ANALYTICITY OF STABLE INVARIANT MANIFOLDS FOR GINZBURG-LANDAU EQUATION

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ABSTRACT. This paper is devoted to prove analyticity of stable invariant manifold in a neighbourhood of an unstable steady-state solution for Ginzburg-Landau equation defined in a bounded domain of dimension not more than three. This investigation is made for possible applications in stabilization theory for semilinear parabolic equation.

INTRODUCTION

In this paper we prove analyticity of stable invariant manifold M_- near unstable steady-state solution of Ginzburg-Landau equation. This result can be used in stabilization theory for semilinear parabolic PDE defined in a bounded domain Ω with feedback Dirichlet control given on the boundary $\partial\Omega$ or on its open part.

This theory for general quasilinear parabolic equation and for Navier-Stokes system was built in [F1], [F2], [F3]. We have to emphasize that the main reason to develop stabilization theory is to provide reliable stable algorithms for numerical stabilization. To construct such algorithms it is very desirable to have a simple description for infinite-dimensional invariant manifold M_- allowing to calculate it easily in arbitrary point. Just such description gives functional-analytic decomposition of M_-.

Using classical description of M_- by means of a map $F(y_-)$ (see [BV], [Hen]), one can look for this map as a serie

$$F(y_-) = \sum_{k=2}^{\infty} F_k(y_-)$$

where $F_k(y_-)$ are maps k-linear in y_-. Using special differential equation in variational derivatives for map F it is possible to obtain recurrent formulae for F_k. These recurrent formulae allow us to prove convergence of serie for $F(y_-)$.

First step in realization of this plan has been made in [F4] where analyticity of stable invariant manifold in a neghborhood of zero steady state solution was proven in the case of one-dimensional semilinear parabolic equation. Moreover, obtained recourrence relations were succesfully used in [K] for numerical calculations.

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Note that under assumptions of [F4] the linearization near steady state solution of the space part of semilinear equation is ordinary differential equation with constant coefficients. Therefore its eigenfunctions are $\sin kx$. This circumstance were used essentially in [F4]. The methods of present paper do not use explicit form of eigenfunctions and therefore can be applied to situation when aforementioned linearization is an elliptic operator with variable coefficients defined in arbitrary bounded domain.

1. Stable Invariant Manifold

In these section we recall certain notions connected with stable invariant manifolds for Ginzburg-Landau equation.

1.1. **Ginzburg-Landau equation.** Let $G \subset \mathbb{R}^n, n = 1, 2, 3$ be a bounded domain with C^∞-boundary ∂G. We consider Ginzburg-Landau equation

$$\partial_t v(t,x) - \nu\Delta v(t,x) - v(t,x) + v^3(t,x) = f(x), \quad x \in G, t > 0 \tag{1.1}$$

with boundary and initial conditions

$$v(t,\cdot)|_{x\in\partial G} = 0, \tag{1.2}$$

$$v(t,x)|_{t=0} = v_0(x), \quad x \in G, \tag{1.3}$$

where $\partial_t v = \partial v/\partial t$, $\nu > 0$ is a parameter, $f(x) \in L_2(G), v_0(x) \in H^2(G) \cap H_0^1(G)$ are given functions. Recall that $H^k(G)$ is the Sobolev space of functions belonging to $L_2(G)$ together with all their derivaties up to the order k, $H_0^1(G) = \{u(x) \in H^1(G) : u|_{\partial G} = 0\}$

As a phase space of the dynamical system generated by (1.1), (1.2) we take the functional space

$$V \equiv V(G) = H^2(G) \cap H_0^1(G) \tag{1.4}$$

Let $\widehat{v}(x) \in V$ be a steady-state solution of (1.1), (1.2), i.e. a solution of the problem

$$-\nu\Delta\widehat{v}(t,x) - \widehat{v}(t,x) + \widehat{v}^3(t,x) = f(x), \quad x \in G, \quad \widehat{v}|_{\partial G} = 0 \tag{1.5}$$

To study the structure of the dynamical system (1.1), (1.2) in a neighborhood of $\widehat{v}(x)$ we make the change of unknown functions in (1.1), (1.2):

$$v(t,x) = \widehat{v}(x) + y(t,x) \tag{1.6}$$

After substitution (1.6) into (1.1)–(1.3) and taking into account (1.5) we get:

$$\partial_t y(t,x) - \nu\Delta y(t,x) - q(x)y(t,x) + B(x, y(t,x)) = 0, \quad x \in G, t > 0 \tag{1.7}$$

$$y(t,\cdot)|_{x\in\partial G} = 0, \tag{1.8}$$

$$y(t,x)|_{t=0} = y_0(x) = v_0(x) - \widehat{v}(x), \quad x \in G, \tag{1.9}$$

where

$$q(x) = 3\widehat{v}^2(x) - 1, \qquad B(x,y) = y^3 + 3\widehat{v}(x)y^2 \tag{1.10}$$

Let

$$\{e_k(x), \lambda_k\}, \qquad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to \infty \quad \text{as} \quad k \to \infty \tag{1.11}$$

be the eigenfunctions and the eigenvalues of the spectral problem

$$Ae \equiv -\nu\Delta e(x) + q(x)e(x) = \lambda e(x), \; x \in G \quad e|\partial G = 0. \tag{1.12}$$

We assume that eigenvalues λ_k of the spectral problem (1.12) satisfy the condition:

$$\lambda_1 \le \cdots \le \lambda_N < 0 < \lambda_{N+1} \le \cdots \le \lambda_k \tag{1.13}$$

Since operator A is symmetric in $L_2(G)$, the set (1.11) of its eigenfunctions $\{e_k\}$ forms orthogonal basis in $L_2(G)$. We can assume (have done normalization) that $\{e_k\}$ is orthonormal basis in $L_2(G)$. It is well-known that usual Sobolev H^2-norm in $V = H^2(G) \cap H_0^1(G)$ is equivalent to the norm

$$\|v\|_V^2 = \sum_{j=1}^{\infty} \lambda_j^2 |v_j|^2, \text{ where } v_j = \int_G v(x)e_j(x)dx, \text{ and} v(x) = \sum_{j=1}^{\infty} v_j e_j(x) \tag{1.14}$$

Evidently, $\{e_j\}$ forms orthogonal basis in V with respect to scalar product defined by norm (1.14). Below we suppose that the phase space V is supplied with the norm (1.14).

In virtue of (1.13) the solutions $e^{-\lambda_k t} e_k(x)$ of the linear equation

$$\frac{\partial y}{\partial t} + Ay = 0 \tag{1.15}$$

tend to infinity as $t \to \infty$ for $k = 1, \dots, N$, and tend to zero as $t \to \infty$ for $k > N$.

We introduce the subspaces

$$V_+ \equiv V_+(G) = [e_1, \dots, e_N], \quad V_- \equiv V_-(G) = [e_{N+1}, e_{N+2} \dots] \tag{1.16}$$

of unstable and stable modes for equation (1.15). Since eigenfunctions (1.11) form orthogonal basis in the phase space (1.4), the following relation is true:

$$V_+(G) \oplus V_-(G) = V(G) \tag{1.17}$$

1.2. **Stable invariant manifold.** It is well-known, that for each $y_0 \in V$ there exists a unique solution $y(t, x) \in C(0, T; V(G))$ of problem (1.7)-(1.10), where $T > 0$ is arbitrary fixed number. We denote by $S(t, y_0)$ the solution operator of the boundary value problem (1.7)-(1.10):

$$S(t, y_0) = y(t, \cdot) \tag{1.18}$$

where $y(t, x)$ is the solution of (1.7)-(1.10).

Recall now some commonly used definitions of stable invariant manifold (see Chapter V in [BV]) adopted for our case.

The origin of the phase space V, i.e. the function $y(x) \equiv 0$, is, evidently, a steady-state solution of problem (1.7)-(1.10).

Definition 1.1. *The set $M_- \subset H$ defined in a neighborhood of the origin is called the stable invariant manifold if for each $y_0 \in M_-$ the solution $S(t, y_0)$ is well-defined and belongs to M_- for each $t > 0$, and*

$$\|S(t, y_0)\|_V \le ce^{-rt} \quad as \quad t \to \infty \tag{1.19}$$

where $0 < r < \lambda_{N+1}$.

The stable invariant manifold can be defined as a graph in the phase space $V = V_+ \oplus V_-$ by the formula

$$M_- = \{y_- + F(y_-), \ y_- \in \mathcal{O}(V_-)\} \tag{1.20}$$

where $\mathcal{O}(V_-)$ is a neighborhood of the origin in the subspace V_-, and

$$F : \mathcal{O}(V_-) \to V_+ \tag{1.21}$$

is a certain map satisfying

$$\|F(y_-)\|_{V_+} / \|y_-\|_{V_-} \to 0 \quad \text{as} \quad \|y_-\|_{V_-} \to 0. \tag{1.22}$$

So, in order to construct the invariant manifold M_- we have to calculate the map (1.21), (1.22).

2. Preliminaries

To get functional-analytic decompozition of the map F that defines stable invariant manifold, we have to derive differential equation for F

2.1. **Equation for F.** First of all we recall derivation of well-known equation for map (1.21) that determines invariant manifold M_-. After that we recall definitions of certain notions that we use later.

Let us introduce several notations. We rewrite equations (1.7),(1.10) using definition (1.12) of operator A as follows:

$$\partial_t y(t) + Ay(t) + B(\cdot, y(t)) = 0 \tag{2.1}$$

Define the orthoprojectors

$$P_+ : V \to V_+, \quad P_- : V \to V_- \tag{2.2}$$

and introduce notations

$$P_+ y = y_+, \quad P_- y = y_-, \quad P_+ S(t, y_0) = S_+(t, y_0), \quad P_- S(t, y_0) = S_-(t, y_0) \tag{2.3}$$

Taking into account that the spaces V_+, V_- are invariant with respect to acting of operator A and using notations (2.3) we can rewrite (2.1) as follows:

$$\begin{aligned} \partial_t y_+(t) + Ay_+(t) + P_+ B(\cdot, y_+(t) + y_-(t)) = 0 \\ \partial_t y_-(t) + Ay_-(t) + P_- B(\cdot, y_+(t) + y_-(t)) = 0 \end{aligned} \tag{2.4}$$

Let $y_0 \in M_-$. Then by (1.20) it has the form $y_0 = y_- + F(y_-)$. By definition of an invariant manifold for each $t \in \mathbb{R}_+$ $S(t, y_0) \in M_-$ or, what is equivalent

$$S_+(t, y_- + F(y_-)) = F(S_-(t, y_- + F(y_-)))$$

We differentiate this equation with respect to t and express t-derivatives with help of equations (2.4). As a result we get:

$$\begin{aligned} &AS_+(t, y_- + F(y_-)) + P_+ B(\cdot, S(t, y_- + F(y_-))) \\ =&\langle F'(S_-(t, y_- + F(y_-))), AS_-(t, y_- + F(y_-)) \\ &+ P_- B(\cdot, S_+(t, y_- + F(y_-)) + S_-(t, y_- + F(y_-)))\rangle \end{aligned} \tag{2.5}$$

where by $\langle F'(z), h\rangle$ we denote the value of derivative $F'(z)$ on vector h. Passing to limit in (2.5) as $t \to 0$ we get the desired equation for F:

$$AF(y_-) + P_+ B(\cdot, y_- + F(y_-)) = \langle F'(y_-), Ay_- + P_- B(\cdot, y_- + F(y_-))\rangle \tag{2.6}$$

2.2. **Analytic maps.** Let H_i be Hilbert spaces with the scalar products $(\cdot,\cdot)_i$ and the norms $\|\cdot\|_i$ where $i = 1, 2$. We denote by $(H_1)^k = H_1 \times \cdots \times H_1$ (k times) the direct product of k copies of H_1 and define by $F_k : (H_1)^k \to H_2$ a k-linear operator $F_k(h_1, \dots, h_k)$, i.e. the operator that is linear with respect to each variable $h_i, i = 1, \dots, k$. Then

$$\|F_k\| = \sup_{\|h_i\|_1 = 1, i=1,\dots,k} \|F_k(h_1, \dots, h_k)\|_2 \tag{2.7}$$

Restriction of k-linear operator $F_k(h_1, \dots, h_k)$ to diagonal $h_1 = \cdots = h_k = h$ is called power operator of order k:

$$F_k(h) = F_k(h, \dots, h) \tag{2.8}$$

Using derivatives one can restore k-linear operator $F_k(h_1, \ldots, h_k)$ by power operator $F_k(h)$.

Denote by $\mathcal{O}(H_1)$ a neighbourhood of origin in the space H_1. The map

$$F : \mathcal{O}(H_1) \to H_2 \tag{2.9}$$

is called analytic if it can be decomposed in the serie

$$F(h) = F_0 + \sum_{k=1}^{\infty} F_k(h) \tag{2.10}$$

where $F_0 \in H_2$ and $F_k(h)$ are power operators of order k. Serie (2.10) converges if the numerical serie $\|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k(h)\|_2$ converges.

Proposition 2.1. *Let norms (2.7) of power operator $F_k(h)$ from (2.10) satisfy*

$$\|F_k\| \le \gamma \rho^{-k} \tag{2.11}$$

where $\gamma > 0, \rho > 0$ do not depend on k. Then serie (2.10) converges for each h belonging to the ball $B_\rho(H_1) = \{h \in H_1 : \|h\|_1 < \rho\}$.

Proof. There exists $\varepsilon > 0$ such that $\|h\|_1 \le \rho - \varepsilon$. Then using (2.7), (2.11) we get

$$\|F(h)\|_2 \le \|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k\| \|h\|_1^k \le \gamma \sum_{k=1}^{\infty} (\frac{\rho - \varepsilon}{\rho})^k < \infty$$

□

2.3. **Operators from equation for F and their kernels.** We consider here operators from equation (2.6).

2.3.1. *Subspapces $V_\pm$ and projectors $P_\pm$.* Subcpaces V_+, V_- of V are defined in (1.16), and projectors $P_\pm$ are defined in (2.2). Orthogonality of decomposition (1.16) as well as orthogonality of projectors (2.2) take place with respect to the scalar product corresponding to norm (1.14).Therefore

$$\|P_+\| \le 1, \qquad \|P_-\| \le 1 \tag{2.12}$$

Kernels $\widehat{P}_\pm$ of operators $P_\pm$, i.e. distributions on $G \times G$ such that

$$(P_\pm v)(x) = \int_G \widehat{P}_\pm(x, \xi) v(\xi) d\xi \quad \forall v(\xi) \in V \tag{2.13}$$

are defined as follows:

$$\widehat{P}_+(x, \xi) = \sum_{k=1}^{N} e_k(x) e_k(\xi), \quad \widehat{P}_-(x, \xi) = \delta(x - \xi) - \sum_{k=1}^{N} e_k(x) e_k(\xi), \tag{2.14}$$

where $\delta(x - \xi)$ is Dirac δ-function. Note that integral in (2.13) in the case $\widehat{P}_-(x, \xi)$ is understood (at each fixed x) as value of distribution $\delta(x - \xi) - \sum_{k=1}^{N} e_k(x) e_k(\xi)$ on the test function $v(x)$. Such notation for values of distributions will be often used below without additional expleinations.

2.3.2. *Analyticity of the map $B(\cdot, y)$.* We intend to decompose operator $B(\cdot, y)$ defined in (1.10) in series (2.10). For this we use that the phase space V is the algebra, i.e. in this space the operation of multiplcation of functions is well-defined.

Define the operator of multiplcation Γ_k as follows:

$$\Gamma_k : V^k \longrightarrow V, \qquad \Gamma_k(v_1, \ldots, v_k)(x) = v_1(x) \cdot \cdots \cdot v_k(x) \tag{2.15}$$

where $V^k = V \times \cdots \times V$ (k times),

Lemma 2.1. *Let $V = H^2(G) \cap H_0^1(G)$, $G \subset \mathbb{R}^n$, $n = 1, 2, 3$. Then operator Γ_k defined in (2.15) is k-linear bounded operator. Moreover, there exists a constatnt $\gamma > 0$ such that for each k*

$$\|\Gamma_k(v_1, \ldots, v_k)\|_V \leq \gamma^{k-1}\|v_1\|_V \cdot \cdots \cdot \|v_k\|_V \tag{2.16}$$

Proof. Since norm (1.14) is equivalent to the norm of Sobolev space $H^2(G)$, we can use H^2-norm. Taking into account that embeddings $H^2(G) \subset C(\overline{G})$ and $H^2(G) \subset W_4^1(G)$ are continious we get:

$$\|v_1 \cdot v_2\|_{H^2(G)} = (\sum_{\|\alpha| \leq 2} \int |D^\alpha(v_1(x)v_2(x))|^2 dx)^{1/2}$$

$$\leq \|v_1\|_{H^2}\|v_2\|_C + \|v_1\|_C\|v_2\|_{H^2} + 2\|v_1\|_{W_4^1}\|v_2\|_{W_4^1} \leq \gamma\|v_1\|_{H^2}\|v_2\|_{H^2}$$

Using this inequality we obtain (2.15) by induction in k □

It follows from Lemma 2.1 and (1.10) that for $y \in V$

$$B(x, y(x)) = \Gamma_3(y, y, y)(x) + 3\widehat{v}(x)\Gamma_2(y, y)(x) \tag{2.17}$$

Therefore operator B is analytic, and relation (2.17) is its decomposition in series (2.10). The kernels of operators from (2.17) are as follows:

$$\widehat{\Gamma}_3(x; \xi_1, \xi_2, xi_3) = \delta(x - \xi_1)\delta(x - \xi_2)\delta(x - \xi_3) \tag{2.18}$$

$$3\widehat{v}(x)\widehat{\Gamma}_2(x; \xi_1, \xi_2) = 3\widehat{v}(x)\delta(x - \xi_1)\delta(x - \xi_2) \tag{2.19}$$

2.4. **Series for operator F.** Let us consider the special case when $H_1 = V_-, H_2 = V_+$ with Hilbert spaces V_-, V_+ defined in(1.16). In this case analytic map (2.9), (2.10) can be rewritten as follows:

$$F : \mathcal{O}(V_-) \to V_+, \quad F(y_-) = \sum_{k=2}^{\infty} F_k(y_-) \tag{2.20}$$

We asume that $F_0 = 0$, $F_1 = 0$ because by (1.22) the map F defining stable invariant manifold M_- has just this form.

Since $F(y_-) \in V_+$, it is a function depending on argument x:

$$F(y_-) \equiv F(x; y_-)$$

Now we define kernels $\widehat{F}_k(x; \xi_1, \ldots, \xi_k)$ of k-linear operator $F_k(\cdot; y_1, \ldots, y_k), y_j \in V_-, j = 1, \ldots, k$. Let

$$V \subset L_2(G) \subset V' \tag{2.21}$$

where V' is the space dual to V with respect to duality generated by scalar product in $L_2(G)$. Define

$$V'_- = \{u(x) \in V' : \int u(x)\varphi(x)dx = 0 \quad \forall\varphi \in V_+\} = V_+^\perp \tag{2.22}$$

Below we use the following notation:

$$\overline{\xi^k} = (\xi_1, \dots, \xi_k), \quad d\overline{\xi^k} = d\xi_1 \dots d\xi_k, \quad \text{where } \xi_j \in G, \ j = 1, \dots, k \tag{2.23}$$

$$y_-(\overline{\xi^k}) = y_-(\xi_1) \cdot \dots \cdot y_-(\xi_k), \quad y(\overline{j^k}; \overline{\xi^k}) = y_{j_1}(\xi_1) \cdot \dots \cdot y_{j_k}(\xi_k) \tag{2.24}$$

The kernel $\widehat{F}_k(x; \overline{\xi^k}), x \in G, \overline{\xi^k} \in G^k \equiv G \times \dots \times G$ (k times) belongs to the space $V_+ \otimes (\overset{k}{\otimes} V'_-)$ where $\overset{k}{\otimes} V'_- = V' \otimes \dots \otimes V'$ (k times), i.e. $\widehat{F}_k(x; \overline{\xi^k})$ is a distribution on G^k with values in V_+, such that for each $y_j \in V, j = 1, \dots, k$ the value

$$F_k(x; y_1, \dots, y_k) = \int \widehat{F}_k(x; \overline{\xi^k}) y(\overline{j^k}; \overline{\xi^k}) \ d\overline{\xi^k} \tag{2.25}$$

of distribution $\widehat{F}_k(x; \overline{\xi^k})$ on test function $y(\xi_1) \cdot \dots \cdot y(\xi_k)$ is well-defined. Moreover, if $y_j \in V_+$ at least for one $j \in \{1, \dots, k\}$ then right hand side of equality (2.25) equals zero. Moreover, since $F_k(\cdot, y_1, \dots, y_k)$ is symmetric with respect to $(y_1, \dots, y_k)$, i.e. $F_k(\cdot, y_1, \dots, y_k) = F_k(\cdot, y_{j_1}, \dots, y_{j_k})$ for each permutation $(j_1, \dots, j_k)$ of $(1, \dots, k)$, we can assume that the distribution $\widehat{F}_k(x; \overline{\xi^k})$ is symmetric with respect to $(\xi_1, \dots, \xi_k)$

Now using (2.25) and (2.24) we can rewrite the series from (2.20) in the form:

$$F(x, y_-) = \sum_{k=2}^{\infty} \int \widehat{F}_k(x; \overline{\xi^k}) y_-(\overline{\xi^k}) d\overline{\xi^k} \tag{2.26}$$

In accordance with (2.7) we define the norm $\|F_k\|$ of $\widehat{F}_k(x; \overline{\xi^k}) \in V_+ \otimes (\overset{k}{\otimes} V'_-)$ by the following way:

$$\begin{aligned} \|F_k\| &= \sup_{\substack{\|y_j\|_{V_-}=1 \\ j=1,\dots,k}} \|F_k(\cdot, y_1, \dots, y_k)\|_{V_+} \\ &= \sup_{\|y_+\|_{V_+}=1} \sup_{\substack{\|y_j\|_{V_-}=1 \\ j=1,\dots,k}} \int y_+(x) \widehat{F}_k(x; \overline{\xi^k}) y(\overline{j^k}; \overline{\xi^k}) \ dx \ d\overline{\xi^k} \end{aligned} \tag{2.27}$$

For each function or distribution $K(\eta_1, \dots, \eta_r)$ defined on G^r we determine the function $\sigma_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r)$ which is simmetric with respect to arbitrary permutation $(\eta_{j_1}, \dots, \eta_{j_r})$ of variables $(\eta_1, \dots, \eta_r)$ by the formula:

$$\sigma_{\overline{\eta^r}} K(\eta_1, \dots \eta_r) = \frac{1}{r!} \sum_{(j_1, \dots, j_r)} K(\eta_{j_1}, \dots, \eta_{j_r}) \tag{2.28}$$

where the sum in the r.h.s. of (2.28) performs over all permutations $(j_1, \dots, j_r)$ of the set $(1, \dots, r)$.

Lemma 2.2. *Let $K(\eta_1, \dots, \eta_r)$ be defined on G^r. Then*

(a) *The following equality is true:*

$$\int K(\overline{\eta^r}) h(\overline{j^r}; \overline{\eta^r}) \ d\overline{\eta^r} = \int \left(\sigma_{\overline{\eta^r}} K(\overline{\eta^r})\right) h(\overline{j^r}; \overline{\eta^r}) \ d\overline{\eta^r} \tag{2.29}$$

for any $h(\overline{j^r}; \overline{\eta^r})$ such that the serie in the l.h.s. converges,

(b) *For any function $G(\eta_1, \dots, \eta_r)$ simmetric in its arguments*

$$G(\overline{\eta^r}) \sigma_{\overline{\eta^r}} K(\overline{\eta^r}) = \sigma_{\overline{\eta^r}} [G(\overline{\eta^r}) K(\overline{\eta^r})] \tag{2.30}$$

(c) *If all distributions* $F_k(x;\overline{\eta^k}) \in V_+ \otimes (\overset{k}{\otimes} V'_-)$ *from (2.26) are symmetric in their arguments* $\overline{\eta^k}$ *then these distributions are defined uniquely by values of analytic functions* $F(y_-),\ y \in V_-$ *from (2.26).*

The proof of this Lemma is evident.

3. Formal construction of the map F

We look for the map defining stable invariant manifold in the form of a series (2.26).In this section we find recurrence relations for kernals $\widehat{F}_k(x;\overline{\xi^k})$.

3.1. **Calculation of** $\widehat{F}_2(x;\overline{\xi^2})$**.** Below using k-linear operators $F_k(x;y,\dots,y) = F_k(x;y)$ (k times) we omitt sometimes variable x writing $F_k(y)$. After substitution (2.26) into (2.6) we get taking into account (1.10) that for each $y = y_- \in V_-$

$$\begin{aligned}&\sum_{q=2}^{\infty} AF_q(y) + P_+\left[\sum_{k=2}^{3} a_k \sum_{j=0}^{k} C_k^j F^j(y) y^{k-j}\right] \\ &= \sum_{q=2}^{\infty} qF_q\left(y,\dots,y,Ay + P_-\left[\sum_{k=2}^{3} a_k \sum_{j=0}^{k} C_k^j F^j(y) y^{k-j}\right]\right)\end{aligned} \tag{3.1}$$

where $C_k^j = k!/(j!(k-j)!)$ and

$$a_3 \equiv a_3(x) \equiv 1, \quad a_2 \equiv a_2(x) = 3\widehat{v}(x) \tag{3.2}$$

Let us equate the terms from (3.1) of the second order with respect to y:

$$AF_2(y,y) + 3P_+(\widehat{v}y^2) = 2F_2(y,Ay)$$

Using kernels of bilinear operator $F_2(y,y)$ we can rewrite this relation as follows:

$$\int [\widehat{F}_2(x;\overline{\xi^2})(A_{\xi_1} + A_{\xi_2})y(\overline{\xi^2})) - A_x\widehat{F}_2(x;\overline{\xi^2})y(\overline{\xi^2})]d\overline{\xi^2} = 3P_+(\widehat{v}y^2)(x) \tag{3.3}$$

where subscript of operator A indicates independent variable of a function to that this operator A is applied. We will use notation:

$$A_{\overline{\xi^k}} = \sum_{j=1}^{k} A_{\xi_j} \tag{3.4}$$

Carrying operator $A_{\overline{\xi^2}}$ from $y(\overline{\xi^2})$ to $\widehat{F}_2(x;\overline{\xi^2})$ and using operator (2.15) in right side of (3.3) we get:

$$\int (A_{\overline{\xi^k}} - A_x)\widehat{F}_2(x;\overline{\xi^2})y(\overline{\xi^2})d\overline{\xi^2} = 3\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{\Gamma}_2(\eta;\overline{\xi^2})y(\overline{\xi^2})d\eta d\overline{\xi^2} \tag{3.5}$$

Since $y \in V_-$ and subspaces V_+, V'_- are invariant with respect of operator A, we obtain from (3.5) the relation determining $\widehat{F}_2$:

$$\widehat{F}_2(x;\overline{\xi^2}) = 3(A_{\overline{\xi^k}} - A_x)^{-1}\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{\Gamma}_2(\eta;\overline{\zeta^2})\widehat{P}_-(\overline{\zeta^2};\overline{\xi^2})d\overline{\zeta^2} \tag{3.6}$$

Note that operator $(A_{\overline{\xi^k}} - A_x)^{-1}$ is well-defined. Moreover, the following assertion hold (recall that we define the norm of the space $V_+ \otimes (\overset{k}{\otimes} V'_-)$ by (2.27)):

Lemma 3.1. *Operator*

$$(A_{\overline{\xi^k}} - A_x)^{-1} : V_+ \otimes (\overset{k}{\otimes} V'_-) \longrightarrow V_+ \otimes (\overset{k}{\otimes} V'_-) \tag{3.7}$$

is well-defined and bounded, and for its norm the following estimate holds:

$$\|(A_{\overline{\xi^k}} - A_x)^{-1}\| \leq b(k+1) \tag{3.8}$$

with certain constant $b > 0$

The proof of this Lemma will be presented in some other place.

At last we write down the recurrence relation for the kernel $\widehat{F}(x;\overline{\xi^3})$ that can be obtained similarly to the formula (3.6)

$$\begin{aligned}
&\widehat{F}_3(x;\overline{\xi^3}) = (A_{\overline{\xi^k}} - A_x)^{-1} S_{\overline{\xi^3}} \left[\int \widehat{P}_+(x,\eta)\widehat{\Gamma}_3(\eta;\overline{\zeta^3})\widehat{P}_-(\overline{\zeta^3};\overline{\xi^3}) d\eta d\overline{\zeta^3} \right. \\
&\quad + 6 \int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{F}_2(\eta;\overline{\zeta^2})\delta(\eta-\zeta_3)\widehat{P}_-(\overline{\zeta^3};\overline{\xi^3}) d\eta d\overline{\zeta^3} \\
&\quad \left. -2 \int \widehat{F}_2(x;\overline{\eta^2})\widehat{P}_-(\eta_1,\xi_3)\widehat{P}_-(\eta_2,s)\widehat{v}(s)\widehat{\Gamma}_2(s;\overline{\zeta^2})\widehat{P}_-(\overline{\zeta^2};\overline{\xi^2}) d\overline{\eta^2} ds d\overline{\zeta^2} \right]
\end{aligned} \tag{3.9}$$

3.2. **Calculation of $\widehat{F}_q(x;\overline{\xi^q})$.** Let us rewrite equality (3.1) as follows:

$$\begin{aligned}
&\sum_{q=2}^{\infty} AF_q(y) + P_+ \left[\sum_{k=2}^{3} a_k y^k \right] + I_1 \\
&= \sum_{q=2}^{\infty} qF_q \left(y,\ldots,y, Ay + P_- \left[\sum_{k=2}^{3} a_k y^k \right] \right) + I_2
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
I_1 =& P_+ \left[\sum_{k=2}^{3} a_k \sum_{j=1}^{k} C_k^j y^{k-j} \sum_{m_1=2}^{\infty} F_{m_1}(y) \cdots \sum_{m_j=2}^{\infty} F_{m_j}(y) \right] \\
I_2 =& \sum_{q=2}^{\infty} qF_q \left(y,\ldots,y, P_- \left[\sum_{k=2}^{3} a_k \sum_{j=1}^{k} C_k^j y^{k-j} \sum_{m_1=2}^{\infty} F_{m_1}(y) \cdots \sum_{m_j=2}^{\infty} F_{m_j}(y) \right] \right)
\end{aligned} \tag{3.11}$$

Writing operators with help of their kernels we get

$$\left(P_+ \sum_{k=2}^{3} a_k y^k \right)(x) = \sum_{k=2}^{3} \int \widehat{P}_+(x,s) a_k(s) \widehat{\Gamma}_k(s;\overline{\xi^k}) y(\overline{\xi^k}) \; ds \; d\overline{\xi^k} \tag{3.12}$$

where, recall, we use notations (2.23), (2.24). Similarly we obtain:

$$
\begin{aligned}
& I_1 \equiv I_1(x) = \sum_{k=2}^{3}\sum_{j=1}^{k}\sum_{p=2j}^{\infty}\sum_{m_1+\dots+m_j=p} C_k^j \\
& \times \int \widehat{P}_+(x,s)a_k(s)(\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{k-j}(s;\cdot))(\overline{\xi^{p+k-j}})y(\overline{\xi^{p+k-j}})\ ds\ d\overline{\xi^{p+k-j}} \\
& \text{(doing change of variables} \qquad (k,j,p)\longrightarrow(q,j,p): \quad q=k-j+p) \\
& = \sum_{q=3}^{\infty}\sum_{(j,q)\in Q_q}\sum_{m_1+\dots+m_j=p} C_{q-p+j}^j \\
& \times \int \widehat{P}_+(x,s)a_{q-p+j}(s)(\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{q-p}(s;\cdot))(\overline{\xi^q})y(\overline{\xi^q})\ ds\ d\overline{\xi^q}
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
& Q_q = \{(j,p)\in\mathbb{N}^2 : 2\le q-p+j\le 3, 1\le j\le q-p+j, p\ge 2j\} \\
& = (\text{if } q\ge 4)\ \{(j,p)\in\mathbb{N}^2 : (1,q-2),(1,q-1),(2,q-1),(2,q),(3,q),(3,q+1)\}, \\
& Q_3 = \{(j,p)\in\mathbb{N}^2 : (1,2),(2,2),(2,3),(3,3),(3,4)\}
\end{aligned}
\tag{3.14}
$$

Besides, we get

$$
\begin{aligned}
\sum_{q=2}^{\infty} qF_q\,(x;y,\dots,y,Ay) &= \sum_{q=2}^{\infty}\int \widehat{F}_q(x;\overline{\xi^q})A_{\overline{\xi^q}}y(\overline{\xi^q})\ d\overline{\xi^q} \\
&= \sum_{q=2}^{\infty}\int A_{\overline{\xi^q}}\widehat{F}_q(x;\overline{\xi^q})y(\overline{\xi^q})\ d\overline{\xi^q}
\end{aligned}
\tag{3.15}
$$

Using notation

$$
\widehat{F_qP_-}(x;s,\overline{\zeta^{q-1}}) = \int \widehat{F}_{q-1}(x;\eta,\overline{\zeta^{q-2}})\widehat{P}_-(\eta,s)d\eta
$$

we can write

$$
\begin{aligned}
& \sum_{q=2}^{\infty} qF_q\left(x;y,\dots,y,P_-\sum_{k=2}^{3}a_ky^k\right) \\
& = \sum_{n=2}^{\infty}\sum_{m=2}^{3}\int n\widehat{F_nP_-}(x;s,\overline{\zeta^{n-1}})a_m(s)\widehat{\Gamma}(s;\overline{\eta^m})y(\overline{\zeta^{n-1}})y(\overline{\eta^m})\ ds\ d\overline{\zeta^{n-1}}\ d\overline{\eta^m} \\
& = \sum_{q=3}^{\infty}\sum_{\substack{n+m=q+1\\ n\ge 2, m=2,3}}\int na_m(s)(\widehat{F_nP_-}(x;s,\cdot)\widehat{\Gamma}_m(s;\cdot))(\overline{\xi^q})y(\overline{\xi^q})\ ds\ d\overline{\xi^q}
\end{aligned}
\tag{3.16}
$$

At last

$$I_2(x) = \sum_{r=2}^{\infty}\sum_{k=2}^{3}\sum_{j=1}^{k} C_k^j \sum_{p=2j}^{\infty} \sum_{m_1+\dots+m_j=p} \int r\widehat{F_r P_-}(x;s,\overline{\zeta^{r-1}})a_k(s)$$
$$(\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{k-j}(s;\cdot))(\overline{\xi^{p+k-j}})y(\overline{\zeta^{r-1}})y(\overline{\xi^{p+k-j}})\ ds\ d\overline{\zeta^{r-1}}\ d\overline{\xi^{p+k-j}}$$
$$\text{(changing variables} \qquad (k,r,j,p) \longrightarrow (q,r,j,p): \quad q = p+r+k-j-1)$$
$$= \sum_{q=4}^{\infty} \sum_{(r,p,j)\in R_q} \sum_{m_1+\dots+m_j=p} C^j_{q-p-r+j+1} \int a_{q-p-r+j+1}(s)$$
$$\times\, r(\widehat{F_r P_-}(x;s,\cdot)\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{q-p-r+1}(s;\cdot))(\overline{\xi^q})y(\overline{\xi^q})\ ds\ d\overline{\xi^q} \tag{3.17}$$

where

$$R_q = \{(r,p,j)\in\mathbb{N}^3 : 1\le q-p-r+j\le 2, 1\le j\le q-p-r+j+1, p\ge 2j, r\ge 2\} \tag{3.18}$$

After substitution (3.12),(3.15)(3.16) into (3.10) and doing some simple transformation we get

$$\sum_{q=2}^{\infty}\int (A_{\overline{\xi^q}} - A_x)\widehat{F}_q(x;\overline{\xi^q})y(\overline{\xi^q})\ d\overline{\xi^q} = \sum_{k=2}^{3}\int \widehat{P}_+(x,s)a_k(s)\widehat{\Gamma}_k(s;\overline{\xi^k})y(\overline{\xi^k})ds d\overline{\xi^k}$$
$$-\sum_{q=3}^{\infty}\sum_{\substack{n+m=q+1\\ n\ge 2, m=2,3}} \int n a_m(s)(\widehat{F_n P_-}(x;s,\cdot)\widehat{\Gamma}_m(s;\cdot))(\overline{\xi^q})y(\overline{\xi^q})\ ds\ d\overline{\xi^q} + I_1(x) - I_2(x) \tag{3.19}$$

where $I_1(x), I_2(x)$ are defined in (3.13),(3.17). In order to derive from (3.19) recurrence relation for $\widehat{F}_q(x;\overline{\xi^q})$ we i) make the change $y(\overline{\xi^q}) = \widehat{P}_-(\overline{\xi^q};\overline{\zeta^q})z(\overline{\zeta^q})$ with arbitrary $z(\zeta)\in V$ where we use the notation

$$\widehat{P}_-(\overline{\zeta^k};\overline{\xi^k}) = \widehat{P}_-(\zeta_1,\xi_1)\cdot\dots\cdot\widehat{P}_-(\zeta_k,\xi_k) \tag{3.20}$$

for kernel of tensor product $P_-\otimes\dots\otimes P_-$ (k times) for projection operator P_-; ii) apply symmetrization operator $\sigma_{\overline{\zeta^q}}$ and avoid $z(\overline{\zeta^q})$; and iii) using Lemma 3.1 invert operator $A_{\overline{\xi^q}} - A_x$. As a result (renaming coordinates) we get the recurrence relation for $\widehat{F}_q(x;\overline{\xi^q})$ with $q\ge 4$:

$$\widehat{F}_q(x;\overline{\xi^q}) = (A_{\overline{\xi^q}} - A_x)^{-1}\sigma_{\overline{\xi^q}}(J_1(x;\overline{\xi^q}) - J_2(x;\overline{\xi^q}) - J_3(x;\overline{\xi^q})) \tag{3.21}$$

where

$$J_1(x;\overline{\xi^q}) = \sum_{(j,q)\in Q_q}\sum_{m_1+\dots+m_j=p} C^j_{q-p+j}\times$$
$$\int \widehat{P}_+(x,s)a_{q-p+j}(s)(\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{q-p}(s;\cdot))(\overline{\zeta^q})\widehat{P}_-(\overline{\zeta^q};\overline{\xi^q})\ ds\ d\overline{\zeta^q} \tag{3.22}$$

$$J_2(x;\overline{\xi^q}) = \sum_{\substack{n+m=q+1\\ n\ge 2, m=2,3}} \int n a_m(s)(\widehat{F_n P_-}(x;s,\cdot)\widehat{\Gamma}_m(s;\cdot))(\overline{\zeta^q})\widehat{P}_-(\overline{\zeta^q};\overline{\xi^q})\ ds\ d\overline{\zeta^q} \tag{3.23}$$

$$J_3(x;\overline{\xi^q}) = \sum_{(r,p,j)\in R_q} \sum_{m_1+\dots+m_j=p} C^j_{q-p-r+j+1} \int a_{q-p-r+j+1}(s) \tag{3.24}$$
$$\times\, r(\widehat{F_r P_-}(x;s,\cdot)\widehat{F}_{m_1}(s;\cdot)\dots\widehat{F}_{m_j}(s;\cdot)\widehat{\Gamma}_{q-p-r+1}(s;\cdot))(\overline{\zeta^q})\widehat{P}_-(\overline{\zeta^q};\overline{\xi^q})\ ds\ d\overline{\zeta^q}$$

Thus we have proven the following Theorem:

Theorem 3.1. *The kernels $\widehat{F}_q(x;\overline{\xi^q})$ from decopmposition (2.26) of the map $F(x;y)$ defining stable invariant manifold are defined in (3.6) (for $q=2$), in recurrence relation (3.9) (for q=3) and in (3.21)-(3.24) (for $q\geq 4$).*

4. Analyticity of the map F

In this section we prove convergence of serie (2.26) for map $F(x;y)$ defining stable invariant manifold.

4.1. Estimate of norm for $\widehat{F}_q(x;\overline{\xi^q})$.

First of all we have to recall that the norm of the space $V_+\otimes(\overset{k}{\otimes}\ V'_-)$ is defined by relation (2.27)) In virtue of (3.21) and Lemmas 2.2, 3.1

$$\|F_q\|\equiv\|\widehat{F}_q\|\leq b(q+1)^{-1}(\|J_1\|+\|J_2\|+\|J_3\|)\quad \text{for}\quad q\geq 4 \tag{4.1}$$

where kernels J_1, J_2, J_3 are defined in (3.22), (3.23), (3.24). Let us estimate now these kernels. Using that $P_+y_+=y_+$, for $y_+\in V_+$, $P_-y_-=y_-$, for $y_-\in V_-$, and taking into accout Lemmas 2.1, 2.2 we get from (3.22)

$$\|J_1\| = \sup_{\|y_+\|_{V_+}=1}\ \sup_{\substack{\|y_r\|_{V_-}=1\\ r=1,\dots,q}} \sum_{(j,q)\in Q_q}\ \sum_{m_1+\dots+m_j=p} C^j_{q-p+j}\times$$
$$\int y_+(x)a_{q-p+j}(x)F_{m_1}(x;y_1,\dots)\dots F_{m_j}(x;\dots,y_p)y_{p+1}(x)\dots y_q(x)\ dx \tag{4.2}$$
$$\leq \sum_{(j,q)\in Q_q}\ \sum_{m_1+\dots+m_j=p} C^j_{q-p+j}\|a_{q-p+j}\|_V\|F_{m_1}\|\dots\|F_{m_j}\|\gamma^{q-p+j}$$

Similarly, we get from (3.23)

$$\|J_2\| = \sup_{\substack{\|y_+\|_{V_+}=1\\ \|y_r\|_{V_-}=1\\ r=1,\dots,q}}\ \sum_{\substack{n+m=q+1\\ n\geq 2,m=2,3}} \int y_+(x)nF_n(x;y_1,\dots,y_{n-1},P_-(a_my_n\dots y_q))\ dx \tag{4.3}$$
$$\leq \sum_{\substack{n+m=q+1\\ n\geq 2,m=2,3}} n\|F_n\|\|a_m\|_V\gamma^m$$

At last we obtain from (3.24)

$$\|J_3\| = \sup_{\substack{\|y\|_{V_+}=1\\ \|y_r\|_{V_-}=1\\ r=1,\dots,q}}\ \sum_{(r,p,j)\in R_q}\ \sum_{m_1+\dots+m_j=p} C^j_{q-p-r+j+1}\int rF_r(x;P_-(y_1\dots y_{q-p-r+1}$$
$$\times\, a_{q-p-r+j+1}F_{m_1}(\cdot;y_{q-p-r+2},\dots)\dots F_{m_j}(\cdot;\dots,y_{q-r+1})),y_{q-r+2},\dots,y_q)y(x)dx$$
$$\leq \sum_{\substack{(r,p,j)\in R_q\\ m_1+\dots+m_j=p}} C^j_{q-p-r+j+1}r\|a_{q-p-r+j+1}\|_V\|F_r\|\|F_{m_1}\|\dots\|F_{m_j}\|\gamma^{q-p-r+j+1} \tag{4.4}$$

Let

$$\|a_k\|_V \le b_0/b \qquad k = 2,3$$

Then summarizing (4.1)–(4.4) we obtain the following estimate for $\|F_q\|$ for $q \ge 4$:

$$\|F_q\| \le \frac{b_0}{q+1} \left(\sum_{(j,q)\in Q_q} \sum_{m_1+\cdots+m_j=p} C^j_{q-p+j} \|F_{m_1}\| \ldots \|F_{m_j}\| \gamma^{q-p+j} \right.$$
$$\left. + \sum_{\substack{n+m=q+1\\ n\ge 2, m=2,3}} n\|F_n\|\gamma^m + \sum_{(r,p,j)\in R_q} \sum_{m_1+\cdots+m_j=p} C^j_\kappa r \|F_r\| \|F_{m_1}\| \ldots \|F_{m_j}\| \gamma^\kappa \right) \tag{4.5}$$

where $\kappa = q - p - r + j + 1$. Similarly we get from (3.6), (3.9):

$$\|F_2\| \le b_0\gamma^2/3 \qquad \|F_3\| \le b_0(\gamma^3 + 4\gamma^2\|F_2\|)/4 \tag{4.6}$$

Thus, we have proven the following Lemma

Lemma 4.1. *The norms $\|F_2\|, \|F_3\|$ satisfy inequality (4.6), and the norms $\|F_q\|$ for $q \ge 4$ satisfy estimate (4.5)*

4.2. **Convergence of series for $F(x,y)$.** Define coefficients φ^q of the series

$$\varphi(\lambda) = \sum_{q=2}^{\infty} \varphi_q \lambda^q \tag{4.7}$$

by the relations

$$\varphi_2 = b_0\gamma^2/3 \qquad \varphi_3 = b_0(\gamma^3 + 4\gamma^2\varphi_2)/4, \tag{4.8}$$

and for $q \ge 4$

$$\varphi_q = b_0(q+1)^{-1} \left(\sum_{(j,q)\in Q_q} \sum_{m_1+\cdots+m_j=p} C^j_{q-p+j} \varphi_{m_1} \ldots \varphi_{m_j} \gamma^{q-p+j} \right.$$
$$\left. + \sum_{\substack{n+m=q+1\\ n\ge 2, m=2,3}} n\varphi_n\gamma^m + \sum_{(r,p,j)\in R_q} \sum_{m_1+\cdots+m_j=p} C^j_\kappa r \varphi_r \varphi_{m_1} \ldots \varphi_{m_j} \gamma^\kappa \right) \tag{4.9}$$

with $\kappa = q - p - r + j + 1$. Evidently,

$$\|F_q\| \le \varphi_q \qquad \text{for each} \quad q \ge 2 \tag{4.10}$$

and therefore to prove convergence of series (2.26) we have to prove convergence of series (4.7).

Theorem 4.1. *The series (4.7) with coefficients φ_q defined in (4.8), (4.9) converges for sufficiently small $|\lambda|$.*

Proof. Multiplying both parts of equality (4.9) on $(q+1)\lambda^q$, and both parts of equalities (4.8) on the same multiplier with $q = 2,3$, and summing obtained equalities over q from 2 to ∞ we get the equality:

$$\frac{\partial}{\partial\lambda}(\lambda\varphi(\lambda) = b_0(S_1(\lambda) + S_2(\lambda) + S_3(\lambda)) \tag{4.11}$$

where

$$S_1(\lambda) = \sum_{k=2}^{3} (\gamma\lambda)^k + \sum_{q=3}^{\infty} \lambda^q \sum_{(j,q)\in Q_q} \sum_{m_1+\cdots+m_j=p} C^j_{q-p+j} \varphi_{m_1} \ldots \varphi_{m_j} \gamma^{q-p+j}, \quad (4.12)$$

$$S_2(\lambda) = \sum_{q=3}^{\infty} \lambda^q \sum_{\substack{n+m=q+1 \\ n\geq 2, m=2,3}} n\varphi_n \gamma^m \quad (4.13)$$

$$S_3(\lambda) = \sum_{q=4}^{\infty} \lambda^q \sum_{(r,p,j)\in R_q} \sum_{m_1+\cdots+m_j=p} C^j_{q-p-r+j+1} r\varphi_r \varphi_{m_1} \ldots \varphi_{m_j} \gamma^{q-r+1} \quad (4.14)$$

Taking into account definition (3.14) of the set Q_q and doing change of variables $(q, j, p) \to (k, j, p): \ k = q + j - p$ in (4.12) we get:

$$\begin{aligned} S_1(\lambda) &= \sum_{k=2}^{3} (\gamma\lambda)^k + \sum_{k=2}^{3} (\gamma\lambda)^k \sum_{j=1}^{k} C^j_k \sum_{p=2j}^{\infty} \sum_{m_1+\cdots+m_j=p} \varphi_{m_1} \ldots \varphi_{m_j} \lambda^{p-j} \\ &= \sum_{k=2}^{3} (\gamma\lambda)^k + \sum_{k=2}^{3} (\gamma\lambda)^k \sum_{j=1}^{k} C^j_k \left(\frac{\varphi(\lambda)}{\lambda}\right)^j = \sum_{k=2}^{3} (\gamma\lambda)^k \left(1 + \frac{\varphi(\lambda)}{\lambda}\right)^k \quad (4.15) \\ &= \sum_{k=2}^{3} \gamma^k (\lambda + \varphi(\lambda))^k \end{aligned}$$

Changing order of summation in (4.13) we obtain:

$$S_2(\lambda) = \sum_{n=2}^{\infty} \sum_{m=2}^{3} n\varphi_n \gamma^m \lambda^{n+m-1} = \varphi'(\lambda) \sum_{m=2}^{3} (\gamma\lambda)^m \quad (4.16)$$

Changing variables $(k, r, j, p) \to (q, r, j, p): \ k = q - p - r + j + 1$ in (4.14) with help of definition (3.18) of the set R_q we get:

$$\begin{aligned} S_3(\lambda) &= \sum_{r=2}^{\infty} \sum_{k=2}^{3} \sum_{j=1}^{k} C^j_k \sum_{p=2j}^{\infty} \sum_{m_1+\cdots+m_j=p} \lambda^{p+r+k-j-1} r\varphi_r \varphi_{m_1} \ldots \varphi_{m_j} \gamma^k \\ &= \sum_{r=2}^{\infty} r\varphi_r \lambda^{r-1} \sum_{k=2}^{3} (\gamma\lambda)^k \sum_{j=1}^{k} C^j_k \left(\sum_{m=1}^{\infty} \varphi_m \lambda^m\right)^j / \lambda^j \quad (4.17) \\ &= \varphi'(\lambda) \sum_{k=2}^{3} (\gamma\lambda)^k \sum_{j=1}^{k} C^j_k \left(\frac{\varphi(\lambda)}{\lambda}\right)^j \end{aligned}$$

Relations (4.16), (4.17) imply:

$$S_2(\lambda) + S_3(\lambda) = \varphi'(\lambda) \sum_{k=2}^{3} (\gamma\lambda)^k \left(1 + \frac{\varphi(\lambda)}{\lambda}\right)^k = \varphi'(\lambda) \sum_{k=2}^{3} \gamma^k \left(\lambda + \varphi(\lambda)\right)^k \quad (4.18)$$

Therefore we get from (4.11), (4.15), (4.18):

$$\frac{\partial}{\partial\lambda}(\lambda\varphi(\lambda) = b_0 \left((1 + \varphi'(\lambda)) \sum_{k=2}^{3} \gamma^k \left(\lambda + \varphi(\lambda)\right)^k\right) \quad (4.19)$$

Doing in (4.19) change of functions

$$\psi(\lambda) = \lambda + \varphi(\lambda) \tag{4.20}$$

we obtain the equality

$$\frac{\partial}{\partial\lambda}(\lambda\psi(\lambda) - \lambda^2) = b_0 \frac{\partial}{\partial\lambda}\left(\sum_{k=2}^{3} \frac{\gamma^k}{k+1}\left(\psi(\lambda)\right)^{k+1}\right) \tag{4.21}$$

Since in (4.21) both expressions under sign of derivative equal zero at $\lambda = 0$, we derive that

$$\lambda\psi(\lambda) - \lambda^2 = b_0 \left(\sum_{k=2}^{3} \frac{\gamma^k}{k+1}\left(\psi(\lambda)\right)^{k+1}\right) \tag{4.22}$$

Doing in (4.22) change

$$\psi(\lambda) = \lambda z(\lambda) \tag{4.23}$$

we obtain the relation

$$F(z(\lambda), \lambda) \equiv z(\lambda) - 1 - b_0 \left(\sum_{k=2}^{3} \frac{\gamma^k \lambda^{k-1}}{k+1} z(\lambda)^{k+1}\right) = 0 \tag{4.24}$$

where the first equality is definition of the function $F(z, \lambda)$. Since (4.20), (4.23) imply that $z(0) = 1$, we get from definition (4.24) that function $F(z, \lambda)$ satisfies:

$$F(z, \lambda)|_{\substack{z=1\\ \lambda=0}} = 0, \qquad F'_z(z, \lambda)|_{\substack{z=1\\ \lambda=0}} = 1$$

Therefore by Implicite Function Theorem there exists a function $z(\lambda)$ analytic in a neighborhood of origin such that $z(0) = 1$ and $F(z(\lambda), \lambda) \equiv 0$. Hence, by (4.20), (4.23) the function $\varphi(\lambda) = \lambda(z(\lambda) - 1)$ defined in (4.7) is also analytic. $\square$

Theorem 4.1 and inequalities (4.10) imply

Theorem 4.2. *Let $F(\cdot, y_-)$ be the map (1.21), (1.22) that defines the stable invariant manifold (1.20). Then decomposition of this map in series (2.25) converges in a neighborhood of origin of the space V_-*

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