$ be a map of class $\mathbb{C}^2$. To solve equation

$$\Psi(y) = z \tag{1.27}$$

one can use the Inverse Map Theorem (see, for instance, [ATF]):

Theorem 1.2. *If $\Psi(0) = 0$, and derivative $\Psi'(0) : Y \to Z$ realizes isomorphism, then for sufficiently small $\rho > 0$ there exists a $\mathbb{C}^2$-map $\Phi : \Theta_\rho(Z) \to Y$ that is inverse to Ψ i.e. $\Psi(y) = z \Leftrightarrow \Phi(z) = y$ for each $z \in \Theta_\rho(Z)$.*

Note that Theorem 1.2 proof contains the proof of the following assertion:

$$\exists \delta(\rho) \to 0 \quad \text{as} \quad \rho \to 0 \quad \text{such that} \quad \Phi(\Theta_\rho(Z)) \subset \Theta_{\delta(\rho)}(Y) \tag{1.28}$$

To prove existence of solution for (1.23) we apply Theorem 1.2 as follows: We take $Y = V^{1,2(-1/2)}, Z = L_2(\mathbb{R}_+; V^{-1/2}) \times V^{1/2}, \Psi(y) = (\partial_t y + A_1 y + \widehat{B}(y), \gamma_0 y)$. As was shown in the very end of previous subsection this map satisfies all conditions of aforementioned theorem and therefore for each $v^0 \in \Theta_\rho(V^{1/2})$ there exists unique solution $y \in V^{1,2(-1/2)}$ of problem (1.23). Moreover, by (1.28)

$$\|y\|_{V^{1,2(-1/2)}} \leq \delta(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \tag{1.29}$$

Note that by definition (1.9) of V^s ellipsoid (1.19) with $\alpha = 1/2$ can be rewritten as follows:

$$El_\rho^{1/2} = \{v^0 \in V^1 : \|v\|_{1/2} < \rho\} = \Theta_\rho(V^{1/2}) \cap V^1 \tag{1.30}$$

and therefore we have proved that for each $v^0 \in El_\rho^{1/2}$ there exists unique solution $y \in V^{1,2(1/2)}$ of problem (1.23). To finish the proof of Theorem 1.1 we have to show that $y \in V^{1,2(0)}$ and v linked with y by relation (1.21) satisfies inequality (1.20).

1.5. **The main estimate.** First of all we note that Lemma 1.1 implies the inequality

$$\|B(y,z)\|_0 \leq c\|y\|_1 \|z\|_{3/2} \tag{1.31}$$

Bound (1.31), (1.12) imply estimate

$$\|B(y,y)\|_{L_2(\mathbb{R}_+;V^0)} \leq c\|y\|_{L_\infty(\mathbb{R}_+;V^1)} \|y\|_{L_2(\mathbb{R}_+;V^{3/2})}) \leq c\|y\|_{V^{1,2(0)})} \|y\|_{V^{1,2(-1/2)})}. \tag{1.32}$$

Inverting operator (1.25) in problem (1.23) we get that solution y of (1.23) satisfies the equality

$$y(t,\cdot) = e^{-A_1 t} v^0 - \int_0^t e^{-A_1(t-\tau)} e^{-\sigma\tau} B(y(\tau,\cdot), y(\tau,\cdot))\, d\tau \tag{1.33}$$

Using boundedness of operator (1.25) and inequalities (1.32),(1.29) we estimate y by right side of (1.33):

$$\|y\|^2_{V^{1,2(0)}} \leq c_1(\|v^0\|_{V^1} + \|B(y,y)\|_{L_2(\mathbb{R}_+;V^0)}) \leq c_1(\|v^0\|_{V^1} + c\delta(\rho)\|y\|_{V^{1,2(0)}}). \tag{1.34}$$

At last assuming that ρ is so small that $1-c_1c\delta(\rho)>0$ we get from (1.34) the final estimate for each $t>0$:

$$\|y(t,\cdot)\|_{V^1}\le c_2\|y\|_{V^{1,2(0)}}\le \frac{c_1c_2}{1-cc_1\delta(\rho)}\|v^0\|_{V^1} \tag{1.35}$$

After substitution into (1.35) expression (1.21) of y by v we obtain desired estimate (1.20).

2. Stable Invariant Manifold

In this section we recall certain notion connected with stable invariant manifolds for Navier-Stokes equations.

2.1. **Input system.** Instead of (1.1)-(1.4) we consider the problem for 3D Navier-Stokes system with right side $f(x)\in V^0(\Omega)$:

$$\partial_t v(t,x)-\Delta v(t,x)+(v,\nabla v)+\nabla p(t,x)=f(x),\quad \operatorname{div} v(t,x)=0, \tag{2.1}$$

$$v(t,x)|_{x\in\partial\Omega}=0,\quad v(t,x)|_{t=0}=v^0(x) \tag{2.2}$$

Similarly to reduction of problem (1.1)-(1.4) to (1.14) we reduce problem (2.1) to the following one applying to both sides of the first equation in (2.1) operator π from (1.13):

$$\partial_t v(t,\cdot)+Av+B(v,v)=f(\cdot),\quad v(t,\cdot)|_{t=0}=v^0 \tag{2.3}$$

where A,B are operators (1.15). To solve this problem we introduce the space similar to (1.11) but composed from vector fields defined on cylinder $Q_T=(0,T)\times\Omega$ bounded in time:

$$V^{1,2(s)}(Q_T)=\{v(t,\cdot)\in L_2(0,T;V^{s+2}):\ \partial_t v(t,\cdot)\in L_2(0,T;V^s\} \tag{2.4}$$

Since $f\in V^0(\Omega), v^0\in V^1(\Omega)$, natural space for solutions of problem (2.1), (2.2) is $V^{1,2(0)}(Q_T)$ and natural phase space for corresponding dynamical system is $V^1(\Omega)$.

Let steady-state solution $\widehat{v}(x)\in V^2(\Omega)$ of (2.1), i.e. solution of problem

$$A\widehat{v}+B(\widehat{v},\widehat{v})=f \tag{2.5}$$

be given.

To study the structure of the dynamical system (2.3) in a neighborhood of $\widehat{v}(x)$ we make the change of unknown functions in (2.3):

$$v(t,x)=\widehat{v}(x)+y(t,x) \tag{2.6}$$

After substitution (2.6) into (2.3) and taking into account (2.5) we get:

$$\partial_t y(t,x)+\widehat{A}y(t,x)+B(y(t,x),y(t,x))=0, \tag{2.7}$$

$$y(t,x)|_{t=0}=y_0(x)=v^0(x)-\widehat{v}(x), \tag{2.8}$$

where

$$\widehat{A}y=Ay+B(\widehat{v},y)+B(y,\widehat{v}). \tag{2.9}$$

Let us consider operator $\widehat{A}:V^0(\Omega)\to V^0(\Omega)$ and its adjoint operator $\widehat{A}^*$. These operators are closed and and their domains of definition are $D(\widehat{A})=D(\widehat{A}^*)=V^2$ (see (1.7)). The spectrum $\Sigma(\widehat{A}),\Sigma(\widehat{A}^*)$ of operators $\widehat{A},\widehat{A}^*$ are discrete subsets of $\mathbb{C}$ consisting of eigenvalues only, that belong to a sector symmetric with respect to $\mathbb{R}$ and containing $\mathbb{R}_+$. Moreover $\Sigma(\widehat{A})=\Sigma(\widehat{A}^*)$ (see [F5]).

The linearization of problem (2.7), (2.8) at zero has the form

$$\partial_t y(t,x)+\widehat{A}y(t,x)=0,\quad y(t,x)|_{t=0}=y_0(x) \tag{2.10}$$

For each $y_0 \in V^0$ the solution $y(t,\cdot)$ of (2.10) is defined by equality $y(t,\cdot) = e^{-\widehat{A}t}y_0$ where $e^{-\widehat{A}t}$ is the resolving semigroup of problem (2.10).

Let $\sigma > 0$ satisfy:

$$\Sigma(\widehat{A}) \cap \{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda \le \sigma\} = \emptyset \tag{2.11}$$

The case when there are certain points of $\Sigma(\widehat{A})$ which are located in the left side of the set $\{0 < \mathrm{Re}\lambda \le \sigma\}$ will be interesting for us because the contrary case is similar to the situation considered in previous section.

Denote by $V_+^0(\widehat{A})$ the subspace of $V^0(\Omega)$ generated by all eigenfunctions and associated functions of operator $\widehat{A}$ corresponding to all eigenvalues of $\widehat{A}$ placed in the set $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \le 0\}$. By $V_+^0(\widehat{A}^*)$ we denote analogous subspace corresponding to operator $\widehat{A}^*$. We denote the orthogonal complement to $V_+^0(\widehat{A}^*)$ in $V^0(\Omega)$ by $V_-^0(\widehat{A}) \equiv V_-^0$:

$$V_-^0 = V^0(\Omega) \ominus V_+^0(\widehat{A}^*) \tag{2.12}$$

For each $s \in \mathbb{R}$ we set

$$V_-^s = V_-^0 \cap V^s \text{ if } s \ge 0; \qquad V_-^s = \text{ closure of } V_-^0 \text{ in } V^s(\Omega) \text{ for } s < 0 \tag{2.13}$$

Let $V_+^s = V_+^0(\widehat{A}),\ s \in \mathbb{R}$, i.e. finite-dimensional space V_+^s as set does not depend on s. It is convenient for us to set on each V_+^s a norm $\|\cdot\|_+$ common for every $s \in \mathbb{R}$. We can do it because on finite-dimensional space all norms are equivalent. (Norm $\|\cdot\|_+$ will be chosen by more concrete way in Appendix II (see (5.11)). Thus

$$V_+^s = V_+^0(\widehat{A}), \quad s \in \mathbb{R} \quad \text{supplied with} \quad \|\cdot\|_+ \tag{2.14}$$

One can show (see [F5]) that subspaces V_+^s, V_-^s are invariant with respect to the action of semigroup $e^{-\widehat{A}t}$ and

$$V^s(\Omega) = V_-^s + V_+^s \quad \forall\, s \in \mathbb{R} \tag{2.15}$$

We change definition (1.9) assuming now that norm in $V^s,\ s \in \mathbb{R}$ is defined as follows:

$$\|v\|_{V^s} = \|v_-\|_{V_-^s} + \|v\|_+, \quad \text{where} \quad v = v_- + v_+, \quad v_- \in V_-^s,\ v_+ \in V_+^s \tag{2.16}$$

and $\|\cdot\|_{V_-^s} = \|\cdot\|_s$ where $\|\cdot\|_s$ is defined in (1.9).

2.2. **Definition of stable invariant manifold.** As we note in previous subsection, natural space for solution of problem (2.7), (2.8) is $V^{1,2(0)}(Q_T)$ (see (2.4)), and in virtue of (1.12) natural phase space for corresponding dynamical system is

$$V = V^1(\Omega) \tag{2.17}$$

It is well-known (see [LS], [BV]), that for each $y_0 \in V$ there exists a unique solution $y(t,x) \in V^{1,2(0)}(Q_{T_{\|v_0\|}})$ of problem (2.7),(2.8), where $0 < T_{\|v_0\|} \to \infty$ as $\|v_0\| \equiv \|v_0\|_V \to 0$. We denote by $S(t,y_0)$ the solution operator of the boundary value problem (2.7),(2.8):

$$S(t,y_0) = y(t,\cdot) \tag{2.18}$$

where $y(t,x)$ is the solution of (2.7),(2.8).

Recall now some commonly used definitions of stable invariant manifold (see Chapter V in [BV]) adopted for our case.

The origin of the phase space V, i.e. the function $y(x) \equiv 0$, is, evidently, a steady-state solution of problem (2.7),(2.8).

Definition 2.1. *The set $M_- \subset V$ defined in a neighborhood of the origin is called the stable invariant manifold if for each $y_0 \in M_-$ the solution $S(t, y_0)$ is well-defined and belongs to M_- for each $t > 0$, and*

$$\|S(t, y_0)\|_V \leq c\|y_0\|_V e^{-\sigma t} \quad \textit{as} \quad t \to \infty \tag{2.19}$$

where quantities $c > 0, \sigma > 0$ does not depend on $y_0 \in M_-$.

In accordance with (2.15), (2.16), (2.17) the following decomposition is true:

$$V = V_- + V_+ \qquad \text{where} \qquad V_- = V_-^1, \quad V_+ = V_+^1 \tag{2.20}$$

The stable invariant manifold can be defined as a graph in the phase space $V = V_+ + V_-$ by the formula

$$M_- = \{y \in V : y = y_- + g(y_-),\ y_- \in \mathcal{O}(V_-)\} \tag{2.21}$$

where $\mathcal{O}(V_-)$ is a neighborhood of the origin in the subspace V_-, and

$$g : \mathcal{O}(V_-) \to V_+ \tag{2.22}$$

is a certain map satisfying

$$\|g(y_-)\|_+/\|y_-\|_{V_-} \to 0 \quad \text{as} \quad \|y_-\|_{V_-} \to 0. \tag{2.23}$$

2.3. **Existence of invariant manifold M_-.** Let us define the metric space where we look for the map g that defines stable invariant manifold M_- by formula (2.21). We assume that $g(y_-)$ is defined on the ball

$$\Theta_\rho(V_-) = \{y_- \in V_- : \|y_-\|_{V_-} \leq \rho\} \tag{2.24}$$

Definition 2.2. *The metric space $G_{\mu,\rho} \equiv G_\mu(\Theta_\rho(V_-))$ is the space of maps $g : \Theta_\rho(V_-) \to V_+$ that are Frechet differentiable for each $y_- \in \Theta_\rho(V_-)$ and their derivatives $g'(y_-)$ satisfy Lipschitz condition*

$$\|g'(y_1) - g'(y_2)\| \leq \mu\|y_1 - y_2\|_{V_-} \quad \forall y_1, y_2 \in \Theta_\rho(V_-). \tag{2.25}$$

where Lipschitz constant μ is unique for all $g \in G_{\rho,\mu}$. Moreover, $g \in G_{\rho,\mu}$ satisfy conditions

$$g(0) = 0,\ g'(0) = 0. \tag{2.26}$$

The metric $d(g_1, g_2)$ in $G_{\mu,\rho}$ is defined as follows:

$$d(g_1, g_2) = \sup_{y \in \Theta_\rho(V_-)\setminus\{0\}} \frac{\|g_1(y) - g_2(y)\|_+}{\|y\|_{V_-}} \tag{2.27}$$

Theorem 2.1. *The metric space $G_{\mu,\rho}$ is complete with respect to metric (2.27).*

This theorem has been proved in Appendix II below.

Now we are in position to formulate existence theorem for invariant manifold M_-:

Theorem 2.2. *There exists unique map $g \in G_{\rho,\mu}$ where $\mu > 0$ is sufficiently large and $\rho > 0$ is sufficiently small* [1] *such that the set M_- defined by formula (2.21) is stable invariant manifold for family of maps $S(t, \cdot)$ defined in (2.18). Moreover,*

$$\|S(t, y_0)\|_V \leq ce^{-\sigma t}\|y_0\|_V \qquad \textit{as} \quad t \to \infty \tag{2.28}$$

where constants $c > 0, \sigma > 0$ do not depend on $y_0 \in M_-$

---

[1] Precise conditions imposed on μ and ρ are written below, in Theorem 5.2 formulation (see (5.19))

This theorem as well as method of its proof is well-known (see [LS], [MM], [Hen], [BV] and references there in). It is proved below in Appendix II (see Theorem 5.2 and Remark 5.1) in a form more convenient for our aims than known for us proofs in literature.

3. Unboundedness of stable invariant manifold M_-

We extend domain $\Theta_\rho(V_-)$ of functions belonging to metric space $G_{\mu,\rho}$ up to the following set:

$$El_\rho^{1/2}(V_-) = El_\rho^{1/2} \cap V_- \tag{3.1}$$

where $El_\rho^{1/2}$ is ellipsoid (1.18) with $\alpha = 1/2$ and $V_- = V_-^1$ is the space (2.13) with $s = 1$. Now we determine the space $G_\mu(El_\rho^{1/2}(V_-))$ of functions defined on $El_\rho^{1/2}(V_-)$ similarly to $G_{\mu,\rho} \equiv G_\mu(\Theta_\rho(V_-))$: new metric space differs from $G_{\mu,\rho}$ only with domain of definition of composing functions. Theorem 2.2 can be strengthen by the following way.

Theorem 3.1. *There exists unique map $g \in G_\mu(El_\rho^{1/2}(V_-))$ where $\mu > 0$ is sufficiently large, $\rho > 0$ is sufficiently small, and $\mu\rho \le 1$* [2] *such that the set M_- defined by formula (2.21) is stable invariant manifold for family of maps $S(t,\cdot)$ defined in (2.17). Moreover*

$$\|S(t,y_0)\|_V \le ce^{-\sigma t}\|y_0\|_V \qquad as \quad t\to\infty \tag{3.2}$$

where constants $c>0, \sigma>0$ do not depend on $y_0 \in M_-$.

Proof. Theorem 2.2 is formulated in the case when phase space of family $S(t,\cdot)$ is $V = V^1$. Opportunity to prove Theorem 2.2 with such choice of phase space is based on the fact that the the corresponding space $V^{1,2(0)}(Q_t)$ of solutions for (2.7),(2.8) satisfies the property: the nonlinear part B of equation (2.7) is continuous in the spaces where its linear part establishes isomorphism:

$$(B,0): V^{1,2(0)}(Q_T) \to L_2(0,T;V^0)\times V^1$$

$$(\partial_t + \widehat{A}, \gamma_0): V^{1,2(0)}(Q_T) \to L_2(0,T;V^0)\times V^1$$

This allows to prove local existence theorem for (2.7),(2.8) in $V^{1,2(0)}(Q_T)$ and after that to prove Theorem 2.2 (see [LS], [BV], and Appendix II below). But as was mentioned in (1.26) not only the space $V^{1,2(0)}(Q_T)$ but also spaces $V^{1,2(-s)}(Q_T)$, for each $s\in[0,1/2]$ possess this property. That is why Theorem 2.2 can be proved when phase space for $S(t,\cdot)$ is $V = V^{1/2}$ which is connected with solutions space $V^{1,2(-1/2)}(Q_T)$ (see Appendix II below).

So invariant manifold can be defined with a certain map $g \in G_\mu(\Theta_\rho(V_-^{1/2}))$. But in virtue of (1.18),(3.1)

$$El_\rho^{1/2}(V_-) = \Theta_\rho(V_-^{1/2}) \cap V^1 \tag{3.3}$$

Therefore a map $g \in G_\mu(El_\rho^{1/2}(V_-))$ that defines the stable invariant manifold M_- by formula (2.21) has been constructed. Moreover by Theorem 2.2 with phase space $V = V^{1/2}$ the inequality

$$\|S(t,y_0)\|_{V^{1/2}} \le ce^{-\sigma t}\|y_0\|_{V^{1/2}} \qquad \text{as} \quad t\to\infty \tag{3.4}$$

---

[2] Bound $\rho\mu \le 1$ follows from condition (5.19) that actually are imposed on μ,ρ in Theorems 2.2 and 3.1

for each $y_0 \in M_-$ is also established. To finish the proof of Theorem 3.1 we have to prove bound (3.2) where $V = V^1$.

Let introduce the projection operators

$$\Pi_- : V^s \to V^s_-, \quad \Pi_+ : V^s \to V^s_+, \quad s \in \mathbb{R} \tag{3.5}$$

Definition of V^s_-, V^s_+ (see (2.12)–(2.16) and few lines around these relations) implies that Π_-, Π_+ does not depend on s. [3] Let denote

$$\Pi_\pm y_0 = y_\pm,\ \Pi_\pm S(t,\cdot) = S_\pm(t,\cdot),\ \Pi_\pm \widehat{A} = \widehat{A}_\pm,\ \Pi_\pm B = B_\pm \tag{3.6}$$

Using definition (2.14)-(2.16) of V^s_+, V^s and inclusion $S_+(t,y_0) \in V^s_+, s = 1, 1/2$ for all $t > 0$ we have taking into account (3.4) that

$$\|S_+(t,y_0)\|_{V^1} = \|S_+(t,y_0)\|_{V^{1/2}} \le \|S(t,y_0)\|_{V^{1/2}} \le ce^{-\sigma t}\|y_0\|_{V^{1/2}} \tag{3.7}$$

That is why in virtue of inequality

$$\|S(t,y_0)\|_{V^1} \le \|S_+(t,y_0)\|_{V^1} + \|S_-(t,y_0)\|_{V^1}$$

in order to prove (3.2) it is enough to estimate $\|S_-(t,y_0)\|_{V^1}$.

We can rewrite (2.7) as follows:

$$\begin{aligned} &\partial_t y_-(t) + \widehat{A}_- y_-(t) + B_-(y_-(t)+y_+(t), y_-(t)+y_+(t)) = 0, \\ &\partial_t y_+(t) + \widehat{A}_+ y_+(t) + B_+(y_-(t)+y_+(t), y_-(t)+y_+(t)) = 0, \end{aligned} \tag{3.8}$$

where $y_\pm(t) = \Pi_\pm y(t) = S_\pm(t,y_0)$. Invariance of manifold (2.21) means that

$$y_+(t) = g(y_-(t)) \quad \forall\, t > 0 \quad \text{if} \quad y_+ = g(y_-) \quad \text{at} \quad t = 0 \tag{3.9}$$

Relations (3.8),(3.9) give closed equation for $y_-(t)$ that describes solution $y(t)$ belonging to M_- of problem (2.7), (2.8):

$$\partial_t y_-(t) + \widehat{A}_- y_-(t) + B_-(y_-(t)+g(y_-(t)), y_-(t)+g(y_-(t))) = 0, \tag{3.10}$$

$$y_-(t)|_{t=0} = y_- \in El^{1/2}_\rho(V_-) \tag{3.11}$$

We make change in equation (3.10) (compare with (1.21)):

$$z(t) = e^{\sigma_1 t} y_-(t) \qquad \text{with} \quad \sigma_1 \in (0,\sigma) \tag{3.12}$$

Then problem (3.10), (3.11) transforms to the following one:

$$\partial_t z(t) + \widehat{A}_1 z(t) + e^{-\sigma_1 t} B_-(w(t),w(t)) = 0, \quad w(t) = z(t) + e^{\sigma_1 t} g(e^{-\sigma_1 t} z(t)) \tag{3.13}$$

$$z(t)|_{t=0} = y_- \tag{3.14}$$

where

$$\widehat{A}_1 z = \widehat{A}_- z - \sigma_1 z \tag{3.15}$$

Let

$$V^{1,2(s)}_- = \Pi_- V^{1,2(s)} \tag{3.16}$$

where $\Pi_-,\ V^{1,2(s)}$ are defined in (3.5), (1.11) correspondingly. Recall that the space V^s_- is defined in (2.13). The following theorem holds.

[3] More precisely, Π_-, Π_+ can be expressed by explicit formulas constructed only with help of duality between V^s and V^{-s} generated by scalar product in $L_2(\Omega)$ and using fixed basis functions of spaces $V^0_+(\widehat{A})$ and $V^0_+(\widehat{A}^*)$

Theorem 3.2. *Let $s \in [-1/2, 0]$. Operator*

$$(\partial_t + \widehat{A}_1, \gamma_0) : V_-^{1,2(s)} \to L_2(\mathbb{R}_+; V_-^s) \times V_-^{1+s} \tag{3.17}$$

is reversible, and its inverse operator is defined by the formula

$$(\partial_t + \widehat{A}_1, \gamma_0)^{-1}(y_-, f) = e^{-\widehat{A}_1 t} y_- + \int_0^t e^{-\widehat{A}_1(t-\tau)} f(\tau, \cdot)\ d\tau \tag{3.18}$$

The proof of this theorem is given below, in Appendix I.

Inverting operator (3.17) in problem (3.13), (3.14)we get

$$z(t) = e^{-\widehat{A}_1 t} y_- - \int_0^t e^{-\widehat{A}_1(t-\tau)} e^{-\sigma_1 \tau} B_-(w(\tau), w(\tau))\ d\tau, \tag{3.19}$$

where

$$w(t) = z(t) + e^{\sigma_1 t} g(e^{-\sigma_1 t} z(t)) \tag{3.20}$$

We will use

Lemma 3.1. *Let $f : X \to Y$ be differentiable map in Banach spaces X, Y with derivative $f'(x)$ satisfying Lipschitz condition*

$$\|f'(x) - f'(y)\| \le \gamma \|x - y\|_X \tag{3.21}$$

with Lipschitz constant γ independent on $x \in X$. Suppose that $f(0) = 0,\ f'(0) = 0$. Then

$$\|f'(x)\| \le \gamma \|x\|_X, \qquad \|f(x)\| \le \frac{\gamma}{2} \|x\|_X^2 \tag{3.22}$$

Proof. The first inequality in (3.22) follows from (3.21) with $y = 0$ and relation $f'(0) = 0$. It is easy to see that

$$f(x) = \int_0^1 f'(\lambda x)[x]\ d\lambda = \lim_{N\to\infty} \sum_{k=1}^N \frac{1}{N} f'\left(\frac{k}{N} x\right)[x]$$

$$= \lim_{N\to\infty} \sum_{k=1}^N \frac{1}{N} \sum_{j=0}^{k-1} \left(f'\left(\frac{k-j}{N} x\right)[x] - f'\left(\frac{k-j-1}{N} x\right)[x] \right)$$

Therefore in virtue of (3.21) we get

$$\|f(x)\|_Y \le \lim_{N\to\infty} \sum_{k=1}^N \frac{1}{N} \sum_{j=0}^{k-1} \left\| f'\left(\frac{k-j}{N} x\right) - f'\left(\frac{k-j-1}{N} x\right) \right\| \|x\|_X$$

$$\le \lim_{N\to\infty} \sum_{k=1}^N \frac{1}{N} \sum_{j=0}^{k-1} \frac{\gamma}{N} \|x\|_X^2 = \gamma \lim_{N\to\infty} \frac{N(N+1)}{2N^2} \|x\|_X^2 = \frac{\gamma}{2} \|x\|_X^2$$

□

Let us show that if $g \in G_\mu(\Theta_\rho(V_-^{1/2})),\ \|z(t)\|_{V_-^{1/2}} \le \rho\ \forall t > 0$ then the function $w(t)$ defined in (3.20) can be estimated as follows

$$\|w(t)\|_s \le (1 + \frac{1}{2} \lambda_1^{\frac{1-2s}{4}}) \|z(t)\|_s \qquad \text{with} \quad s = \frac{1}{2}, 1, \frac{3}{2} \tag{3.23}$$

where λ_1 is the first eigenvalue from problem (1.8). Indeed, using Lemma 3.1, inclusion $g \in G_\mu(\Theta_\rho(V_-^{1/2}))$, and relation $\mu\rho \le 1$ we get

$$\|w(t)\|_1 \le \|z\|_1 + \frac{e^{-\sigma_1 t}\mu}{2}\|z(t)\|_{1/2}^2 \le \|z(t)\|_1\left(1+\frac{\mu\rho\lambda_1^{-1/4}}{2}\right) \le (1+\frac{1}{2}\lambda_1^{-1/4})\|z(t)\|_1$$

Bounds (3.23) with $s = 1/2$ and $s = 3/2$ are proved similarly.

Using boundedness of inverse operator for (3.17), Lemma 1.1, and relations (3.4), (3.12), (3.23) with $s = 1/2, s = 3/2$ we get for $z(t)$ from (3.19)

$$\begin{aligned}
&\|z\|_{V^{1,2(-1/2)}} \le c(\|y_-\|_{1/2} + \|B(w,w)\|_{L_2(\mathbb{R}_+;V^{-1/2})}) \\
&\le c(\rho + c_1\|w\|_{L_\infty(\mathbb{R}_+;V^{1/2})}\|w\|_{L_2(\mathbb{R}_+;V^{3/2})}) \\
&\le c(\rho + \frac{3c_1}{4}(2+\lambda_1^{-1/2})\|z\|_{L_\infty(\mathbb{R}_+;V^{1/2})}\|z\|_{L_2(\mathbb{R}_+;V^{3/2})}) \\
&\le c(\rho + c_2\sup_{t\ge 0} e^{(\sigma_1-\sigma)t}\|y_-\|_{1/2}\|z\|_{V^{1,2(-1/2)}}) \le c(\rho + c_2\rho\|z\|_{V^{1,2(-1/2)}})
\end{aligned} \tag{3.24}$$

Assuming that ρ is so small that $1 - cc_2\rho > 0$ we obtain from (3.24)

$$\|z\|_{V^{1,2(-1/2)}} \le \frac{c\rho}{1-cc_2\rho} \tag{3.25}$$

Similarly to (3.24) using (1.31), (3.23) with $s = 1$ and $s = 3/2$ we get

$$\begin{aligned}
&\|z\|_{V^{1,2(0)}} \le c(\|y_-\|_1 + \|B(w,w)\|_{L_2(\mathbb{R}_+;V^0)}) \\
&\le c(\|y_-\|_1 + c_1\|w\|_{L_\infty(\mathbb{R}_+;V^1)}\|w\|_{L_2(\mathbb{R}_+;V^{3/2})}) \\
&\le c(\|y_-\|_1 + \frac{c_1}{4}(2+\lambda_1^{-1/4})(2+\lambda_1^{-1/2})\|z\|_{L_\infty(\mathbb{R}_+;V^1)}\|z\|_{L_2(\mathbb{R}_+;V^{3/2})}) \\
&\le c(\|y_-\|_1 + c_2\|z\|_{V^{1,2(0)}}\|z\|_{V^{1,2(-1/2)}}) \le c\|y_-\|_1 + \frac{c^2c_2\rho}{1-cc_2\rho}\|z\|_{V^{1,2(-1/2)}}
\end{aligned} \tag{3.26}$$

Assuming that ρ is so small that $c^2c_2\rho < (1 - cc_2\rho)$ we obtain from (3.26)

$$\|z(t)\|_{V^1} \le c_3\|z\|_{V^{1,2(0)}} \le cc_3\left(1-\frac{c^2c_2\rho}{1-cc_2\rho}\right)^{-1}\|y_-\|_{V^1} \tag{3.27}$$

Relations (3.27), (3.12), (3.7) imply (3.2). □

4. Appendix I. Proof of Theorem 3.2

The proof of operator (3.17) reversibility is, evidently, reduced to establishing of operator (3.18) boundedness. We first estimate the second term in the right side of (3.18).

4.1. **Estimate of the last term in (3.18).** First of all we bound resolvent for operator $-\widehat{A}_1$.

Lemma 4.1. *Let $s \in [-1/2, 0]$.* [4] *There exists a constant $c > 0$ such that for each $\xi \in \mathbb{R}$*

$$\|(i\xi I + \widehat{A}_1)^{-1}f\|_{V_-^{s+2}} + \xi^2\|(i\xi I + \widehat{A}_1)^{-1}f\|_{V_-^s} \le c\|f\|_{V_-^s}^2 \tag{4.1}$$

where $\widehat{A}_1$ is operator (3.15), and I is identity operator.

[4] Restriction on s in this Lemma is connected with smoothness condition imposed on $\widehat{v}(x)$ from definition (3.15),(2.9) of operator $\widehat{A}_1$. In fact for $\widehat{v} \in V^2$ assertion of Lemma 4.1 holds for less restriction on s than $s \in [-1/2, 0]$.

Proof. Instead of $\widehat{A}_1$ we consider first the operator $A = -\pi\Delta$. Since $\|\cdot\|_{V_-^s} = \|\cdot\|_s$, by definition (1.9) of the norm $\|\cdot\|_s$ of the space V^s we get:

$$\|(i\xi I + A)y\|_s^2 = \sum_{j=1}^{\infty} |i\xi + \lambda_j|^2 \lambda_j^s |y_j|^2 = \|y\|_{s+2}^2 + \xi^2\|y\|_s^2 \tag{4.2}$$

In virtue of (3.15), (2.9), (1.15)

$$\widehat{A}_1 y - Ay = B(\widehat{v}, y) + B(y, \widehat{v}) - \sigma_1 y. \tag{4.3}$$

Applying Lemma 1.1 we get

$$\|\widehat{A}_1 y - Ay\|_s = \|B(\widehat{v}, y) + B(y, \widehat{v}) - \sigma_1 y\|_s \leq c\|y\|_{s+3/2} \tag{4.4}$$

with $c > 0$ independent of y. Then (4.4) implies

$$\|(i\xi + A)y\|_s^2 \leq (\|(i\xi + \widehat{A}_1)y\|_s + \|\widehat{A}_1 y - Ay\|_s)^2 \leq 2\|(i\xi + \widehat{A}_1)y\|_s^2 + 2c^2\|y\|_{s+3/2}^2. \tag{4.5}$$

Since by definition (1.9) of $\|\cdot\|_s$

$$c^2\|y\|_{s+3/2}^2 \leq c^2\|y\|_s^{1/2}\|y\|_{s+2}^{3/2} \leq \frac{27c^8}{4}\|y\|_s^2 + \frac{1}{4}\|y\|_{s+2}^2$$

we obtain from (4.5), (4.2) that

$$\begin{aligned} \|(i\xi + \widehat{A}_1)y\|_s^2 &\geq \frac{1}{2}\|(i\xi + A)y\|_s^2 - c^2\|y\|_{s+3/2}^2 \geq \frac{1}{4}\|y\|_{s+2}^2 + \left(\frac{\xi^2}{2} - \frac{27c^8}{4}\right)\|y\|_s^2 \\ &\geq \frac{1}{4}(\|y\|_{s+2}^2 + \xi^2\|y\|_s^2) \qquad \text{if} \quad |\xi| > \sqrt{27}c^4 \end{aligned} \tag{4.6}$$

Since by definition of operator $\widehat{A}_1$ the set $\{\lambda \in \mathbb{C} : \lambda = i\xi, \xi \in [-\sqrt{27}c^4, \sqrt{27}c^4]\}$ belongs to resolvent set of $-\widehat{A}_1$ there exists a constant $c_1 > 0$ such that

$$\|(i\xi + \widehat{A}_1)y\|_s^2 \geq c_1\|y\|_{s+2}^2 \quad \text{if} \quad |\xi| \leq \sqrt{27}c^4. \tag{4.7}$$

By definition (1.9) $\|v\|_{s+2} \geq \lambda_1\|v\|_s$ where λ_1 is the minimal eigenvalue of problem (1.8). That is why

$$\|y\|_{s+2}^2 \geq \frac{1}{2}(\|y\|_{s+2}^2 + \frac{\lambda_1^2}{27c^8}\xi^2\|y\|_s^2) \geq \min\{\frac{1}{2}, \frac{\lambda_1^2}{54c^8}\}(\|y\|_{s+2}^2 + \xi^2\|y\|_s^2) \quad \text{if} \quad |\xi| \leq \sqrt{27}c^4 \tag{4.8}$$

Inequalities (4.6), (4.7), (4.8) imply (4.1) □

Lemma 4.2. *For operator*

$$(Rf)(t) = \int_0^t e^{-\widehat{A}_1(t-\tau)} f(\tau)\, d\tau \tag{4.9}$$

the following estimate holds:

$$\|Rf\|_{V_-^{1,2(s)}} \leq c\|f\|_{L_2(\mathbb{R}_+; V_-^s)}, \quad s \in [-1/2, 0] \tag{4.10}$$

where $c > 0$ does not depend on f.

Proof. Let

$$F(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}; \qquad E(t) = \begin{cases} e^{-\widehat{A}_1 t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then by (4.9)

$$Rf(t) = (E * F)(t) \tag{4.11}$$

Applying to both parts of (4.11) Fourier transform $\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-it\xi} g(t)\, dt$ we get:

$$(\widehat{Rf})(\xi) = (\widehat{E * F})(\xi) = \widehat{E}(\xi)\widehat{F}(\xi) = (i\xi I + \widehat{A}_1)^{-1}\widehat{F}(\xi), \tag{4.12}$$

since $\widehat{E}(\xi) = (i\xi I + \widehat{A}_1)^{-1}$. [5] Then in virtue of (4.11), (4.12), Parseval equality, and Lemma 4.1 we get

$$\|Rf\|_{V_-^{1,2(s)}} = \int_{-\infty}^{\infty} \left(\|(i\xi I + \widehat{A}_1)^{-1}\widehat{F}(\xi)\|_{s+2}^2 + \xi^2 \|(i\xi I + \widehat{A}_1)^{-1}\widehat{F}(\xi)\|_{s+2}^2 \right) d\xi$$

$$\leq c_1 \int_{-\infty}^{\infty} \|\widehat{F}(\xi)\|_s^2 \, d\xi \leq c\|f\|_{L_2(\mathbb{R}_+;V_-^s)}^2$$

□

4.2. **Estimate of the first term in r.s.of (3.18).** Operator $e^{-\widehat{A}_1 t}$ can be determined by the formula (see [Hen], [F5], p. 275):

$$e^{-\widehat{A}_1 t} = (2\pi i)^{-1} \int_\gamma (\lambda I + \widehat{A}_1)^{-1} e^{\lambda t}\, d\lambda \tag{4.13}$$

where γ is a contour belonging to resolvent set $\rho(-\widehat{A}_1)$ of $-\widehat{A}_1 : V_-^s \to V_-^s$, $s \in [-1/2, 0]$ such that $\arg \lambda = \pm\theta$ for $\lambda \in \gamma, |\lambda| > N$ for certain $\theta \in (\pi/2, \pi)$ and for sufficiently large N. Moreover, $\gamma \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\, \lambda < 0\}$ and γ surrounds spectrum $\Sigma(-\widehat{A}_1)$ of $-\widehat{A}_1 : V_-^s \to V_-^s$ from the right.

Lemma 4.3. *Let $s \in [-1, 0]$. For resolvent of operator $-\widehat{A}_1 : V_-^s \to V_-^s$ the following estimate holds:*

$$\|(\lambda I + \widehat{A}_1)^{-1} f\|_{V_-^{s+2}} \leq \frac{c}{|\lambda| + 1} \|f\|_{V_-^{1+s}}, \qquad \lambda \in \gamma \tag{4.14}$$

where γ is the contour from (4.13), and constant $c > 0$ does not depend on $f \in V_-^{s+1}$ and on $\lambda \in \gamma$.

For $s = -1$ this lemma has been proved in [F5], p.289-290. This proof can be easily generalized on the case $s \in [-1, 0]$ with help of arguments of Lemma 4.1.

Lemma 4.4. *Let $\widehat{A}_1 : V_-^0 \to V_-^0$ be operator defined in (3.15), (2.9), (1.15). Then for $s \in [-1/2, 0]$*

$$\|e^{-\widehat{A}_1 t} v_0\|_{V_-^{1,2(s)}} \leq c\|v_0\|_{V_-^{1+s}} \tag{4.15}$$

where c does not depend on $v_0 \in V_-^{s+1}$

Proof. Let γ be the contour from (4.13) defined before Lemma 4.3 formulation. Then $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\begin{aligned} &\gamma_1 = \{\lambda = -\mu(1 + i\nu) : \mu > N\}, \quad \gamma_2 = \{\lambda = -\mu(1 - i\nu) : \mu > N\}, \\ &\gamma_3 = \gamma \cap \{|\lambda| \leq N|1 + i\nu|\} \end{aligned} \tag{4.16}$$

where $\nu \in (0, 1)$ is a fixed number as well as sufficiently large $N > 0$. Then

$$e^{-\widehat{A}_1 t} = \sum_{j=1}^{3} I_j(t), \text{ where } I_j(t) = (2\pi i)^{-1} \int_{\gamma_j} (\lambda I + \widehat{A}_1)^{-1} e^{\lambda t}\, d\lambda,\ j = 1, 2, 3 \tag{4.17}$$

[5]Derivation of this formula is based on equality $\frac{d}{dt} e^{-\widehat{A}_1 t} = -\widehat{A}_1 e^{-\widehat{A}_1 t}$ and the property that spectrum of operator $\widehat{A}_1 : V_-^0 \to V_-^0$ belongs to the set $\{\lambda \in \mathbb{C} : \mathrm{Re}\, \lambda > 0\}$.

Using (4.17), (4.14) we get

$$\int_0^\infty \|I_1(t)v_0\|^2_{s+2}\,dt \le c\int_0^\infty \left(\int_N^\infty \|(-\mu(1+i\nu)I+\widehat{A}_1)^{-1}v_0\|_{s+2}e^{-\mu t}\,d\mu\right)^2 dt$$
$$\le c_1\|v_0\|^2_{s+1}\int_0^\infty \left(\int_N^\infty \frac{e^{-\mu t}}{1+\mu}\,d\mu\right)^2 dt$$
$$\le c_1\|v_0\|^2_{s+1}\int_N^\infty \frac{d\mu}{(1+\mu)^{3/2}}\int_N^\infty\int_0^\infty e^{-2\mu t}\,dt\frac{d\mu}{(1+\mu)^{1/2}} \le c_2\|v_0\|^2_{s+2} \tag{4.18}$$

Similarly we obtain

$$\int_0^\infty \|I_2(t)v_0\|^2_{s+2}\,dt \le c\|v_0\|^2_{s+1} \tag{4.19}$$

Since contour γ_3 is compact and we can choose $\lambda_0>0$ such that $\operatorname{Re}\lambda \le -\lambda_0$ for each $\lambda\in\gamma_3$, the following bound is true by (4.14):

$$\int_0^\infty \|I_3(t)v_0\|^2_{s+2}\,dt \le c\int_0^\infty e^{-\lambda_0 t}\left(\int_{\gamma_3}\|(\lambda I+\widehat{A}_1)^{-1}v_0\|_{s+2}\,|d\lambda|\right)^2 dt \le c\|v_0\|^2_{s+1} \tag{4.20}$$

Relations (4.18)-(4.20) imply

$$\int_0^\infty \|e^{-\widehat{A}_1 t}v_0\|^2_{s+2}\,dt \le c\|v_0\|^2_{s+1} \tag{4.21}$$

Since $\frac{d}{dt}e^{-\widehat{A}_1 t}v_0 = -\widehat{A}_1 e^{-\widehat{A}_1 t}v_0$, we get by (4.3),(4.21):

$$\begin{aligned}\int_0^\infty \|\frac{d}{dt}e^{-\widehat{A}_1 t}v_0\|^2_s\,dt &\le c\int_0^\infty (\|Ae^{-\widehat{A}_1 t}v_0\|^2_s + \|e^{-\widehat{A}_1 t}v_0\|^2_{s+1})\,dt\\ &\le c\int_0^\infty \|e^{-\widehat{A}_1 t}v_0\|^2_{s+2}\,dt \le c\|v_0\|^2_{s+1}\end{aligned} \tag{4.22}$$

The bound (4.15) follows from (4.21), (4.22). □

Now Theorem 3.2 follows from Lemmas 4.2, 4.4.

5. Appendix II. Existence of stable invariant manifold

Here we prove existence of stable invariant manifold in such phase spaces that are more convenient for proof of the main Theorem 3.1 than spaces used in literature, for instance in [BV].

5.1. **Completeness of metric space $G_{\mu,\rho}$.** Recall that spaces V^s_-, V^s_+, V^s are defined in (2.13) - (2.16) correspondingly. We suppose that $V=V^{s+1}, V_-=V^{s+1}_-, V_+=V^{s+1}_+$ with arbitrary fixed $s\in[-1/2,0]$, and in particular $G_{\mu,\rho}\equiv G_\mu(\Theta_\rho(V_-))$, where $V_-=V^{s+1}_-$ (See Definition 2.2)

Lemma 5.1. *For each $g\in G_{\mu,\rho}$*

$$\|g(u+h)-g(u)-g'(u)[h]\|_+ \le \frac{\mu}{2}\|h\|^2_{V_-} \quad \forall u, u+h\in\Theta_\rho(V_-) \tag{5.1}$$

Proof. Evidently

$$g(u+h)-g(u)-g'(u)[h]=\int_0^1\left(\frac{d}{d\lambda}g(u+\lambda h)-g'(u)[h]\right)\,d\lambda$$
$$=\int_0^1 (g'(u+\lambda h))[h]-g'(u)[h])\,d\lambda \tag{5.2}$$

Estimate of (5.2) with help of (2.25) yields (5.1) □

Proof. of Theorem 2.1. Suppose that $\{g_k\}$ is a Cauchy sequence in $G_{\mu,\rho}$ with respect of metric (2.27). Then there exists unique $g\in C(\Theta_\rho(V_-);V_+)$ satisfying

$$\|g-g_k\|_{C(\Theta_\rho(V_-);V_+)}=\sup_{v\in\Theta_\rho(V_-)}\|g-g_k\|_+\to 0\quad\text{as}\quad k\to\infty, \tag{5.3}$$

and $g(0)=0$ since $g_k(0)=0$ for each k. Let UJ be a countable dense subset of $\Theta_\rho(V_-)$. Since by Lemma 3.1 $\|g_k'(u)\|_{\mathcal{L}(V_-;V_+)}\le\mu\|u\|_{V_-}\le\mu\rho$ for $u\in\Theta_\rho(V_-)$ one can choose subsequence $\{n\}\subset\{k\}$ such that

$$g_n'(u_j)[h]\longrightarrow\widehat{g}(u_j)[h]\quad\text{in } V_+\quad\text{as } n\to\infty\quad\forall\, u_j\in UJ,\ \forall\, h\in V_-\quad k\to\infty, \tag{5.4}$$

where $\widehat{g}(u_j):V_-\to V_+$ is bounded linear operator satisfying by (2.25) Lipschitz condition for each $u_i,u_j\in UJ$:

$$\|\widehat{g}(u_i)-\widehat{g}(u_j)\|_{\mathcal{L}(V_-;V_+)}\le\mu\|u_i-u_j\|_{V_-} \tag{5.5}$$

By (5.5) $\widehat{g}$ can be uniquely extended in to continuous operator-function $\widehat{g}(u)\in C(\Theta_\rho(V_-);\mathcal{L}(V_-;V_+))$ that satisfies (5.5) for every $u_i,u_j\in\Theta_\rho(V_-)$. Relations (5.4) and $g_n'(0)=0$ imply $\widehat{g}(0)=0$. By (5.3), (5.4), (5.5), and Lemma 5.1 for each $u,u+h\in\Theta_\rho(V_-)$ and for arbitrary small $\delta>0$

$$\|g(u+h)-g(u)-\widehat{g}(u)[h]\|_+\le\|g(u+h)-g_n(u+h)\|_++\|g(u)-g_n(u)\|_+$$
$$+\|\widehat{g}(u)[h]-g_n'(u)[h]\|_++\|g_n(u+h)-g_n(u)-g'(u)_n[h]\|_+\le\delta+\frac{\mu}{2}\|h\|^2_{V_-}.$$

if n is sufficiently large. Hence $g(u)$ is differentiable and $g'(u)=\widehat{g}(u)$. In virtue of (2.27), (5.3), and Lemma 3.1 for arbitrary $0<\delta<\rho$

$$d(g,g_k)\le\sup_{0<\|u\|\le\delta}\left(\frac{\|g\|}{\|u\|}+\frac{\|g_k(u)\|}{\|u\|}\right)+\sup_{\delta<\|u\|<\rho}\frac{\|g(u)-g_k(u)\|}{\|u\|}$$
$$\le\mu\delta+\delta^{-1}\|g-g_k\|_{C(\Theta_\rho(V_-);V_+)}\longrightarrow\mu\delta\quad\text{as}\quad k\to\infty$$

Hence $\lim_{k\to\infty}d(g,g_k)=0$. □

5.2. **Preliminaries.** We begin with formulation of the following existence theorem for problem (2.7)-(2.9).

Theorem 5.1. *Let $s\in[-1/2,0]$. For any $T>0$ there exists ρ_0 such that for each $y_0\in\Theta_{\rho_0}(V^{s+1})$ there exists unique solution $y(t,x)\in V^{1,2(s)}(Q_T)$ of problem (2.7)-(2.9).*

The proof of this (local) theorem consists of reduction to Inverse Map Theorem (see, for example [LS]).

The shift operator corresponding to problem (2.7)-(2.9) we denote as follows:

$$S(t,y_0)=y(t,\cdot) \tag{5.6}$$

where $y(t,x)$ is solution of (2.7)-(2.9).

Let $V = V^{s+1}$ with fixed $s \in [-1/2, 0]$ be the phase space for $S(t,\cdot) : \Theta_{\rho_0}(V) \to V$. Inverse Map Theorem implies that for each $t \in (0,T), y_0 \in \Theta_{\rho_0}(V)$ operator $S(t, y_0)$ possesses differential Frchet with respect to y_0 (moreover, $S(t, y_0)$ is analytic in y_0). Let

$$S(t, y_0) = e^{-\widehat{A}t} y_0 + N(t, y_0) \tag{5.7}$$

where $e^{-\widehat{A}t} y_0$ is solution of (2.10), and $N(t, y_0)$ is defined by (5.7). Recall that subspaces V_+ and V_- are invariant with respect of $\widehat{A}$, and by (3.15) (4.15)

$$\|e^{-\widehat{A}t} u\|_{V_-} \leq c e^{-\sigma_1 t} \|u\|_{V_-} \tag{5.8}$$

with $c > 0$ independent of $u \in V_-$. Therefore for each $\alpha \in (0,1)$ there exists $T > 0$ such that

$$\|e^{-\widehat{A}T} u\|_{V_-} \leq \alpha \|u\|_{V_-}. \tag{5.9}$$

Thus, for a given $\alpha \in (0,1)$ we choose T satisfying (5.9), and after that we choose ρ_0 satisfying conditions of Theorem 5.1. We set

$$e^{-\widehat{A}T}|_{V_-} = L_-, \quad e^{-\widehat{A}T}|_{V_+} = L_+, \quad e^{-\widehat{A}T} \equiv L = L_- + L_+ \tag{5.10}$$

Note that $\dim V_+ < \infty$ and finite-dimensional operator $L_+ : V_+ \to V_+$ is invertible. In virtue of Theorem1 in Section 2 of Chapter IV in [BV] we can choose in V_+ such norm $\|\cdot\|_+$ that

$$\|L_+^{-1} v_+\|_+ \leq 2 \|v_+\|_+ \tag{5.11}$$

Everywhere below we use only this norm $\|\cdot\|_+$ on V_+ which is concrete definition of the norm (2.14). By (2.12)-(2.16) on $V = V^{1+s}$ the following norm is defined:

$$\|v\|_V = \|v_-\|_{V_-} + \|v_+\|_+, \quad \text{where} \quad V \ni v = v_- + v_+, \quad v_- \in V_-,\ v_+ \in V_+ \tag{5.12}$$

Then norms of projection operators (3.5) satisfy relations

$$\|\Pi_-\| \leq 1, \quad \|\Pi_+\| \leq 1. \tag{5.13}$$

Recall that (5.9), (5.10) imply

$$\|L_- y_0\|_{V_-} \leq \alpha \|y_0\|_{V_-}. \tag{5.14}$$

We set

$$S(y_0) = S(T, y_0), \qquad N(y_0) = N(T, y_0) \tag{5.15}$$

where $S(T, y_0), N(T, y_0)$ are defined in (5.6), (5.7). Then (5.15) implies

$$N(0) = 0, \qquad N'(0) = 0 \tag{5.16}$$

At last, denote by $b > 0$ the Lipschitz constant for $N'(v)$:

$$\|N'(v_1) - N'(v_2)\| \leq b \|v_1 - v_2\| \qquad \forall\, v_1, v_2 \in \Theta_{\rho_0}(V). \tag{5.17}$$

5.3. **The main result.** Remind that a set M_- containing $\{0\}$ is called stable invariant manifold for the map $S : \mathcal{O}(V) \to V$ if inclusion $y_0 \in M_-$ implies $S(y_0) \in M_-$ and

$$\|S^j(y_0)\| \leq c\gamma^j \|y_0\| \quad \text{as} \quad j \to \infty \quad \text{with } \gamma < 1 \qquad \forall\, y_0 \in M_- \tag{5.18}$$

We construct stable invariant manifold for map S in a form (2.21)-(2.23) where $g(u) \in G_{\mu,\rho} = G_\mu(\Theta_\rho(V_-))$

Theorem 5.2. *Suppose that constant α from (5.14) and indexes μ, ρ of $G_{\mu,\rho}$ satisfy conditions*

$$\alpha \leq 1/3, \quad \mu \geq 10b, \quad \mu\rho \leq 1/5, \quad \rho \leq \rho_0 \tag{5.19}$$

where b is the constant from (5.17), and ρ_0 is determined in Theorem 5.1 formulation. Then there exists unique $g(u) \in G_{\mu,\rho}$ such that the set (2.21) is stable invariant manifold for the map S. Moreover, estimate (5.18) holds with $c = 1, \gamma < 3\alpha/2 + 1/100$ under additional assumption $\|L_+\|\mu\rho \leq \alpha$.

First of all we derive equation for the map g forming M_- with help of invariance assumption for M_-. We will use notations:

$$S_- = \Pi_- S, \quad S_+ = \Pi_+ S, \quad N_- = \Pi_- N, \quad N_+ = \Pi_+ N \tag{5.20}$$

Relations (5.7), (5.10), (5.15), (5.20) imply

$$S = L + N, \quad S = S_+ + S_-, \quad S_- = L_- + N_-, \quad S_+ = L_+ + N_+ \tag{5.21}$$

Invariance of M_- from (2.21) with respect to S means that

$$\text{if} \quad M_- \ni v = u + g(u) \quad \text{then} \quad M_- \ni S(v) = S_-(v) + g(S_-(v)) \tag{5.22}$$

where $u \in \Theta_\rho(V_-)$. In virtue of (5.21), (5.10)

$$S_-(u+g(u)) = L_-u+N_-(u+g(u)); \quad S_+(u+g(u)) = L_+g(u)+N_+(u+g(u)) \tag{5.23}$$

By (5.21), (5.22) $S_+(v) = g(S_-(v))$. After substitution into this equation relations (5.22), (5.23) and applying to obtained equality operator L_+^{-1} we get the desired equation for g:

$$g(u) = L_+^{-1} g(L_-u + N_-(u + g(u))) - L_+^{-1} N_+(u + g(u)) \equiv F(g(u)) \tag{5.24}$$

where the last equality is definition of the map F.

Lemma 5.2. *Let parameters α, μ, b, ρ satisfy conditions (5.19). Then for each $g \in G_{\mu,\rho}$ inclusion $u \in \Theta_\rho(V_-)$ implies $L_-u + N_-(u + g(u)) \in \Theta_\rho(V_-)$.*

Proof. Using (5.13), (5.14), (5.17), (5.19) and Lemma 3.1 we get

$$\|L_-u + N_-(u + g(u))\|_{V_-} \leq \alpha\rho + \frac{b}{2}\|u + g(u)\|^2 \leq \rho\left(\alpha + \frac{b\rho}{2}\left(1 + \frac{\mu\rho}{2}\right)^2\right) \leq \rho$$

□

Lemma 5.3. *Let parameters α, μ, b, ρ satisfy (5.19). Then inclusion $g \in G_{\mu,\rho}$ implies $F(g) \in G_{\mu,\rho}$ where F is map (5.24).*

Proof. If $g \in G_{\mu,\rho}, \ u \in \Theta_\rho(V_-)$ then by Lemma 5.2 $g(L_-u + N_-(u + g(u)))$ is well-defined and hence $F(g(u))$ is also well-defined. We intend to calculate Holder constant for derivative $F(g(u))'_u[h]$. By (5.24) we have

$$F(g(u))'[h] = I(u)[h] + J(u)[h] \quad \text{where} \quad J(u)[h] = L_+^{-1} N_+'(u + g(u))[h + g'(u)[h]] \tag{5.25}$$

and

$$I(u)[h] = L_+^{-1} g'(L_-u + N_-(u + g(u)))[L_-h + N_-'(u + g(u))[h + g'(u)[h]]] \tag{5.26}$$

The following equality is true:

$$I(u_1)[h] - I(u_2)[h] = I_1(u_1, u_2)[h] + I_2(u_1, u_2)[h] + I_3(u_1, u_2)[h] \tag{5.27}$$

where

$$I_1(u_1,u_2)[h] = L_+^{-1} g'(L_- u_1 + N_-(u_1+g(u_1)))[L_- h] \\ - L_+^{-1} g'(L_- u_2 + N_-(u_2+g(u_2)))[L_- h] \tag{5.28}$$

$$I_2(u_1,u_2)[h] = I_1(u_1,u_2)[N_-'(u_1+g(u_1))[h]] \\ + L_+^{-1} g'(L_- u_2 + N_-(u_2+g(u_2)))[N_-'(u_1+g(u_1))[h] - N_-'(u_2+g(u_2))[h]] \tag{5.29}$$

$$I_3(u_1,u_2)[h] = I_2(u_1,u_2)[g'(u_1)[h]] \\ + L_+^{-1} g'(L_- u_2 + N_-(u_2+g(u_2)))[N_-'(u_2+g(u_2))[g'(u_1)[h] - g'(u_2)[h]]] \tag{5.30}$$

All estimates of this lemma obtained below are based on Lemmas 5.2, 3.1, and on Lagrange Theorem.[6] We begin from calculation of Holder constant for map J defined in (5.25). We get using (5.11), (5.20), (5.13) and inclusion $g \in G_{\mu,\rho}$:

$$\|J(u_1) - J(u_2)\| \le 2b(\|u_1 - u_1\| + \|g(u_1) - g(u_2)\|)(1 + \|g'(u_1)\|)) \\ + 2\|N_+'(u_2 + g(u_2))\| \|g'(u_1) - g'(u_2)\| \le 2b\|u_1 - u_2\|(1 + \mu\rho(1+\mu\rho)) \\ + 2b(\rho + \mu\rho^2/2)\mu\|u_1 - u_2\| \le \mu\|u_1 - u_2\|\beta \tag{5.31}$$

where

$$\beta = \frac{2b(1+\mu\rho)^2}{\mu} + 2b\rho\left(1 + \frac{\mu\rho}{2}\right) \tag{5.32}$$

Using similar reasons as above and Lemma 5.2 we get from (5.28):

$$\|I_1(u_1,u_2)\| \le 2\alpha\mu(\alpha\|u_1 - u_2\| + b(\rho + \frac{\mu\rho^2}{2})(\|u_1 - u_2\| + \mu\rho\|u_1 - u_2\|)) \le \mu\|u_1 - u_2\|\beta_1 \tag{5.33}$$

where

$$\beta_1 = 2\alpha(\alpha + b\rho(1+\mu\rho/2)(1+\mu\rho)) \tag{5.34}$$

Using (5.33), (5.31) we obtain as above from (5.29):

$$\|I_2(u_1,u_2)\| \le \|I_1(u_1,u_2)\| b(\rho + \frac{\mu\rho^2}{2}) + 2\mu\rho b(1+\mu\rho)\|u_1 - u_2\| \le \mu\|u_1 - u_2\|\beta_2 \tag{5.35}$$

where

$$\beta_2 = \beta_1 b\rho(1+\mu\rho/2) + 2\rho b(1+\mu\rho) \tag{5.36}$$

At last using (5.35) we get from (5.30)

$$\|I_3(u_1,u_2)\| \le \|I_2(u_1,u_2)\|\mu\rho + 2\mu\rho b(\rho + \frac{\mu\rho^2}{2})\mu\|u_1 - u_2\| \le \mu\|u_1 - u_2\|\beta_3 \tag{5.37}$$

where

$$\beta_3 = \beta_2\mu\rho + 2\mu\rho b\rho(1+\mu\rho/2) \tag{5.38}$$

The map $F(g(u))$ belongs to $G_{\mu,\rho}$ if Holder constant for $F(g(u))'_u$ is not more than μ. This condition satisfies if

$$\beta + \beta_1 + \beta_2 + \beta_3 \le 1 \tag{5.39}$$

where $\beta, \beta_1, \beta_2, \beta_3$ are constants (5.32), (5.34), (5.36), (5.38). Condition (5.39) follows from (5.19). □

[6] I.e. on the following estimate: $\|f(u_1) - f(u_2)\| \le \sup_{w\in[u_1,u_2]} \|f'(w)\|\|u_1 - u_2\|$.

Lemma 5.4. *Let α, μ, b, ρ satisfy (5.19). Then operator F defined in (5.24) satisfies inequality*

$$d(F(g_1), F(g_1)) \leq c d(g_1, g_2) \tag{5.40}$$

with $c < 1$ where metric d is defined in (2.27).

Proof. Using definition (5.24) of map F, Lagrange Theorem, and Lemmas 3.1, 5.2 we get:

$$\begin{aligned}
&\frac{\|F(g_1(u)) - F(g_2(u))\|_+}{2\|u\|_{V_-}} \\
&\leq \frac{\|g_1(L_-u + N_-(u + g_1(u))) - g_1(L_-u + N_-(u + g_2(u)))\|_+}{\|u\|_{V_-}} \\
&+ \frac{\|g_1(L_-u + N_-(u + g_2(u))) - g_2(L_-u + N_-(u + g_2(u)))\|_+}{\|L_-u + N_-(u + g_2(u))\|_{V_-}} \\
&\times \frac{\|L_-u + N_-(u + g_2(u))\|_{V_-}}{\|u\|_{V_-}} \\
&+ \frac{\|N_+(u + g_1(u)) - N_+(u + g_2(u))\|_{V_-}}{\|u\|_{V_-}} \leq \mu\rho b \left(\rho + \frac{\mu\rho^2}{2}\right) \frac{\|g_1(u) - g_2(u)\|_+}{\|u\|_{V_-}} \\
&+ d(g_1, g_2) \left(\alpha + \frac{b\rho}{2}\left(1 + \frac{\mu\rho}{2}\right)^2\right) + b\left(\rho + \frac{\mu\rho^2}{2}\right) \frac{\|g_1(u) - g_2(u)\|_+}{\|u\|_{V_-}}
\end{aligned} \tag{5.41}$$

Taking supremum in (5.41) over $u \in \Theta_\rho(V_-) \setminus \{0\}$ we get inequality (5.40) with the following constant c:

$$c = 2\mu\rho b \left(\rho + \frac{\mu\rho^2}{2}\right) + 2\left(\alpha + \frac{b\rho}{2}\left(1 + \frac{\mu\rho}{2}\right)^2\right) + 2b\left(\rho + \frac{\mu\rho^2}{2}\right)$$

Conditions (5.19) imply inequality $c < 1$ □

Proof. of Theorem 5.2. In virtue of Lemma 5.3 operator F defined in (5.24) maps $G_{\mu,\rho}$ into $G_{\mu,\rho}$. Lemma 5.4 implies that the map $F : G_{\mu,\rho} \to G_{\mu,\rho}$ is contraction. By Contraction Mapping Principle there exists unique $g \in G_{\mu,\rho}$ satisfying equation (5.24). Hence, the set M_- defined in (2.21) by this g is invariant with respect of the map S defined in (5.15), (5.6).

Note that by Lemma 3.1 and (5.19)

$$\|v + g(v)\| \geq \|g\|(1 - \mu\rho/2) \geq \frac{9}{10}\|v\|$$

If $u \in M_-$ then $u = v + g(v)$ with $v \in V_-$, and using (5.14), (5.17), Lemma 3.1, (5.19), and assumption $\|L_+\|\mu\rho \leq \alpha$ we get

$$\|S(u)\| = \|L_-v + L_+g(v) + N(v + g(v))\|$$

$$\leq (\alpha + \|L_+\|\mu\rho/2)\|v\| + b(\rho + \mu\rho^2/2)\|v + g(v)\| \leq (5\alpha/3 + 11/500)\|v + g(v)\|$$

Several iterations of this inequality imply bound (5.18) with $c = 1, \gamma \leq 5\alpha/3 + 11/500 < 1$. □

Remark 5.1. *Theorem 2.2 follows from Theorem 5.2 with help of simple repeating the proof of Theorem 4 from Section 2 in Chapter V of* [BV].

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