

Feedback Stabilization for Oseen Fluid Equations: A Stochastic Approach

Jinqiao Duan and Andrei V. Fursikov

Abstract. The authors consider stochastic aspects of the stabilization problem for two and three-dimensional Oseen equations with help of feedback control defined on a part of the fluid boundary. Stochastic issues arise when inevitable unpredictable fluctuations in numerical realization of stabilization procedures are taken into account and they are supposed to be independent identically distributed random variables. Under this assumption the solution to the stabilization problem obtained via boundary feedback control can be described by a Markov chain or a discrete random dynamical system. It is shown that this random dynamical system possesses a unique, exponentially attracting, invariant measure, namely, this random dynamical system is ergodic. This gives adequate statistical description of the stabilization process on the stage when stabilized solution has to be retained near zero (i.e. near unstable state of equilibrium).

Mathematics Subject Classification (2000). 93D15, 76D05.

Keywords. Oseen fluid equations, stabilization, extension operator, Markov chain, random dynamical system (RDS), invariant (stationary) measure, ergodicity.

1. Introduction

This paper is devoted to study some statistical aspects of the stabilization problem for two and three-dimensional Oseen equations with help of feedback control defined on a part of the boundary that restricts a domain where the equations are determined. Since for stabilization problem the case of unstable equations is interesting, we assume that Oseen equations possess a solution that is exponentially growing as time $t \rightarrow \infty$, i.e. the solution is unstable. New approach to the problem of stabilization by feedback control was proposed by one of the authors of this paper in [8]–[14]. Namely, construction of stabilization from a part of boundary for parabolic equation, 2D Oseen equations as well as for 2D and 3D Navier–Stokes system was created in [8], [9], [11], [12]. This construction reduces solution of a stabilization problem to solving a mixed boundary value problem defined on an extended domain with zero Dirichlet boundary conditions and with initial condition

This work was partially supported by a COBASE grant from the National Research Council and the NSF grant DMS-0209326.

belonging to a stable invariant manifold defined in an neighborhood of steady-state solution, near which we stabilize our system. In the case of (linear) Oseen equations, steady state solution equals to zero and invariant manifold is replaced on subspace X_σ invariant with respect to resolving semigroup and such that the solution going out initial condition $w_0 \in X_\sigma$ tends to zero as time $t \rightarrow \infty$ with the rate $e^{-\sigma t}$ or faster.

Evidently, aforementioned mixed boundary value problem is not stable because if $w_0 \notin X_\sigma$ then solution $w(t, \cdot)$ outgoing w_0 goes away from X_σ (and goes away from zero) even if w_0 is arbitrarily close to X_σ . That is why straightforward application of proposed construction of stabilization to numerical simulation may not be successful, because unpredictable fluctuations inevitably arise in real calculations. This situation was analysed in [10], [13], and [14] with help of the concept of real process introduced there. The method of damping of unpredictable fluctuations by some feedback mechanism was worked out in these papers and an estimate of stabilization for real process was obtained.

This estimate is informative only when a norm of stabilized real process is not too small. But when this norm has the same order as norms of unpredictable fluctuations, aforementioned estimate became uninformative. Actually in this situation behavior of stabilized real process became chaotic. The goal of this paper is just to investigate the behavior of stabilized real process in small neighborhood of zero. More precisely, we solve the problem of retention of stabilized flow near unstable state of equilibrium. To do this we impose additional assumption on unpredictable fluctuations. We suppose that they are independent identically distributed (i.i.d.) random variables with probability distribution supported in a small neighborhood of origin for phase space. Then the real process is described by a random dynamical system and forms a Markov chain.

Our aim is to prove that this Markov chain is ergodic, i.e. it possesses unique stationary measure invariant with respect to the corresponding Markov semigroup. This gives us the possibility to calculate by well known formulas the statistical characteristics of stabilized real process and to make clear its behavior using these formulas.

During the last few years uniqueness of invariant measures for 2D Navier–Stokes equations have been proved by F. Flandoli, B.Maslovski [7], S. Kuksin, A. Shirikyan [20], [21], [22], W. E, J. Mattingly, Ya. Sinai [6], S. Kuksin [18], Duan and Goldys [5], and other authors.

To prove ergodicity of indicated Markov chain we use recent results of S. Kuksin, A. Shirikyan [21] and S. Kuksin [18] on uniqueness of invariant measures for 2D Navier–Stokes equations with random kick-forces where coupling approach was applied.

Actually, it is enough for us to verify that random dynamical system arising in stabilization construction indicated above satisfies all conditions imposed in [21], [18] on random dynamical systems.

In Section 2 we recall necessary information on stabilization method. In Sec-

tion 3 we formulate the main results and present conditions imposed in [21], [18] on random dynamical systems in a form convenient for our situation. In Sections 4–6 we verify that these conditions fulfil for RDS corresponding to stabilization procedure.

We thank S. B. Kuksin and A. Shirikyan for useful discussion on ergodicity of Navier–Stokes equations. We thank also M. S. Agranovich for allowing us to read the proof of his result announced in [1] and formulated in Lemma 4.2 below before its publication.

2. Preliminaries to the stabilization theory

In this section we recall briefly results of [8]–[14] used below.

2.1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $\partial\Omega \in C^\infty$, $Q = \mathbb{R}_+ \times \Omega$. We consider the Oseen equations:

$$\partial_t v(t, x) - \Delta v + (a(x), \nabla)v + (v, \nabla)a + \nabla p(t, x) = 0, \quad \operatorname{div} v(t, x) = 0 \quad (2.1)$$

with initial condition

$$v(t, x)|_{t=0} = v_0(x). \quad (2.2)$$

Here $(t, x) = (t, x_1, \dots, x_d) \in Q$, $v(t, x) = (v_1, \dots, v_d)$ is a velocity of fluid flow, $p(t, x)$ is pressure, $a(x) = (a_1(x), \dots, a_d(x))$ is a given solenoidal vector field.

We suppose that $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_0$, $\Gamma \neq \emptyset$ where Γ, Γ_0 are open sets (in topology of $\partial\Omega$). Here, as usual, the over line means the closure of a set. We define $\Sigma = \mathbb{R}_+ \times \Gamma$, $\Sigma_0 = \mathbb{R}_+ \times \Gamma_0$, and set:

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u \quad (2.3)$$

where u is a control, supported on Σ .

Let $\sigma > 0$ be given. The problem of stabilization with rate σ of a solution to problem (2.1)–(2.3) is to construct a control u defined on Σ such that the solution $v(t, x)$ of boundary value problem (2.1)–(2.3) satisfies:

$$\|v(t, x)\|_{L_2(\Omega)}^2 \leq c e^{-\sigma t} \quad (2.4)$$

where $c > 0$ depends on v_0, σ and Γ_0 .

2.2. The main idea of the stabilization method

Let $\omega \subset \mathbb{R}^d$ be a bounded domain such that $\Omega \cap \omega = \emptyset$, $\overline{\Omega} \cap \overline{\omega} = \overline{\Gamma}$. We set

$$G = \operatorname{Int}(\overline{\Omega} \cup \overline{\omega}) \quad (2.5)$$

(the notation $\operatorname{Int} A$ means, as always, the interior of the set A).

We suppose that $\partial G \in C^\alpha$ where $\alpha > 2$ is fixed and in all points except $\bar{\Gamma} \setminus \Gamma \equiv \partial\Gamma$ it possesses the C^∞ smoothness. For the construction of ω and detailed description of ∂G in a neighborhood of $\partial\Gamma$, please see [9], [13].

We extend problem (2.1)–(2.2) from Ω to G . Let us assume that

$$a(x) \in V^2(G) \cap (H_0^1(G))^d, \quad a(x) \text{ is real valued}, \quad (2.6)$$

where, as usually, $H^k(G)$, $k \in \mathbb{N}$, is the Sobolev space of scalar functions, defined and square integrable on G together with all its derivatives up to order k and $(H^k(G))^d$ is the analogous space of vector fields. Besides, $H_0^1(G) = \{f(x) \in H^1(G) : f(x)|_{x \in \partial G} = 0\}$ and

$$V^k(\Omega) = \{v(x) = (v_1, \dots, v_d) \in (H^k(\Omega))^d : \operatorname{div} v = 0\}.$$

The extension of (2.1)–(2.2) from Ω to G can be written as follows:

$$\partial_t w(t, x) - \Delta w + (a(x), \nabla)w + (w, \nabla)a + \nabla p(t, x) = 0, \quad \operatorname{div} w(t, x) = 0, \quad (2.7)$$

$$w(t, x)|_{t=0} = w_0(x), \quad w|_S = 0, \quad (2.8)$$

where $S = \mathbb{R}_+ \times \partial G$. Note that, actually, w_0 from (2.8) will be some special extension of $v_0(x)$ from (2.2) such that the solution $w(t, x)$ of problem (2.7), (2.8) satisfies the inequality

$$\|w(t, \cdot)\|_{V^0(G)} \leq ce^{-\sigma t} \|w_0\|_{V^0(G)} \quad \text{for } t \geq 0. \quad (2.9)$$

For vector fields defined on G we denote by γ_Ω the operator of restriction on Ω and by γ_Γ we denote the operator of restriction on Γ :

$$\gamma_\Omega : V^k(G) \longrightarrow V^k(\Omega), \quad \gamma_\Gamma : V^k(G) \longrightarrow V^{k-1/2}(\Gamma), \quad k \geq 0. \quad (2.10)$$

Evidently, these operators are well-defined and bounded (see [23]).

Definition 2.1. A control $u(t, x)$ in (2.1)–(2.3) is called feedback¹ if

$$v(t, \cdot) = \gamma_\Omega w(t, \cdot), \quad u(t, \cdot) = \gamma_\Gamma w(t, \cdot) \quad \forall t \geq 0 \quad (2.11)$$

where $(v(t, \cdot), u(t, \cdot))$ is the solution of stabilization problem (2.1)–(2.3) and $w(t, \cdot)$ is the solution of boundary value problem (2.7)–(2.8).

Evidently, if the solution w of (2.7)–(2.8) satisfies (2.9), the pair (v, u) defined in (2.11) satisfies (2.4). Since (v, u) satisfies (2.1)–(2.3) as well, it forms a solution of the initial stabilization problem (2.1)–(2.4).

2.3. Description of “correct” initial conditions

First of all we describe the set of initial conditions $\{w_0\}$ such that solutions $w(t, x)$ of (2.7)–(2.8) satisfy (2.9).

¹ It will be clear later why defined control really possesses feedback property.

Let G be domain (2.5) and

$$V_0^0(G) = \{v(x) \in V^0(G) : (v, \nu)|_{\partial\Omega} = 0\}, \quad V_0^1(G) = V^1(G) \cap (H_0^1(G))^d \quad (2.12)$$

where $\nu(x)$ is the vector-field of outer unit normals to ∂G . Evidently,

$$\|v\|_{V^0(G)} = \|v\|_{V_0^0(G)} := \|v\|_{(L_2(G))^d}; \quad \|v\|_{V_0^1(G)} := \|\nabla v\|_{(L_2(G))^{d^2}}.$$

Denote by

$$\hat{\pi} : (L_2(G))^2 \longrightarrow V_0^0(G) \quad (2.13)$$

the operator of orthogonal projection. We consider the Oseen steady state operator

$$Av \equiv -\hat{\pi}\Delta v + \hat{\pi}[(a(x), \nabla)v + (v, \nabla)a] : V_0^0(G) \longrightarrow V_0^0(G) \quad (2.14)$$

and its adjoint operator A^* . These operators are closed and have the domain $\mathcal{D}(A) = V^2(G) \cap (H_0^1(G))^2$. Emphasize that $\mathcal{D}(A)$ consists of vector fields equal to zero on ∂G . The spectrums $\Sigma(A), \Sigma(A^*)$ of operators A and A^* are discrete subsets of a complex plane \mathbb{C} which belong to a sector symmetric with respect to \mathbb{R} and containing \mathbb{R}_+ . In other words, A is a sectorial operator. So spectrums $\Sigma(A), \Sigma(A^*)$ contain only eigenvalues of A, A^* , respectively. In virtue of (2.6) they are symmetric with respect to \mathbb{R} , and moreover $\Sigma(A) = \Sigma(A^*)$.

We rewritten the boundary value problem (2.7)–(2.8) for Oseen equations in the following form

$$\frac{dw(t, \cdot)}{dt} + Aw(t, \cdot) = 0, \quad w|_{t=0} = w_0, \quad (2.15)$$

where A is the operator (2.14). Then for each $w_0 \in V_0^0(G)$ the solution $w(t, \cdot)$ of (2.15) is defined by $w(t, \cdot) = e^{-At}w_0$ where e^{-At} is the resolving semigroup of problem (2.15).

Let $\sigma > 0$ satisfy:

$$\Sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \sigma\} = \emptyset. \quad (2.16)$$

The case when there are certain points of $\Sigma(A)$ which are in the left of the line $\{\operatorname{Re}\lambda = \sigma\}$ will be interesting for us.

Denote by $X_\sigma^+(A)$ the subspace of $V_0^0(G)$ generated by all eigenfunctions and associated functions of operator A corresponding to all eigenvalues of A placed in the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \sigma\}$. By $X_\sigma^+(A^*)$ we denote analogous subspace corresponding to adjoint operator A^* . We denote the orthogonal complement to $X_\sigma^+(A^*)$ in $V_0^0(G)$ by $X_\sigma(A) \equiv X_\sigma$:

$$X_\sigma = V_0^0(G) \ominus X_\sigma^+(A^*). \quad (2.17)$$

One can show that subspaces $X_\sigma^+(A), X_\sigma$ are invariant with respect to the action of semigroup e^{-At} , and $X_\sigma + X_\sigma^+(A) = V_0^0(G)$.

Theorem 2.1. *Suppose that A is operator (2.14) and $\sigma > 0$ satisfies (2.16). Then for each $w_0 \in X_\sigma$ the inequality (2.9) holds. Besides, the solution of problem (2.15)*

with such initial conditions are defined by the formula

$$w(t, \cdot) = e^{-At} w_0 = (2\pi i)^{-1} \int_{\gamma} (A - \lambda I)^{-1} e^{-\lambda t} w_0 d\lambda. \quad (2.18)$$

Here γ is a contour belonging to $\rho(A) := \mathbb{C} \setminus \Sigma(A)$ such that $\arg \lambda = \pm\theta$ for $\lambda \in \gamma$, $|\lambda| \geq N$ for certain $\theta \in (0, \pi/2)$ and for sufficiently large N . Moreover, γ encloses from the left the part of the spectrum $\Sigma(A)$ placed right of the line $\{\operatorname{Re} \lambda = \sigma\}$. The complementary part of the spectrum $\Sigma(A)$ is placed left of the contour γ .

Such contour γ exists, of course.

Proof. See [9], [10]. □

2.4. Theorem on extension

To complete the construction of stabilization for Oseen equations (2.1), (2.2) we have to construct the operator E extending initial condition v_0 from (2.2) from Ω in G such that $E v_0 := w_0 \in X_\sigma$. This $w_0(x)$ we take as initial value in (2.8). We consider here direct analog of construction from [13].

Introduce the space

$$V^0(\Omega, \Gamma_0) = \{u(x) \in V^0(\Omega) : u \cdot \nu|_{\Gamma_0} = 0, \exists v \in V_0^0(G) : u = \gamma_\Omega v\} \quad (2.19)$$

supplied with the norm:

$$\|u\|_{V^0(\Omega, \Gamma_0)} = \inf_{w \in V_0^0(G) : \gamma_\Omega w = u} \|w\|_{V_0^0(G)}$$

where γ_Ω is the restriction operator defined in (2.10) and $V_0^0(G)$ is defined in (2.12).

We consider also the following closed subspace of $V_0^0(G)$:

$$V_0^0(G, \Omega) = \{w \in V_0^0(G) : w|_\Omega = 0\}.$$

For $u \in V^0(\Omega, \Gamma_0)$ denote by \hat{u} the element of the quotient space $V_0^0(G)/V_0^0(G, \Omega)$ such that for each $v \in \hat{u}$, $\gamma_\Omega v = u$. Now we define the extension operator

$$L : V^0(\Omega, \Gamma_0) \rightarrow V_0^0(G) \quad \text{by} \quad Lu \in \hat{u} \quad \text{and} \quad Lu \perp_{V_0^0(G)} V_0^0(G, \Omega) \quad (2.20)$$

(i.e. Lu is orthogonal to $V_0^0(G, \Omega)$ in the space $V_0^0(G)$). Evidently, $\|L\| = 1$.

It is known (see [8], [10], [11]) that in the space $X_\sigma^+(A^*)$ one can choose a basis $(d_1(x), \dots, d_K(x))$ such that restriction $(d_1(x)|_\omega, \dots, d_K(x)|_\omega)$ on an arbitrary subdomain $\omega \subset G$ forms a linear independent set of vector fields. We can define space (2.17) by the following equivalent form:

$$X_\sigma = \left\{ v(x) \in V_0^0(G) : \int_G v(x) \cdot d_j(x) dx = 0, \quad j = 1, \dots, K \right\}. \quad (2.21)$$

Theorem 2.2 ([12], [13]). *There exists a linear bounded extension operator*

$$E : V^0(\Omega, \Gamma_0) \rightarrow X_\sigma, \quad (2.22)$$

i.e. $(Ev)(x) \equiv v(x)$ for $x \in \Omega$.

Proof. Let subset $\omega_1 \subset G \setminus \Omega$ be a domain with C^∞ -boundary $\partial\omega_1$ such that $\text{Int}(\partial\omega_1 \cap \partial G) \neq \emptyset$. In this set we consider the Stokes problem:

$$-\Delta w(x) + \nabla p(x) = v(x), \quad \text{div } w(x) = 0, \quad x \in \omega_1; \quad w|_{\partial\omega_1} = 0.$$

As is well known, for each $v \in V^0(\omega_1)$ there exists a unique solution $w \in V_0^1(\omega_1) \cap V^2(\omega_1)$ of this problem. The resolving operator to this problem we denote as follows: $(-\hat{\pi}\Delta)_{\omega_1}^{-1}v = w$. Extension of $(-\hat{\pi}\Delta)_{\omega_1}^{-1}v$ from ω_1 in G by zero we also denote as $(-\hat{\pi}\Delta)_{\omega_1}^{-1}v$. Evidently, $(-\hat{\pi}\Delta)_{\omega_1}^{-1}v \in V_0^1(G)$.

We look for the extension operator E in the form

$$Ev(x) = (Lv)(x) + \left[\sum_{j=1}^K c_j (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j \right](x), \quad (2.23)$$

where L is the operator (2.20), c_j are constants which should be determined. Evidently, $Ev(x) = v(x)$ if $x \in \Omega$ for any c_j . Besides, $Ev \in V_0^1(G)$. To define constants c_j we note that by (2.21) $Ev \in X_\sigma$ if

$$\int_G d_k(x) \left[\sum_{j=1}^K c_j (-\hat{\pi}\Delta)_{\omega_1}^{-1} d_j(x) \right] dx = - \int_G d_k(x) (Lv)(x) dx \quad (2.24)$$

for $k = 1, \dots, K$. As in [12], [13] one can prove that this system of linear equations has a unique solution. \square

Aforementioned results imply the following result on stabilization (see [8]–[12]).

Theorem 2.3. *Let domains Ω and G satisfy (2.5). Then for each initial value $v_0(x) \in V^1(\Omega, \Gamma_0)$ and for each $\sigma > 0$ there exists a feedback control u defined on Σ such that the solution $v(t, x)$ of (2.1)–(2.4) satisfies the inequality*

$$\|v(t, \cdot)\|_{V_0^0(\Omega)} \leq ce^{-\sigma t} \quad \text{as } t \rightarrow \infty. \quad (2.25)$$

2.5. Real processes

Recall results from [13], [14] on justification of numerical simulation of stabilization construction described above. In virtue of definition (2.11) stabilization problem (2.1)–(2.4) is reduced to problem (2.7), (2.8), and (2.9). Its numerical simulation is what we have to justify.

Let $e^{-At} = S(t)$ be resolving operator of problem (2.7), (2.8). Suppose that we calculate this problem in discrete time instants

$$t_1 < t_2 < \dots < t_k < \dots$$

where $t_k = k\tau$ and $\tau > 0$ is fixed. Denote $S = S(\tau)$. Let \tilde{w}^k be the result of our calculations at time instant t_k . Since numerical calculations can not be exact, we have

$$\tilde{w}^k = S\tilde{w}^{k-1} + \varphi^k \quad (2.26)$$

where φ^k is an error of calculation which is unknown for us before time t_k . The sequence $\{\tilde{w}^k\}$ defined by (2.26) is called real process. We suppose that we can estimate the error of our calculations a priori:

$$\|\varphi^k\|_{V_0^0(G)} \leq \hat{\varepsilon} \ll 1, \quad k > 0, \quad (2.27)$$

where $\hat{\varepsilon} > 0$ is a known quantity. Note also that at time t_k the vector \tilde{w}^k is completely known (completely observable) since it is result of our calculations.

Formulae (2.26) is supplemented with initial condition

$$\tilde{w}^0 = w_0, \quad w_0 \in X_\sigma. \quad (2.28)$$

(We assume for simplicity that $\varphi^0 = 0$.)

Equations (2.26), (2.28) imply that

$$\tilde{w}^k = S^k w_0 + \sum_{j=0}^{k-1} S^j \varphi^{k-j}. \quad (2.29)$$

In virtue of (2.29), (2.28) the estimate $\|\tilde{w}^k\| \leq ce^{-k\sigma\tau}$ is not true. Indeed, although $S^k w_0 \in X_\sigma$ and therefore $S^k w_0$ satisfies estimate of such kind, the fluctuations φ^k possess nonzero components belonging to $X_\sigma^+(A)$. Hence, $\|S^j \varphi^{k-j}\|_{V_0^0(G)}$ grows exponentially as $k \rightarrow \infty$ that these terms destroy all stabilization construction described in previous subsection.

To save this stabilization construction we use feedback mechanism (see [13], [14]): in the time moment t_k when \tilde{w}^k from (2.26) is calculated we act on it by a special projection operator $\Pi : V_0^0(G) \rightarrow X_\sigma$ that damps undesirable properties of fluctuations φ^k . This operator has to satisfy the following properties:

$$\text{i) } \Pi\varphi = \varphi, \quad \forall \varphi \in X_\sigma, \quad \text{ii) } \forall \varphi \in V_0^0(G) \quad (\Pi\varphi)(x) = \varphi(x) \quad \forall x \in \Omega.$$

The operator satisfying these properties can be defined by the formula analogous to (2.23):

$$\Pi\varphi(x) = \varphi(x) + \left[\sum_{j=1}^K c_j (-\pi\Delta)_{\omega_1}^{-1} d_j \right] (x), \quad (2.30)$$

where constants c_j are defined by (2.24) with $Lv = \varphi$.

Applying to both parts of (2.26), (2.28) operator Π . Taking into account that X_σ is invariant with respect to action of operator S and using notation $w^k = \Pi\tilde{w}^k$, we get the recurrent formulae for controlled real process:

$$w^{k+1} = Sw^k + \Pi\varphi^{k+1}, \quad \text{for } k \geq 0; \quad w^0 = w_0. \quad (2.31)$$

In [13], [14] the following estimate for controlled real process has been proved:

Theorem 2.4. *Suppose that unpredictable fluctuations φ^k satisfy (2.27), and operator Π of projection on X_σ is defined by (2.30). Then controlled real process $\{w^k\}$ defined by (2.31) satisfies the estimate*

$$\|w^k\|_{V^0(G)} \leq (\gamma_0 e^{-\sigma k \tau} \|w_0\|_{V^0(G)} + \|\Pi\| \hat{\varepsilon} / (1 - e^{-\sigma \tau})), \quad (2.32)$$

where constant $\gamma_0 \in (0, 1)$ depends only on σ if $\tau > \tau_0$ and τ_0 is a fixed magnitude,² $\|\Pi\|$ is the norm of operator (2.30), and $\hat{\varepsilon} > 0$ is the magnitude from inequality (2.27).

Note that outside a small neighborhood of the origin, estimate (2.32) is equivalent to estimate (2.9). Since in contrast to definition (2.18) of solution $w(t, \cdot)$ to problem (2.7), (2.8), in definition (2.31) of controlled real process fluctuations φ^k permanently arise, therefore estimates (2.32), (2.9) can not be equivalent in a small neighborhood of the origin. Investigation of behavior for w^k in this neighborhood is the main goal of this paper.

3. Statistical problem in stabilization theory

3.1. Primary considerations

We continue our investigation of the real process defined in Subsection 2.5. above.

It is clear from the definition (2.26), (2.28) of real process that its behavior is determined by its values in the instants $t_k = k\tau$ when the unpredictable fluctuations φ^k arise. In the following we restrict ourselves by considering the real process only in these points and thus we obtain a discrete real process. So we consider iterated sequence (2.31) with a fixed initial vector $w_0 = w_0(x) \in V_0^0(G)$ and unpredictable fluctuation $\varphi^{k+1} \in V_0^0(G)$ satisfying (2.27).

It is reasonable to assume that φ^k is a random variable (for each k) defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, taking values in $V_0^0(G)$. We also assume that the random sequence φ^k is independently identically distributed (i.i.d.). Thus (2.31) defines a random dynamical system (RDS).

For each k , φ^k has distribution μ , whose probability measure $\mu(\omega)$ is defined on the Borel σ -algebra $\mathcal{B}(V_0^0(G))$ of the space $V_0^0(G)$, and is supported in a neighborhood of the origin:

$$\text{supp } \mu \subset B_{\hat{\varepsilon}} = \{v \in V_0^0(G) : \|v\|_{V_0^0(G)} \leq \hat{\varepsilon}\}. \quad (3.1)$$

Let us consider the random dynamical system (2.31) from the point of view of stabilization. The solution of the stabilization problem can be derived in two stages: (a) To reach zero (or in general setting, to reach a steady state); (b) To keep the controlled solution w^k near zero.

The solution of the stage (a) was explained above in Section 2. In particular, estimate (2.32) of stabilized real process in G was obtained. Here and hereafter,

² This property of constant γ_0 is obtained in formula (4.13) below.

we denote $\|w^k\| = \|w^k\|_{V_0^0(G)}$, unless otherwise noted. The stage (a) is the part of the process when w^k does not reach the ball B_{r_0} with $r_0 = \|\Pi\|\hat{\varepsilon}/(1 - ce^{-\sigma\tau})$ where $c \in (0, 1)$ is the constant from (2.32).

Note that if $w^k \notin B_{r_0}$, then using (4.7) (see below) for estimating S we get

$$\begin{aligned} \|Sw^k - w^k\| &> \|w^k\| - \|Sw^k\| \geq (1 - \gamma_0 e^{-\sigma\tau})\|w^k\| \\ &\geq (1 - \gamma_0 e^{-\sigma\tau}) \frac{\|\Pi\|\hat{\varepsilon}}{1 - \gamma_0 e^{\sigma\tau}} = \|\Pi\|\hat{\varepsilon}. \end{aligned} \quad (3.2)$$

This estimate means that at stage (a), the distance between w^k and Sw^k is more than the norm of the fluctuation $\Pi\varphi^k$, i.e., the deterministic component of random process w^k is prevailing. Therefore, in stage (a) the behavior of w^{k+1} is determined mainly by the term Sw^k .

The situation in stage (b) when $w^k \in B_{r_0}$ is different. Now the terms Sw^k and $\Pi\varphi^k$ from the right hand side of (2.31) have the equivalent order, and the motion of the realization w^k of the RDS (2.31) becomes ‘‘irregular or chaotic’’.

The goal of this paper is to understand the behavior of the controlled stabilized sequence w^k in the stage (b). We will do it with help of the modern theory of Markov chains or random dynamical systems. We begin with some preliminaries.

3.2. Gauss measures

We recall some information about Gauss measures which will be used below. Let H be a Hilbert space with scalar product (\cdot, \cdot) and norm $\|\cdot\|$, and let $\mathcal{B}(H)$ be the σ -algebra of Borel subsets of H . A measure $G(du)$ defined on $\mathcal{B}(H)$ is called a Gauss measure if its Fourier transform $\tilde{G}(v)$ is of the form

$$\tilde{G}(v) = \int e^{i(u,v)} G(du) = e^{i(v,a) - \frac{1}{2}(Kv,v)}, \quad v \in H, \quad (3.3)$$

where $a \in H$ is the mean vector, and $K : H \rightarrow H$ is a linear self-adjoint positive³ trace class operator, called correlation operator of G :

$$K^* = K > 0, \quad \text{Trace}(K) = \sum_{j=1}^{\infty} \lambda_j < \infty, \quad (3.4)$$

with $\{\lambda_j\}$ the set of eigenvalues of K . Differentiation of (3.3) with respect to v implies that

$$a = \int u G(du), \quad (Kv_1, v_2) = (v_1, a)(v_2, a) - \int (v_1, u)(v_2, u) G(du).$$

Therefore a is the mathematical expectation of the Gauss measure G .

³ Positiveness condition can be weakened a little bit in the case of problem considered here: see below footnote in the Subsection 5.2.

In particular, when $H = \mathbb{R}^m$ and image $\text{Im } K = \mathbb{R}^m$, then for each $\Gamma \in \mathcal{B}(\mathbb{R}^m)$,

$$G(\Gamma) = \int p(dy)dy, \quad (3.5)$$

where density $p(y)$, $y \in \mathbb{R}^m$, is defined by

$$p(y) = \frac{1}{(2\pi)^{m/2} \det K} \exp \left[-\frac{1}{2}(K^{-1}(y - a), (y - a)) \right]. \quad (3.6)$$

Let $A : H \rightarrow H$ be a continuous map in Hilbert space H . As well-known ([24]), each measure μ induces a new measure $A^*\mu(du)$ defined by

$$A^*\mu(\omega) = \mu(A^{-1}\omega), \quad \omega \in \mathcal{B}(H), \quad (3.7)$$

where $A^{-1}\omega = \{x \in H : Ax \in \omega\}$. This definition is equivalent to

$$\int f(v)A^*\mu(dv) = \int f(Au)\mu(du) \quad (3.8)$$

for arbitrary f for which at least one side in equation (3.8) is defined. Thus (3.3) and (3.8) imply that if $\mu = G$ is the Gauss measure defined above and map A is linear, then A^*G is a Gauss measure with mathematical expectation $a_1 = Aa$ and correlation operator $K_1 = AK A^*$.

3.3. Distribution of φ^k

We consider the right hand side of (2.31) where φ^k is an i.i.d. random sequence. We suppose that the distribution $\mathcal{D}(\varphi^k)$ of φ^k has the form

$$\mathcal{D}(\varphi^k) = c\chi_{\hat{\varepsilon}}(u)G(du) = \nu(du), \quad (3.9)$$

where $c = (\int_{B_{\hat{\varepsilon}}} G(du))^{-1}$, and

$$\chi_{\hat{\varepsilon}}(u) = \begin{cases} 1, & \|u\| \leq \hat{\varepsilon} \\ 0, & \|u\| > \hat{\varepsilon} \end{cases} \quad (3.10)$$

is the characteristic function of the ball $B_{\hat{\varepsilon}}$ defined in (3.1), $G(du)$ is the Gauss measure with mathematical expectation $a = 0$ and correlation operator K satisfying conditions (3.4)

In virtue of definition (3.9),

$$\nu(\omega) = cG(B_{\hat{\varepsilon}} \cap \omega), \quad \omega \in \mathcal{B}(V_0^0(G)) \quad (3.11)$$

and therefore $\nu(\omega)$ is a probability measure on $\mathcal{B}(V_0^0(G))$ supported on the ball $B_{\hat{\varepsilon}}$.

3.4. The main result

Since φ^k are i.i.d., $\Pi\varphi^k$ are i.i.d. as well, and $D(\Pi\varphi^k) \in X_\sigma$. Thus RDS (2.31) defines a family of Markov chains in X_σ with transition function

$$P(k, w_0, \Gamma) = \mathbb{P}\{w^k(w_0) \in \Gamma\}, \quad \Gamma \in \mathcal{B}(X_\sigma), \quad (3.12)$$

where $w^k = w^k(w_0)$ is defined by (2.31) and \mathbb{P} is probability measure defined on σ -algebra \mathcal{A} of subsets of probability space Ω . Let $\mathcal{P}(X_\sigma)$ be the space of Borel probability measures on $\mathcal{B}(X_\sigma)$ and $C_b(X_\sigma)$ be the space of continuous bounded functions on X_σ . Moreover, we denote by \mathfrak{P}_k and \mathfrak{P}_k^* the corresponding Markov semigroups acting in $C_b(X_\sigma)$ and $\mathcal{P}(X_\sigma)$, respectively:

$$\mathfrak{P}_k f(v) = \mathbb{E}f(w^k(w_0)) \equiv \int f(z)P(k, w_0, dz), \quad f \in C_b(X_\sigma),$$

$$\mathfrak{P}_k^* \mu(\Gamma) = \int_H P(k, w_0, \Gamma) \mu(dw_0), \quad \mu \in \mathcal{P}(X_\sigma),$$

where \mathbb{E} is for the mathematical expectation, and $P(k, w_0, \Gamma)$ is defined in (3.12).

A continuous function $f(u)$ on X_σ is called Lipschitz if

$$\sup_{v \in X_\sigma, \|v\| \leq 1} \frac{|f(u+v) - f(u)|}{\|v\|} \equiv \text{lip} f(u) < \infty, \quad u \in X_\sigma.$$

We denote $\text{Lip} f = \|\text{lip} f(\cdot)\|_{C(X_\sigma)}$.

A measure $\mu \in \mathcal{P}(X_\sigma)$ is called a stationary measure for the RDS (2.31) if $\mathfrak{P}_k^* \mu = \mu, \forall k$. The main theorem of this paper is as follows.

Theorem 3.1. *The random dynamical system (2.31) has a unique stationary measure $\hat{\mu}$. Moreover, there exists a constant $\gamma \in (0, 1)$ such that*

$$\left| \int f(z)P(k, w_0, dz) - \int f(z)\hat{\mu}(dz) \right| \leq c\gamma^k, \quad k = 1, 2, 3, \dots, \quad (3.13)$$

for every Lipschitz function f on X_σ such that $\|f\|_{C_b(X_\sigma)} \leq 1$ and $\text{Lip} f \leq 1$. The constant c depends only on initial state $\|w_0\|$.

Note that the uniqueness of the stationary measure means that the RDS (2.31) is ergodic. The exponential convergence (3.13) means that RDS (2.31) possesses the property of exponential mixing.

Theorem 3.1 provides us the possibility of calculating easily the probability characteristics of Markov chain (2.31). Indeed, in numerical simulation we actually obtain certain realizations $w^k = w^k(\omega)$ of RDS (2.31), where $\omega \in \Omega$ is a random sample. By the strong law of large numbers

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^N w^k(\omega) \rightarrow \int w \mu(dw), \quad \text{as } N \rightarrow \infty, \quad (3.14)$$

where $\mu(dw)$ is an invariant measure of RDS (2.31). In virtue of Theorem 3.1, $\mu(dw) = \hat{\mu}(dw)$. So while calculating $w^k(\omega)$, we can simultaneously obtain mathematical expectation of $\hat{\mu}$. Moreover, (3.13) gives us the convergence rate in (3.14).

The topic connected with (3.14) will be studied in detail in some other place. Here we only note that strong law of large numbers was derived from ergodicity of Navier–Stokes equations in [19] in the case when random force is white noise. Of course in the case of kick forces this derivation can be made as well.

3.5. Ergodic theorem

In order to prove Theorem 3.1, we use a result in [21, 18] on ergodicity of a Markov chain, proved by coupling techniques. Let us formulate this result in a form suitable for our problem. Consider a Markov chain (or a RDS) in a Hilbert space H

$$u(k) = Tu(k-1) + \eta_k, \quad u(0) = u_0, \quad (3.15)$$

where $u_0 \in H$, $T : H \rightarrow H$ is a linear bounded operator such that

$$\|Tu\| \leq \gamma_0 \|u\|, \quad \forall u \in H, \quad (3.16)$$

for a constant $\gamma_0 \in (0, 1)$.

Assume also that there exists an orthonormal basis $\{e_j\}$ in H (true for any separable Hilbert space H) and a sequence of subspaces $H_1 \subset H_2 \subset \dots \subset H_k \subset \dots$ such that

$$H_k = \text{span}\{e_1, \dots, e_{r_k}\} \quad \text{where } r_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Denote by H_k^\perp the orthogonal complement to H_k in H : $H = H_k \oplus H_k^\perp$, and designate by

$$Q_k : H \longrightarrow H_k, \quad Q_k^\perp : H \longrightarrow H_k^\perp$$

orthogonal projectors. Suppose that

$$\|Q_k^\perp Tu\| \leq \gamma_k \|u\|, \quad \forall u \in H, \quad \text{and } \gamma_k \rightarrow 0 \text{ as } k \rightarrow \infty, \quad (3.17)$$

At last assume that in (3.15), η_k is an i.i.d. random sequence with distribution $\mu(\omega), \omega \in H$, such that the projection $Q_k^* \mu$ on H_k , has a continuous density $R(x)$ with a compact support:

$$Q_k^* \mu(dx) = R(x) dx$$

where $H_k \ni x = \sum x_i e_i$, $dx = dx_1 \cdots dx_{r_k}$. Moreover, this density $R(x)$ satisfies the condition:

$$\int_{H_k} |R(x - v_1) - R(x - v_2)| dx \leq c \|v_1 - v_2\|_{H_N} \quad (3.18)$$

where the constant $c > 0$ does not depend on $v_1, v_2 \in H_N$.

Under aforementioned assumptions in [21, 18], the following theorem has been proved (see [18]).

Theorem 3.2. *The RDS (3.15) has a unique stationary measure $\hat{\mu}$. Moreover, there exists a constant $\gamma \in (0, 1)$ such that*

$$\left| \int f(z)P(k, u, dz) - \int f(z)\hat{\mu}(dz) \right| \leq c\gamma^k, \quad k = 1, 2, \dots,$$

for every Lipschitz function f on H satisfying $|f|_{C_b(H)} \leq 1$ and $\text{Lip } f \leq 1$. Here $P(k, u, dz)$ is the transition function (see (3.12)) corresponding to RDS (3.15).

4. Check of assumptions (3.16), (3.17)

To prove Theorem 3.1, we have to check that RDS (2.31) satisfies conditions (3.16), (3.17), and (3.18) of Theorem 3.2. In this section we check the first and the second of them.

4.1. Subspaces of $V_0^0(G)$

First we introduce an orthogonal decomposition of $V_0^0(G)$ in order to define analogs of subspaces H_k from Subsection 3.5. Recall that an important subspace of $V_0^0(G)$ is (see definition in (2.21))

$$X_\sigma = \left\{ v \in V_0^0(G) : \int_G v(x)d_j(x)dx = 0, j = 1, \dots, m \right\}, \quad (4.1)$$

where $\{d_j(x), j = 1, \dots, m\}$ is the basis of $X_\sigma^+(A^*)$, constructed in [9] from eigenvectors and associated vectors of the operator A^* , corresponding to all eigenvalues λ_j satisfying $\text{Re } \lambda_j > \sigma$.

Let $\sigma < \sigma_1 < \dots < \sigma_k \rightarrow \infty$ as $k \rightarrow \infty$ be a sequence of numbers which satisfy, as σ , the condition

$$\{\lambda \in \mathbb{C} : \text{Re } \lambda = \sigma_k\} \cap \sigma(A) = \emptyset \quad \forall k. \quad (4.2)$$

Analogously to (4.1), we can define the spaces

$$X_{\sigma_k} = \left\{ v \in V_0^0(G) : \int_G v(x)d_j(x)dx = 0, j = 1, \dots, n_k \right\}, \quad (4.3)$$

where basis $\{d_j(x), j = 1, \dots, n_k\}$ in $X_{\sigma_k}^+(A^*)$ is constructed from eigenvectors and associated vectors of the operator A^* , corresponding to all eigenvalues λ_j satisfying $\text{Re } \lambda_j > \sigma_k$. This basis is an extension of the basis in $X_\sigma^+(A^*)$ from (4.1), and it is constructed by the same rules as the basis from (4.1).

Since $\sigma_i > \sigma_j > \sigma$ for $i > j$ we have that $n_i > n_j > m$ and therefore $X_{\sigma_i} \subset X_{\sigma_j} \subset X_\sigma$. Moreover, we introduce the following subspaces of $V_0^0(G)$. Let $X_{\sigma\sigma_1}$ be an orthogonal complement in X_σ for the subspace X_{σ_1} , and $X_{\sigma_k\sigma_{k+1}}$ be an orthogonal complement in X_{σ_k} for the subspace $X_{\sigma_{k+1}}$. In other words, $X_{\sigma\sigma_1}$ and

$X_{\sigma_k \sigma_{k+1}}$ are subspaces satisfying

$$X_{\sigma_1} \oplus X_{\sigma \sigma_1} = X_\sigma \quad X_{\sigma_{k+1}} \oplus X_{\sigma_k \sigma_{k+1}} = X_{\sigma_k}. \quad (4.4)$$

We also define the subspace $X_\sigma^\perp \subset V_0^0(G)$, which is the orthogonal complement of X_σ in $V_0^0(G)$:

$$X_\sigma \oplus X_\sigma^\perp = V_0^0(G). \quad (4.5)$$

Evidently, $X_\sigma^\perp = X_\sigma^+(A^*)$. The subspace X_σ will play the role of space H in Subsection 3.5. Likewise,

$$X_{\sigma \sigma_k} := X_{\sigma \sigma_1} \oplus \cdots \oplus X_{\sigma_{k-1} \sigma_k} \quad (4.6)$$

will play the role of H_k and X_{σ_k} will play the role of H_k^\perp . Recall that subspace X_σ is invariant with respect to the operator S in RDS (2.31). That is why we put in (3.15)

$$T = S|_{X_\sigma}.$$

Now we construct the basis $\{e_j\}$ from Subsection 3.5. Let $\{e_1, \dots, e_m\}$ be orthogonalization of basis d_1, \dots, d_m in $X_\sigma^+(A^*) = X_\sigma^\perp$. Evidently, $\{e_1, \dots, e_m\}$ forms a orthonormal basis in X_σ^\perp . Continuing orthogonalization process for

$$\{d_{m+1}, \dots, d_{n_1}\}, \dots, \{d_{n_{k-1}+1}, \dots, d_{n_k}\}, \dots$$

we get orthonormal basis $\{e_{m+1}, \dots, e_{n_1}\}$ in $X_{\sigma \sigma_1}$ and $\{e_{n_{k-1}+1}, \dots, e_{n_k}\}$ in $X_{\sigma_{k-1} \sigma_k}$ for $k = 2, 3, \dots$. We have to prove now that countable orthonormal system $\{e_1, \dots, e_j, \dots\}$ forms a basis in $V_0^0(G)$. For this it is enough to establish that this system is dense in $V_0^0(G)$. But in virtue of Keldysh Theorem (see [17], [16]), the system $\{d_j, j \in \mathbb{N}\}$ constructed by eigenfunctions and associated functions of operator A^* adjoint to operator (2.14) is dense in $V_0^0(G)$. Hence, system $\{e_j, j \in \mathbb{N}\}$ obtained from $\{d_j, j \in \mathbb{N}\}$ by orthogonalization process is also dense. Therefore the system $\{e_j, j = m+1, \dots, m+k, \dots\}$ forms basis in the space $H = X_\sigma$.

In the next subsection we establish inequality (3.16).

4.2. Certain properties of RDS (2.31)

We prove the following assertion.

Lemma 4.1. *Let S be the operator in (2.31) and $\sigma > 0$ be given. Then for each $\gamma_0 \in (0, 1)$, there exists $\tau > 0$ such that for $S = S(\tau)$ the following estimate holds*

$$\|Su\| \leq \gamma_0 \|u\|, \quad \forall u \in X_\sigma. \quad (4.7)$$

Proof. Recall that the basic space is $V_0^0(G)$ and therefore we use the notation $\|\cdot\| = \|\cdot\|_{V_0^0}$. Inequality (4.7) follows from the bound established in [9]:

$$\left\| \int_{\gamma_\sigma} (A - \lambda I)^{-1} e^{-\lambda \tau} d\lambda \right\| = \left\| \int_{-\gamma_\sigma} (A + \lambda I)^{-1} e^{\lambda \tau} d\lambda \right\| \leq ce^{-\sigma \tau}, \quad (4.8)$$

where A is the infinitesimal generator for $S(t)$, and the contour $-\gamma_\sigma$ is defined as follows:

$$-\gamma_\sigma = \gamma_\sigma^1 \cup \gamma_\sigma^2$$

where

$$\begin{aligned} \gamma_\sigma^1 &= \{ \lambda \in \mathbb{C}, \operatorname{Re} \lambda = -\sigma, \operatorname{Im} \lambda \in [-(\sigma + \theta) \tan(\pi - \psi), (\sigma + \theta) \tan(\pi - \psi)] \} \\ \gamma_\sigma^2 &= \left\{ \lambda \in \mathbb{C}, \operatorname{Re} \lambda < -\sigma, \lambda = \gamma e^{\pm i\psi} + \theta, \text{ for } \gamma \in \left[\frac{\sigma + \theta}{|\cos \psi|}, \infty \right) \right\} \end{aligned} \quad (4.9)$$

with $\theta > 0$ and $\pi/2 < \psi < \pi$ fixed. We have to prove that c in (4.8) can be chosen independent of $\sigma > 0$. Since γ_σ belongs to the resolvent set of the operator A , we can get as in [9, Lemma 4.7] that

$$\|(\lambda I + A)^{-1}\| \leq \frac{M_1}{1 + |\lambda|}, \quad \lambda \in -\gamma_\sigma, \quad (4.10)$$

with $M_1 > 0$ independent of $\lambda \in -\gamma_\sigma$. We see that

$$\left\| \int_{-\gamma_\sigma} (\lambda I + A)^{-1} e^{\lambda\tau} d\lambda \right\| \leq I_1 + I_2$$

where

$$I_1 = \int_{\gamma_\sigma^1} \|(\lambda I + A)^{-1}\| |e^{\lambda\tau} d\lambda| \quad I_2 = \int_{\gamma_\sigma^2} \|(\lambda I + A)^{-1}\| |e^{\lambda\tau} d\lambda|. \quad (4.11)$$

Using (4.10) we get

$$\begin{aligned} I_1 &\leq \int_{-(\sigma+\theta)\tan(\pi-\psi)}^{(\sigma+\theta)\tan(\pi-\psi)} \frac{M_1 e^{-\sigma\tau} dx}{1 + \sqrt{\sigma^2 + x^2}} \leq M_1 e^{-\sigma\tau} \int_{-(\sigma+\theta)\tan(\pi-\psi)}^{(\sigma+\theta)\tan(\pi-\psi)} \frac{dx/\sigma}{\sqrt{1 + (x/\sigma)^2}} \\ &= M_1 e^{-\sigma\tau} \int_{-(1+\theta/\sigma)\tan(\pi-\psi)}^{(1+\theta/\sigma)\tan(\pi-\psi)} \frac{dy}{\sqrt{1 + y^2}} \leq M_1 c e^{-\sigma\tau}, \end{aligned}$$

where c does not depend on $\sigma \geq 1$ and $\tau > 0$. If $\sigma \in (0, 1)$, then

$$I_1 \leq \int_{-(1+\theta)\tan(\pi-\psi)}^{(1+\theta)\tan(\pi-\psi)} \frac{M_1 e^{-\sigma\tau} dx}{1 + |x|} = M_1 c e^{-\sigma\tau},$$

with c not depending on $\sigma \in (0, 1)$ and $\tau > 0$. Moreover, by change of variables $x = (\gamma |\cos \psi| - \theta)$, we get

$$\begin{aligned} I_2 &\leq 2M_1 \int_{\frac{\sigma+\theta}{|\cos \psi|}}^{\infty} \frac{\exp[-(\gamma |\cos \psi| - \theta)\tau] d\gamma}{1 + \sqrt{(\gamma |\cos \psi| - \theta)^2 + \gamma^2 \sin^2 \psi}} \\ &= \frac{2M_1}{|\cos \psi|} \int_{\sigma}^{\infty} \frac{\exp[-x\tau] dx}{1 + \sqrt{x^2 + (x + \theta)^2 \tan^2 \psi}} \leq \frac{2M_1}{|\cos \psi|} \int_{\sigma}^{\infty} \frac{\exp(-x\tau) dx}{1 + x} \\ &= \frac{2M_1}{|\cos \psi|} \int_{\sigma\tau}^{\infty} \frac{\exp(-y) dy}{\tau + y} = \frac{2M_1 \exp(-\sigma\tau)}{|\cos \psi|} \int_0^{\infty} \frac{\exp(-z) dz}{\tau(1 + \sigma) + z}, \end{aligned}$$

where in the final step, we used the change of variables $y = z + \sigma\tau$.

Note that

$$\int_0^\infty \frac{\exp(-z) dz}{\tau(1+\sigma) + z} \leq 1 \quad \text{if } \tau \geq 1.$$

For $\tau \in (0, 1)$ we have

$$\int_0^\infty \frac{\exp(-z) dz}{\tau(1+\sigma) + z} \leq \int_0^\infty \frac{\exp(-z) dz}{\tau + z} \leq \int_0^{1-\tau} \frac{dz}{\tau + z} + \int_0^\infty e^{-z} dz = 1 - \ln \tau.$$

So we have

$$I_2 \leq \frac{2M_1}{|\cos \psi|} e^{-\sigma\tau} c_1, \quad (4.12)$$

where c_1 does not depend on $\tau > \tau_0$ and $\sigma > 0$.

Thus we have proved that c in (4.8) does not depend on $\sigma > 0$ and $\tau > \tau_0$. So for $u \in X_\sigma$,

$$\|S(\tau)u\| \leq cM_1 e^{-\sigma\tau}, \quad (4.13)$$

where $c > 0$ does not depend on $\sigma > 0$ and $\tau > \tau_0$. Therefore for given γ_0 and $\sigma > 0$, we can take $\tau > 0$ such that $ce^{-\sigma\tau} = \gamma_0$. \square

Inequality (3.16) evidently follows from (4.7).

4.3. Estimate $S(\tau)$ on X_{σ_k} for large k

We check here bound (3.17) for $T = S(\tau)$ and $Q_k^\perp H = X_{\sigma_k}$. Simultaneously we make more precise the choice of σ_k . Together with operator A defined in (2.14) we consider the Stokes operator

$$A_0 = -\hat{\pi}\Delta : V_0^0(G) \rightarrow V_0^0(G) \quad (4.14)$$

where $\hat{\pi}$ is projector (2.13). Operator A_0 is positive self-adjoint with domain $\mathcal{D}(A_0) = \mathcal{D}(A) = V^2(G) \cap V_0^1(G)$ and with discrete spectrum. Let $\{\varepsilon_j\}$ be eigenvectors of A_0 forming an orthonormal basis in $V_0^0(G)$ and $0 < \mu_1 \leq \mu_2 \leq \dots$ be corresponding eigenvalues, taking into account of their multiplicities. Then for each $q \in \mathbb{R}$ the power A_0^q can be defined by the formula $A_0^q v = \sum \mu_j^q(v, \varepsilon_j)_{V_0^0(G)} \varepsilon_j$. Eigenvalues μ_j possess the following asymptotic for large j :

$$\mu_j = \beta_0 j^{2/d} + O(j^{2/d}/\ln j) \quad \text{as } j \rightarrow \infty \quad (4.15)$$

where $\beta_0 > 0$, d is dimension of G (i.d. $d = 2$ or 3), $O(j^{2/d}/\ln j)$ is a function satisfying the estimate $|O(j^{2/d}/\ln j)| \leq c j^{2/d}/\ln j$ as $j \rightarrow \infty$ with $c > 0$ independent on j . Asymptotics (4.15) was obtained by K. I. Babenko [4].⁴

⁴ Actually, K. I. Babenko proved (4.15) in [4] in the case $d = \dim G = 3$ only. But his proof can be extended in the case $d = 2$ as well.

We can write operator A from (2.14) as follows:

$$A = A_0 + A_1 \quad (4.16)$$

where $A_1 v = \hat{\pi}[(a(x), \nabla)v + (v, \nabla)a]$. Using [15, Ch. 3, Lemma 4.5] one can easily get the following bound:

$$\|A_1 A_0^{-1/2}\| = b < \infty. \quad (4.17)$$

Now we describe one result of M. S. Agranovich announced in [1, Bound (6.61)]. Although it is obtained for general abstract operators we formulate it in the case of Oseen operator. Its proof will be published in [2]. This result consists of the choice of sequence $\sigma_k \rightarrow \infty$ as $k \rightarrow \infty$ such that on segments

$$\Gamma_k = \{\lambda = \sigma_k + i\gamma; |\gamma| \leq b'\sigma_k^{1/2}\}, \quad b' > 0 \text{ does not depend on } k, \quad (4.18)$$

resolvent $(A - \lambda I)^{-1}$ possesses some optimal bound. Numbers σ_k are found on segments⁵

$$\Delta_k = [e^{2k/d}, e^{2(k+1)/d}], \quad d = \dim G = 2 \text{ or } 3. \quad (4.19)$$

Lemma 4.2. *Suppose that an operator A has the form (4.16) where A_0 is self-adjoint positive operator with discrete spectrum and eigenvalues satisfying (4.15), and A_1 satisfies (4.17). Then there exists $\sigma_k \in \Delta_k$ such that for $\lambda \in \Gamma_k$ the following estimate holds:*

$$\|(A - \lambda I)^{-1}\| \leq c_1 c_2 \sigma_k^{\frac{d-1}{2}} \sigma_k^{-1/2} \quad (4.20)$$

where segments Γ_k, Δ_k are defined in (4.18), (4.19), respectively.

Using Lemma 4.2 we can check (3.17). Recall that $S(t) = e^{-At}$ is the resolving semigroup of problem (2.15) where A is Oseen operator (2.14), and $S = S(\tau)$ where τ is a fixed number chosen in Lemma 4.1 such that (4.7) is true. Note that if we would increase τ , (4.7) is true as well. Recall that the space X_{σ_k} defined in (4.3) is invariant with respect to the operator $S = S(\tau)$.

Theorem 4.1. *Let A be Oseen operator (2.14), $S = S(\tau) = e^{-A\tau}$, and sequence $\sigma_k \rightarrow \infty$ as $k \rightarrow \infty$ be chosen in Lemma 4.2. Then there exists $\tau_0 > 0$ such that for each $\tau > \tau_0$ on the spaces X_{σ_k} the following estimate hold:*

$$\|S(\tau)u\|_{X_{\sigma_k}} \leq \gamma_k \|u\|_{X_{\sigma_k}}, \quad \text{where } \gamma_k \rightarrow 0 \text{ as } k \rightarrow \infty, \quad (4.21)$$

where γ_k 's do not depend on $u \in X_{\sigma_k}$.

⁵ Actually, Agronovitch's result was proved in [2] under assumption that remainder term in (4.15) has the form $O(j^r)$ with $r < 2/d$ (that is stronger than $O(j^{2/d}/\ln j)$ from (4.15)), and σ_k are looked for in segments $[k^{2\rho/d}, (k+1)^{2\rho/d}]$ with some $\rho > 4$. But if we change these segments on segments (4.19) then a straightforward repeating of the corresponding proof from [2] leads to the desired estimate.

Proof. It is clear that for $u \in X_{\sigma_k}$

$$S(\tau)u = (2\pi i)^{-1} \int_{\gamma_{\sigma_k}} (A - \lambda I)^{-1} u e^{-\lambda \tau} d\lambda = -(2\pi i)^{-1} \int_{-\gamma_{\sigma_k}} (A + \lambda I)^{-1} u e^{\lambda \tau} d\lambda$$

where $-\gamma_{\sigma_k} = \gamma_{\sigma_k}^1 \cup \gamma_{\sigma_k}^2$ and similarly to (4.9)

$$\begin{aligned} \gamma_{\sigma_k}^1 &= \{ \lambda \in \mathbb{C}, \operatorname{Re} \lambda = -\sigma_k, \operatorname{Im} \lambda \in [-(\sigma_k + \theta) \tan(\pi - \psi), (\sigma_k + \theta) \tan(\pi - \psi)] \} \\ \gamma_{\sigma_k}^2 &= \left\{ \lambda \in \mathbb{C}, \operatorname{Re} \lambda < -\sigma_k, \lambda = \gamma e^{\pm i\psi} + \theta, \text{ for } \gamma \in \left[\frac{\sigma_k + \theta}{|\cos \psi|}, \infty \right) \right\}. \end{aligned}$$

Doing calculation as in proof of Lemma 4.1 we get the same formulas where σ is changed to σ_k only. Then estimation of the term I_2 gives as in (4.12):

$$I_2 \leq \frac{2M_1}{|\cos \psi|} e^{-\sigma_k \tau} c_1, \quad (4.22)$$

where $c_1 = c_1(\tau_0)$ does not depend on $\tau > \tau_0$. To estimate I_1 we use (4.20) instead of (4.10). More precisely we use the estimate

$$\|(A + \lambda I)^{-1}\| \leq c_1 e^{\sigma_k \ln c_2} \sigma_k^{-1/2}, \quad \lambda \in -\tilde{\Gamma}_k \quad (4.23)$$

where $\tilde{\Gamma}_k = \{ \lambda = \sigma_k + i\gamma; |\gamma| \leq (\sigma_k + \theta) \tan(\pi - \psi) \}$. If $\lambda \in -\Gamma_k$, estimate (4.23) directly follows from (4.20). For $\lambda \in -\{\tilde{\Gamma}_k \setminus \Gamma_k\}$ situation is easier and one can prove estimate similar to (4.10) in right side of which $|\lambda|$ is changed on $|\lambda|^{1/2}$ that is stronger than (4.23). (This has been done in [2].) Applying (4.23) to I_2 defined in (4.11) (with σ changed on σ_k) we get:

$$I_1 \leq \int_{-(\sigma_k + \theta) \tan(\pi - \psi)}^{(\sigma_k + \theta) \tan(\pi - \psi)} c_1 e^{-\sigma_k(\tau - \ln c_2)} \sigma_k^{1/2} dx \leq \tilde{c} e^{\sigma_k(\tau - \ln c_2)} \quad (4.24)$$

where \tilde{c} does not depend on k . If we choose $\tau > \ln c_2$ then (4.22), (4.24) imply that

$$I_1 + I_2 \leq \gamma_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This proves (4.21). \square

Inequality (3.17) follows from (4.21).

5. Reduction to the finite dimensional case

Now we need only to check the condition (3.18). The rest part of the paper is devoted to prove that the distribution $\Pi \varphi^k$ of random forces in RDS (2.31) satisfies this property. This will then complete this paper. First, we project $\Pi \varphi^k$ on finite-dimensional subspace.

5.1. Calculation of the projection for probability distribution $\Pi\varphi^k$

In this subsection we calculate probability distribution for random variable $Sw^k + \Pi\varphi^{k+1}$ in RDS (2.31) under assumption that Sw^k is a fixed vector. Since $Sw^k \in X_\sigma$ and $\Pi : V_0^0 \rightarrow X_\sigma$ is a projection on X_σ , the probability distribution $\mathcal{D}(Sw^k + \Pi\varphi^{k+1})$ is supported on X_σ . It is enough to calculate $\mathcal{D}(\Pi\varphi^{k+1})$ because $\mathcal{D}(Sw^k + \Pi\varphi^{k+1})$ is simply the shift of $\mathcal{D}(\Pi\varphi^{k+1})$ along the vector Sw^k . In virtue of (3.9)–(3.11), $\mathcal{D}(\varphi^{k+1}) = \nu$, where the measure ν is defined by

$$\nu(\omega) = cG(B_\varepsilon \cap \omega), \quad \forall \omega \in \mathcal{B}(V_0^0), \quad (5.1)$$

where G is the Gauss measure with mathematical expectation $a = 0$ and correlation operator K satisfying (3.4). Clearly, $\mathcal{D}(\Pi\varphi^{k+1}) = \Pi^*\nu$. Since

$$\Pi : V_0^0 \rightarrow X_\sigma \quad (5.2)$$

is a linear projection on X_σ , we have by (3.8) for $\omega \in \mathcal{B}(X_\sigma)$:

$$\begin{aligned} \Pi^*\nu(\omega) &= c \int_{B_\varepsilon} \chi_\omega(\Pi u) G(du) = c \int_{\Pi^{-1}\omega \cap B_\varepsilon} G(du) \\ &= c \int_{\hat{\Pi}^{-1}(\omega \cap \Pi B_\varepsilon)} G(du) = c \int_{\omega \cap \Pi B_\varepsilon} \Pi^* G(du). \end{aligned} \quad (5.3)$$

We have already mentioned that Π^*G is the Gauss measure supported on X_σ with mathematical expectation $a = 0$ and correlation operator $K_1 = \Pi K \Pi^*$. Note that in (5.3), $\chi_\omega(v)$ is the characteristic function of the set ω , and

$$\Pi^{-1}\omega = \{x \in V_0^0 : \Pi x \in \omega\}, \quad \hat{\Pi}^{-1}\omega_1 = \{x \in V_0^0 \cap B_\varepsilon : \Pi x \in \omega_1\}. \quad (5.4)$$

Hence to define completely the measure $\Pi^*\nu(\omega)$ from (5.3), we need to calculate ΠB_ε . Let us consider the decomposition (4.5) of V_0^0 . Since $\Pi : V_0^0 \rightarrow X_\sigma \subset V_0^0$ is a linear projection, for each $y \in V_0^0$ decomposed as $y = y' + y''$ with $y' \in X_\sigma$, $y'' \in X_\sigma^\perp$, we obtain

$$x = \Pi y = y' + Ay'' \quad \text{where} \quad A = \Pi|_{X_\sigma^\perp}. \quad (5.5)$$

So

$$A : X_\sigma^\perp \rightarrow X_\sigma, \quad A^* : X_\sigma \rightarrow X_\sigma^\perp,$$

where A^* is the operator adjoint to A . (We identify Hilbert spaces X_σ, X_σ^\perp with their dual spaces.) Note that

$$B_\varepsilon = \{\|y\|^2 \leq \varepsilon^2\} = \{\|y'\|^2 + \|y''\|^2 \leq \varepsilon^2\}.$$

Thus by (5.5),

$$\begin{aligned} \Pi B_\varepsilon &= \{x \in X_\sigma : \text{There exists } y'' \in B_\varepsilon \cap X_\sigma^\perp \\ &\quad \text{such that } \|x - Ay''\|^2 + \|y''\|^2 \leq \varepsilon^2\}. \end{aligned} \quad (5.6)$$

To make this more precise, we consider the extreme problem

$$f(y'') \equiv \|x - Ay''\|^2 + \|y''\|^2 \rightarrow \inf, \quad y'' \in X_\sigma^\perp. \quad (5.7)$$

The solution \hat{y} of this problem exists, is unique and satisfies

$$(f'(\hat{y}), h) = 2\{-(x - A\hat{y}, Ah) + (\hat{y}, h)\} = 2(\hat{y} + A^*A\hat{y} - A^*x, h) = 0, \forall h \in X_\sigma^\perp.$$

Since the operator A^*A is nonnegative, the equation

$$A^*Ay'' + y'' = A^*x$$

for each $x \in X_\sigma$ has a unique solution $\hat{y} = \hat{y}(x) \in X_\sigma^\perp$ and it is the solution to the extreme problem (5.7). It is clear that the map $x \mapsto \hat{y}(x) : X_\sigma \rightarrow X_\sigma^\perp$ is a bounded linear operator and $\hat{y}(x) = (A^*A + E)^{-1}A^*x$. Thus the definition of ΠB_ε in (5.6) can be rewritten as follows

$$\Pi B_\varepsilon = \{x \in X_\sigma : \|x - A^*\hat{y}(x)\|^2 + \|\hat{y}(x)\|^2 \leq \varepsilon^2\}. \quad (5.8)$$

The set (5.8) is an ellipsoid. Thus (5.3) and (5.8) define the measure $\Pi^*\nu$, and hence the RDS (2.31) is completely defined as well.

5.2. Finite-dimensional measure

In order to prove that the RDS (2.31) is ergodic, we study a map of the measure $\Pi^*\nu$. We introduce the operator of orthogonal projection Q connected with projection operator Q_k from Subsection 3.5

$$Q : V_0^0 \rightarrow X_{\sigma_k}^\perp \equiv X_\sigma^\perp \oplus X_{\sigma\sigma_k}, \quad (5.9)$$

where subspaces X_σ^\perp , $X_{\sigma\sigma_k}$ are defined in (4.5), (4.6). We have the following property for the operator Q .

Lemma 5.1. *Let Π, Q be projection operators defined in (2.30) and (5.9), respectively. Then*

$$Q\Pi = Q\Pi Q. \quad (5.10)$$

Proof. Each $x \in V_0^0$ can be decomposed as follows

$$x = y_1 + y_2 + y_3$$

with $y_1 \in X_\sigma^\perp$, $y_2 \in X_{\sigma\sigma_k}$ and $y_3 \in X_{\sigma_k}$. Then

$$Q\Pi x = Q(\Pi y_1 + y_2 + y_3) = Q\Pi y_1 + y_2, \quad Q\Pi Q x = Q(\Pi y_1 + y_2) = Q\Pi y_1 + y_2.$$

□

The random dynamical system (2.31) generates naturally the measure $\Pi^*\nu$. We now study the measure $Q^*\Pi^*\nu$. By (5.10) we show that this study can be reduced to the study of a measure defined on a finite dimensional space.

Theorem 5.1. *Let Q be the orthogonal projection in (5.9), Π be the projection in (2.30), and $\nu(\omega)$ be a probability measure on $\mathcal{B}(V_0^0)$. Then*

$$Q^*\Pi^*\nu = Q^*\Pi^*Q^*\nu. \quad (5.11)$$

Proof. By (3.8) and (5.10), we get

$$\begin{aligned} \int f(u)(Q^*\Pi^*\nu)(du) &= \int f(Q\Pi v)\nu(dv) \\ &= \int f(Q\Pi Qw)\nu(dw) = \int f(u)(Q^*\Pi^*Q^*\nu)(du). \end{aligned} \quad (5.12)$$

This proves the theorem. \square

The relation (5.11) allows us to reduce our investigation to the case of measures defined on finite dimensional space. Let us calculate the measure $Q^*\nu$. Taking into account the definitions (3.9) of $\nu(du)$ and (5.9) of Q , we get, analogous to (5.3), that for each $\Gamma \in \mathcal{B}(X_{\sigma_k}^\perp)$,

$$Q^*\nu(\Gamma) = c \int_{\Gamma \cap QB_\varepsilon} Q^*G(du), \quad (5.13)$$

where Q^*G is the Gauss measure supported on $X_{\sigma_k}^\perp$, with mathematical expectation zero and correlation operator QKQ . We make the following identification taking into account identity in (5.9): using in $X_{\sigma_k}^\perp$ an orthonormal basis $\{e_j, j = 1, \dots, n\}$ ($n = n_k$) introduced in Subsection 4.1, we can write

$$y = \sum_{j=1}^n y_j e_j \in X_{\sigma_k}^\perp, \quad u = \sum_{j=1}^m u_j e_j \in X_\sigma^\perp, \quad v = \sum_{j=m+1}^n v_j e_j \in X_{\sigma\sigma_k}, \quad (5.14)$$

and take the following identifications:

$$X_{\sigma_k}^\perp \cong \mathbb{R}^n = \{\vec{y} = (y_1, \dots, y_n)\}, \quad X_\sigma^\perp \cong \mathbb{R}^m = \{\vec{u}\}, \quad X_{\sigma\sigma_k} \cong \mathbb{R}^{n-m} = \{\vec{v}\}. \quad (5.15)$$

We restrict correlation operator QKQ on $X_{\sigma_k}^\perp$. Then QKQ can be regarded as a $n \times n$ matrix. By (3.4) this matrix is non-degenerate because for each $0 \neq u \in X_{\sigma_k}^\perp$ $(u, QKQu) = (Qu, KQu) = (u, Ku) > 0$ and therefore $\ker QKQ = 0$ ⁶ We denote $\hat{K} = (QKQ)^{-1}$. By (3.5)–(3.6), we conclude that

$$Q^*G(dy) := \hat{G}(dy) = g(y)dy, \quad (5.16)$$

where $g(y) = \frac{\det \hat{K}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}(\hat{K}y, y)}$. Note that

$$QB_\varepsilon = \left\{ y = \sum_{j=1}^n y_j e_j \in X_{\sigma_k}^\perp : \sum_{j=1}^n y_j^2 \leq \varepsilon^2 \right\} := B. \quad (5.17)$$

⁶ Actually we can weaken the first condition in (3.4) assuming that $K^* = K \geq 0$ and $\ker QKQ = 0$ where Q is orthogonal projector on $X_{\sigma_k}^\perp$ with big enough σ_k .

We hence obtain

$$\hat{\nu}(dy) := Q^*\nu(dy) = \hat{c}\hat{\chi}_\varepsilon(y)\hat{G}(dy), \quad (5.18)$$

where $\hat{c} = (\int_B \hat{G}(dx))^{-1}$ and $\hat{\chi}_\varepsilon$ is the characteristic function of the ball B .

We introduce the projection operator

$$\pi = Q\Pi : X_{\sigma_k}^\perp \rightarrow X_{\sigma\sigma_k}. \quad (5.19)$$

In virtue of (5.18), for each $\omega \in \mathcal{B}(X_{\sigma_k}^\perp)$,

$$Q^*\Pi^*Q^*\nu(\omega) = \pi^*\hat{\nu}(\omega). \quad (5.20)$$

6. Density $P(x)$ of the measure $\pi^*\hat{\nu}$

6.1. Preliminaries

Recall that the space $X_{\sigma_k}^\perp$ admits the orthogonal decomposition

$$X_{\sigma_k}^\perp = X_\sigma^\perp \oplus X_{\sigma\sigma_k}. \quad (6.1)$$

Below in order to emphasize belonging $u \in X_\sigma^\perp, v \in X_{\sigma\sigma_k}$ we write $u \oplus v$ instead of $u + v$.

The projection operator π defined in (5.19) can be represented as follows

$$\pi = (\alpha, E), \quad (6.2)$$

where E is the identity operator in $X_{\sigma\sigma_k}$ and $\alpha : X_\sigma^\perp \rightarrow X_{\sigma\sigma_k}$. For $x \in X_{\sigma\sigma_k}$ we denote by π_x , the affine plane in $X_{\sigma_k}^\perp$:

$$\pi_x = \pi^{-1}x = \{y \equiv u \oplus v \in X_\sigma^\perp \oplus X_{\sigma\sigma_k} = X_{\sigma_k}^\perp : \alpha u + v = x\}. \quad (6.3)$$

In particular, when $x = 0$,

$$\pi_0 = \pi^{-1}0 = \{y \equiv u \oplus v \in X_\sigma^\perp \oplus X_{\sigma\sigma_k} = X_{\sigma_k}^\perp : \alpha u + v = 0\}. \quad (6.4)$$

Since B is the support of the measure $\hat{\nu}(dy) \equiv \hat{\nu}(du, dv)$ defined in (5.18), the ellipsoid πB is the support of the measure $\pi^*\hat{\nu}$.

For each $f \in C(\pi B)$ we have

$$\int_{\pi B} f(x)\pi^*\hat{\nu}(dx) = \int_B f(\alpha u + v)\hat{\nu}(du, dv) = \int_{\pi B} f(x)dx \int_{\pi_x \cap B} \Gamma(w, x)dw, \quad (6.5)$$

where the first equality is via the definition of the measure $\pi^*\hat{\nu}$ and the second equality follows from the change of variables

$$w = u - \alpha u \in \pi_0, \quad x = \alpha u + v \in X_{\sigma_k}^\perp. \quad (6.6)$$

The calculation of $\Gamma(w, x)$ will be done later. The formulae (6.5) gives the expression of the density $P(x)$ for the measure $\pi^*\hat{\nu}(dx)$:

$$P(x)dx = \pi^*\hat{\nu}(dx), \quad \text{where } P(x) = \int_{\pi_x \cap B} \Gamma(w, x)dw. \quad (6.7)$$

6.2. Change of variables $(w, x) \rightarrow (u, v)$

In order to calculate the kernel functional $\Gamma(w, x)$ in (6.7), we need to consider the following change of variables, i.e., the inverse of (6.6)

$$u = u(w, x), \quad v = v(w, x). \quad (6.8)$$

We introduce an orthonormal basis $\{b_j, j = 1, \dots, n\}$ in $X_{\sigma_k}^\perp$. Let

$$\{b_j, j = 1, \dots, m\} \text{ be the orthonormal basis of } X_\sigma^\perp \subset X_{\sigma_k}^\perp, \quad (6.9)$$

composed of eigenvectors of the operator $E + \alpha^* \alpha : X_\sigma^\perp \rightarrow X_\sigma^\perp$. Suppose that $1 \leq \mu_1, \dots, 1 \leq \mu_m$ are eigenvalues corresponding to eigenvectors b_1, \dots, b_m . Assume that

$$\mu_1 > 1, \dots, \mu_s > 1, \mu_{s+1} = \dots = \mu_m = 1. \quad (6.10)$$

Lemma 6.1. *The following statements hold:*

- (i) $b_j \in \ker \alpha, j = s+1, \dots, m$;
- (ii) $\{\alpha b_j, j = 1, \dots, s\}$ form an orthogonal basis in $\text{Im } \alpha \subset X_{\sigma\sigma_k}$.

Proof. (i) For $j = s+1, \dots, m$, $(E + \alpha^* \alpha)b_j = b_j$ iff $\alpha^* \alpha b_j = 0$. Since $\ker \alpha^* \perp \text{Im } \alpha$, $\alpha b_j \neq 0$ implies $\alpha^* \alpha b_j \neq 0$. Hence $b_j \in \ker \alpha$.

(ii) Note that for $i, j = 1, \dots, s$, $(\alpha b_i, \alpha b_j) = (\alpha^* \alpha b_i, b_j) = (\mu_i - 1)(b_i, b_j) = 0$ and $\mu_i \neq 1$. Hence $\alpha b_i \perp \alpha b_j$. If $\gamma \in \text{Im } \alpha$ then for a certain $b \in X_\sigma^\perp$, we have $\gamma = \alpha b = \alpha \sum_{j=1}^m c_j b_j = \sum_{j=1}^s c_j \alpha b_j$. Therefore $\{\alpha b_j, j = 1, \dots, s\}$ form a basis in $\text{Im } \alpha$. \square

Since $\ker \alpha^* \oplus \text{Im } \alpha = X_{\sigma\sigma_k}$ and $\dim \text{Im } \alpha = s$ due to Lemma 6.1, we see that $\dim \ker \alpha^* = n - m - s$. Let $\{b_{m+s+1}, \dots, b_n\}$ be an orthonormal basis for $\ker \alpha^*$. Then by Lemma 6.1 again, the vectors

$$b_{j+m} = \frac{\alpha b_j}{\|\alpha b_j\|}, j = 1, \dots, s, \quad b_{m+s+1}, \dots, b_n \quad (6.11)$$

form an orthonormal basis in $X_{\sigma\sigma_k}$. In virtue of (6.9)–(6.11),

$$\text{Vectors } b_1, \dots, b_n \text{ form an orthonormal basis of } X_{\sigma_k}^\perp. \quad (6.12)$$

On the plane π_0 defined in (6.4), we consider the vectors

$$\theta_j = \frac{b_j \oplus (-\alpha b_j)}{(1 + \|\alpha b_j\|^2)^{\frac{1}{2}}}, j = 1, \dots, m. \quad (6.13)$$

Lemma 6.2. *Vectors (6.13) form an orthonormal basis of the plane π_0 .*

Proof. By the definitions (6.2), (6.4) and (6.13), we see that $\theta_j \in \pi_0$, as

$$\pi \theta_j = (1 + \|\alpha b_j\|^2)^{-\frac{1}{2}} [\alpha b_j - \alpha b_j] = 0.$$

By Lemma 6.1, for $i \neq j$,

$$\begin{aligned} (\theta_i, \theta_j)_{\pi_0} &= \left(\frac{b_i \oplus (-\alpha b_i)}{\sqrt{1 + \|\alpha b_i\|^2}}, \frac{b_j \oplus (-\alpha b_j)}{\sqrt{1 + \|\alpha b_j\|^2}} \right)_{X_{\sigma_k}^\perp} \\ &= (1 + \|\alpha b_i\|^2)^{-\frac{1}{2}} (1 + \|\alpha b_j\|^2)^{-\frac{1}{2}} [(b_i, b_j)_{X_{\sigma}^\perp} + (-\alpha b_i, -\alpha b_j)_{X_{\sigma\sigma_k}}] \\ &= 0 \end{aligned} \quad (6.14)$$

and therefore the system (6.13) is orthonormal. As the rank of the matrix for (6.2) equals to $n - m$,

$$\dim \pi_0 = \dim \ker \pi \equiv \dim \ker(\alpha, E) = m.$$

Hence the system (6.13) forms an orthonormal basis of the plane π_0 . \square

Define

$$\theta_j = b_j, \quad j = m + 1, \dots, n. \quad (6.15)$$

Then the vectors

$$\theta_j, \quad j = 1, \dots, n \quad (6.16)$$

defined in (6.13) and (6.15) form a basis of $X_{\sigma_k}^\perp$.

Let $R = (R_{ij})$ be the $n \times n$ matrix with components R_{ij} defined as follows.

$$\begin{aligned} R_{ii} &= (1 + \|\alpha b_i\|^2)^{-\frac{1}{2}}; & R_{i,m+1} &= \frac{-\alpha}{\sqrt{1 + \|\alpha b_i\|^2}}, \quad i = 1, \dots, s; \\ R_{ii} &= 1, \quad i = s + 1, \dots, n; & R_{ij} &= 0 \text{ for other } i, j. \end{aligned} \quad (6.17)$$

It can be checked directly that the matrix R transforms the basis $\vec{b} = (b_1, \dots, b_n)$ to the basis $\vec{\theta} = (\theta_1, \dots, \theta_n)$:

$$\vec{\theta} = R\vec{b}. \quad (6.18)$$

Now we can calculate the change of variables (6.8), the inverse of (6.6). Let $y \in X_{\sigma_k}^\perp$ admits decompositions $y = u + v = w + x$ with $u \in X_{\sigma}^\perp$, $v \in X_{\sigma\sigma_k}$, $w \in \pi_0$ and $x \in X_{\sigma\sigma_k}$. Define

$$\begin{aligned} y &= \sum_{j=1}^n y_j b_j = \sum_{j=1}^n z_j \theta_j, \\ u &= \sum_{j=1}^m u_j b_j, & v &= \sum_{j=m+1}^n v_{j-m} b_j, \\ w &= \sum_{j=1}^m w_j \theta_j, & x &= \sum_{j=m+1}^n x_{j-m} \theta_j. \end{aligned} \quad (6.19)$$

We introduce notations

$$\begin{aligned}\vec{y} &= (y_1, \dots, y_n), \vec{z} = (z_1, \dots, z_n), \\ \vec{u} &= (u_j \equiv y_j, j = 1, \dots, m), \quad \vec{v} = (v_j \equiv y_{j+m}, j = 1, \dots, n-m), \\ \vec{w} &= (w_j \equiv z_j, j = 1, \dots, m), \quad \vec{x} = (x_j \equiv z_{j+m}, j = 1, \dots, n-m).\end{aligned}\quad (6.20)$$

Then the change of variables (6.8) is rewritten as

$$\vec{u} = u(\vec{w}, \vec{x}), \quad \vec{v} = v(\vec{w}, \vec{x}), \quad (6.21)$$

or in more compact form

$$\vec{y} = y(\vec{z}). \quad (6.22)$$

Theorem 6.1. *The transformation (6.22) can be calculated as follows*

$$\vec{y} = y(\vec{z}) = R^* \vec{z}, \quad (6.23)$$

where R^* is the conjugate matrix of the matrix R defined in (6.17)–(6.18). Note that the matrix R transforms the basis $\{b_j\}$ to the basis $\{\theta_j\}$.

Proof. In fact, the relation (6.23) follows from (6.18)–(6.19). \square

Using (6.23) and (6.17), we obtain the Jacobian of the transformation (6.23):

$$J = \det D\vec{y}/D\vec{z} = \det R^* = \prod_{i=1}^s (1 + \|\alpha b_i\|^2)^{-\frac{1}{2}}. \quad (6.24)$$

We see that the Jacobian J depends only on the operator α . Now we make more precise the expression for density $P(x)$ in (6.7). Note that in (6.5) we made just the change of variables (6.22) or the equivalent (6.23). Taking into account of the definition $\vec{y} = (\vec{u}, \vec{v})$ and $\vec{z} = (\vec{w}, \vec{x})$, and using the facts (5.16), (5.18), (6.23) and (6.24), we obtain that the integrand $\Gamma(w, x)$ in the expression of density $P(x)$ in (6.7):

$$\Gamma(w, x) = \prod_{i=1}^s (1 + \|\alpha b_i\|^2)^{\frac{1}{2}} \det \hat{K} (2\pi)^{-n/2} \exp \left[-\frac{1}{2} (\hat{K} Q R^* \vec{z}, Q R^* \vec{z}) \right], \quad (6.25)$$

with $\vec{z} = (w, x)$. We suppress the arrow \rightarrow on top of w and x here.

7. Smoothness of the density $P(x)$

Formulas (6.5) and (6.7) imply that the density $P(x)$ is supported in the ellipsoid $\pi B \subset X_{\sigma\sigma_k}$. So $P(x) \equiv 0$ for $x \in X_{\sigma\sigma_k} \setminus \pi B$. We now investigate the smoothness of $P(x)$ for $x \in \partial(\pi B)$ and for $x \in \text{Int}(\pi B)$, respectively.

7.1. Smoothness of the density $P(x)$ on boundary $\partial(\pi B)$

It is clear that the set $B \cap \pi_x$ is \emptyset if $x \notin \pi B$, it is a single point if $x \in \partial(\pi B)$, and it is a ball in the $n - m$ dimensional plane π_x if $x \in \text{Int}(\pi B)$. We first calculate the center and radius of this ball.

Lemma 7.1. *Let the ball B be defined in (5.17), $x \in \text{Int}(\pi B)$, and the plane π_x be defined in (6.3). Then the center of the ball $B \cap \pi_x$ is*

$$\tilde{w} = \hat{u} \oplus \hat{v} = [(E + \alpha^* \alpha)^{-1} \alpha^* x] \oplus [x - \alpha(E + \alpha^* \alpha)^{-1} \alpha^* x] \quad (7.1)$$

and the radius of the ball $B \cap \pi_x$ is

$$r = (\varepsilon^2 - \|(E + \alpha^* \alpha)^{-1} \alpha^* x\|_{X_\sigma^\perp}^2 - \|x - \alpha(E + \alpha^* \alpha)^{-1} \alpha^* x\|_{X_{\sigma\sigma_k}}^2)^{\frac{1}{2}}, \quad (7.2)$$

where, recall that, ε is the radius of the ball B .

Proof. Evidently, the center $\{\hat{u}, \hat{v}\}$ is the solution of the extreme problem

$$\|u\|_{X_\sigma^\perp}^2 + \|v\|_{X_{\sigma\sigma_k}}^2 \rightarrow \inf, \quad \{u, v\} \in \pi_x. \quad (7.3)$$

By definition (6.3), $\{u, v\} \in \pi_x$ iff $\alpha u + v = x$, i.e., $v = x - \alpha u$. Substituting this into (7.3) and solving the extreme problem, we obtain the solution (7.1). The radius r follows from the Pythagoras theorem. \square

Let us consider the following extreme problem: Given $x \in \pi B$, find $h \in X_{\sigma\sigma_k}$ such that

$$F(h) = \|(E + \alpha^* \alpha)^{-1} \alpha^* (x + h)\|^2 + \|(E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) (x + h)\|^2 \rightarrow \inf, \quad (7.4)$$

$$\|h\|^2 = \gamma_0^2, \quad (7.5)$$

with $\gamma_0 > 0$ a given sufficiently small parameter. Recall that each $x \in X_{\sigma\sigma_k}$ admits the decomposition

$$x = x_0 + \sum_{j=1}^s x_j \alpha b_j, \quad x_0 \in \ker \alpha^*, \quad x_j \in \mathbb{R}, \quad (7.6)$$

where $\{b_j\}$ is the basis in (6.9). Note also that, by Lemma 6.1(ii), $\{\alpha b_j, j = 1, \dots, s\}$ is a basis of $\text{Im } \alpha$.

Lemma 7.2. *Suppose that $x \in \pi B$ and it has decomposition (7.6). If $\gamma_0 > 0$ is small enough, then there exists a unique solution \hat{h} of the extreme problem (7.4)–(7.5). The solution \hat{h} is determined by*

$$\hat{h} = h_0 + \sum_{j=1}^s h_j \alpha b_j, \quad h_0 \in \ker \alpha^*, \quad h_j \in \mathbb{R}, \quad (7.7)$$

where

$$h_0 = -\frac{x_0}{1 + \lambda(\gamma_0)}, \quad h_j = -\frac{x_j}{1 + \lambda(\gamma_0)\mu_j}, \quad j = 1, \dots, s \quad (7.8)$$

and $x_0, x_j, j = 1, \dots, s$ are defined in (7.6), $\mu_j > 1, j = 1, \dots, s$ are eigenvalues (6.10) of the operator $E + \alpha^* \alpha$, and $\lambda(\gamma_0)$ is the unique solution of the equation

$$\frac{\|x_0\|^2}{(1 + \lambda)^2} + \sum_{j=1}^s \frac{x_j^2 \|\alpha b_j\|^2}{(1 + \lambda \mu_j)^2} = \gamma_0^2. \quad (7.9)$$

Proof. The existence of a solution of the finite-dimensional problem (7.4)–(7.5) is evident. We now prove the uniqueness of this solution h .

Let $\mathcal{L}(h, \lambda) = F(h) + \lambda(\|h\|^2 - \gamma_0^2)$ be the Lagrange function for the extreme problem (7.4)–(7.5). By the Lagrange principle, if h is a solution of this problem, then there exists $\lambda \in \mathbb{R}$ such that

$$(\mathcal{L}'_h(h, \lambda), \delta) = (F'(h), \delta) + 2\lambda(h, \delta) = 0, \quad \forall \delta \in X_{\sigma\sigma_k}. \quad (7.10)$$

Substitution of expression $F(h)$ in (7.4) into (7.10) yields

$$\begin{aligned} & ((E + \alpha^* \alpha)^{-1} \alpha^*(x + h), (E + \alpha^* \alpha)^{-1} \alpha^* \delta) \\ & + ((E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)(x + h), (E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) \delta) + \lambda(h, \delta) = 0. \end{aligned}$$

We transform operators from right multiplies in scalar products to the left multipliers. Noting that

$$\alpha(E + \alpha^* \alpha)^{-2} \alpha^* + (E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)^2 = E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*, \quad (7.11)$$

we obtain equations for h and λ :

$$(E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)(x + h) + \lambda h = 0. \quad (7.12)$$

Substituting the decompositions (7.6)–(7.7) for x and h into (7.12), we arrive at the following system of equations

$$x_0 + h_0(1 + \lambda) = 0, \quad (7.13)$$

$$\left[1 - \frac{\mu_j - 1}{\mu_j}\right] (x_j + h_j) + \lambda h_j = 0, \quad (7.14)$$

and therefore

$$h_0 = -\frac{x_0}{1 + \lambda}, \quad h_j = -\frac{x_j}{1 + \lambda \mu_j}. \quad (7.15)$$

By (7.5), (7.7) and (7.15), we finally get

$$r(\lambda) \equiv \|h\|^2 = \|h_0\|^2 + \sum_{j=1}^s h_j^2 \|\alpha b_j\|^2 = \frac{\|x_0\|^2}{(1 + \lambda)^2} + \sum_{j=1}^s \frac{x_j^2 \|\alpha b_j\|^2}{(1 + \lambda \mu_j)^2} = \gamma_0^2. \quad (7.16)$$

Since $r(0) = \|x\|^2$, $r(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ and $r'(\lambda) < 0$, there exists a unique solution $\lambda = \lambda(\gamma_0)$ if $\gamma_0^2 < \|x\|^2$. \square

Theorem 7.1. *Let $x \in \partial\pi B$, $x + h \in \text{Int}(\pi B)$ and $\|h\| < \|x\|$. Then $P(x) = 0$ and for some positive constant c ,*

$$|P(x + h)| \leq c\|h\|^{\frac{m}{2}} \text{ as } \|h\| \rightarrow 0. \quad (7.17)$$

Proof. If $x \in \partial(\pi B)$, then

$$B \cap \pi_x = \hat{u} \oplus \hat{v},$$

where the point $\hat{u} \oplus \hat{v}$ is defined in (7.1). Hence by (6.7), $P(x) = 0$. Denote $\|h\| = \gamma_0$ and take the solution \hat{h} of the problem (7.4)–(7.5). By (7.2), (7.4) and (7.5), the radius $r(x + \hat{h})$ of the ball $B \cap \pi_{x+\hat{h}}$ is maximal in the sets of radius $r(x + h)$ of $B \cap \pi_{x+h}$ corresponding to vectors h such that $\|h\| = \gamma_0$. We calculate $r(x + \hat{h})$.

Since $x \in \partial(\pi B)$, we have $\|\hat{u}(x)\|^2 + \|\hat{v}(x)\|^2 = \varepsilon^2$ with $\{\hat{u}, \hat{v}\}$ defined in (7.1). In (7.2), taking $x \rightarrow x + \hat{h}$ and substituting $\varepsilon^2 = \|\hat{u}(x)\|^2 + \|\hat{v}(x)\|^2$, and taking into account (7.6)–(7.8), we get

$$\begin{aligned} & r^2(x + \hat{h}) \\ &= -((E + \alpha^* \alpha)^{-1} \alpha^* \hat{h}, (E + \alpha^* \alpha)^{-1} \alpha^* (2x + \hat{h})) \\ &\quad - ((E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) \hat{h}, (E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) (2x + \hat{h})) \\ &= \left((E + \alpha^* \alpha)^{-1} \alpha^* \sum_{j=1}^s \frac{x_j \alpha b_j}{1 + \lambda \mu_j}, (E + \alpha^* \alpha)^{-1} \alpha^* \sum_{l=1}^s x_l \alpha b_l \left(2 - \frac{1}{1 + \lambda \mu_j} \right) \right) \\ &\quad + \left(\frac{x_0}{1 + \lambda} + \sum_{j=1}^s (E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) \alpha b_j \frac{x_j}{1 + \lambda \mu_j}, \right. \\ &\quad \left. x_0 \left(2 - \frac{1}{1 + \lambda} \right) + \sum_{l=1}^s (E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*) \alpha b_l x_l \left(2 - \frac{1}{1 + \lambda \mu_j} \right) \right) \\ &= \sum_{j,l=1}^s \frac{x_j (\mu_j - 1)}{\mu_j (1 + \lambda \mu_j)} \frac{x_l (\mu_l - 1)}{\mu_l} \left(2 - \frac{1}{1 + \lambda \mu_l} \right) (b_j, b_l) + \frac{\|x_0\|^2 (1 + 2\lambda)}{(1 + \lambda)^2} \\ &\quad + \sum_{j,l=1}^s x_j x_l (\alpha b_j, \alpha b_l) \left(1 - \frac{\mu_j - 1}{\mu_j} \right) \frac{1}{1 + \lambda \mu_j} \left(1 - \frac{\mu_l - 1}{\mu_l} \right) \left(2 - \frac{1}{1 + \lambda \mu_l} \right) \\ &= \sum_{j=1}^s \frac{x_j^2 (\mu_j - 1)^2 (1 + 2\lambda \mu_j)}{\mu_j^2 (1 + \lambda \mu_j)^2} + \frac{\|x_0\|^2 (1 + 2\lambda)}{(1 + \lambda)^2} + \sum_{j=1}^s x_j^2 \|\alpha b_j\|^2 \frac{1 + 2\lambda \mu_j}{\mu_j^2 (1 + \lambda \mu_j)^2}. \end{aligned} \quad (7.18)$$

We rewrite this as follows:

$$r^2(x + \hat{h}) = \frac{\|x_0\|^2 (1 + 2\lambda)}{(1 + \lambda)^2} + \sum_{j=1}^s \frac{x_j^2 \|\alpha b_j\|^2}{(1 + \lambda \mu_j)^2} \frac{1 + 2\lambda \mu_j}{\mu_j^2} \left(1 + \frac{(\mu_j - 1)^2}{\|\alpha b_j\|^2} \right). \quad (7.19)$$

There exist constants $0 < c_1 < c_2/2$ such that for each $j = 1, \dots, s$ and for every $\lambda > 0$, we have

$$c_1(1 + \lambda) \leq \frac{1 + 2\lambda\mu_j}{\mu_j^2} \left(1 + \frac{(\mu_j - 1)^2}{\|\alpha b_j\|^2}\right) \leq \frac{c_2}{2}(1 + \lambda). \quad (7.20)$$

Comparing (7.9) and (7.19), we thus get

$$c_1\gamma_0^2(1 + \lambda) \leq r^2(x + \hat{h}) \leq c_2\gamma_0^2(1 + \lambda). \quad (7.21)$$

Let $A_1(x) = \left(\|x_0\|^2 + \sum_{j=1}^s \frac{x_j^2 \|\alpha b_j\|^2}{\mu_j^2}\right)^{\frac{1}{2}}$. Then using (7.9) we get

$$\gamma_0^2 \geq \frac{\|x_0\|^2}{(1 + \lambda)^2} + \sum_{j=1}^s \frac{x_j^2 \|\alpha b_j\|^2}{\mu_j^2 (1 + \lambda)^2} = \frac{A_1(x)^2}{(1 + \lambda(\gamma_0))^2}.$$

and therefore

$$1 + \lambda(\gamma_0) \geq \frac{A_1(x)}{\gamma_0}. \quad (7.22)$$

Let now $A_2(x) = \left(\|x_0\|^2 + \sum_{j=1}^s x_j^2 \|\alpha b_j\|^2\right)^{\frac{1}{2}}$. Then by (7.9) we obtain

$$\gamma_0^2 \leq \frac{A_2(x)^2}{(1 + \lambda)^2}.$$

Hence,

$$1 + \lambda(\gamma_0) \leq \frac{A_2(x)}{\gamma_0}. \quad (7.23)$$

Substituting (7.22), (7.23) into (7.21), we get

$$c_1 A_1(x) \gamma_0 \leq r^2(x + \hat{h}) \leq c_2 A_2(x) \gamma_0. \quad (7.24)$$

By (6.25), there exists constants $0 < \hat{c}_1 < \hat{c}_2$ such that

$$\hat{c}_1 \leq \Gamma(w, x) \leq \hat{c}_2, \quad (7.25)$$

for each $(w, x) = \vec{z}$ such that $\|QR^* \vec{z}\|^2 \equiv \|\vec{y}\|^2 \leq \varepsilon^2$, i.e., on the ball B . Hence, by (6.7),

$$\hat{c}_1 V(\pi_{x+h} \cap B) \leq P(x + h) = \int_{\pi_{x+h} \cap B} \Gamma(w, x) dw \leq \hat{c}_2 V(\pi_{x+h} \cap B), \quad (7.26)$$

where $V(\pi_{x+h} \cap B)$ is the volume of the ball $\pi_{x+h} \cap B$.

Note that $r(x + h) \leq r(x + \hat{h})$ for each $h \in X_{\sigma\sigma_k}$ such that $\|h\| = \|\hat{h}\| = \gamma_0$ and $x + h \in \text{Int}(\pi B)$. Thus (7.24), (7.26) and the fact that $V(\pi_{x+h} \cap B) = c_m r(x + h)^m$ (where c_m is the volume of the unit ball in \mathbb{R}^m) implies (7.17). \square

Remark. The inequalities in (7.26) imply that for each $x \in \partial(\pi B)$ and for \hat{h} defined in the proof of Theorem 7.1, the following estimate holds

$$P(x + \hat{h}) \geq c \|\hat{h}\|^{\frac{m}{2}}.$$

This inequality means that if $m = 1$, $P(x)$ is not differentiable at x on the boundary $\partial(\pi B)$.

7.2. Smoothness of the density $P(x)$ in the interior $\text{Int}(\pi B)$

We first reformulate the definition of $P(x)$ in (6.7). By (6.3) and (6.4), for each $x \in X_{\sigma\sigma_k}$, we see that

$$\pi_x = 0 \oplus x + \pi_0, \text{ with } \pi_0 = \{u \oplus (-\alpha u), u \in X_\sigma^\perp\} \subset X_\sigma^\perp \oplus X_{\sigma\sigma_k}.$$

Therefore the ball

$$\pi_x \cap B = \{w = u \oplus v \in \pi_x : \|w - \hat{w}\|^2 \leq r^2\},$$

where the center $\hat{w} = \hat{w}(x)$ and the radius $r = r(x)$ are defined in (7.1)–(7.2), can be rewritten as

$$\pi_x \cap B = 0 \oplus x + \mathcal{D}(x).$$

Here

$$\mathcal{D}(x) = \{w = u \oplus (-\alpha u) \in \pi_0 : \|w - \hat{w}(x)\|^2 \leq r^2(x)\} \quad (7.27)$$

with $r(x)$ being defined in (7.2) and

$$\hat{w}(x) = [(E + \alpha^* \alpha)^{-1} \alpha^* x] \oplus [-\alpha(E + \alpha^* \alpha)^{-1} \alpha^* x]. \quad (7.28)$$

We now decompose w and \hat{w} in the basis θ_j of π_0 , defined in (6.13):

$$w = \sum_{j=1}^m w_j \theta_j, \quad \hat{w}(x) = \sum_{j=1}^m \hat{w}_j \theta_j,$$

and consider the integral

$$P(x) = \int_{B \cap \pi_x} \Gamma(w, x) dw = \int_{\mathcal{D}(x)} \Gamma(w, x) dw, \quad (7.29)$$

with $w = (w_1, \dots, w_m) \in \mathbb{R}^m$.

Let $x \in \text{Int}(\pi B)$. To investigate the smoothness of $P(x)$ at x , we consider the difference

$$P(x+h) - P(x) = \int_{\mathcal{D}(x+h)} \Gamma(w, x+h) dw - \int_{\mathcal{D}(x)} \Gamma(w, x) dw. \quad (7.30)$$

By (6.25), the function $\Gamma(w, x)$ is infinitely differentiable in x and in w . By Taylor expansion

$$\Gamma(w, x+h) = \Gamma(w, x) + (\Gamma'_x(w, x), h) + o(w, x, h), \quad (7.31)$$

where

$$|o(w, x, h)| \leq c \|h\|^2 \quad (7.32)$$

with c depending on (w, x) . As usual, in this case, we denote $o(w, x, h)$ as $O(\|h\|^2)$. Inserting (7.31) into (7.30), we conclude that

$$\begin{aligned} P(x+h) - P(x) &= \int_{\mathcal{D}(x+h) \setminus \mathcal{D}(x)} \Gamma(w, x) dw - \int_{\mathcal{D}(x) \setminus \mathcal{D}(x+h)} \Gamma(w, x) dw \\ &\quad + \int_{\mathcal{D}(x+h)} (\Gamma'_x(w, x), h) dw + o(\|h\|^2). \end{aligned} \quad (7.33)$$

We now calculate the Gâteaux derivative of $P(x)$ at x . Denote $e = h/\|h\|$ and $\lambda = \|h\|$. Divide (7.33) by λ and take limit as $\lambda \rightarrow 0$. For the third term in the right hand side, we have

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathcal{D}(x+\lambda e)} (\Gamma'_x(w, x), \lambda e) dw = \int_{\mathcal{D}(x)} (\Gamma'_x(w, x), e) dw. \quad (7.34)$$

To find the similar limit for the first and second terms in the right hand side, we introduce a kind of “polar” coordinates in the sets $\mathcal{D}(x+h) \setminus \mathcal{D}(x)$ and $\mathcal{D}(x) \setminus \mathcal{D}(x+h)$ when they are not empty. Suppose that

$$\mathcal{D}(x+h) \setminus \mathcal{D}(x) \neq \emptyset. \quad (7.35)$$

Let b be the running point on the part of the sphere $\partial B(x+h)$ which is the part of boundary for the set (7.35). Define

$$a = \overline{w(x)b} \cap \partial B(x), \quad (7.36)$$

where $\overline{w(x)b}$ is the vector with end points $w(x)$ and b . Denote by ψ the magnitude of the angle $\angle bw(x)w(x+h)$. By the cosine theorem in the triangle $\triangle bw(x)w(x+h)$, we see that

$$\|\overline{w(x+h)b}\|^2 = \|\overline{w(x)b}\|^2 + \|\overline{w(x)w(x+h)}\|^2 - 2\|\overline{w(x)b}\|\|\overline{w(x)w(x+h)}\| \cos \psi. \quad (7.37)$$

Define $\|\overline{w(x)w(x+h)}\| = \rho(h)$ and $\|\overline{w(x)b}\| = z$. Note that $\|\overline{w(x+h)b}\| = r(x+h)$. Then (7.37) may be reformulated as a quadratic equation with respect to z :

$$z^2 - 2\rho z \cos \psi + \rho^2 - r^2(x+h) = 0$$

and therefore $z = \rho \cos \psi + \sqrt{r^2(x+h) - \rho^2 \sin^2 \psi}$. We choose the positive sign because for $\psi = 0$, we should get $z = \rho + r(x+h)$. As a result,

$$\|\overline{ab}(\psi)\| = z - r(x) = \rho \cos \psi + \sqrt{r^2(x+h) - \rho^2 \sin^2 \psi} - r(x). \quad (7.38)$$

We introduce polar coordinates in the set (7.35):

$$\mathcal{D}(x+h) \setminus \mathcal{D}(x) \ni w \rightarrow (\psi, \omega, \gamma)$$

with (ψ, ω) the spherical coordinate: ψ is the angle in Figure 1 and ω is complement spherical coordinate; the coordinate γ is the distance from a to the point that has

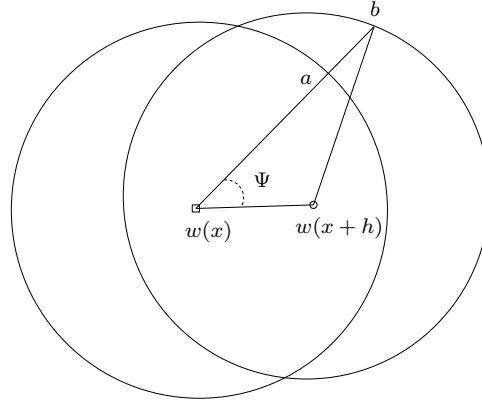


FIG. 1

coordinate w where a is the following point: $a = \overline{w(x)w} \cap \partial B(x)$. As is well-known, the Jacobian of the transformation $w = w(\psi, \omega, \gamma)$ is equal to $(r(x) + \gamma)^{m-1}$. So

$$dw = (r(x) + \gamma)^{m-1} d\psi d\omega d\gamma. \tag{7.39}$$

Let us estimate (7.38) for small $\|h\|$. By (7.28),

$$\rho^2(h) = \|(E + \alpha^* \alpha)^{-1} \alpha^* h\|^2 + \|\alpha(E + \alpha^* \alpha)^{-1} \alpha^* h\|^2 \tag{7.40}$$

and using (7.2),

$$\begin{aligned} r^2(x+h) &= r^2(x) - 2((E + \alpha^* \alpha)^{-1} \alpha^* x, (E + \alpha^* \alpha)^{-1} \alpha^* h) \\ &\quad - 2(x - \alpha(E + \alpha^* \alpha)^{-1} \alpha^* x, h - \alpha(E + \alpha^* \alpha)^{-1} \alpha^* h) + O^2(h). \end{aligned}$$

Hence by Taylor expansion and by (7.11),

$$\|\overline{ab}(\psi)\|(x, h) = \rho(h) \cos \psi - \frac{1}{r(x)} ((E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)x, h) + o(\|h\|^2). \tag{7.41}$$

Let $\partial_1(x, h)$ be the part of the boundary for the set $\mathcal{D}(x+h) \setminus \mathcal{D}(x)$, composed of the points on sphere $\partial \mathcal{D}(x)$. Changing to polar coordinates, and applying the Taylor expansion for Γ and using (7.39) and (7.41), we get

$$\begin{aligned} \int_{\mathcal{D}(x+h) \setminus \mathcal{D}(x)} \Gamma(w, x) dw &= \int_{\partial_1(x, h)} \int_0^{\|\overline{ab}(\psi)\|(x, h)} \Gamma(\psi, \omega, \gamma, x) (r(x) + \gamma)^{m-1} d\gamma d\psi d\omega \\ &= \int_{\partial_1(x, h)} \Gamma(\psi, \omega, 0, x) \Psi_1(x, \psi, e) d\psi d\omega \|h\| + O(\|h\|^2), \end{aligned} \tag{7.42}$$

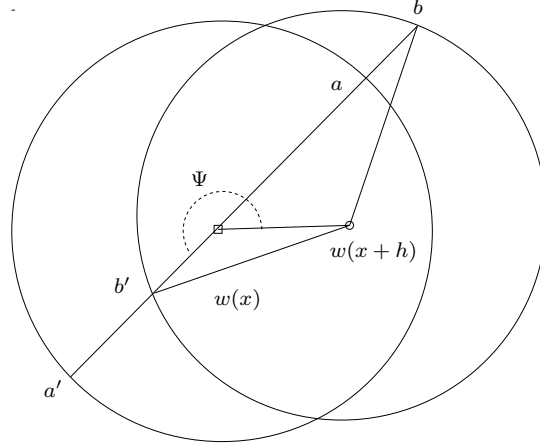


FIG. 2

where, recall, $e = h/\|h\|$, and the function $\Psi(x, \psi, e)$ is defined by

$$\begin{aligned} \int_0^{\|\overline{ab}(\psi)\|(x,h)} (r(x) + \gamma)^{m-1} d\gamma &= \frac{1}{m} ((r(x) + \|\overline{ab}(\psi)\|(x, h))^m - r(x)^m) \\ &= \Psi(x, \psi, e) \|h\| + o(\|h\|^2). \end{aligned}$$

In other words, by (7.41),

$$\Psi(x, \psi, e) = r^{m-1}(x) (\rho(e) \cos \psi - \frac{1}{r(x)} ((E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)x, e)). \quad (7.43)$$

To calculate the integral over $\mathcal{D}(x) \setminus \mathcal{D}(x+h)$ in (7.33) we use notations of points $w(x)$, $w(x+h)$, a' , b' and angle ψ from Figure 2. Similarly to the case $\mathcal{D}(x+h) \setminus \mathcal{D}(x)$ we define $\|w(x)w(x+h)\| = \rho(h)$, $\|w(x)b'\| = z$, $\|w(x+h)b'\| = r(x+h)$. Then by Cosine Theorem in triangle $\triangle b'w(x)w(x+h)$ we have the equality $z^2 + \rho^2 - 2z\rho \cos \psi - r^2(x+h) = 0$ and therefore $z = \rho \cos \psi + \sqrt{r^2(x+h) - \rho^2 \sin^2 \psi}$ (We put sign “plus” before square root because for $\psi = \pi$ the equality $z = r(x+h) - \rho$ holds). Then

$$\|\overline{a'b'}\| = r(x) - z = -\rho(h) \cos \psi + r^{-1}(x) ((E - \alpha(E + \alpha^* \alpha)^{-1} \alpha^*)x, h)$$

and similarly to (7.42), (7.43) we get

$$- \int_{\mathcal{D}(x) \setminus \mathcal{D}(x+h)} \Gamma(w, x) dw = \int_{\partial_2(x,h)} \Gamma(\psi, \omega, 0, x) \Psi(x, \psi, e) d\psi d\omega \|h\| + o(\|h\|^2), \quad (7.44)$$

where $\Psi(x, \psi, e)$ is defined in (7.43) and $\partial_2(x, h)$ is the part of the boundary for the set $\mathcal{D}(x) \setminus \mathcal{D}(x+h)$ composed of the points on sphere $\partial\mathcal{D}(x)$.

Substituting (7.42), (7.43), (7.44) into (7.33), dividing the obtained equation by $\lambda = \|h\|$ and taking the limit $\lambda \downarrow 0$, we conclude

$$\lim_{\lambda \downarrow 0} \frac{P(x + \lambda e) - P(x)}{\lambda} = \int_{\partial \mathcal{D}(x)} \Gamma(\psi, \omega, 0, x) \Psi(x, \psi, e) d\psi d\omega + \int_{\mathcal{D}(x)} (\Gamma'_x(w, x), e) dw.$$

This equality shows that the density $P(x)$ possesses the first variation⁷ for each x on $\text{Int}(\pi B)$, and moreover

$$P'(x, h) = \int_{\partial \mathcal{D}(x)} \Gamma(\psi, \omega, 0, x) \Psi(x, \psi, h) d\psi d\omega + \int_{\mathcal{D}(x)} (\Gamma'_x(w, x), h) dw. \quad (7.45)$$

Theorem 7.2. *The first variation $P'(x, h)$ is the Lagrange variation. Moreover, the function*

$$x \rightarrow \sup_{\|e\|_{X_{\sigma\sigma_k}}} |P'(x, e)| \quad \text{is continuous for } x \in \text{Int}(\pi B). \quad (7.46)$$

Proof. Note that the second terms in right sides of (7.45), (7.43) are linear with respect to h (to e). Calculation of (7.43) in the case of $-e$ gives that the first term in right side of (7.43) is equal $r^{m-1}(x)\rho(-e)\cos\psi_1$ where $\psi_1 = \psi + \pi$ and ψ is the angle from (7.43). Hence,

$$r^{m-1}(x)\rho(-e)\cos\psi_1 = r^{m-1}(x)\rho(e)\cos(\psi + \pi) = -r^{m-1}(x)\rho(e)\cos\psi.$$

Therefore $P'(x, -h) = -P'(x, h)$.

The assertion (7.46) follows directly from the explicit formulas (7.45), (7.43). \square

Now we are in a position to prove that $P(x)$ satisfy (3.18).

Lemma 7.3. *For every $v_1, v_2 \in X_{\sigma\sigma_k}$*

$$\int_{X_{\sigma\sigma_k}} |P(x - v_1) - P(x - v_2)| dx \leq c \|v_1 - v_2\|_{X_{\sigma\sigma_k}} \quad (7.47)$$

where $c > 0$ does not depend on v_1, v_2 .

Proof. For each $x, v_1, v_2 \in X_{\sigma\sigma_k}$

$$P(x - v_1) - P(x - v_2) = \int_0^1 \frac{dP(x + v_2 + \theta(v_2 - v_1))}{d\theta} d\theta. \quad (7.48)$$

Note that derivative $dP/d\theta$ is well defined for every $\theta \in [0, 1]$ except, maybe, one value such that $x + v_2 + \theta(v_2 - v_1) \in \partial\pi B$ because $P = 0$ outside πB and P

⁷ Recall (see, for instance, [3]) that $P(x)$ possesses the first variation at a point x if for each $h \in X_{\sigma\sigma_k}$ there exists a limit $\lim_{\lambda \downarrow 0} (P(x + \lambda h) - P(x))/\lambda := P'(x, h)$. Evidently, $P'(x, h)$ is positively homogeneous on h : $P'(x, \lambda h) = \lambda P'(x, h)$, $\forall \lambda > 0$. The first variation $P'(x, h)$ called Lagrange variation if $P'(x, -h) = -P'(x, h)$.

possesses Lagrange variation inside πB . Besides, by Theorems 7.2 and 7.1 the function $\theta \rightarrow dP/d\theta$ is integrable and, hence, (7.48) is well defined. Therefore (7.48) implies

$$\begin{aligned} & \int_{X_{\sigma si_k}} |P(x - v_1) - P(x - v_2)| dx \\ & \leq \|v_2 - v_1\|_{X_{\sigma\sigma_k}} \int_{X_{\sigma\sigma_k}} \int_0^1 \left| P' \left(x + v_2 + \theta(v_2 - v_1), \frac{v_2 - v_1}{\|v_2 - v_1\|_{X_{\sigma\sigma_k}}} \right) \right| d\theta dx \\ & \leq \|v_2 - v_1\|_{X_{\sigma\sigma_k}} \int_{X_{\sigma\sigma_k}} \sup_{\|e\|_{X_{\sigma\sigma_k}}=1} |P'(x, e)| dx. \end{aligned}$$

This implies inequality (7.47) □

Thus, we have proved that RDS (2.31) is a particular case of RDS (3.15) and therefore it satisfies all conditions of Theorem 3.2. Indeed, Lemma 7.3 implies that random dynamical system (2.31) satisfies the condition (3.18). As was shown above, conditions (3.16), (3.17) are true for RDS (2.31) in virtue of Lemma 4.1. So assertion of Theorem 3.2 is true for RDS (2.31). This proves Theorem 3.1 and completes our investigation in this paper.

Acknowledgement. A part of this work was done while A. V. Fursikov was visiting Department of Applied Mathematics, Illinois Institute of Technology, Chicago, USA. This visit was sponsored by the National Research Council.

References

- [1] M. S. AGRANOVICH, Elliptic operators on closed manifolds, in: *Partial Differential Equations VI*, V. YU. EGOROV, M. A. SHUBIN (eds.), 1–130, Encyclopedia of Math. Sciences **63**, Springer, Berlin, 1994.
- [2] M. S. AGRANOVICH, *Operators with discrete spectrum* (to appear).
- [3] V. M. ALEKSEEV, V. M. TIKHOMIROV and S. V. FOMIN, *Optimal Control*, Consultants Bureau, New York, 1987.
- [4] K. I. BABENKO, On the asymptotic behavior of eigenvalues of linearized Navier–Stokes equations, *Dokl. Akad. Nauk SSSR* **263** (1982), no. 3, 521–525. English translation: *Soviet Math. Dokl.* **25** (1982), no. 2, 359–364.
- [5] J. DUAN and B. GOLDYS, Ergodicity of Stochastically Forced Large Scale Geophysical Flows, *International J. Math. Math. Sci.* **28** (2001), 313–320.
- [6] W. E, J. C. MATTINGLY and YA. G. SINAY, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, *Comm. Math. Phys.* **224** (2001), 83–106.
- [7] F. FLANDOLY and B. MASLOVSKI, Ergodicity of the 2D Navier–Stokes equation under random perturbations, *Comm. Math. Phys.* **172** (1995), 119–141.
- [8] A. V. FURSIKOV, Stabilizability of quasilinear parabolic equation by feedback boundary control, *Sbornik: Mathematics*, **192**:4 (2001), 593–639.
- [9] A. V. FURSIKOV, Stabilizability of two-dimensional Navier–Stokes equations with help of boundary feedback control, *J. of Math. Fluid Mechanics* **3** (2001), 259–301.

- [10] A. V. FURSIKOV, Feedback stabilization for the 2D Navier–Stokes equations, in: *The Navier–Stokes equations: theory and numerical methods*, 179–196, Lecture Notes in pure and appl. Math. **223**, Marcel Dekker, Inc., New York, Basel, 2001.
- [11] A. V. FURSIKOV, Feedback stabilization for the 2D Oseen equations: additional remarks, in: *Proceedings of the 8th Conference on Control of Distributed Parameter Systems*, 169–187, International series of numerical mathematics **143**, Birkhäuser Verlag, 2002.
- [12] A. V. FURSIKOV, Stabilization for the 3D Navier–Stokes system by feedback boundary control, *Discrete and Cont. Dyn. Syst.* **10** (2004), 289–314.
- [13] A. V. FURSIKOV, Real process corresponding to 3D-Navier–Stokes system and its feedback stabilization from boundary, *Amer. Math. Soc. Transl. Series 2* **206**, *Advances in Math. Sciences-51. PDE M. Vishik seminar* (2002), 95–123.
- [14] A. V. FURSIKOV, Real Processes and Realizability of a Stabilization Method for Navier–Stokes Equations by Boundary Feedback Control, in: *Nonlinear Problems in Mathematical Physics and Related Topics II, In Honor of Professor O. A. Ladyzhenskaya*, 137–177 (127–164 in Russian edition), Kluwer/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.
- [15] A. V. FURSIKOV, *Optimal control of Distributed Systems. Theory and Applications*, Transl. of Math. Monographs **187**, AMS, Providence, Rhode Island, 2000, XIV+305 pp.
- [16] I. C. GOHBERG and M. G. KREIN, *Introduction to the theory of linear nonselfadjoint operators*, Trans. of Math. Monographs **18**, AMS, Providence, RI, 1969, XV+378 pp.
- [17] M. V. KELDYSH, On completeness of eigenfunctions for certain classes of not self-adjoint linear operators, *Russian Math. Surveys* **26**(4) (1971), 15–41 (in Russian).
- [18] S. KUKSIN, On exponential convergence to a stationary measure for nonlinear PDE’s, perturbed by random kick-forces, and the turbulence-limit, in: *Partial Differential Equations, American Math. Soc. Translations Series 2* **206** (2002), 161–176.
- [19] S. KUKSIN, Ergodic Theorems for 3D Statistical Hydrodynamics, *Reviews in Math. Physics* **14**, no 6 (2002), 1–16.
- [20] S. KUKSIN and A. SHIRIKYAN, Stochastic dissipative PDE’s and Gibbs measures, *Comm. Math. Phys.* **213** (2000), 291–330.
- [21] S. KUKSIN and A. SHIRIKYAN, A coupling approach to randomly forced nonlinear PDE’s. I, *Comm. Math. Phys.* **221** (2001), 351–366.
- [22] S. KUKSIN and A. SHIRIKYAN, Ergodicity for the randomly forced 2D Navier–Stokes equations, *Math. Phys. Anal. Geom.* **4** (2001), 147–195.
- [23] R. TEMAM, *Navier–Stokes Equations – Theory and Numerical Analysis*, Elsevier, Amsterdam, 1984.
- [24] M. I. VISHIK and A. V. FURSIKOV, *Mathematical Problems of Statistical Hydromechanics*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1988.

Jinqiao Duan
 Department of Applied Mathematics
 Illinois Institute of Technology
 Chicago
 IL 60616
 USA
 e-mail: duan@iit.edu

Andrei V. Fursikov
 Department of Mechanics and Mathematics
 Moscow State University
 119899 Moscow
 Russia
 e-mail: fursikov@mtu-net.ru

(accepted: March 27, 2004)



To access this journal online:
<http://www.birkhauser.ch>
