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Exact controllability of the Navier–Stokes and Boussinesq equations

A. V. Fursikov and O. Yu. Imanuvilov[1]

Contents

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[1] His name is also spelt Émanuilov.

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Introduction

The paper deals with the study of the exact controllability of the Navier–Stokes and Boussinesq equations, which describe the flow of a viscous incompressible fluid without and with heat processes taken into account, respectively. To give various statements of the exact controllability problem, let us consider the Cauchy problem for the abstract evolution equation

$$\frac{\partial v(t)}{\partial t} + A(v(t)) = Bu(t) + f(t), \tag{0.1}$$

$$v(t,\,\cdot\,)\big|_{t=0} = v_0, \tag{0.2}$$

where $v(t)$ is a phase function defined for $t \in [0,T]$ and taking values in a phase space X, $u(t)$ is a control taking values in a space U, A is a non-linear operator, $B\colon U \to X$ is a linear operator, and $f(t)$ is a given right-hand side. Suppose that we are given a solution $\widehat{v}(t)$ of the equation

$$\frac{\partial \widehat{v}(t)}{\partial t} + A(\widehat{v}(t)) = f(t) \tag{0.3}$$

with the same operator A and right-hand side f as in (0.1), but without the control u. Furthermore, we assume that $\widehat{v}(0) \neq v_0$. The exact controllability problem is as follows: construct a control $u(t)$ such that the solution of the Cauchy problem (0.1), (0.2) with this control coincides at $t = T$ with the given solution $\widehat{v}(t)$:

$$v(t)\big|_{t=T} = \widehat{v}(t)\big|_{t=T}. \tag{0.4}$$

If it is known in addition that v_0 is sufficiently close to $\widehat{v}(0,\,\cdot\,)$ in the norm of X,

$$\|\widehat{v}(0,\,\cdot\,) - v_0\|_X \leqslant \varepsilon, \tag{0.5}$$

where $\varepsilon \equiv \varepsilon(\widehat{v})$ is sufficiently small, then the problem of constructing a control u such that the solution v of the Cauchy problem (0.1), (0.2) satisfies (0.4) is called the local exact controllability problem for equation (0.1).

We note that if on the right-hand side of (0.4) one substitutes an arbitrary element v_1 instead of the value $\widehat{v}\big|_{t=T}$ of the solution of (0.3), then for the case in which (0.1) is not reversible in time the exact controllability problem (0.1), (0.2), (0.4) will not be soluble in general. The Navier–Stokes and Boussinesq systems are irreversible in time. (For the case of reversible equations (0.1), say, for hyperbolic systems, one can substitute an arbitrary sufficiently smooth function v_1 on the right-hand side of (0.4).)

For irreversible evolution equations, the approximate controllability problem is often considered, stated as follows: for the controlled system (0.1), (0.2), an arbitrary given element $v_1 \in X$, and an arbitrarily small $\varepsilon > 0$, construct a control $u_\varepsilon \in U$ such that the solution of problem (0.1), (0.2) at time $t = T$ satisfies the condition

$$\|v(T,\,\cdot\,) - v_1\|_X \leqslant \varepsilon. \tag{0.6}$$

Controllability theory for evolution partial differential equations began developing in the 1960s. The foundations of this theory were laid by Egorov [18], Russell [113]–[114], and Fattorini [24] (see also Seidman [116] and Littman [102]). In particular, in these papers the moment method was developed, which reduces the solution of the exact controllability problem to problems in the theory of exponential series (for the modern state of this method, see [1], [3]), and the duality principle was introduced, which reduces the controllability problem for an evolution equation to the observability problem for the adjoint equation. A general review of the state of the theory up to 1978 can be found in Russell's paper [113].

Starting from the mid-1980s, interest in controllability theory increased substantially. At that time, the case of hyperbolic equations was mainly studied. In 1986, Ho [51] found sufficient observability conditions for a second-order hyperbolic equation by the method of multipliers. Simultaneously, J.-L. Lions introduced the Hilbert uniqueness method, which enables one to derive the solubility of the controllability problem for the original equation from the uniqueness theorem for the adjoint equation. These methods have been intensively developed in numerous papers. The most comprehensive review of the literature can be found in the monographs by J.-L. Lions [98], Lagnese [81], Lagnese and Lions [82], and Komornik [74], as well as in the survey by Lions [97] (see also [75], [83]–[85]).

An important step in the development of boundary controllability theory was made by Bardos, Lebeau, and Rauch [4], who used theorems on propagation of singularities for the solution of boundary controllability problems for hyperbolic equations; this approach enabled one to obtain sufficient solubility conditions (close to necessary conditions) in terms of the non-trapping condition (see [5], [58], [90]).

Interesting results on controllability for hyperbolic and close-to-hyperbolic equations were obtained in [83]–[89], [47]–[52], [128], [130].

We have already noted that it is possible to derive results on controllability of the original equation from uniqueness for a certain Cauchy problem for the adjoint equation. Carleman estimates provide one of the most powerful methods for proving uniqueness for the Cauchy problem. Hence we hope that the brief survey below of the development of the theory of Carleman estimates will be useful.

Since Hörmander's fundamental results [53], [54] at the beginning of the 1960s and Isakov's subsequent papers [65]–[68], the theory of Carleman estimates has been developing in several directions. We mention the theory of Carleman estimates in the spaces L^p with $p \neq 2$ (see [69], [70], [72], [73], [118], [119], [123]) and the theory of Carleman estimates with singular weight functions [69]. Carleman estimates for elliptic and parabolic equations with non-smooth right-hand sides were obtained in [19], [21], [64]. The case of a hyperbolic equation was considered by Ruiz [112]. The most general results were obtained by Tataru [120]–[123].

Since the beginning of the 1990s, Carleman estimates have been widely used in exact boundary controllability problems. Using these estimates, Kazemi and Klibanov [71] solved the observability problem for the wave equation, and Lasiecka and Triggiani [86] studied the controllability of a system of hyperbolic equations. The controllability of hyperbolic equations in domains with non-smooth boundary was considered by Grisvard [46] and Heibig and Moussaoui [50].

A vast majority of the above papers deals with controllability problems for linear evolution equations. Much less is known for non-linear equations. For example, the exact boundary controllability problem for the elementary one-dimensional hyperbolic equation $y_{tt} - y_{xx} + y^3 = 0$ remains open. Some results are known on local controllability (see [113]) and on null-controllability in a time depending on the energy of the initial data. For a one-dimensional hyperbolic equation with a non-linearity growing at infinity no faster than $|y| \ln^+ |y|$, the solubility of the controllability problem is proved in [128]; for non-linearities of the form e^y or $-y^3$ the corresponding result is given in [57].

In [93], [94] Lions conjectured the global boundary or locally distributed controllability of the Navier–Stokes equations. After these papers, from the start of the 1990s controllability has been intensively studied for parabolic equations with elementary non-linearities and for equations describing fluid flows.

The approximate controllability of the Stokes equations was studied in [93], [94], [96], [32]–[35], [19]–[21]. The problem of approximate controllability by a local external force of constant direction (stated by Lions in [94]) was studied by Diaz and Fursikov [17] and by Lions and Zuazua [100].

The approximate controllability for a semilinear parabolic equation with a non-linearity of at most linear growth at infinity was established by Fabre, Puel, and Zuazua [22], [23]; in [27], [129] it was proved for the case in which the non-linear term contains the gradient.

For a non-linear parabolic equation with a quadratic or higher-order non-linearity, the situation is completely different. A priori estimates that readily imply the non-existence of solutions of the approximate controllability problem for a parabolic equation for some initial data were obtained in [15], [59] for a semilinear equation with a non-linearity of type y^3 and in [33]–[35] for the Burgers equation. (A detailed analysis for the case of a one-dimensional semilinear parabolic equation can be found in [41].)

So far, the most powerful known method for proving exact controllability of non-linear parabolic equations is the method in which the solution is constructed with the help of an extremal problem and then Carleman estimates are applied. The foundations of this method were laid by Fursikov and Imanuvilov [35], [36], who studied the local exact controllability of the Burgers equation and the two-dimensional Helmholtz equation. The Carleman estimate was first obtained in [60]. The exact controllability for semilinear parabolic equations with a non-linearity growing at most linearly at infinity was established in [61], [10], [64]. In [25], [26] this result was generalized to the case of a semilinear parabolic equation with a non-linear term growing at infinity no faster than $|y| \ln^+ |y|$. We also mention the paper [127], where the controllability of a one-dimensional parabolic equation with an analytic non-linearity was established with the help of the Cauchy–Kowalewski theorem.

A new approach to the proof of solubility of the controllability problem for linear parabolic equations with coefficients independent of time was suggested by Lebeau and Robbiano [91]. Interesting results for the one-dimensional heat equation with rapidly oscillating coefficients were obtained in a recent paper by Lopez and Zuazua [105]. Controllability issues for the Burgers equation were also considered in [56], [6], [7].

The exact null-controllability of the Navier–Stokes equations was established in [36], [28].

The local exact controllability of the Navier–Stokes equations and the Boussinesq system was proved by Fursikov and Imanuvilov [37]–[42] for the case in which the control is distributed over the boundary or a part of the boundary, and also for the case of a locally distributed control.

The local exact controllability of the Navier–Stokes equations and the Boussinesq system with a locally distributed control and with slip boundary conditions was studied in [62]. The case of a locally distributed control for the Navier–Stokes equations with zero boundary conditions was considered by Imanuvilov [57] under additional restrictions on the prescribed velocity.

The approximate controllability of the two-dimensional Euler equation and the two-dimensional Navier–Stokes system with slip boundary conditions and with boundary control was established by Coron [11]–[13]. Later this result was extended in [43] to the three-dimensional Euler equation.

The global exact controllability of the Navier–Stokes equations with a local distributed control on a closed two-dimensional manifold is proved in [14].

The main object of study in the present paper is the controllability problem for the Boussinesq system on the cylinder $Q = (0,T) \times \Pi$, where $(0,T)$ is a time interval and $\Pi = \mathbb{R}^n / L\mathbb{Z}^n$ is the n-dimensional torus, that is, the direct product of n circles of length L. The dimension n of the torus Π is assumed to be 2 or 3. Furthermore, we assume that the control u is distributed with support contained in the cylinder $Q^\omega = (0,T) \times \omega$, where ω is an arbitrary given subdomain of Π. Results for the Navier–Stokes equations are obtained as simple corollaries of the corresponding results for the Boussinesq system. However, for simplicity we discuss in the introduction the results for the example of the Navier–Stokes equations on the cylinder Q:

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(t,x) + u(t,x), \quad \operatorname{div} v = 0, \tag{0.7}$$

$$v(t,x)\big|_{t=0} = v_0(x), \tag{0.8}$$

where $\operatorname{supp} u \subset Q^\omega$.

We prove the exact controllability of system (0.7), (0.8) for $Q^\omega = (0,T) \times \omega$ with arbitrarily small T and with any open subset $\omega \subset \Pi$ containing the support of the control u.

The exact controllability problem on the interval $[0,T]$ for problem (0.7), (0.8) obviously reduces to the solution of the approximate controllability problem on $[0,T_1]$ ($T_1 < T$) followed by the solution of the local exact controllability problem on $[T_1,T]$. We note that $X = V^1(\Pi) = \{v \in (H^1(\Pi))^n : \operatorname{div} v = 0\}$ for problem (0.7), (0.8), where $H^1(\Pi)$ is the Sobolev space of vector fields on the torus Π. Thus, inequalities (0.5), (0.6), as well as the notions of local exact controllability and approximate controllability, are well defined.

We prove the approximate controllability of problem (0.7), (0.8) on the interval $[0,T_1]$, where $T_1 \equiv T_1(\varepsilon)$ and ε is the number from inequality (0.6). Furthermore, $T_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

It follows from our proof of the theorem on the local exact controllability on an interval $[T_1,T]$ that the interval $[T_1,T]$ can be chosen arbitrarily small provided that the ε in inequality (0.5) is sufficiently small. As a result, we obtain a theorem on the (non-local) exact controllability with control u supported on $Q^\omega = (0,T) \times \omega$, where T can be arbitrarily small and ω is an arbitrary open subset of Π.

The exact controllability theorem implies that it is possible to stabilize steady-state unstable solutions. Also, it implies that there are arbitrarily complicated (chaotic) solutions of the Navier–Stokes and Boussinesq equations and that these systems have a certain reversibility property (see § 1.5).

Finally, we note that, as is shown in § 1, the exact controllability theorem for system (0.1), (0.2) on the torus readily implies the controllability theorem for the Navier–Stokes equations in an arbitrary bounded domain Ω with control distributed over the entire boundary $\partial\Omega$.

§ 1. Statement of the problem. Main results

1.1. Exact boundary controllability of the Navier–Stokes equations. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with C^∞ boundary $\partial\Omega$, $T > 0$, $\mathcal{C} = (0,T) \times \Omega$, and $\Sigma = (0,T) \times \partial\Omega$ the lateral surface of $\mathcal{C}$. We consider the following mixed boundary-value problem for the Navier–Stokes equations:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p = f(t,x), \qquad (t,x) \in \mathcal{C}, \tag{1.1}$$

$$\operatorname{div} v(t,x) \equiv \sum_{i=1}^n \partial_i v_i(t,x) = 0, \qquad (t,x) \in \mathcal{C}, \tag{1.2}$$

$$v\big|_\Sigma = \alpha(t,x), \tag{1.3}$$

$$v(t,x)\big|_{t=0} = v_0(x), \tag{1.4}$$

where $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_i = \dfrac{\partial}{\partial x_i}$, $x = (x_1, \dots, x_n) \in \Omega$, $v(t,x) = (v_1(t,x), \dots, v_n(t,x))$ is the velocity vector field of the fluid, $\nabla p(t,x)$ is the pressure gradient in the fluid, Δ is the Laplace operator, $(v,\nabla)v = \sum_{i=1}^n v_i \partial_i v$, $f(t,x) = (f_1(t,x), \dots, f_n(t,x))$ is a given density of external forces, v_0 is a given initial vector field, and the vector field α defined on the boundary Σ is not prescribed in advance, but rather can be used as a control.

The exact controllability problem for the Navier–Stokes equations with control defined on the boundary Σ, that is, the exact boundary controllability problem, is as follows. Suppose that we are given a solution $(\widehat{v}(t,x), \nabla\widehat{p}(t,x))$ of system (1.1), (1.2):

$$\partial_t \widehat{v}(t,x) - \Delta\widehat{v} + (\widehat{v},\nabla)\widehat{v} + \nabla\widehat{p} = f(t,x), \qquad \operatorname{div}\widehat{v} = 0. \tag{1.5}$$

It is required to find a control $\alpha(t,x)$ defined on Σ such that the solution $v(t,x)$ of problem (1.1)–(1.4) at $t = T$ coincides with $\widehat{v}(T,x)$:

$$v(t,x)\big|_{t=T} \equiv \widehat{v}(T,x). \tag{1.6}$$

To state the problem precisely and present the main result, we need some function spaces. By $H^k(\Omega)$, where k is a positive integer, we denote the Sobolev space of scalar functions square integrable on Ω together with all derivatives of order $\leqslant k$; by $(H^k(\Omega))^n$ we denote the corresponding Sobolev space of vector fields. We set

$$V^k(\Omega) = \{v(x) = (v_1, \dots, v_n) \in (H^k(\Omega))^n : \operatorname{div} v = 0\}, \tag{1.7}$$

$$H^{1,2}(\mathcal{C}) = \{v(t,x) \in L_2(0,T; H^2(\Omega)) : \partial_t v \in L_2(0,T; H^0(\Omega))\}, \tag{1.8}$$

$$V^{1,2}(\mathcal{C}) = \{v \in (H^{1,2}(\mathcal{C}))^n : \operatorname{div} v = 0\}. \tag{1.9}$$

More generally, for $1 \leqslant p \leqslant \infty$, $k \geqslant 0$, we set

$$V_p^k(\Omega) = \{v(x) \in (W_p^k(\Omega))^n : \operatorname{div} v = 0\}, \tag{1.10}$$

where $W_p^k(\Omega)$ is the Sobolev space of functions pth-power integrable together with the derivatives of order $\leqslant k$. The definition of the Sobolev spaces $H^k(\Omega)$ and $W_p^k(\Omega)$ with fractional k can be found in [99], [117]. Moreover, we shall use the function spaces $C^{k,\alpha}(\Omega)$, where k is a non-negative integer and $\alpha \in (0,1)$, of k times continuously differentiable functions on $\overline{\Omega}$ all of whose kth derivatives satisfy the Hölder condition with exponent α.

One of the main results of the present paper is given by the following theorem.

Theorem 1.1. *Let $f \in L_2(0,T; V^2(\Omega))$ and $v_0 \in V^4(\Omega)$, and let*

$$(\widehat{v}, \widehat{p}) \in C^1(0,T; V^4(\Omega)) \times L_2(0,T; H^1(\Omega))$$

be a solution of system (1.5). *Suppose that the relations*

$$\int_{\Gamma_i} (\widehat{v}(t,x), \nu(x))\, d\sigma = 0 \quad a.e., \quad t \in [0,T], \qquad \int_{\Gamma_i} (v_0(x), \nu(x))\, d\sigma = 0, \tag{1.11}$$

where $\nu(x)$ is the outward normal vector field on $\partial\Omega$, are satisfied on each connected component Γ_i of the boundary $\partial\Omega$. Then there is a solution $(v, \nabla p, \alpha) \in V^{1,2}(\mathcal{C}) \times (L_2(\mathcal{C}))^n \times L_2(0,T; (H^{3/2}(\partial\Omega))^n)$ of problem (1.1)–(1.4), (1.6).

The proof of this theorem can be reduced to the proof of the statement given in the next subsection.

1.2. Local exact distributed controllability of the Navier–Stokes equations. Let $L > 0$ be some number, $\Pi = \mathbb{R}^n / L\mathbb{Z}^n$ the n-dimensional torus ($n = 2, 3$) that is the product of circles of length L, and $\omega \subset \Pi$ an open subset. We set $Q = (0,T) \times \Pi$ and $Q^\omega = (0,T) \times \omega$. On the cylinder Q, we consider the Navier–Stokes equations

$$\partial_t v(t,x) - \Delta v + (v, \nabla) v + \nabla p = f(t,x) + u(t,x), \qquad (t,x) \in Q, \tag{1.12}$$

$$\operatorname{div} v = 0, \tag{1.13}$$

with the initial condition

$$v(t,x)\big|_{t=0} = v_0(x), \qquad x \in \Pi. \tag{1.14}$$

Here f and v_0 are given vector fields an $u(t,x)$ is a control concentrated on the cylinder Q^ω. We note that relations (1.12)–(1.14) on Π mean precisely that these relations hold for every $x \in \mathbb{R}^n$ and that all vector fields occurring in (1.12)–(1.14), that is, $v(t,x)$, $\nabla p(t,x)$, $f(t,x)$, $u(t,x)$, $v_0(x)$, where $x=(x_1,\dots,x_n)$, are L-periodic in each of the variables x_i. For example,

$$v(t,x_1,\dots,x_j+L,\dots,x_n) = v(t,x_1,\dots,x_j,\dots,x_n) \qquad \forall\, j=1,\dots,n.$$

By analogy with (1.7)–(1.10), we introduce the spaces $V^k(\Pi)$ and $V_p^k(\Pi)$ on the torus Π and the spaces $H^{1,2}(Q)$ and $V^{1,2}(Q)$ on the cylinder $Q=(0,T)\times\Pi$.

We define the control space by

$$U(\omega) = \bigl\{u(t,x) \in (L_2(Q))^n : \operatorname{supp} u \subset Q^\omega\bigr\}. \tag{1.15}$$

The following theorem holds on local exact distributed controllability of the Navier–Stokes equations (1.12), (1.13).

Theorem 1.2. *Let* $f \in L_2(0,T;V^0(\Pi))$ *and* $v_0 \in V^1(\Pi)$. *Suppose that* $(\widehat{v},\nabla\widehat{p}) \in C^1(0,T;V^0(\Pi)\cap(C^{2,\alpha}(\Pi))^n)\times(L_2(Q))^n$ *is a solution of system* (1.5) *on* Q. *Then there is a solution*

$$(v,\nabla p,u) \in V^{1,2}(Q)\times(L_2(Q))^n\times U(\omega)$$

of problem (1.12)–(1.14), (1.6).

We derive Theorem 1.1 from Theorem 1.2.

Proof of Theorem 1.1. Let us translate Ω by an appropriate vector $x_0\in\mathbb{R}^n$. Then we can assume that Ω is a subset of the cube

$$K = \bigl\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : 0\leqslant x_j\leqslant L,\ j=1,\dots,n\bigr\},$$

where $L>0$ is some number depending on Ω. By identifying the opposite facets of K, that is, the sets

$$\bigl\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_j=0\bigr\} \quad\text{and}\quad \bigl\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_j=L\bigr\}$$

for each $j\in\{1,\dots,n\}$, we obtain the torus Π. Thus, we have constructed an embedding $\Omega\subset\Pi$. With regard to (1.11), it follows from Proposition 2.3 in [40] that the vector field $\widehat{v}\in C^1(0,T;V^4(\Omega))$ can be extended to a vector field $R\widehat{v}\in C^1(0,T;V^4(K))\subset C^1(0,T;V^0(K)\cap C^{2,\alpha}(K))$, and the vector field $v_0\in V^4(\Omega)$ can be extended to a vector field $Rv_0\in V^4(K)\subset V^0(K)\cap C^{2,\alpha}(K)$ with some $\alpha\in(0,1)$. Moreover, the method used in the proof of Proposition 2.3 in [40] enables one to choose $R\widehat{v}$ and Rv_0 such that $R\widehat{v}=0$ in a neighbourhood of $(0,T)\times\partial K$ and $Rv_0=0$ in a neighbourhood of ∂K. This property enables one to extend $R\widehat{v}$ and Rv_0 periodically from K to $\mathbb{R}^n$, that is, assume that $R\widehat{v}$ and Rv_0 are defined on Q and Π, respectively, and moreover, $R\widehat{v}\in C^1(0,T;V^0(\Pi)\cap C^{2,\alpha}(\Pi))$ and $Rv_0\in V^0(\Pi)\cap C^{2,\alpha}(\Pi)$. By substituting $R\widehat{v}$ into (1.5), we find that

$$\partial_t R\widehat{v}(t,x) - \Delta R\widehat{v} + (R\widehat{v},\nabla)R\widehat{v} = h(t,x), \tag{1.16}$$

where $h(t,x)$ is a vector field satisfying the condition

$$h\big|_{(0,T)\times\Omega} = f - \nabla\widehat{p}. \tag{1.17}$$

Since $R\widehat{v} \in C^1(0,T;V^0(\Pi)\cap C^{2,\alpha}(\Pi))$, it follows from (1.16) that $h(t,x) \in (C(\overline{Q}))^n \subset (L_2(Q))^n$. By applying the Weyl decomposition to $h(t,x)$ for almost all $t\in(0,T)$, we obtain

$$h(t,x) = h_\sigma(t,x) + \nabla\widehat{q}(t,x), \tag{1.18}$$

where $h_\sigma(t,x) \in L_2(0,T;V^0(\Pi)), \nabla\widehat{q}(t,x) \in (L_2(Q))^n$. It follows from (1.16) and (1.18) that

$$\partial_t(R\widehat{v}) - \Delta(R\widehat{v}) + ((R\widehat{v}),\nabla)(R\widehat{v}) + \nabla\widehat{q} = h_\sigma(t,x), \qquad \operatorname{div} R\widehat{v} = 0. \tag{1.19}$$

Let us replace f by h_σ and v_0 by Rv_0 in problem (1.12)–(1.14). Then the assumptions of Theorem 1.2 are satisfied with $\omega = \Pi\backslash\Omega$ and with the pair $(\widehat{v},\widehat{p})$ replaced by the pair $(R\widehat{v},\widehat{q})$, which satisfies (1.19). Hence, by Theorem 1.2, there is a solution $(v,\nabla p,u)$ of problem (1.12)–(1.14) with $f = h_\sigma$ and $v_0 = Rv_0$ such that

$$v(T,x) \equiv R\widehat{v}(T,x), \qquad x\in\Pi. \tag{1.20}$$

The restriction of (1.20) to $\mathcal{C}$ coincides with (1.6), and the restriction of (1.12) with $f = h_\sigma$ to $\mathcal{C}$ can be represented by virtue of (1.18), (1.17) in the form

$$\partial_t v - \Delta v + (v,\nabla)v + \nabla p + \nabla\widehat{p} + \nabla\widehat{q} = f.$$

We set $\nabla p + \nabla\widehat{p} + \nabla\widehat{q} = \nabla p_1$ and write $v\big|_{(0,T)\times\partial\Omega} = \alpha$. Then the triple $(v,\nabla p_1,\alpha)$ satisfies the assertion of Theorem 1.1.

1.3. Exact controllability of the Boussinesq system. We consider the Boussinesq system

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \theta(t,x)\vec{e} + \nabla p(t,x) = f(t,x), \quad \operatorname{div} v = 0, \tag{1.21}$$

$$\partial_t\theta(t,x) - \Delta\theta + (v,\nabla)\theta = g(t,x), \tag{1.22}$$

$$v(t,x)\big|_\Sigma = \alpha(t,x), \qquad \theta(t,x)\big|_\Sigma = \beta(t,x), \tag{1.23}$$

$$v(t,x)\big|_{t=0} = v_0(x), \qquad \theta(t,x)\big|_{t=0} = \theta_0(x), \tag{1.24}$$

in the cylinder $\mathcal{C} = (0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^n$, $n = 2,3$, is a bounded domain with boundary $\partial\Omega\subset C^\infty$, $\Sigma = (0,T)\times\partial\Omega$, $v(t,x) = (v_1(t,x),\dots,v_n(t,x))$ is the fluid velocity vector field, p is the pressure, $\theta(t,x)$ is the fluid temperature, $\vec{e}$ is the gravitational force direction, $f(t,x) = (f_1(t,x),\dots,f_n(t,x))$ is the external force density, $g(t,x)$ is the heat source density, v_0,θ_0 are the initial data, and α, β are the boundary data.

The statement of the exact boundary controllability problem for the Boussinesq system is similar to the corresponding statement for the Navier–Stokes equations. Let $(\widehat{v}(t,x),\nabla\widehat{p}(t,x),\widehat{\theta}(t,x)) \in C^1(0,T;V^4(\Omega))\times(L_2(\mathcal{C}))^n\times C^1(0,T;H^4(\Omega))$ satisfy equations (1.21), (1.22); it is required to find a control

$$(\alpha,\beta)\in L_2\big(0,T;(H^{3/2}(\partial\Omega))^n\big)\times L_2\big(0,T;H^{3/2}(\partial\Omega)\big)$$

such that the solution (v,p,θ) of problem (1.21)–(1.24) satisfies the condition

$$v(T,x) = \widehat{v}(T,x), \qquad \theta(T,x) = \widehat{\theta}(T,x). \tag{1.25}$$

The following theorem holds.

Theorem 1.3. *Let $n=2,3$. Let $f\in L_2(0,T;V^0(\Omega))$ and $g\in L_2(0,T;L_2(\Omega))$ and suppose that a triple of functions $(\widehat{v},\widehat{p},\widehat{\theta})\in C^1(0,T;V^4(\Omega))\times(L_2(\mathcal{C}))^n\times C^1(0,T;H^4(\Omega))$ satisfying equations* (1.21) *and* (1.22) *and condition* (1.11) *is given. Then for an arbitrary initial condition $(v_0,\theta_0)\in V^4(\Omega)\times H^4(\Omega)$ satisfying* (1.11) *there is a boundary control $(\alpha,\beta)\in L_2(0,T;(H^{3/2}(\partial\Omega))^n)\times L_2(0,T;H^{3/2}(\partial\Omega))$ such that problem* (1.21)–(1.24) *has a solution $(v,\nabla p,\theta)$ in the space $V^{1,2}(\mathcal{C})\times(L_2(\mathcal{C}))^n\times H^{1,2}(\mathcal{C})$, and this solution satisfies conditions* (1.25).

Let us now state a result on the local exact distributed controllability of the Boussinesq system. Let $\Pi=\mathbb{R}^n/L\mathbb{Z}$ be a torus and $\omega\subset\Pi$ an open subset. We write $Q=(0,T)\times\Pi$ and $Q^\omega=(0,T)\times\omega$.

On the cylinder Q we consider the Boussinesq system with a locally distributed control:

$$N(v,\theta)\equiv\partial_t v(t,x)-\Delta v+(v,\nabla)v+\theta(t,x)\vec{e}=\nabla p+f(t,x)+u'(t,x),\quad \operatorname{div} v=0, \tag{1.26}$$

$$R(v,\theta)\equiv\partial_t\theta(t,x)-\Delta\theta+(v,\nabla\theta)=g(t,x)+u_{n+1}(t,x), \tag{1.27}$$

and with the initial conditions (1.24) on the torus Π. Here the function

$$u(t,x)\equiv(u'(t,x),u_{n+1}(t,x))\equiv(u_1,\dots,u_n,u_{n+1})$$

is the control. By analogy with (1.15), we introduce the control space by the formula

$$U(\omega)=U(\omega;0,T)=\left\{u(t,x)=(u',u_{n+1})\in(L_2(Q))^{n+1}:\operatorname{supp}u\subset Q^\omega\right\} \tag{1.28}$$

($0,T$ indicate the time interval $(0,T)$ on which the control $u\in U(\omega;0,T)$ is concentrated).

Let a triple

$$(\widehat{v},\nabla\widehat{p},\widehat{\theta})\in C^1(0,T;V^0(\Pi)\cap(C^{2,\alpha}(\Pi))^n)\times(L_2(Q))^n\times C^1(0,T;C^{2,\alpha}(\Pi)) \tag{1.29}$$

with some $\alpha\in(0,1)$ satisfy (1.21) and (1.22) on Q. We must find a control $u=(u',u_{n+1})\in U(\omega;0,T)$ such that the solution $(v,\nabla p,\theta)$ of problem (1.26), (1.27), (1.24) on Π satisfies (1.25).

Theorem 1.4. *Suppose that $n=2$ or 3 and the right-hand sides $f\in L_2(0,T;V^0(\Pi))$ and $g\in L_2(0,T;L_2(\Pi))$, as well as the solution* (1.29) *of equations* (1.21), (1.22) *on the cylinder Q with these right-hand sides, are given. Then for arbitrary initial conditions $v_0\in V^0(\Pi)\cap(C^{2,\alpha}(\Pi))^n$, $\theta_0\in C^{2,\alpha}(\Pi)$ there is a solution*

$$(v,\nabla p,\theta,u)\in V^{1,2}(Q)\times(L_2(Q))^n\times H^{1,2}(Q)\times U(\omega;0,T)$$

of problem (1.26), (1.27), (1.24) *such that relations* (1.25) *are satisfied on the torus Π at time T.*

Let us show that Theorem 1.4 readily implies Theorem 1.3.

Proof of Theorem 1.3. Let Π be the torus constructed with the help of Ω in the same way as in the proof of Theorem 1.1. Next, let $R\widehat{v}$ be the extension of $\widehat{v}$ from $\mathcal{C}$ to Q and Rv_0 the extension of v_0 from Ω to Π constructed in the proof of the same theorem. Let $R\widehat{\theta} \in C(0,T;W^1_\infty(\Pi))$ be a continuation of θ from $\mathcal{C}$ to Q. By substituting $R\widehat{v}, R\widehat{\theta}$ into (1.21), (1.22), we obtain, as in (1.16),

$$\partial_t(R\widehat{v}) - \Delta R\widehat{v} + (R\widehat{v},\nabla)R\widehat{v} + R\widehat{\theta}\vec{e} = h(t,x), \tag{1.30}$$
$$\partial_t R\widehat{\theta} - \Delta R\widehat{\theta} + (R\widehat{v},\nabla R\widehat{\theta}) = Rg,$$

where Rg is some extension of g from $\mathcal{C}$ to Q and h satisfies (1.17).

Let us substitute the Weyl decomposition (1.18) for the vector field h into (1.30) and let $\omega = \Pi \setminus \Omega$. Then we see that system (1.26), (1.27) with $f = h_\sigma$ and with Rg instead of g satisfies the assumptions of Theorem 1.4. By applying this theorem, we complete the proof of Theorem 1.3.

Thus, our goal is to prove Theorems 1.2 and 1.4. The proofs are similar apart from the fact that, for obvious reasons, the proof of Theorem 1.4 is somewhat more complicated. Hence we prove Theorem 1.4 in detail; the simplifications needed to obtain the proof of Theorem 1.2 are obvious.

In the following subsection, we reduce the proof of Theorem 1.4 to that of local exact controllability and of approximate controllability of the Boussinesq system.

1.4. Exact local controllability and approximate controllability of the Boussinesq system. We weaken the notion of exact controllability in two ways by introducing the notions of local exact and approximate controllability.

i) Let the triple (1.29) satisfy the Boussinesq equation (1.21), (1.22) on Q. The Boussinesq system (1.26), (1.27) is said to be *locally exactly controllable* with respect to the control space (1.28) if there is an $\varepsilon_0 > 0$ such that for every ε in the interval $0 < \varepsilon < \varepsilon_0$ and every initial condition $(v_0,\theta_0) \in V^4(\Pi) \times H^4(\Pi)$ satisfying the inequality

$$\|\widehat{v}(0,\,\cdot\,) - v_0\|_{V^1(\Pi)} + \|\widehat{\theta}(0,\,\cdot\,) - \theta_0\|_{H^1(\Pi)} \leqslant \varepsilon \tag{1.31}$$

there is a control $u \in U(\omega;0,T)$ such that problem (1.26), (1.27), (1.24) with the above initial conditions and control has a solution $(v,\nabla p,\theta)$ in the space $V^{1,2}(Q) \times L_2(0,T;H^1(\Pi)) \times H^{1,2}(Q)$, and moreover, this solution satisfies condition (1.25).

Theorem 1.5. *The Boussinesq system is locally exactly controllable with respect to the control space* (1.28). *Moreover, the parameter ε_0 is a continuous monotonically decreasing function of*

$$\frac{1}{T} + \|\widehat{v}\|_{C^1(0,T;V^2_\infty(\Pi))} + \|\widehat{\theta}\|_{C^1(0,T;W^2_\infty(\Pi))}.$$

ii) Suppose that the initial conditions (1.24) and a pair

$$(v_1,\theta_1) \in \bigl(V^0(\Pi) \cap (C^{2,\alpha}(\Pi))^n\bigr) \times C^{2,\alpha}(\Pi)$$

are given. Sometimes it will be useful to consider the set of pairs such that

$$\sum_{i=0}^{1}\bigl(\|v_i\|^2_{(V^0(\Pi)\cap(C^{2,\alpha}(\Pi))^n)} + \|\theta_i\|^2_{C^{2,\alpha}(\Pi)}\bigr) \leqslant K. \tag{1.32}$$

The Boussinesq system (1.26), (1.27), (1.24) is said to be *approximately controllable* with respect to the control space (1.28) if for every $\varepsilon > 0$ and every pair (v_1, θ_1) in the set (1.32) there exist a time $T = T_{\varepsilon,K}$ and a solution

$$(v, p, \theta, u) \in V^{1,2}(Q_{T_{\varepsilon,K}}) \times L_2(0, T_{\varepsilon,K}; H^1(\Pi)) \times H^{1,2}(Q_{T_{\varepsilon,K}}) \times U(\omega; 0, T_{\varepsilon,K})$$

of problem (1.26), (1.27), (1.24) such that

$$\|v(T_{\varepsilon,K}, \cdot\,) - v_1\|_{V^1(\Pi)} + \|\theta(T_{\varepsilon,K}, \cdot\,) - \theta_1\|_{H^1(\Pi)} \leqslant \varepsilon. \tag{1.33}$$

Here $Q_{T_{\varepsilon,K}} = (0, T_{\varepsilon,K}) \times \Pi$.

Theorem 1.6. *The Boussinesq system* (1.26), (1.27), (1.24) *is approximately controllable with respect to the control space* (1.28). *Moreover, for every $\varepsilon > 0$ and for any given $K > 0$, the time $T = T_{\varepsilon,K}$ can be chosen so that*

$$T_{\varepsilon,K} \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Theorems 1.5 and 1.6 will be proved in the remaining part of the paper, and now we derive Theorem 1.4 from them.

Proof of Theorem 1.4. To construct the desired control, we divide the interval $[0, T]$ into two intervals, $[0, S]$ and $[S, T]$, where $S < T/2$. Let

$$\varepsilon_0 = \varepsilon_0 \left(\frac{1}{T} + \|\widehat{v}\|_{C^1(0,T;V^2_\infty(\Pi))} + \|\widehat{\theta}\|_{C^1(0,T;W^2_\infty(\Pi))} \right)$$

be the continuous monotonically decreasing function in Theorem 1.5. We take an $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon &< \varepsilon_0 \left(\frac{2}{T} + \|\widehat{v}\|_{C^1(0,T;V^2_\infty(\Pi))} + \|\widehat{\theta}\|_{C^1(0,T;W^2_\infty(\Pi))} \right) \\ &\leqslant \varepsilon_0 \left(\frac{1}{T-S} + \|\widehat{v}\|_{C^1(S,T;V^2_\infty(\Pi))} + \|\widehat{\theta}\|_{C^1(S,T;W^2_\infty(\Pi))} \right), \end{aligned} \tag{1.34}$$

where the second inequality holds since $\varepsilon_0(\lambda)$ is a monotone function. At the initial time $S < T/2$, we prescribe the initial condition

$$v(t, x)\big|_{t=S} = v_S, \qquad \theta(t, x)\big|_{t=S} = \theta_S \tag{1.35}$$

with $(v_S, \theta_S) \in V^4(\Pi) \times H^4(\Pi)$ satisfying the inequality

$$\|\widehat{v}(S, \cdot\,) - v_S\|_{V^1(\Pi)} + \|\widehat{\theta}(S, \cdot\,) - \theta_S\|_{H^1(\Pi)} \leqslant \varepsilon,$$

where ε is the number in (1.31). Then by Theorem 1.5 there is a control $u \in U(\omega; S, T)$ such that the solution $(v, \nabla p, \theta)$ of problem (1.26), (1.27), (1.35) satisfies (1.25). Thus Theorem 1.6 will be proved once we take an $S < T/2$ and choose a control on $(0, S)$ such that the solution $(v, \nabla p, \theta)$ of problem (1.26), (1.27), (1.24) satisfies the inequality

$$\|\widehat{v}(S, \cdot) - v(S, \cdot)\|_{V^1(\Pi)} + \|\widehat{\theta}(S, \cdot) - \theta(S, \cdot)\|_{H^1(\Pi)} \leqslant \varepsilon, \tag{1.36}$$

where ε is the number in (1.34).

Then it remains to apply Theorem 1.5 with the initial condition $(v_S, \theta_S) = (v(S, \cdot), \theta(S, \cdot))$ at $t = S$.

Since

$$(\widehat{v}, \widehat{\theta}) \in C^1(0, T; V^2_\infty(\Pi)) \times C^1(0, T; W^2_\infty(\Pi)) \subset C(0, T; V^1(\Pi)) \times C(0, T; H^1(\Pi)),$$

it follows that there is a $\delta > 0$ such that

$$\|\widehat{v}(\tau, \cdot) - \widehat{v}(0, \cdot)\|_{V^1(\Pi)} + \|\widehat{\theta}(\tau, \cdot) - \widehat{\theta}(0, \cdot)\|_{H^1(\Pi)} \leqslant \varepsilon/2 \quad \forall\, 0 < \tau < \delta, \tag{1.37}$$

where ε is the number occurring in (1.34). By Theorem 1.6, there is an $\varepsilon_1 < \varepsilon/2$ such that $T_{\varepsilon_1} < \delta$ and on the interval $(0, T_{\varepsilon_1})$ there is a control $u \in U(\omega; 0, T_{\varepsilon_1})$ such that the solution $(v, \nabla p, \theta)$ of problem (1.26), (1.27), (1.24) satisfies the inequality

$$\|v(T_{\varepsilon_1}, \cdot) - \widehat{v}(0, \cdot)\|_{V^1(\Pi)} + \|\theta(T_{\varepsilon_1}, \cdot) - \widehat{\theta}(0, \cdot)\|_{H^1(\Pi)} \leqslant \varepsilon_1 < \varepsilon/2. \tag{1.38}$$

It follows from (1.37), (1.38) that (1.36) holds with $S = T_{\varepsilon_1}$.

1.5. Some applications. Let us give some applications of the above controllability theorems.

1. *Stabilizability of unstable steady-state flows.* Let $(\widehat{v}(x), \nabla\widehat{p}(x)) \in V^2(\Omega) \times (L_2(\Omega))^n$ be a steady-state solution of the Navier–Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$-\Delta\widehat{v}(x) + (\widehat{v}, \nabla)\widehat{v} + \nabla\widehat{p}(x) = f(x), \qquad \operatorname{div}\widehat{v} = 0,$$
$$\widehat{v}\big|_{\partial\Omega} = 0.$$

Suppose that this solution is not stable. Thus, for each $\varepsilon > 0$ there is an initial condition $v_0(x) \in \{v : \|v - \widehat{v}\|_{V^1(\Omega)} < \varepsilon\}$ such that the solution $(v(t, x), \nabla p(t, x))$ of the Navier–Stokes equations (1.1)–(1.4) with $f(t, x) \equiv f(x)$ and $\alpha(t, x) \equiv 0$ does not tend to $\widehat{v}$ as $t \to \infty$:

$$\|v(t, \cdot) - \widehat{v}\|_{V^1(\Omega)} \not\to 0 \quad \text{as} \quad t \to \infty.$$

One can stabilize the unstable steady-state solution $(\widehat{v}, \nabla\widehat{p})$ by replacing the no-slip boundary conditions by the boundary control (1.3). Indeed, by Theorem 1.1, for any initial condition $v_0(x) \in V^1(\Omega)$ there is a boundary control $\alpha(t, x)$, $(t, x) \in \Sigma$, such that the solution $(v(t, x), \nabla p(t, x))$ of problem (1.1)–(1.4) satisfies the condition

$$v(T, x) \equiv \widehat{v}(x).$$

The Boussinesq system also has a similar property of stabilization of steady-state flows by a boundary control.

2. *The existence of chaotic flows.* Let us consider the Boussinesq system (1.26), (1.27), (1.24) on the cylinder $Q=(0,T)\times\Pi$, where Π is the torus and the control $u(t,x)$ is supported in $Q^\omega=(0,T)\times\omega$ for sufficiently 'small' $\omega\subset\Pi$ (for example, ω can be a ball of sufficiently small radius). Suppose that $f(t,x)\equiv 0$ and $g(t,x)\equiv 0$ in (1.26), (1.27). Let us show that this problem has an arbitrarily complicated (chaotic) solution. Namely, let us consider a partition $0=t_0<t_1<\dots<t_N=T$ of the time interval $(0,T)$. We assign an arbitrary pair $(v_j,\theta_j)\in V^1(\Pi)\times H^1(\Pi)$ to each t_j and specify a number $\varepsilon>0$. It follows from Theorem 1.6 on the approximate controllability of the Boussinesq system that there is a control $u\in U(\omega;0,T)$ such that the solution $(v(t,x),\nabla p(t,x),\theta(t,x))$ of problem (1.26), (1.27), (1.24) satisfies the condition

$$\|v(t_j,\,\cdot\,)-v_j\|^2_{V^1(\Pi)}+\|\theta(t_j,\,\cdot\,)-\theta_j\|^2_{H^1(\Pi)}<\varepsilon \qquad \forall\, j=1,\dots,N. \tag{1.39}$$

We note that the right-hand side of system (1.26), (1.27) is zero for $(t,x)\in Q\setminus Q^\omega$, while the solution of this system can be arbitrarily complicated on the entire Q in the sense that relations (1.39) are valid.

Needless to say, the Navier–Stokes equations have a similar property.

3. *Reversibility.* Let $(v_i(t,x),\nabla p_i(t,x))$, $(t,x)\in Q$, $i=1,2$, be two solutions of the Navier–Stokes equations (1.5) (with the same right-hand side $f(t,x)$, say, $f(t,x)\equiv 0$). Then for any $\omega\in\Pi$ and $(t_1,t_2)\subset(0,T)$ there is a control $u(t,x)$ supported in $(t_1,t_2)\times\omega$ and taking the solution $(v_1,\nabla p_1)$ to $(v_2,\nabla p_2)$. Namely, under this control u the solution $(v,\nabla p)$ of system (1.12) has the following property:

$$\begin{aligned}(v(t,x),p(t,x))&\equiv(v_1(t,x),p_1(t,x))\quad\text{for}\quad t\in(0,t_1),\\ (v(t,x),p(t,x))&\equiv(v_2(t,x),p_2(t,x))\quad\text{for}\quad t\in(t_2,T).\end{aligned}$$

This property is referred to as *reversibility.* By virtue of Theorem 1.2, the Navier–Stokes equations possess this property.

In a similar way, one can state the reversibility of the Boussinesq system; this follows from Theorem 1.4.

The reversibility of the Navier–Stokes equations and the Boussinesq system are of interest in connection with certain issues in climate theory. (See Lions [93], [94], [96] and Coron–Fursikov [14] for more details.)

§ 2. Carleman estimates

The proof of solubility of the exact controllability problem for a linear evolution system can often be reduced to studying the corresponding observability problem for the adjoint system. This method is implemented in §§ 2–4 of the present paper.

In this section we prove some Carleman type estimates for solutions of the system adjoint to the linearized Boussinesq equations.

2.1. Preliminaries. Just as before, Π is the n-dimensional torus and $\omega\subset\Pi$ is a subdomain with boundary $\partial\omega\in C^\infty$. We assume that $\omega'\Subset\omega$ and that ω' is star-shaped with respect to some point $x_0\in\omega'$.

Lemma 2.1. *There is a function* $\psi(x) \in C^2(\Pi)$ *such that*

$$|\nabla \psi(x)| > 0 \qquad \forall\, x \in \overline{\Pi \setminus \omega'}, \tag{2.1}$$

where $\omega' \Subset \omega \subset \Pi$. *Moreover,*

$$\psi(x) \geqslant 1. \tag{2.2}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$ and the torus Π is obtained from the cube $K = \{x = (x_1, \dots, x_n),\ -L/2 \leqslant x_j \leqslant L/2,\ j = 1, \dots, n\}$ by identifying opposite facets. We set

$$\psi(x) = c + \sum_{j=1}^{n} \beta(x_j),$$

where $c > 0$ is a constant, $\beta(x)$ is a periodic function such that

$$\beta(x) \in C^\infty(-L/2, L/2), \qquad \beta(x) = \begin{cases} x, & x \in [\varepsilon, L/2], \\ x + L, & x \in [-L/2, -\varepsilon], \end{cases}$$

and ε is sufficiently small. Obviously, the function ψ thus constructed satisfies condition (2.1) with

$$\omega' = \bigl\{x = (x_1, \dots, x_n),\ |x_j| \leqslant \varepsilon,\ j = 1, \dots, n\bigr\}.$$

We can readily ensure condition (2.2) by choosing a sufficiently large constant $c > 0$.

Let $\gamma(t) \in C^\infty(0, T)$ be a function satisfying the conditions

$$0 < \gamma(t) \leqslant T - t, \qquad \gamma(t) = \begin{cases} t, & t \in [0, T/4], \\ T - t, & t \in [3T/4, T]. \end{cases} \tag{2.3}$$

Furthermore, we assume that $\gamma(t)$ is monotonically increasing for $t \in (0, T/2)$ and monotonically decreasing for $t \in (T/2, T)$. We introduce the functions

$$\varphi = \varphi_\lambda(t, x) = \frac{e^{\lambda \psi(x)}}{\gamma(t)}, \qquad \alpha \equiv \alpha_\lambda(t, x) = \frac{e^{\lambda^2 \|\psi\|_{C(\Pi)}} - e^{\lambda \psi(x)}}{\gamma(t)}, \tag{2.4}$$

where $\lambda > 1$ is a parameter and $\psi(x)$ and $\gamma(t)$ are the above-introduced functions.

On the cylinder Q we consider the backward heat equation

$$\partial_t z(t, x) + \Delta z(t, x) = f(t, x). \tag{2.5}$$

Lemma 2.2. *Suppose that* $f \in L_2(Q)$ *and* $z \in L_2(Q)$ *satisfy* (2.5). *Then there is a* $\widehat{\lambda} > 1$ *such that for every* $\lambda > \widehat{\lambda}$ *there is an* $s_0(\lambda)$ *with the following property. For every* $s \geqslant s_0(\lambda)$, *the Carleman estimate*

$$\int_Q \biggl((s\varphi)^{-1} |\partial_t z|^2 + (s\varphi)^{-1} \sum_{i,j=1}^{n} |\partial^2_{x_i x_j} z|^2 + s\varphi |\nabla z|^2 + (s\varphi)^3 z^2 \biggr) e^{-s\alpha_\lambda}\, dx\, dt$$
$$\leqslant c \biggl(\int_Q |f(t, x)|^2 e^{-s\alpha_\lambda}\, dx\, dt + \int_{Q^{\omega'}} s^3 \varphi^3 z^2 e^{-s\alpha_\lambda}\, dx\, dt \biggr) \tag{2.6}$$

holds, where $c > 0$ *is a constant independent of* f, z, *and* s.

Let us consider the Poisson equation on Π:

$$\Delta z = f. \tag{2.7}$$

Lemma 2.3. *Suppose that $z \in H^2(\Pi)$ and $f \in L_2(\Pi)$ satisfy* (2.7). *Then there is a $\widehat{\lambda} > 1$ such that for each $\lambda > \widehat{\lambda}$ there is an $s_0(\lambda)$ with the following property*: *for every $s \geqslant s_0(\lambda)$, the estimate*

$$\int_\Pi \biggl((s\varphi)^{-1} \sum_{i,j=1}^n |\partial^2_{x_i x_j} z|^2 + s\varphi|\nabla z|^2 + s^3\varphi^3|z|^2\biggr) e^{-s\alpha_\lambda}\,dx \leqslant c\biggl(\int_\Pi |f|^2 e^{-s\alpha_\lambda}\,dx + \int_{\omega'} s^3\varphi^3|z|^2 e^{-s\alpha_\lambda}\,dx\biggr) \tag{2.8}$$

holds, where $c > 0$ is a constant independent of f, z, and s.

Lemmas 2.2 and 2.3 were proved in [10], [30], [38], [40], [41], [61] for the case of a bounded domain. For the torus Π the proof is similar.

We replace the constants $\widehat{\lambda}$ and $s_0(\lambda)$ in Lemmas 2.2 and 2.3 by the maximum of these constants and assume from now on that they are the same in both lemmas.

We consider the following system of equations on the cylinder Q:[1]

$$N^*(y,\tau) = -\partial_t y - \Delta y - (\widehat{v},\nabla)y + ((y,\nabla)\widehat{v})^* + \tau\nabla\widehat{\theta} + \nabla p = f, \quad \operatorname{div} y = 0, \tag{2.9}$$

$$R^*(y,\tau) = -\partial_t \tau - \Delta\tau - (\nabla\tau, \widehat{v}) + (\vec{e}, y) = g, \tag{2.10}$$

$$y(t,x)\big|_{t=T} = y_T(x), \qquad \tau(t,x)\big|_{t=T} = \tau_T(x), \tag{2.11}$$

where $\widehat{v}$, $\widehat{\theta}$, f, g, y_T, and τ_T are given, (y,τ) are the unknown functions, and

$$((y,\nabla)\widehat{v})^* = ((\partial_{x_1}\widehat{v}, y), \dots, (\partial_{x_n}\widehat{v}, y)).$$

The following lemma implies that problem (2.9)–(2.11) is well posed.

Lemma 2.4. *Let $\widehat{v} \in L_\infty(0,T;V^1_\infty(\Pi))$ and $\widehat{\theta} \in L_\infty(0,T;W^1_\infty(\Pi))$. Then for any initial data $y_0 \in V^1(\Pi)$, $\tau_0 \in H^1(\Pi)$, $f \in (L_2(Q))^n$, and $g \in L_2(Q)$ there is a unique solution $(y,\nabla p,\tau) \in V^{1,2}(Q) \times (L^2(Q))^n \times H^{1,2}(Q)$ of problem* (2.9)–(2.11). *This solution satisfies the estimate*

$$\|y\|^2_{V^{1,2}(Q)} + \|\nabla p\|^2_{(L_2(Q))^n} + \|\tau\|^2_{H^{1,2}(Q)} \leqslant c\bigl(\|y_T\|^2_{V^1(\Pi)} + \|\tau_T\|^2_{H^1(\Pi)} + \|f\|^2_{(L_2(Q))^n} + \|g\|^2_{L_2(Q)}\bigr), \tag{2.12}$$

where the constant $c > 0$ is independent of y_T, τ_T, f, and g.

Scheme of the proof. Methods by which assertions of this type can be proved are well known (e.g., see [77], [124]). To make the exposition self-contained, we recall the main stages of the proof. First we obtain an energy inequality for the solution of problem (2.9)–(2.11). We take the inner product of (2.9) by y in $(L_2(\Pi))^n$ and the

[1] This system is the formal adjoint of the linearized Boussinesq system, which will be introduced in § 3 (see (3.1), (3.2)).

inner product of (2.10) by τ in $L_2(\Pi)$. By summing the resulting inequalities, integrating by parts, and performing simple manipulations (in particular, integrating over time from t to T), we obtain

$$\|y(t,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\Pi)} + 2\int_t^T \bigl(\|\nabla y(s,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\nabla \tau(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds$$
$$\leqslant c_2 \int_t^T \bigl(\|y(s,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(s,\,\cdot\,)\|^2_{L_2(\Pi)} + \|f(s,\,\cdot\,)\|^2_{V^0(\Pi)} + \|g(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds$$
$$+ \|y_T\|^2_{V^0(\Pi)} + \|\tau_T\|^2_{L_2(\Pi)}.$$

An application of the Gronwall lemma to this inequality yields the energy estimate

$$\|y(t,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\Pi)} + 2\int_t^T \bigl(\|\nabla y(s,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\nabla \tau(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds$$
$$\leqslant c\biggl(\|y_T\|^2_{V^0(\Pi)} + \|\tau_T\|^2_{L_2(\Pi)} + \int_t^T \bigl(\|f(s,\,\cdot\,)\|^2_{V^0(\Pi)} + \|g(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds\biggr). \tag{2.13}$$

Taking the projection of equation (2.9) on the subspace of solenoidal vector fields and using Galerkin methods, we derive from inequality (2.13) the existence and uniqueness of a solution

$$(y,\tau) \in \bigl(L_\infty(0,T;V^0(\Pi)) \cap L_2(0,T;V^1(\Pi))\bigr) \times \bigl(L_\infty(0,T;L_2(\Pi)) \cap L_2(0,T;H^1(\Pi))\bigr).$$

Let $P\colon (L_2(\Pi))^n \to V^0(\Pi)$ be the orthogonal projection. Using the explicit formula for P for the case of the torus (see [126]), we readily see that $P\Delta = \Delta P$ and that the restriction of P to $(H^k(\Pi))^n$ acts into $V^k(\Pi)$, $P\big|_{(H^k(\Pi))^n}\colon (H^k(\Pi))^n \to V^k(\Pi)$. Hence the projection of (2.9) on $V^0(\Pi)$ has the form

$$-\partial_t y - \Delta y = P\bigl[f + (\widehat{v},\nabla)y + ((y,\nabla)\widehat{v})^* + \tau\nabla\widehat{\theta}\bigr] \equiv q, \tag{2.14}$$

and moreover, we obviously have $q \in L_2(0,T;V^0(\Pi))$. By applying the theorem on the smoothness of solutions of the Stokes problem to problem (2.14), (2.11) with right-hand side q, we find that $y \in V^{1,2}(Q)$ and

$$\|y\|^2_{V^{1,2}(Q)} \leqslant c\Bigl(\|y_T\|^2_{V^1(\Pi)} + \|q\|^2_{L_2(0,T;V^0(\Pi))}\Bigr)$$
$$\leqslant c_1\Bigl(\|y_T\|^2_{V^1(\Pi)} + \|y\|^2_{L_2(0,T;V^1(\Pi))}$$
$$+ \|\tau\|^2_{L_2(0,T;H^1(\Pi))} + \|y\|^2_{(L_2(Q))^n}\Bigr). \tag{2.15}$$

Inequalities (2.15) and (2.13) imply the desired estimate for $\|y\|^2_{V^{1,2}(Q)}$. The estimate for $\|\tau\|^2_{H^{1,2}(Q)}$ can be obtained in a similar way. By expressing ∇p from (2.9), we obtain the estimate for $\|\nabla p\|^2_{(L_2(Q))^n}$.

2.2. Carleman estimates for solutions of system (2.9), (2.10). First, let us obtain additional information on the smoothness of the pressure p from (2.9). To this end, we subject $\widehat{v}$, $\widehat{\theta}$, and f to stronger smoothness assumptions than in Lemma 2.4. Suppose that $\operatorname{div} f \in L_2(Q)$ and

$$\widehat{v} \in L_\infty(0,T;V^2_\infty(\Pi)), \qquad \widehat{\theta} \in L_\infty(0,T;W^2_\infty(\Pi)). \tag{2.16}$$

We apply the operator div on both sides of (2.9). Since the function y is divergence-free, we obtain

$$\Delta p = \operatorname{div}\bigl((\widehat{v},\nabla)y - ((y,\nabla)\widehat{v})^* - \tau\nabla\widehat{\theta} + f\bigr). \tag{2.17}$$

By (2.16) and the inclusions $y \in L_2(0,T;V^2(\Pi))$ and $\tau \in L_2(0,T;H^2(\Pi))$, which follow from Lemma 2.4, we have

$$\operatorname{div}\bigl((\widehat{v},\nabla)y - ((y,\nabla)\widehat{v})^* - \tau\nabla\widehat{\theta}\bigr) \in L_2(Q).$$

Hence it follows from (2.17) and Sobolev's embedding theorem that

$$p \in L_2(0,T;H^2(\Pi)) \subset L_2(0,T;C(\Pi)),$$

and hence the value of p at x_0, that is, $p(t,x_0)$, is well defined as an element of $L_2(0,T)$. To specify the function p in (2.9) uniquely, we set

$$p(t,x_0) = 0, \tag{2.18}$$

where $x_0 \in \omega'$ and ω' is the subdomain, star-shaped with respect to x_0, introduced in Lemma 2.1.

Theorem 2.1. *Let $\widehat{v}$ and $\widehat{\theta}$ satisfy* (2.16), *and let*

$$f \in (L_2(Q))^n, \quad \operatorname{div} f \in L_2(Q), \quad g \in L_2(Q). \tag{2.19}$$

Then there is a $\widehat{\lambda} > 1$ such that for each $\lambda > \widehat{\lambda}$ there is an $s_0(\lambda)$ with the following property: *for each $s \geqslant s_0(\lambda)$, the solution of problem* (2.9)–(2.11) *satisfies the estimate*

$$\begin{aligned} I(y,\tau,s) \equiv \int_Q \biggl(& (\varphi s)^{-1}(|\partial_t y|^2 + |\partial_t \tau|^2) + (s\varphi)^{-1} \sum_{i,j=1}^n (|\partial^2_{x_i x_j} y|^2 + |\partial^2_{x_i x_j}\tau|^2) \\ & + s\varphi(|\nabla y|^2 + |\nabla\tau|^2) + s^3\varphi^3(|y|^2+|\tau|^2)\biggr) e^{-s\alpha_\lambda}\,dx\,dt \\ \leqslant c\biggl(& \int_Q (|f(t,x)|^2 + |\operatorname{div} f|^2 + |g(t,x)|^2) e^{-s\alpha_\lambda}\,dx\,dt \\ & + \int_{Q^{\omega'}} s^3\varphi^3(|y|^2+|\tau|^2+|p|^2)e^{-s\alpha_\lambda}\,dx\,dt\biggr), \end{aligned} \tag{2.20}$$

where $c > 0$ is independent of f, g, and s.

Proof. We set

$$\tilde{f}(t,x) = f(t,x) + (\widehat{v},\nabla)y - ((y,\nabla)\widehat{v})^* - \tau\nabla\widehat{\theta}, \tag{2.21}$$

$$\tilde{g}(t,x) = g(t,x) - (\vec{e},y) + (\nabla\tau,\widehat{v}), \tag{2.22}$$

$$q(t,x) = \tilde{f} - \nabla p. \tag{2.23}$$

It follows from (2.9) and (2.23) that $\operatorname{div} q = 0$. Using the notation (2.21)–(2.23), we can rewrite (2.9) and (2.10) in the form

$$-\frac{\partial y}{\partial y} - \Delta y = q, \quad \operatorname{div} y = 0; \quad -\frac{\partial \tau}{\partial y} - \Delta\tau = \tilde{g}. \tag{2.24}$$

Let us apply Lemma 2.2 to each equation in (2.24) and sum the resulting estimates. Then we obtain the inequality

$$I(y,\tau,s) \leqslant c\biggl(\int_Q (|q|^2+|\tilde{g}|^2)e^{-s\alpha_\lambda}\,dx\,dt + \int_{Q^{\omega'}} s^3\varphi^3(|y|^2+|\tau|^2)e^{-s\alpha_\lambda}\,dx\,dt\biggr), \tag{2.25}$$

where $I(y,\tau,s)$ is the left-hand side of (2.20) and c is independent of q, $\tilde{g}$, and $s > s_0(\lambda)$.

Using definition (2.21)–(2.23) of the functions q and $\tilde{g}$, we estimate the first integral on the right-hand side in (2.25) as follows:

$$\int_Q (|q|^2 + |\tilde{g}|^2)e^{-s\alpha_\lambda}\,dx\,dt$$
$$\leqslant c_1 \int_Q \bigl(|f|^2 + |g|^2 + |\nabla p|^2 + |\nabla y|^2 + |\nabla\tau|^2 + |y|^2 + |\tau|^2\bigr)e^{-s\alpha_\lambda}\,dx\,dt. \tag{2.26}$$

Now from (2.25) and (2.26), taking a larger s where necessary, we obtain the inequality

$$I(y,\tau,s) \leqslant c_2\biggl(\int_Q (|f|^2 + |g|^2 + |\nabla p|^2)e^{-s\alpha_\lambda}\,dx\,dt + \int_{Q^{\omega'}} s^3\varphi^3(|y|^2 + |\tau|^2)e^{-s\alpha_\lambda}\,dx\,dt\biggr). \tag{2.27}$$

An application of Lemma 2.3 to (2.17) yields the estimate

$$\int_Q \biggl((s\varphi)^{-1}\sum_{i,j=1}^n |\partial^2_{x_ix_j}p|^2 + s\varphi|\nabla p|^2 + s^3\varphi^3|p|^2\biggr)e^{-s\alpha_\lambda}\,dx\,dt$$
$$\leqslant c\biggl(\int_Q |\operatorname{div}((\widehat{v},\nabla)y - ((y,\nabla)\widehat{v})^* - \tau\nabla\widehat{\theta} + f)|^2 e^{-s\alpha_\lambda}\,dx\,dt + \int_{Q^{\omega'}} s^3\varphi^3|p|^2e^{-s\alpha_\lambda}\,dx\,dt\biggr)$$
$$\leqslant c_1\biggl(\int_Q (|\nabla y|^2 + |\nabla\tau|^2 + |y|^2 + |\tau|^2 + |\operatorname{div} f|^2)e^{-s\alpha_\lambda}\,dx\,dt + \int_{Q^{\omega'}} s^3\varphi^3|p|^2e^{-s\alpha_\lambda}\,dx\,dt\biggr). \tag{2.28}$$

We estimate ∇p in (2.27) with the help of (2.28) and increase s where necessary, thus obtaining the estimate (2.20).

Corollary 2.1. *For* $(\widehat{v},\widehat{\theta})$ *in* (2.16) *let* $(y,\tau)\in L_2(0,T;V^1(\Pi))\times L^2(0,T;H^1(\Pi))$ *be a solution of problem* (2.9)–(2.11) *with* $f\equiv 0$ *and* $g\equiv 0$ *such that* $(y,\tau)\big|_{Q^\omega}\equiv 0$. *Then*

$$(y,\tau)\equiv 0 \quad \text{on} \quad Q.$$

Proof. Indeed, $\nabla p\big|_{Q^\omega}\equiv 0$ by (2.9). Consequently,

$$p(t,x)=p(t) \quad \forall\,(t,x)\in Q^\omega,$$

and $p\big|_{Q^\omega}\equiv 0$ by (2.18). Thus, our assertion follows from (2.20).

Now we want to get rid of the term $|p|^2$ on the right-hand side of the estimate (2.20). To this end, we use the following property of the function α_λ, which is valid for $\lambda\geqslant\lambda_0$, where λ_0 is sufficiently large, by virtue of the second relation in (2.4) and (2.2):

$$\widehat{\alpha}_\lambda(t)<\frac{10}{9}\tilde{\alpha}(t) \quad \text{for} \quad \widehat{\alpha}_\lambda(t)=\max_{x\in\Omega}\alpha_\lambda(t,x), \quad \tilde{\alpha}(t)=\min_{x\in\Omega}\alpha_\lambda(t,x). \tag{2.29}$$

Theorem 2.2. *Let* $\widehat{v}$, $\widehat{\theta}$, f, *and* g *satisfy* (2.16) *and* (2.19). *Suppose that* $\omega'\Subset\omega\Subset\Pi$, *where* $\partial\omega\in C^\infty$ *and* ω' *is a set star-shaped with respect to the point* x_0 *in* (2.18). *Then there is a* $\widehat{\lambda}>1$ *such that for each* $\lambda>\widehat{\lambda}$ *there is an* $s_0(\lambda)$ *with the following property*: *for all* $s\geqslant s_0(\lambda)$ *the solution of problem* (2.9)–(2.11) *satisfies the estimate*

$$\begin{aligned} I(y,\tau,s)\leqslant c\bigg(&\int_Q(|f|^2+|\operatorname{div}f|^2+|g|^2)e^{-s\alpha_\lambda}\,dx\,dt\\ &+\int_0^T e^{-9s\widehat{\alpha}_\lambda/10}\big(\|f(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n}+\|\operatorname{div}f(t,\,\cdot\,)\|^2_{H^{-1}(\omega)}\big)\,dt\\ &+\int_{Q^\omega}(|y|^2+|\tau|^2+|\partial_t y|^2)e^{-9s\widehat{\alpha}_\lambda/10}\,dx\,dt\bigg), \end{aligned}\tag{2.30}$$

where $c>0$ *is independent of* f *and* g, *and* $I(y,\tau,s)$ *is defined in* (2.20).

Proof. Let us express ∇p using (2.9):

$$\nabla p=f+\partial_t y+\Delta y+(\widehat{v},\nabla)y-((y,\nabla)\widehat{v})^*-\tau\nabla\widehat{\theta}\equiv f_1. \tag{2.31}$$

By applying the operator div to both sides of this relation, we obtain

$$\begin{aligned} \Delta p=\sum_{i,k=1}^n\frac{\partial\widehat{v}_i}{\partial x_k}\frac{\partial y_k}{\partial x_i}-\sum_{i=1}^n\bigg(\frac{\partial^2\widehat{v}}{\partial x_i^2},y\bigg)&\\ -\sum_{i,k=1}^n\frac{\partial\widehat{v}_k}{\partial x_i}\frac{\partial y_k}{\partial x_i}-\tau\Delta\widehat{\theta}-(\nabla\tau,\nabla\widehat{\theta})+\operatorname{div}f\equiv f_2,& \end{aligned}\tag{2.32}$$

where the last equality is just the definition of the function f_2. Let us represent p as the sum

$$p(t,x) = p_1(t,x) + p_2(t,x), \qquad t \in (0,T), \quad x \in \omega, \tag{2.33}$$

where

$$\Delta p_1(t,x) = f_2(t,x), \qquad x \in \omega, \quad p_1\big|_{\partial\omega} = 0, \tag{2.34}$$

$$\Delta p_2(t,x) = 0, \qquad x \in \omega, \quad p_2\big|_{\partial\omega} = p\big|_{\partial\omega}. \tag{2.35}$$

As shown above (see (2.18)), $p\big|_{\partial\omega}$ is well defined for almost all $t \in (0,T)$. By applying the estimate for solutions of elliptic boundary-value problems to (2.34) and using definition (2.32) of the right-hand side f_2 and (2.16), we obtain

$$\|\nabla p_1(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} \leqslant c\|f_2(t,\,\cdot\,)\|^2_{H^{-1}(\omega)} \leqslant c_1\big(\|y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\omega)} + \|\operatorname{div} f(t,\,\cdot\,)\|^2_{(H^{-1}(\omega))^n}\big). \tag{2.36}$$

By (2.31) and (2.33),

$$\nabla p_2 = \nabla p - \nabla p_1 = f_1 - \nabla p_1,$$

and hence, with regard to (2.36) and (2.16), we have

$$\|\nabla p_2(t,\,\cdot\,)\|^2_{H^{-2}(\omega)} \leqslant c_2\big(\|y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\omega)} + \|\partial_t y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|f(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n} + \|\operatorname{div} f(t,\,\cdot\,)\|^2_{H^{-1}(\omega)}\big). \tag{2.37}$$

The differentiation of (2.35) yields

$$\Delta\nabla p_2(t,x) = 0, \qquad x \in \omega.$$

Now ∇p_2 satisfies the assumptions of the following lemma, which can readily be derived, say, with the help of the cut-off function technique, from results due to Lions and Magenes [99].

Lemma 2.5. *Let $w(x) \in H^{-2}(\omega)$ be a harmonic function, that is, $\Delta w(x) = 0$, $x \in \omega$, and let $\omega' \Subset \omega$. Then*

$$\|w\|_{L_2(\omega')} \leqslant c\|w\|_{H^{-2}(\omega)}, \tag{2.38}$$

where c is independent of w.

By applying the estimates (2.38) and (2.37) to ∇p_2, we obtain

$$\|\nabla p_2(t,\,\cdot\,)\|^2_{(L_2(\omega'))^n} \leqslant c\big(\|y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\omega)} + \|\partial_t y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|f(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n} + \|\operatorname{div} f(t,\,\cdot\,)\|^2_{H^{-1}(\omega)}\big). \tag{2.39}$$

Since, by assumption, the set ω' is star-shaped with respect to the point x_0 in (2.18), it follows from (2.18), (2.33), (2.36), and (2.39) that

$$\|p(t,\,\cdot\,)\|^2_{L_2(\omega')} \leqslant c\|\nabla p\|^2_{(L_2(\omega'))^n} \leqslant c_1\big(\|y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|\tau(t,\,\cdot\,)\|^2_{L_2(\omega)} + \|\partial_t y(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|f(t,\,\cdot\,)\|^2_{(L_2(\omega))^n}\big).$$

By applying this estimate to the right-hand side of (2.20) and by taking account of (2.29), we have

$$
\begin{aligned}
I(y,\tau,s) \leqslant c\Bigg(\int_0^T e^{-9s\widehat{\alpha}_\lambda/10}\bigg(\int_\omega (|y|^2+|\tau|^2+|p|^2)\,dx + \|f(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n} \\
+ \|\operatorname{div} f(t,\,\cdot\,)\|^2_{H^{-1}(\omega)}\bigg)dt + \int_Q (|f|^2+|\operatorname{div} f|^2+|g|^2)e^{-s\alpha_\lambda}\,dx\,dt\Bigg) \\
\leqslant c\bigg(\int_{Q^\omega}(|y|^2+|\tau|^2+|\partial_t y|^2+|f|^2)e^{-9s\widehat{\alpha}_\lambda/10}\,dx\,dt \\
+ \int_0^T e^{-9s\widehat{\alpha}_\lambda/10}\big(\|f(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n} + \|\operatorname{div} f(t,\,\cdot\,)\|^2_{H^{-1}(\omega)}\big)dt \\
+ \int_Q \big(|f|^2+|\operatorname{div} f|^2+|g|^2\big)e^{-s\alpha_\lambda}\,dx\,dt\bigg).
\end{aligned}
$$

This proves (2.30).

2.3. Definitive estimates. Let us now eliminate the term $\partial_t y$ on the right-hand side of inequality (2.30).

Theorem 2.3. *Suppose that f, g, $\widehat{v}$, and $\widehat{\theta}$ satisfy* (2.19) *and* (2.16), *and moreover,*

$$
\partial_t\widehat{v} \in L_\infty(0,T;V^2_\infty(\Pi)), \qquad \partial_t\widehat{\theta} \in L_\infty(0,T;W^2_\infty(\Pi)). \tag{2.40}
$$

Next, let the sets ω' and ω satisfy the assumptions of Theorem 2.2. *Then there is a $\widehat{\lambda}>1$ such that for each $\lambda>\widehat{\lambda}$ there is an $s_0(\lambda)$ with the following property*: *for all $s\geqslant s_0(\lambda)$, the solution of problem* (2.9)–(2.11) *satisfies the estimate*

$$
\begin{aligned}
\int_Q \frac{1}{s\varphi}\big(|y(t,x)|^2+|\tau(t,x)|^2\big)e^{-s\alpha_\lambda(t,x)}\,dx\,dt \\
\leqslant c\bigg(\int_Q\big(|f(t,x)|^2+|\operatorname{div} f|^2+|g(t,x)|^2\big)e^{-s\alpha_\lambda(t,x)}\,dx\,dt \\
+ \int_{Q^\omega}\big(|f(t,x)|^2+|y(t,x)|^2+|\tau(t,x)|^2\big)e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt \\
+ \|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\bigg),
\end{aligned} \tag{2.41}
$$

where $c>0$ is independent of f, g, y, τ, and s.

Proof. We introduce the functions

$$
\begin{aligned}
\tilde{y}(t,x) &= \int_{T/2}^t y(s,x)\,ds, \qquad \tilde{\tau}(t,x) = \int_{T/2}^t \tau(s,x)\,ds, \\
\tilde{p}(t,x) &= \int_{T/2}^t p(s,x)\,ds, \qquad \tilde{f}(t,x) = \int_{T/2}^t f(s,x)\,ds, \\
\tilde{g}(t,x) &= \int_{T/2}^t g(s,x)\,ds.
\end{aligned} \tag{2.42}
$$

Let us integrate (2.9) and (2.10) with respect to time from $T/2$ to t. Then we find that the functions (2.42) satisfy the relations

$$-\partial_t\tilde{y} - \Delta\tilde{y} + \nabla\tilde{p} = \tilde{f} - y(T/2,x) + \int_{T/2}^{t}\Big((\widehat{v}(s,x),\nabla)y(s,x) - ((y,\nabla),\widehat{v})^*(s,x) - \tau(s,x)\nabla\widehat{\theta}(s,x)\Big)\,ds = 0, \quad (2.43)$$

$$-\partial_t\tilde{\tau} - \Delta\tilde{\tau} + (\vec{e},\tilde{y}) = \tilde{g} + \int_{T/2}^{t}\big(\nabla\tau(s,x),\widehat{v}(s,x)\big)\,ds - \tau(T/2,x). \quad (2.44)$$

Let us transform the integrals in (2.43) and (2.44). We integrate by parts and obtain

$$\int_{T/2}^{t}\big(\nabla\tau(s,x),\widehat{v}(s,x)\big)\,ds = \big(\nabla\tilde{\tau}(t,x),\widehat{v}(t,x)\big) - \int_{T/2}^{t}(\nabla\tilde{\tau}(s,x),\partial_s\widehat{v}(s,x))\,ds;$$

in the same way, one can prove that

$$\int_{T/2}^{t}(\widehat{v},\nabla)y\,ds = \big(\widehat{v}(t,x),\nabla\big)\tilde{y}(t,x) - \int_{T/2}^{t}\big(\partial_t\widehat{v}(s,x),\nabla\big)\tilde{y}\,ds,$$

$$\int_{T/2}^{t}\big((y,\nabla)\widehat{v}\big)^*\,ds = \big((\tilde{y}(t,x),\nabla)\widehat{v}(t,x)\big)^* - \int_{T/2}^{t}\big((\tilde{y},\nabla)\partial_s\widehat{v}\big)^*\,ds,$$

$$\int_{T/2}^{t}\tau\nabla\widehat{\theta}\,ds = \tilde{\tau}(t,x)\nabla\widehat{\theta}(t,x) - \int_{T/2}^{t}\tilde{\tau}\partial_s\nabla\widehat{\theta}\,ds.$$

By substituting this into (2.43) and (2.44), we find that $\tilde{y}$, $\tilde{\tau}$, and $\tilde{p}$ satisfy (2.9) and (2.10), where the right-hand sides f and g are replaced, respectively, by

$$F = \tilde{f} - y(T/2,x) - \int_{T/2}^{t}\Big((\partial_s\widehat{v},\nabla)\tilde{y} - ((\tilde{y},\nabla)\partial_t\widehat{v})^* - \tilde{\tau}\nabla\partial_s\widehat{\theta}\Big)\,ds, \quad (2.45)$$

$$G = \tilde{g} - \int_{T/2}^{t}(\nabla\tilde{\tau},\partial_s\widehat{v})\,ds - \tau(T/2,x). \quad (2.46)$$

An application of Theorem 2.2 to these equations gives the following analogue of the estimate (2.30):

$$I(\tilde{y},\tilde{\tau},s) \leqslant c\bigg(\int_Q\big(|F(t,x)|^2 + |\operatorname{div}F(t,x)|^2 + |G(t,x)|^2\big)e^{-s\alpha_\lambda(t,x)}\,dx\,dt + \int_0^T e^{-9s\widehat{\alpha}_\lambda/10}\big(\|F(t,\cdot)\|^2_{(H^{-2}(\omega))^n} + \|\operatorname{div}F(t,\cdot)\|^2_{H^{-1}(\omega)}\big)\,dt + \int_{Q^\omega}\big(|\tilde{y}|^2 + |\tilde{\tau}|^2 + |\tilde{y}_t|^2\big)e^{-9s\widehat{\alpha}_\lambda/10}\,dx\,dt\bigg). \quad (2.47)$$

Let $h(t,x)$ be a function. We write

$$\tilde{h}(t,x) = \int_{T/2}^{t} h(\xi,x)\,d\xi.$$

Let us prove the inequality

$$\int_Q |\tilde{h}(t,x)|^2 e^{-s\alpha_\lambda(t,x)}\,dx\,dt \leqslant c\int_Q |h(t,x)|^2 e^{-s\alpha_\lambda(t,x)}\,dx\,dt, \tag{2.48}$$

where c is independent of h and s. By applying the Cauchy–Schwarz–Bunyakovskii inequality and then Fubini's theorem, we obtain

$$\begin{aligned}\int_Q |\tilde{h}(t,x)|^2 e^{-s\alpha_\lambda(t,x)}\,dx\,dt &\leqslant c\biggl(\int_0^{T/2}\int_\Pi\int_t^{T/2} |h(\xi,x)|^2\,d\xi\, e^{-s\alpha_\lambda(t,x)}\,dx\,dt \\ &\qquad + \int_{T/2}^{T}\int_\Pi\int_{T/2}^{t} |h(\xi,x)|^2 d\xi\, e^{-s\alpha_\lambda(t,x)}\,dx\,dt\biggr) \\ &= c\biggl(\int_0^{T/2}\int_\Pi |h(\xi,x)|^2\int_0^{\xi} e^{-s\alpha_\lambda(t,x)}\,dt\,dx\,d\xi \\ &\qquad + \int_{T/2}^{T}\int_\Pi |h(\xi,x)|^2\biggl(\int_\xi^T e^{-s\alpha_\lambda(t,x)}\,dt\biggr)\,dx\,d\xi\biggr) \\ &\leqslant c_1\int_Q |h(t,x)|^2 e^{-s\alpha_\lambda(t,x)}\,dx\,ds,\end{aligned}$$

where the last inequality follows from definition (2.4) of the function α_λ and the fact that the function $\gamma(t)$ is monotonically increasing for $t\in(0,T/2)$ and monotonically decreasing for $t\in(T/2,T)$. Thus, we have proved (2.48).

By (2.48),

$$\begin{aligned}\int_Q (|F|^2 + |\operatorname{div} F|^2)e^{-s\alpha_\lambda}\,dx\,dt &\leqslant c\biggl(\int_Q |f|^2 e^{-s\alpha_\lambda}\,dx\,dt \\ &\quad + \int_\Pi |y(T/2,x)|^2 e^{-s\alpha_\lambda(T/2,x)}\,dx + \int_Q (|\nabla\tilde{y}|^2 + |\tilde{y}|^2 + \tilde{\tau}^2)e^{-s\alpha_\lambda}\,dx\,dt\biggr), \end{aligned}\tag{2.49}$$

$$\begin{aligned}\int_Q |G|^2 e^{-s\alpha_\lambda}\,dx\,dt &\leqslant c\biggl(\int_Q |g|^2 e^{-s\alpha_\lambda}\,dx\,dt \\ &\quad + \int_\Pi |\tau(T/2,x)|^2 e^{-s\alpha_\lambda(T/2,x)}\,dx + \int_Q |\nabla\tilde{\tau}|^2 e^{-s\alpha_\lambda}\,dx\,dt\biggr),\end{aligned}\tag{2.50}$$

where the constant c is independent of s.

Since the vector field $\partial_t\widehat{v}$ is divergence-free, we see that

$$\int_{T/2}^{t} (\partial_s\widehat{v},\nabla)\tilde{y}\,ds = \sum_{j=1}^{n}\int_{T/2}^{t}\partial_j(\partial_s\widehat{v}_j,\tilde{y})\,ds.$$

Hence it follows from (2.45) that

$$\begin{aligned}\|F(t,\,\cdot\,)\|^2_{(H^{-2}(\omega))^n} + \|\operatorname{div} F(t,\,\cdot\,)\|^2_{H^{-1}(\omega)} &\leqslant c\biggl(\|\tilde{f}(t,\,\cdot\,)\|^2_{(L_2(\omega))^n} \\ &\quad + \int_{T/2}^{t}\bigl(\|\tilde{y}(s,\,\cdot\,)\|^2_{(L_2(\omega))^n} + \|\tilde{\tau}(s,\,\cdot\,)\|^2_{L_2(\omega)}\bigr)\,ds + \|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)}\biggr).\end{aligned}\tag{2.51}$$

Let us estimate the right-hand side of inequality (2.47) with the help of (2.49)–(2.51). Then we have

$$
\begin{aligned}
I(\tilde{y},\tilde{\tau},s) \leqslant c\biggl(\int_Q \bigl(&|f(t,x)|^2+|\operatorname{div} f(t,x)|^2+|g(t,x)|^2+|\nabla\tilde{y}(t,x)|^2+|\tilde{y}(t,x)|^2\\
&+|\nabla\tilde{\tau}(t,x)|^2+|\tilde{\tau}(t,x)|^2\bigr)e^{-s\alpha_\lambda(t,x)}\,dx\,dt\\
&+\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)}+\|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\\
&+\int_{Q^\omega}\bigl(|\tilde{y}|^2+|\tilde{\tau}|^2+|\partial_t\tilde{y}|^2\bigr)e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt\biggr),
\end{aligned}
\tag{2.52}
$$

where c is independent of s. Using definition (2.20) of the functional $I(\tilde{y},\tilde{\tau},s)$ and transposing the terms containing $|\nabla\tilde{y}(t,x)|$, $|\tilde{y}(t,x)|$, $|\nabla\tilde{\tau}(t,x)|$, and $|\tilde{\tau}(t,x)|$ from the integral over Q on the right-hand side to the left-hand side, we obtain (increasing $s_0(\lambda)$ where necessary)

$$
\begin{aligned}
I(\tilde{y},\tilde{\tau},s) \leqslant C_1\biggl(&\int_Q \bigl(|f(t,x)|^2+|\operatorname{div} f(t,x)|+|g(t,x)|^2\bigr)e^{-s\alpha_\lambda(t,x)}\,dx\,dt\\
&+\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)}+\|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\\
&+\int_{Q^\omega}\bigl(|\tilde{y}|^2+|\tilde{\tau}|^2+|\partial_t\tilde{y}|^2\bigr)e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt\biggr).
\end{aligned}
$$

Next, in the integral over Q^ω we replace the vector field $\partial_t\tilde{y}$ by y and estimate the terms containing $|\tilde{y}|$ and $|\tilde{\tau}|$ with the help of an obvious analogue of inequality (2.48). As a result, we arrive at (2.44).

Let us now eliminate the terms containing $y(T/2,\,\cdot\,)$ and $\tau(T/2,\,\cdot\,)$ on the right-hand side in (2.41).

Theorem 2.4. *Let the hypotheses of Theorem* 2.3 *be valid, and let* $\widehat{\lambda}$ *and* $s_0(\lambda)$ *be defined in Theorem* 2.3. *Then the solution of problem* (2.9)–(2.11) *satisfies the estimate*

$$
\begin{aligned}
&\int_Q \frac{1}{s\varphi}\bigl(|y(t,x)|^2+|\tau(t,x)|^2\bigr)e^{-s\alpha_\lambda(t,x)}\,dx\,dt\\
&\quad+\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)}+\|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\\
&\leqslant c\biggl(\int_Q (|f(t,x)|^2+|\operatorname{div} f(t,x)|^2+|g(t,x)|^2)e^{-s\alpha_\lambda(t,x)}\,dx\,dt\\
&\quad+\int_{Q^\omega}\bigl(|\tau(t,x)|^2+|f(t,x)|^2+|y(t,x)|^2\bigr)e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt\biggr)\quad \forall\, s\geqslant s_0(\lambda),
\end{aligned}
\tag{2.53}
$$

where c is independent of f and g.

Proof. Let $\rho(t)\in C^\infty(T/2,3T/4)$, $\rho(T/2)=1$, $\rho(3T/4)=0$, and let (y,p,τ) be a solution of system (2.9), (2.10). Then, obviously, for $(t,x)\in(T/2,3T/4)\times\Pi$ the triple $(\rho y,\rho p,\rho\tau)$ satisfies system (2.9), (2.10) with right-hand sides f,g replaced by $\rho f + y\partial_t\rho$ and $\rho g + \tau\partial_t\rho$, respectively. Moreover, for $t = 3T/4$ the functions ρy and $\rho\tau$ satisfy zero initial conditions. Hence, by Lemma 2.4 and the trace theorem,

$$\begin{aligned}
&\|y(T/2,\,\cdot\,)\|^2_{V^1(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^1(\Pi)} \\
&\quad\leqslant c\Big(\|f\|^2_{(L_2((\frac{T}{2},\frac{3T}{4})\times\Pi))^n} + \|g\|^2_{L_2((\frac{T}{2},\frac{3T}{4})\times\Pi)} \\
&\qquad + \|y\|^2_{(L_2((\frac{T}{2},\frac{3T}{4})\times\Pi))^n} + \|\tau\|^2_{L_2((\frac{T}{2},\frac{3T}{4})\times\Pi)}\Big) \\
&\quad\leqslant c_s\bigg(\int_{T/2}^{3T/4}\int_\Pi\bigg[(|f|^2+|g|^2) + \frac{1}{s\varphi}(|y|^2+|\tau|^2)\bigg]e^{-s\alpha_\lambda(t,x)}\,dx\,dt\bigg), \qquad (2.54)
\end{aligned}$$

where the last inequality is valid since for $t \in (T/2, 3T/4)$ the functions $s\varphi$ and $e^{-s\alpha_\lambda(t,x)}$ are bounded above and below by positive constants that depend on s and λ but are independent of t and x. Let us multiply both sides of (2.54) by $(2c_s)^{-1}$ and add the result to (2.41). By transposing the term

$$\frac{1}{2}\int_{T/2}^{3T/4}\int_\Pi \frac{1}{s\varphi}(|y|^2+|\tau|^2)e^{-\alpha_\lambda(t,x)}\,dx\,dt$$

in the resulting inequality from the right- to the left-hand side, we obtain the estimate

$$\begin{aligned}
&\int_Q \frac{1}{\varphi}(|y|^2+|\tau|^2)e^{-s\alpha_\lambda(t,x)}\,dx\,dt + \|y(T/2,\,\cdot\,)\|^2_{V^1(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^1(\Pi)} \\
&\quad\leqslant c_1\bigg(\int_Q (|f|^2 + |\operatorname{div} f|^2 + |g|^2)e^{-s\alpha_\lambda(t,x)}\,dx\,dt \\
&\qquad + \int_{Q^\omega}(|f|^2+|y|^2+|\tau|^2)e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt \\
&\qquad + \|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\bigg). \qquad (2.55)
\end{aligned}$$

Let us show that (2.55) implies inequality (2.53). Suppose the contrary. Then there is a sequence of quintuples $(y_k, \nabla p_k, \tau_k, f_k, g_k)$ with the following properties. For each k they satisfy system (2.9), (2.10) and, by virtue of the preceding, the estimate (2.55). Moreover, substitution of (y_k, τ_k, f_k, g_k) in (2.53) gives 1 on the left-hand side of the inequality, while the right-hand side tends to zero as $k \to \infty$.

Proceeding to a subsequence if necessary, we can assume that

$$(y_k, \tau_k) \to (\widehat{y}, \widehat{\tau}) \text{ weakly in the space}^2\ \big(L_2(Q, e^{-s\alpha_\lambda}/(s\varphi))\big)^{n+1}, \qquad (2.56)$$

[2] Here $L_2(e^{-s\alpha_\lambda}/(s\varphi), Q)$ is a weighted L_2 space, that is, the space with the norm

$$\|z\|^2_{L_2(e^{-s\alpha_\lambda}/(s\varphi),Q)} = \int_Q z^2 e^{-s\alpha_\lambda}/(s\varphi)\,dx\,dt.$$

and

$$(f_k, g_k) \to (\widehat{f}, \widehat{g}) \equiv (0,0) \text{ in } \big(L_2(Q, e^{-s\alpha_\lambda})\big)^{n+1}, \tag{2.57}$$

$$\int_{Q^\omega} (|y_k|^2 + |\tau_k|^2 + |\widehat{f}_k|^2) e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt$$
$$\to 0 \equiv \int_{Q^\omega} (|\widehat{y}|^2 + |\widehat{\tau}|^2 + |f|^2) e^{-9s\widehat{\alpha}_\lambda(t)/10}\,dx\,dt, \tag{2.58}$$

$$\big(y_k(T/2,\,\cdot\,), \tau_k(T/2,\,\cdot\,)\big) \to \big(\widehat{y}(T/2,\,\cdot\,), \widehat{\tau}(T/2,\,\cdot\,)\big) \text{ weakly in } V^1(\Pi) \times H^1(\Pi), \tag{2.59}$$

and consequently, since the embedding $V^1(\Pi) \times H^1(\Pi) \Subset V^0(\Pi) \times H^0(\Pi)$ is compact,

$$\big(y_k(T/2,\,\cdot\,), \tau_k(T/2,\,\cdot\,)\big) \to \big(\widehat{y}(T/2,\,\cdot\,), \widehat{\tau}(T/2,\,\cdot\,)\big) \text{ in } V^0(\Pi) \times H^0(\Pi). \tag{2.60}$$

Since the left-hand sides of inequalities (2.53) and (2.55) coincide, we substitute (y_k, τ_k, f_k, g_k) into (2.55) and pass to the limit as $k \to \infty$, thus obtaining the inequality

$$1 \leqslant c_1\big(\|\widehat{y}(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\widehat{\tau}(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\big). \tag{2.61}$$

Obviously, there is a $\widehat{p}$ such that the triple $(\widehat{y}, \nabla\widehat{p}, \widehat{\tau})$ is a weak solution of system (2.9), (2.10) with $f = 0$ and $g = 0$. By the theorem on smoothness of weak solutions of system (2.9), (2.10) (which can be proved by methods close to those used in the proof of Lemma 2.4),

$$\widehat{y} \in V^{1,2}((0, T-\varepsilon) \times \Pi), \qquad \widehat{\tau} \in H^{1,2}((0, T-\varepsilon) \times \Pi) \tag{2.62}$$

for every $\varepsilon > 0$.

On the other hand, $(\widehat{y}, \widehat{\tau}) \equiv 0$ in Q^ω by (2.58), and hence it follows from Corollary 2.1 that $(\widehat{y}, \widehat{\tau})\big|_Q \equiv 0$, which contradicts (2.61).

Now we introduce the following functions instead of α_λ and $\widehat{\alpha}_\lambda$:

$$\eta \equiv \eta(t,x) \equiv s_0(\widehat{\lambda}) \frac{e^{\widehat{\lambda}^2\|\psi\|_{C(\Pi)}} - e^{\widehat{\lambda}\psi(x)}}{T-t}, \qquad \widehat{\eta}(t) = \tfrac{9}{10} s_0(\widehat{\lambda}) \frac{\widehat{\alpha}(t)\gamma(t)}{T-t}, \tag{2.63}$$

where ψ is the function in Lemma 2.1, $\widehat{\alpha}(t)$ is defined in (2.29), and $s_0(\widehat{\lambda})$ and $\widehat{\lambda}$ are the numbers in Theorem 2.4. Let us prove an analogue of the estimate (2.53) in which the function $\alpha_{\widehat{\lambda}}$ is replaced by η.

Theorem 2.5. *Let the hypotheses of Theorem* 2.3 *be satisfied, and let the functions* η *and* $\widehat{\eta}$ *be defined in* (2.63). *Then the solution of problem* (2.9)–(2.11) *satisfies the estimate*

$$J(y,\tau) \equiv \int_Q (T-t)(|y(t,x)|^2 + |\tau(t,x)|^2) e^{-\eta(t,x)}\,dx\,dt$$
$$+ \|y(0,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(0,\,\cdot\,)\|^2_{H^0(\Pi)}$$
$$\leqslant c\bigg(\int_Q (|f(t,x)|^2 + |\operatorname{div} f(t,x)|^2 + |g(t,x)|^2) e^{-\eta(t,x)}\,dx\,dt$$
$$+ \int_{Q^\omega} (|f(t,x)|^2 + |\tau(t,x)|^2 + |y(t,x)|^2) e^{-9\widehat{\eta}(t)/10}\,dx\,dt\bigg), \tag{2.64}$$

where c *is independent of* f *and* g.

Proof. By definitions (2.3), (2.4), and (2.63) of the functions $\gamma(t)$, $\alpha_\lambda(t,x)$, and $\eta(t,x)$, we have

$$s_0(\widehat{\lambda})\alpha_{\widehat{\lambda}}(t,x) \equiv \eta(t,x) \quad \text{for} \quad t \in (3T/4, T).$$

This equation, together with the boundedness above and below of the functions $\eta(t,x)$ and $\alpha_\lambda(t,x)$ for $t \in (T/2, T-T_0)$, implies the estimate

$$\int_{T/2}^{T}\int_\Pi (T-t)(|y|^2+|\tau|^2)e^{-\eta(t,x)}\,dx\,dt \leqslant c\int_{T/2}^{T}\int_\Pi (T-t)(|y|^2+|\tau|^2)e^{-s_0(\widehat{\lambda})\alpha_{\widehat{\lambda}}(t,x)}\,dx\,dt. \tag{2.65}$$

Since the functions $(T-t)$ and $\eta(t,x)$ are bounded above and below on the set $(0,T/2)\times\Pi$, we can apply the energy inequality (2.13) and obtain

$$\begin{aligned}
&\int_0^{T/2}\int_\Pi (|y|^2+|\tau|^2)e^{-\eta(t,x)}\,dx\,dt + \|y(0,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(0,\,\cdot\,)\|^2_{H^0(\Pi)} \\
&\quad\leqslant c \sup_{0\leqslant t\leqslant T/2}\big(\|y(t,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(t,\,\cdot\,)\|^2_{H^0(\Pi)}\big) \\
&\quad\leqslant c\big(\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)} \\
&\qquad + \|f\|^2_{L_2(0,T/2;V^0(\Pi))} + \|g\|^2_{L_2(0,T/2;H^0(\Pi))}\big) \\
&\quad\leqslant c_1\bigg(\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)} \\
&\qquad + \int_0^{T/2}\int_\Pi (|f|^2+|g|^2)e^{-\eta(t,x)}\,dx\,dt\bigg).
\end{aligned} \tag{2.66}$$

By summing inequalities (2.65) and (2.66) and by applying the estimate (2.53) to the right-hand side of the resulting inequality, we obtain the following inequalities in view of the fact that the functions $(T-t)$ and $\eta(t,x)$ are bounded above and below for $(t,x)\in(0,T/2)\times\Pi$:

$$\begin{aligned}
&\int_Q (|y|^2+|\tau|^2)e^{-\eta(t,x)}\,dx\,dt + \|y(0,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(0,\,\cdot\,)\|^2_{H^0(\Pi)} \\
&\quad\leqslant c\bigg(\int_Q \frac{1}{s_0\varphi}(|y|^2+|\tau|^2)e^{-s_0(\widehat{\lambda})\alpha_{\widehat{\lambda}}}\,dx\,dt \\
&\qquad + \big(\|y(T/2,\,\cdot\,)\|^2_{V^0(\Pi)} + \|\tau(T/2,\,\cdot\,)\|^2_{H^0(\Pi)}\big) + \int_Q (|f|^2+|g|^2)e^{-\eta}\,dx\,dt\bigg) \\
&\quad\leqslant c\bigg(\int_Q (|f|^2+|\operatorname{div} f|^2+|g|^2)e^{-\eta}\,dx\,dt + \int_{Q^\omega}(|f|^2+|g|^2+|y|^2)e^{-\widehat{\eta}}\,dx\,dt\bigg).
\end{aligned}$$

The proof of the theorem is complete.

§ 3. Solubility of the exact controllability problem for the linearized Boussinesq system

3.1. Statement of the problem. We linearize the Boussinesq system (1.26), (1.27) at a point $\widehat{v}$, $\widehat{p}$, $\widehat{\theta}$:

$$N'(\widehat{v},\widehat{\theta})(y,\tau) \equiv \partial_t y - \Delta y + (\widehat{v},\nabla)y + (y,\nabla)\widehat{v} + \tau\vec{e}$$
$$= -\nabla p + f + u', \qquad \operatorname{div} y = 0, \tag{3.1}$$
$$R'(\widehat{v},\widehat{\theta})(y,\tau) \equiv \partial_t \tau - \Delta\tau + (\widehat{v},\nabla\tau) + (y,\nabla\widehat{\theta}) = g + u_{n+1}. \tag{3.2}$$

System (3.1), (3.2), like (1.26), (1.27), is considered under periodic boundary conditions; that is, $y = y(t,x)$ and $\tau = \tau(t,x)$, where $(t,x) \in Q \equiv (0,T)\times\Pi$, and Π is the n-dimensional torus ($n = 2,3$). We supplement (3.1), (3.2) with the initial conditions

$$y(t,x)\big|_{t=0} = y_0(x), \qquad \tau(t,x)\big|_{t=0} = \tau_0(x). \tag{3.3}$$

Lemma 3.1. *Let* $(\widehat{v},\widehat{\theta}) \in L_\infty(0,T;V^1_\infty(\Pi))\times L_\infty(0,T;W^1_\infty(\Pi))$. *Then for any* $y_0 \in V^1(\Pi)$, $\tau_0 \in H^1(\Pi)$, $f \in (L^2(Q))^n$, *and* $g \in L^2(Q)$ *and any function* $u \in U(\omega) \subset (L^2(Q))^{n+1}$ *there is a unique solution* $(y,\tau,\nabla p) \in V^{1,2}(Q)\times H^{1,2}(Q)\times (L^2(Q))^n$ *of problem* (3.1)–(3.3), *and this solution satisfies the estimate*

$$\|y\|^2_{V^{1,2}(Q)} + \|\tau\|^2_{H^{1,2}(Q)} + \|\nabla p\|^2_{(L_2(Q))^n}$$
$$\leqslant c\big(\|y_0\|^2_{V^1(\Pi)} + \|\tau_0\|^2_{H^1(\Pi)} + \|f\|^2_{(L_2(Q))^n} + \|g\|^2_{L_2(Q)} + \|u\|^2_{(L_2(Q))^{(n+1)}}\big). \tag{3.4}$$

This lemma can be proved in the same way as Lemma 2.4.

We pose a controllability problem for system (3.1), (3.2) by supplementing (3.3) with the following conditions at time T:

$$y(T,x) \equiv 0, \qquad \tau(T,x) \equiv 0. \tag{3.5}$$

Here the unknown functions include not only $(y,\tau,\nabla p)$, but also the control $u = (u', u_{n+1})$.

To state the controllability problem precisely, we introduce appropriate function spaces for the initial data and solutions of problem (3.1)–(3.3), (3.5).

Just as before, let $\omega \subset \Pi$ be a subdomain of the torus Π, and let

$$\chi_\omega(x) = \begin{cases} 1, & x \in \overline{\omega}, \\ 0, & x \in \omega, \end{cases}$$

be the characteristic function of ω. We introduce the weight functions

$$\rho_1(t,x) = e^{\widehat{\eta}(t)}\chi_\omega(x) + \frac{e^{\eta(t,x)}}{T-t}\big(1 - \chi_\omega(x)\big), \tag{3.6}$$
$$\rho(t) = e^{\widehat{\eta}(t)}, \tag{3.7}$$

where $\eta(t,x)$ and $\widehat{\eta}(t)$ are the functions defined in (2.63).

We set

$$L_2(\rho, Q) = \Big\{ y(t,x) \in L_2(Q) : \|y\|^2_{L_2(\rho,Q)} = \int_Q \rho y^2\, dx\, dt < \infty \Big\}. \tag{3.8}$$

The space $L_2(\rho_1, Q)$ is defined in a similar way.

Let us introduce the space of right-hand sides for equation (3.1):

$$\begin{aligned} F(Q) = \Big\{ & f \in L_2(0,T;(L_2(\Pi))^n) : \exists\, f_1 \in (L_2(\rho_1,Q))^n \\ & \text{and } \exists\, f_2 \in L_2(0,T;H^1(\Pi)) \text{ such that } f = f_1 + \nabla f_2; \\ & \|f\|_{F(Q)} = \inf_{f=f_1+\nabla f_2} \big(\|f_1\|^2_{(L_2(\rho_1,Q))^n} + \|f_2\|^2_{L_2(0,T;H^1(\Pi))} \big)^{1/2} \Big\}. \end{aligned} \tag{3.9}$$

We define the space of solutions of system (3.1), (3.2) as follows:

$$\begin{aligned} \Theta = \big\{ & (y,\tau) \in V^{1,2}(Q) \times H^{1,2}(Q) : L(y,\tau) \in F(Q) \times L_2(\rho_1, Q), \\ & y e^{9\widehat{\eta}/21} \in V^{1,2}(Q),\ \tau e^{9\widehat{\eta}/21} \in H^{1,2}(Q) \big\}, \end{aligned} \tag{3.10}$$

where $L(y,\tau) = (N'(\widehat{v},\widehat{\theta})(y,\tau), R'(\widehat{v},\widehat{\theta})(y,\tau))$, and N' and R' are the operators defined in (3.1), (3.2). We define a norm on the space Θ by setting

$$\|(y,\tau)\|^2_\Theta = \|L(y,\tau)\|^2_{F(Q)\times L_2(\rho_1,Q)} + \|y e^{\frac{9}{21}\widehat{\eta}}\|^2_{V^{1,2}(Q)} + \|\tau e^{\frac{9}{21}\widehat{\eta}}\|^2_{H^{1,2}(Q)}. \tag{3.11}$$

The control space is defined to be

$$U(\rho,\omega) = \big\{ u = (u', u_{n+1}) \in (L_2(\rho,Q))^{n+1},\ \operatorname{supp} u \subset Q^\omega = (0,T)\times\omega \big\}. \tag{3.12}$$

The following theorem is the main assertion in this section.

Theorem 3.1. *Let $(\widehat{v},\widehat{\theta}) \in W^1_\infty(0,T;V^2_\infty(\Pi)) \times W^1_\infty(0,T;W^2_\infty(\Pi))$. Then for any $y_0 \in V^1(\Pi)$, $\tau_0 \in H^1(\Pi)$, $f \in F(Q)$, and $g \in L_2(\rho_1,Q)$ there is a solution $(y,p,\tau,u) \in V^{1,2}(Q) \times L_2(0,T;H^1(\Pi)) \times H^{1,2}(Q) \times U(\rho,\omega)$ of problem* (3.1)–(3.3), (3.5).

The rest of the section deals with the proof of this theorem.

3.2. An auxiliary extremal problem. To prove theorem 3.1, we apply a version of the penalty method. Let us consider the extremal problem

$$J_k(y,\tau,u) = \frac{1}{2}\int_Q \rho(t)(|y|^2 + \tau^2)\, dx\, dt + \frac{1}{2}\int_Q m_k(t,x)|u|^2\, dx\, dt \to \inf, \tag{3.13}$$

$$L(y,\tau) = (-\nabla p + f + u', g + u_{n+1}), \qquad \operatorname{div} y = 0, \tag{3.14}$$

$$y(t,x)\big|_{t=0} = y_0(x), \qquad \tau(t,x)\big|_{t=0} = \tau_0(x), \tag{3.15}$$

where $L(y,\tau) = (N'(\widehat{v},\widehat{\theta})(y,\tau), R'(\widehat{v},\widehat{\theta})(y,\tau))$, and the operators N' and R' are defined in (3.1) and (3.2), respectively. Furthermore, the function $\rho(t)$ is defined in (3.7), and $m_k(t,x)$ is given by the formula

$$m_k(t,x) = \chi_\omega(x) e^{\frac{T-t}{T-t+1/k}\widehat{\eta}} + \big(1 - \chi_\omega(x)\big)k, \tag{3.16}$$

where $\chi_\omega(x)$ is the characteristic function of ω, $\widehat{\eta}(t)$ is the function in (2.63), and k is a positive integer.

Lemma 3.2. *$f \in (L_2(\rho_1, Q))^n$, $g \in L_2(\rho_1, Q)$, $y_0 \in V^1(\Pi)$, and $\tau_0 \in H^1(\Pi)$. For each positive integer k there is a unique solution $(y, \tau, \nabla p, u) \in V^{1,2}(Q) \times H^{1,2}(Q) \times L_2(0, T; (L_2(\Pi))^n) \times (L_2(Q))^{n+1}$ of problem* (3.13)–(3.15). *Moreover, $(y, \tau) \in (L_2(\rho, Q))^n \times L_2(\rho, Q)$.*

Proof. We recall that an *admissible element* of problem (3.13)–(3.15) is a quadruple $(y, \tau, \nabla p, u) \in V^{1,2}(Q) \times H^{1,2}(Q) \times (L_2(Q))^n \times (L_2(Q))^{n+1}$ that satisfies relations (3.14) and (3.15) and on which the functional in (3.13) is finite. The set of admissible elements will be denoted by $\mathcal{A}$. Let $(y, \tau, \nabla p) \in V^{1,2}(Q) \times H^{1,2}(Q) \times (L_2(Q))^n$ be a solution of problem (3.1)–(3.3) with $u \equiv 0$, and let $\mu(t) \in C^\infty[0, T]$ be a function equal to 1 in a neighbourhood of zero and zero in a neighbourhood of T. Let $u(t, x) = \left(\frac{d\mu}{dt} y - f + \mu f, \frac{d\mu}{dt} \tau - g + \mu g\right)$. Then the quadruple $(\mu(t)y, \mu(t)\tau, \mu(t)p, u)$ belongs to $\mathcal{A}$.

Since $\mathcal{A} \neq \varnothing$, it follows that there is a minimizing sequence $(y_m, \tau_m, \nabla p_m, u_m) \in \mathcal{A}$ for the functional J_k:

$$J_k(y_m, \tau_m, \nabla p_m, u_m) \to \inf_{(y, \tau, \nabla p, u) \in \mathcal{A}} J_k(y, \tau, \nabla p, u). \tag{3.17}$$

It follows from (3.17) and the estimate (3.4) that

$$\|y_m\|_{(V^{1,2}(Q) \cap (L_2(\rho, Q))^n)} + \|\tau_m\|_{H^{1,2}(Q) \cap L_2(\rho, Q)} + \|\nabla p_m\|_{(L_2(Q))^n} + \|u_m\|_{(L_2(Q))^{n+1}} \leqslant c,$$

where c is independent of m. Hence the minimizing sequence contains a subsequence weakly convergent in $(V^{1,2}(Q) \cap (L_2(\rho, Q))^n) \times (H^{1,2}(Q) \cap L_2(\rho, Q)) \times (L_2(Q))^n \times (L_2(Q))^{n+1}$ to some quadruple $(\widehat{y}, \widehat{\tau}, \nabla \widehat{p}, \widehat{u})$ of functions. We can readily see that this quadruple is a solution of problem (3.13)–(3.15). The uniqueness of the solution follows by a standard argument from the strict convexity of the functional (3.13).

Let us derive an optimality system for problem (3.13)–(3.15).

Lemma 3.3. *Let the hypotheses of Lemma* 3.2 *be satisfied, and let*

$$(y_k, \tau_k, \nabla p_k, u_k) \in (V^{1,2}(Q) \cap (L_2(\rho, Q))^n) \times (H^{1,2}(Q) \cap L_2(\rho, Q)) \times (L_2(Q))^n \times (L_2(Q))^{n+1}$$

be a solution of problem (3.13)–(3.15). *Then there is a triple*

$$(z_k, r_k, \nabla q_k) \in L_2(0, T; V^0(\Pi)) \times L_2(Q) \times (L_2(Q))^n$$

such that

$$L(y_k, \tau_k) = (-\nabla p_k + f + u_k', g + u_{k,n+1}), \ y_k(0, x) = y_0(x), \ \tau_k(0, x) = \tau_0(x), \tag{3.18}$$

$$L^*(z_k, r_k) = (\nabla q_k - \rho y_k, -\rho \tau_k) \ \textit{in } Q, \tag{3.19}$$

$$(z_k, r_k) - m_k u_k = 0 \ \textit{in } Q, \tag{3.20}$$

where

$$L^*(z_k, r_k) = \big(N^*(z_k, r_k), R^*(z_k, r_k)\big),$$

and the operators N^* *and* R^* *are defined in* (2.9) *and* (2.10). *Furthermore, the pair* (z_k, r_k) *satisfies the estimate*

$$\int_Q (T-t)(|z_k|^2+|r_k|^2)e^{-\eta}\,dx\,dt+\|z_k(0,\,\cdot\,)\|^2_{V^0(\Pi)}+\|r_k(0,\,\cdot\,)\|^2_{L_2(\Pi)}$$

$$\leqslant C_2\int_Q \rho^2(|y_k|^2+|\tau_k|^2)e^{-\eta}\,dx\,dt+\int_{Q^\omega}(m_k^2|u_k|^2+\rho^2|y_k|^2)e^{-\widehat{\eta}}\,dx\,dt, \tag{3.21}$$

where C_2 *is independent of* (y_k, τ_k, u_k).

Proof. Relations (3.18) hold by virtue of (3.14) and (3.15).

To derive (3.19) and (3.20), we use Lagrange's principle for an abstract smooth problem of the form

$$J(x)\to\inf,\qquad Fx=0, \tag{3.22}$$

where $J\colon X\to\mathbb{R}^1$ is a continuously differentiable functional and $F\colon X\to Z$ is a continuous linear operator between the Hilbert spaces X and Z. Alekseev, Tikhomirov, and Fomin [2] and Fursikov [30] showed that if $\widehat{x}$ is a local extremum of problem (3.22) and the operator F is a map onto Z, then there is a $z\in Z^*=Z$ such that the Lagrange functional $\mathcal{L}(x,z)=J(x)+(Fx,z)_Z$ satisfies the relation

$$\mathcal{L}'_x(\widehat{x},z)[h]=0\quad\forall\,h\in X. \tag{3.23}$$

In problem (3.13)–(3.15) we have $x=(y,\tau,\nabla p,u)$ and

$$X=(V^{1,2}(Q)\cap(L_2(\rho,Q))^n)\times(H^{1,2}(Q)\cap L_2(\rho,Q))\\ \times L_2(0,T;\nabla H^1(\Pi))\times(L_2(Q))^{n+1},$$

where $\nabla H^1(\Pi)=\{\nabla p(x),\,p\in H^1(\Pi)\}$, $J(x)=J_k(y,\tau,u)$ is the functional in (3.13), $Z=(L_2(Q))^{n+1}\times V^1(\Pi)\times H^1(\Pi)$, and

$$F(x)=(L(y,\tau)-(\nabla p,0)-u,\,y|_{t=0},\,\tau|_{t=0}),$$

where $L(y,\tau)$ is the operator (3.14). To prove that the operator $F\colon X\to Z$ is onto, we establish the solubility of the operator

$$L(y,\tau)-(\nabla p,0)-u=(f,g),\qquad y\big|_{t=0}=\tilde{y}_0,\quad \tau\big|_{t=0}=\tilde{\tau}_0 \tag{3.24}$$

for any $(f,g,\tilde{y}_0,\tilde{\tau}_0)\in(L_2(\rho_1,Q))^{n+1}\times V^1(\Pi)\times H^1(\Pi)$. Indeed, by substituting an arbitrary pair $(y,\tau)\in(V^{1,2}(Q)\cap(L_2(\rho,Q))^n)\times(H^{1,2}(Q)\cap L_2(\rho,Q))$, satisfying (3.24_2) and (3.24_3) into (3.24_1), we can find u from the resulting equation by setting $p=0$. Obviously, $u=(u',u_{n+1})\in(L_2(Q))^{n+1}$. Thus, the Lagrange principle applies to problem (3.13)–(3.15). The Lagrange function in this case has the form

$$\mathcal{L}(y,\tau,\nabla p,u,z,r,\phi_1,\phi_2)=J_k(y,\tau,u)\\ +\int_Q\big[N'(\widehat{v},\widehat{\theta})(y,\tau)\cdot z+R'(\widehat{v},\widehat{\theta})(y,\tau)r-\nabla p\cdot z-u'\cdot z-u_{n+1}r\big]\,dx\,dt\\ +\big(y(0,\,\cdot\,)-\tilde{y}_0,\phi_1\big)_{V^0(\Pi)}+\big(\tau(0,\,\cdot\,)-\tilde{\tau}_0,\phi_2\big)_{H^0(\Pi)}, \tag{3.25}$$

where $(z, r, \phi_1, \phi_2) \in (L_2(Q))^{n+1} \times V^{-1}(\Pi) \times H^{-1}(\Pi)$ is an element of the Hilbert space Z^* and $(N', R') = L$ are the operators (3.1) and (3.2). By applying (3.23) with differentiation with respect to the variable $x = (y, \tau, 0, 0, 0)$ to the Lagrange function (3.25), we obtain (3.19); the application of (3.23) with differentiation with respect to the variable $x = (0, 0, 0, u', u_{n+1})$ yields (3.20). By differentiating the Lagrange function with respect to ∇p, we obtain $\operatorname{div} z_k = 0$. Let us apply inequality (2.64) with $y = z_k$, $\tau = r_k$, $\nabla p = \nabla q_k$, $f = -\rho y_k$, and $g = -\rho \tau_k$ to relation (3.19) and then use (3.20) in the integral over Q^ω on the right-hand side. This gives us inequality (3.21).

3.3. Proof of the main result. In what follows we prove Theorem 3.1 by passing to the limit as $k \to \infty$ in problem (3.13)–(3.15). To this end, we first estimate $J_k(y_k, \tau_k, u_k)$.

Lemma 3.4. *Let* $f \in (L_2(\rho_1, Q))^n$, $f\big|_{Q^\omega} \equiv 0$, $g \in L_2(\rho_1, Q)$, $g\big|_{Q^\omega} \equiv 0$, $y_0 \in V^1(\Pi)$, *and* $\tau_0 \in H^1(\Pi)$, *and let* $(y_k, \tau_k, \nabla p_k, u_k)$ *be the solution of problem* (3.13)–(3.15) *constructed in Lemma* 3.2. *Then there is a constant* $c > 0$ *independent of* k *such that*

$$J_k(y_k, \tau_k, u_k) \leqslant c\big(\|f\|^2_{(L_2(\rho_1,Q))^n} + \|g\|^2_{L_2(\rho_1,Q)} + \|y_0\|^2_{V^0(\Pi)} + \|\tau_0\|^2_{H^0(\Pi)}\big). \quad (3.26)$$

Proof. Let (z_k, r_k) be the functions constructed in Lemma 3.3 and satisfying the estimate (3.21). Using the definitions (3.16) and (3.7) of the functions $m_k(t, x)$ and $\rho(t)$, we obtain

$$|\chi_\omega(x) m_k(t, x) e^{-\widehat{\eta}}| \leqslant c, \qquad \rho^2(t) e^{-\widehat{\eta}(t)} = \rho(t), \quad \rho^2(t) e^{-\eta(t,x)} \leqslant c\rho(t),$$

where c is a constant independent of k.

By applying these relations to the right-hand side of inequality (3.21) and by taking account of definition (3.13) of the functional J_k, we can readily derive the following estimate from (3.21):

$$\int_Q (T - t)(|z_k|^2 + |r_k|^2) e^{-\eta} \, dx \, dt + \|z_k(0, \cdot\,)\|^2_{V^0(\Pi)} + \|r_k(0, \cdot\,)\|^2_{L_2(\Pi)} \leqslant c_1 J_k(y_k, \tau_k, u_k), \quad (3.27)$$

where the constant c_1 is independent of k. Let us take the inner product of (3.19) by (y_k, τ_k) in $(L_2(Q))^n \times L_2(Q)$, integrate by parts with respect to x and t, and use (3.18). Then we obtain

$$-\int_Q \rho(|y_k|^2 + |\tau_k|^2) \, dx \, dt = (z_k(0, \cdot\,), y_0)_{V^0(\Pi)} + (r_k(0, \cdot\,), \tau_0)_{L^2(\Pi)} + \int_Q ((f + u'_k, z_k) + (g + u_{n+1,k}) r_k) \, dx \, dt.$$

This equation, by virtue of (3.20) and definition (3.13) of the functional J_k, implies that

$$
\begin{aligned}
&2J_k(y_k,\tau_k,u_k)\\
&\quad= -(z_k(0,\cdot),y_0)_{V^0(\Pi)} - (r_k(0,\cdot),\tau_0)_{L^2(\Pi)} - \int_Q ((f,z_k)+gr_k)\,dx\,dt\\
&\quad\leqslant \left(\|z_k(0,\cdot)\|^2_{V^0(\Pi)} + \|r_k(0,\cdot)\|^2_{L_2(\Pi)} + \int_Q (T-t)(|z_k|^2+|r_k|^2)e^{-\eta}\,dx\,dt\right)^{1/2}\\
&\qquad\times \big(\|y_0\|^2_{V^0(\Pi)} + \|\tau_0\|^2_{L_2(\Pi)} + \|f\|^2_{(L_2(\rho_1,Q))^n} + \|g\|^2_{L_2(\rho_1,Q)}\big)^{1/2},
\end{aligned}
$$

where ρ_1 is the weight from (3.6) (here we have used our assumption that $f\big|_{Q^\omega}=0$ and $g\big|_{Q^\omega}=0$). By applying the estimate (3.27) to the right-hand side of the last inequality, we obtain

$$
\begin{aligned}
2J_k(y_k,\tau_k,u_k) \leqslant c_1\big(&\|f\|^2_{(L_2(\rho_1,Q))^n} + \|g\|^2_{L_2(\rho_1,Q)}\\
&+ \|y_0\|^2_{V^0(\Pi)} + \|\tau_0\|^2_{L_2(\Pi)}\big)^{1/2} J_k^{1/2}(y_k,\tau_k,u_k),
\end{aligned}
$$

whence (3.26) follows.

Proof of Theorem 3.1. First, we assume that $f\big|_{Q^\omega}\equiv 0$ and $g\big|_{Q^\omega}\equiv 0$. Let us consider a sequence $(y_k,\tau_k,\nabla p_k,u_k)$ of solutions of problem (3.13)–(3.15) as the parameter k goes to infinity. It follows from (3.26), (3.13), and (3.16) that

$$
\|y_k\|^2_{(L_2(\rho,Q))^n} + \|\tau_k\|^2_{L_2(\rho,Q)} + k\int_{Q\setminus Q^\omega} |u_k|^2\,dx\,dt + \int_{Q^\omega} e^{\widehat{\eta}}|u_k|^2\,dx\,dt \leqslant c, \tag{3.28}
$$

where c is independent of k. By virtue of (3.28) and the estimate (3.4), we have

$$
\|y_k\|^2_{V^{1,2}(\Pi)} + \|\tau_k\|^2_{H^{1,2}(\Pi)} + \|\nabla p_k\|^2_{L_2(Q)} \leqslant c_1, \tag{3.29}
$$

where c_1 is also independent of k. By virtue of (3.28) and (3.29), the sequence of solutions contains a subsequence (which will also be denoted by $\{(y_k,\tau_k,\nabla p_k,u_k)\}$) such that

$$
\begin{gathered}
y_k\to\widehat{y} \text{ weakly in } V^{1,2}(Q)\cap(L_2(\rho,Q))^n, \quad \tau_k\to\widehat{\tau} \text{ weakly in } H^{1,2}(Q)\cap L_2(\rho,Q),\\
\nabla p_k\to\nabla\widehat{p} \text{ weakly in } (L_2(Q))^n, \quad u_k\to\widehat{u} \text{ weakly in } (L_2(Q))^{n+1}.
\end{gathered} \tag{3.30}
$$

Moreover, by (3.28),

$$
\begin{gathered}
\big(1-\chi_\omega(x)\big)u_k\to 0 \ \text{ in } \ (L_2(Q))^n,\\
\varlimsup_{k\to\infty}\int_Q m_k|u_k(t,x)|^2\,dx\,dt \leqslant c,
\end{gathered} \tag{3.31}
$$

where $c>0$ is the same constant as in (3.28), and hence

$$
\int_0^{T-\varepsilon}\int_\omega e^{\widehat{\eta}}|\widehat{u}(t,x)|^2\,dx\,dt \leqslant c \qquad \forall\,\varepsilon>0,
$$

which readily implies that

$$\int_{Q^\omega} |\widehat{u}(t,x)|^2 e^{\widehat{\eta}}\,dx\,dt \leqslant c. \tag{3.32}$$

It follows from (3.31) and (3.32) that $\widehat{u}(t,x) \in U(\rho,\omega)$ (see (3.12)). Using (3.30), we pass to the limit as $k \to \infty$ in (3.18). As a result, we find that the quadruple $(\widehat{y},\widehat{\tau},\nabla\widehat{p},\widehat{u}) \in (V^{1,2}(Q)\cap(L_2(\rho,Q))^n)\times(H^{1,2}(Q)\cap L_2(\rho,Q))\times(L_2(Q))^n\times U(\rho,\omega)$ satisfies relations (3.18) and hence also (3.1)–(3.3). Let $\widehat{\eta}(t)$ be the function in (3.7) defined in (2.63). We make the substitution $\tilde{y} = e^{9\widehat{\eta}/21}\widehat{y}$, $\tilde{\tau} = e^{9\widehat{\eta}/21}\widehat{\tau}$, $\tilde{u} = e^{9\widehat{\eta}/21}\widehat{u}$, $\tilde{p} = e^{9\widehat{\eta}/21}\widehat{p}$ in (3.1)–(3.3) and find that the functions $(\tilde{y},\tilde{\tau},\nabla\tilde{p},\tilde{u})$ satisfy the system of equations

$$N'(\widehat{v},\widehat{\theta})(\tilde{y},\tilde{\tau}) = \nabla\tilde{p} + e^{9\widehat{\eta}/21}f + \tilde{u}' + \frac{9}{21}\,\frac{d\widehat{\eta}(t)}{dt}\,\tilde{y}, \qquad \operatorname{div}\tilde{y} = 0, \tag{3.33}$$

$$R'(\widehat{v},\widehat{\theta})(\tilde{y},\tilde{\tau}) = e^{9\widehat{\eta}/21}g + \tilde{u}_{n+1} + \frac{9}{21}\,\frac{d\widehat{\eta}(t)}{dt}\,\tilde{\tau}. \tag{3.34}$$

We have already proved that the right-hand side of system (3.33), (3.34) belongs to the space $L_2(0,T;V^0(\Pi))\times L_2(0,T;H^0(\Pi))$. Hence $(\tilde{y},\tilde{\tau},\nabla\tilde{p}) \in V^{1,2}(Q)\times H^{1,2}(Q)\times(L_2(Q))^n$ by Lemma 3.2. Thus, we have proved that, by virtue of the properties of the function ρ in (3.6), $(\widehat{y},\widehat{\tau},\nabla\widehat{p},\widehat{u})$ is a solution of the linear exact controllability problem (3.1)–(3.3), (3.5) and belongs to the space $\Theta\times(L_2(Q))^n\times U(\rho,\omega)$. Finally, let (f,g) be an arbitrary function from the space $F(Q)\times L_2(\rho_1,Q)$ (see (3.11)). By the definition of the space $F(Q)$, there are $f_1 \in (L_2(\rho_1,Q))^n$ and $\nabla f_2 \in (L_2(Q))^n$ such that $f = f_1 + \nabla f_2$. We have already proved that for any initial data $((1-\chi_\omega)f_1,(1-\chi_\omega)g,y_0,\tau_0) \in (L_2(\rho_1,Q))^{(n+1)}\times V^1(\Pi)\times H^1(\Pi)$ there is a solution $(y,\tau,\nabla p,u) \in \Theta\times(L_2(Q))^n\times U(\rho,\omega)$ of problem (3.1)–(3.5). One can readily see that the function $(y,\tau,\nabla p - \nabla f_2, u + (\chi_\omega f,\chi_\omega g))$ is a solution of problem (3.1)–(3.3), (3.5) with the initial data (f_1,g,y_0,τ_0).

§ 4. Local exact controllability of the Boussinesq system

In this section we prove Theorem 1.5. We seek the solution of problem (1.26), (1.27), (1.24), (1.25) in the form

$$v(t,x) = \widehat{v}(t,x) + y(t,x), \quad \theta(t,x) = \widehat{\theta}(t,x) + \tau(t,x), \quad \nabla p = \nabla\widehat{p} + \nabla q, \tag{4.1}$$

where $(\widehat{v},\widehat{\theta},\widehat{p})$ is the solution of the Boussinesq system (1.26), (1.27) with $u \equiv 0$ given under the assumptions of Theorem 1.4. By substituting (4.1) in (1.26) and (1.27) and subtracting (1.26) and (1.27) for $(\widehat{v},\nabla\widehat{p},\widehat{\tau})$ from the resulting equations, we arrive at the following system for the new independent functions $(y,\tau,\nabla q,u)$:

$$N(y,\tau,q,u) = \partial_t y - \Delta y + (\widehat{v},\nabla)y + (y,\nabla)\widehat{v} + \tau\vec{e} + \nabla q + u' = 0, \ \operatorname{div} y = 0, \tag{4.2}$$

$$R(y,\tau,u) = \partial_t\tau - \Delta\tau + (\widehat{v},\nabla\tau) + (y,\nabla\widehat{\theta}) + (y,\nabla\tau) + u_{n+1} = 0, \tag{4.3}$$

$$y(0,\,\cdot\,) = v_0 - \widehat{v}(0,\,\cdot\,), \qquad \tau(0,\,\cdot\,) = \theta_0 - \widehat{\theta}(0,\,\cdot\,), \tag{4.4}$$

$$y(T,\,\cdot\,) = 0, \qquad \tau(T,\,\cdot\,) = 0. \tag{4.5}$$

Let us introduce a map $A(y,\tau,q,u)$ by setting

$$A(y,\tau,q,u)=(N(y,\tau,q,u),R(y,\tau,q,u),y(0,\,\cdot\,),\tau(0,\,\cdot\,)). \tag{4.6}$$

To analyze problem (4.2)–(4.6), we use the following version of the implicit function theorem (see [2]).

Theorem 4.1. *Let X and Z be Banach spaces, and let $A\in C^1(X;Z)$ be a continuously differentiable map from X to Z. Assume that*

$$A(x_0)=z_0 \tag{4.7}$$

for some $x_0\in X$ and $z_0\in Z$ and the derivative $A'(x_0)\colon X\to Z$ of A at x_0 is an epimorphism. Then there is an $\varepsilon>0$ such that for each $z\in\{z\in Z:\|z-z_0\|<\varepsilon\}$ there is a solution $x\in X$ of the equation $A(x)=z$.

Proof of Theorem 1.5. The change of variables (4.1) reduces problem (1.24)–(1.27) to problem (4.2)–(4.5). We prove the solubility of the latter problem for small $(v_0-\widehat{v}(0,\,\cdot\,),\theta_0-\widehat{\theta}(0,\,\cdot\,))$ with the help of Theorem 4.1. To this end, we set

$$X=\Theta\times L_2(0,T;\nabla H^1(\Pi))\times U(\rho,\omega), \tag{4.8}$$

$$Z=F(Q)\times L_2(\rho_1,Q)\times V^1(\Pi)\times H^1(\Pi), \tag{4.9}$$

where $L_2(\rho,Q)$, $F(Q)$, Θ, and $U(\rho,\omega)$ are the spaces (3.8), (3.9), (3.10), and (3.12), respectively. We define the map A by formulae (4.6), (4.2), and (4.3) and the spaces X and Z by (4.8) and (4.9). We note that relations (4.5) are then satisfied automatically, since $x=(y,\tau,q,u)\in X$, where X is the space (4.8). We set $x_0=(0,0,0,0)$ and $z_0=(0,0,0,0)$; then condition (4.7) is obviously satisfied. By elementary estimates we establish that the operator $A(x)\colon X\to Z$ is continuously differentiable. The surjectivity of the map $A'(0)\colon X\to Z$ has already been proved in Theorem 3.1. Thus, the operator (4.6), (4.2), (4.3), acting in the spaces (4.8) and (4.9), satisfies all assumptions of Theorem 4.1, and hence the conclusion of this theorem is valid for it. Therefore, we have established the solubility of problem (4.2)–(4.5) for small $(v_0-\widehat{v}(0,\,\cdot\,),\theta_0-\widehat{\theta}(0,\,\cdot\,))$, and the proof of Theorem 1.5 is complete.

§ 5. Approximate controllability of the Boussinesq system: reduction to a linear system of a special form

5.1. The idea of the proof. In this and the next sections, we prove the approximate controllability of the Boussinesq system (1.26), (1.27), that is, the system

$$\partial_t v-\Delta v+(v,\nabla)v+\theta\vec{e}-\nabla p=f+u',\quad \operatorname{div}v=0, \tag{5.1}$$

$$\partial_t\theta-\Delta\theta+(v,\nabla\theta)=g+u_{n+1}, \tag{5.2}$$

where (f,g) are given external forces and the control $u=(u',u_{n+1})$ is supported in $Q^\omega=(0,T)\times\omega$. Just as before, system (5.1), (5.2) is considered in the cylinder $Q=(0,T)\times\Pi$, where Π is the n-dimensional torus, $n=2,3$. We assume that the initial conditions

$$v\big|_{t=0}=v_0,\qquad \theta\big|_{t=0}=\theta_0 \tag{5.3}$$

are given for $t = 0$, where $v_0 \in V^0(\Pi) \cap (C^{2,\alpha}(\Pi))^n$ and $\theta_0 \in C^{2,\alpha}(\Pi)$ are given functions. We recall that the approximate controllability problem is stated as follows: for any $v_1 \in V^0(\Pi) \cap (C^{2,\alpha}(\Pi))^n$, $\theta_1 \in C^{2,\alpha}(\Pi)$ and any $\varepsilon > 0$, construct a control $u = (u', u_{n+1}) \in U(\omega; 0, T)$,[3] such that the restriction of the solution $(v, \nabla p, \theta)$ of problem (5.1)–(5.3) to $t = T$ satisfies the condition

$$\|v(T,\,\cdot\,) - v_1\|^2_{V^1(\Pi)} + \|\theta(T,\,\cdot\,) - \theta_1\|^2_{H^1(\Pi)} < \varepsilon^2. \tag{5.4}$$

Let $m(t,x) = (m_1, \dots, m_n) \in C^\infty(Q)$ be a vector field on Q such that

$$\operatorname{div} m(t,x) = \sum_{i=1}^n \partial_i m_i = 0,\ \ (t,x) \in Q, \quad \text{and} \quad m(t,x) = \nabla\gamma(t,x),\ \ (t,x) \in Q \setminus Q^\omega, \tag{5.5}$$

where $\gamma(t,x)$ is a function on $Q \setminus Q^\omega$, which by (5.5) is harmonic on $\Pi \setminus \omega$ for every $t \in (0,T)$.

We intend to reduce the approximate controllability problem (5.1)–(5.4) to the proof of exact controllability of a system of linear first-order equations with coefficients that we choose ourselves. These coefficients will be determined by some vector field m of the form (5.5). We seek the solution (v, p, θ) in the form

$$v = z + m, \quad \theta = r; \qquad \operatorname{div} z = 0, \quad \operatorname{div} m = 0 \tag{5.6}$$

(the resulting pressure p will be written out later). By substituting (5.6) into (5.1), (5.2), we obtain

$$\begin{aligned} &\partial_t z + (m,\nabla)z + (z,\nabla)m + (z,\nabla)z - \Delta z + \partial_t m \qquad (5.7)\\ &\qquad - \Delta m + (m,\nabla)m - \nabla p + r\vec{e} = f + u', \quad \operatorname{div} z = 0,\\ &\qquad \partial_t r + (m,\nabla)r + (z,\nabla)r - \Delta r = g + u_{n+1}. \qquad (5.8) \end{aligned}$$

We note that the main difficulty is in the construction of the desired functions (z,r) on the set $Q \setminus Q^\omega$. Indeed, once we have constructed (z,r) on $Q \setminus Q^\omega$, we can extend (z,r) to Q^ω arbitrarily, preserving only the condition $\operatorname{div} z = 0$ and the prescribed values of (z,r) for $t = T, x \in \omega$. We then find the control u by substituting (z,r) in the left-hand side of (5.7), (5.8).

Let us show that, by the second equation in (5.5), the expression $\partial_t m - \Delta m + (m,\nabla)m$ on $Q \setminus Q^\omega$ is the gradient of some function q_1:

$$\partial_t m - \Delta m + (m,\nabla)m = \nabla q_1. \tag{5.9}$$

Indeed, $\partial_t m - \Delta m = \nabla(\partial_t\gamma - \Delta\gamma)$ by the second equation in (5.5). Next,

$$(m,\nabla)m = \sum_{j=1}^n m_j \partial_j m = \sum_{j=1}^n \partial_j\gamma\,\partial_j\nabla\gamma = \frac{1}{2}\nabla|\nabla\gamma|^2 \quad \text{in} \quad Q \setminus Q^\omega.$$

[3] The control space $U(\omega; 0, T)$ is defined in (1.28).

In what follows we shrink the time coordinate $t \to t/\delta$ so that the terms $(z,\nabla)z$, Δz, $r\vec{e}$, and f in (5.7) and $(z,\nabla)r$, Δr, and g in (5.8) will be small. Retaining the leading terms in (5.7), (5.8), we obtain the system

$$\partial_t z + (m,\nabla)z + (z,\nabla)m - \nabla q = u', \quad \operatorname{div} z = 0, \tag{5.10}$$

$$\partial_t r + (m,\nabla)r = u_{n+1}, \tag{5.11}$$

where $\nabla q = \nabla p - \nabla q_1$, and ∇q_1 is defined in (5.9). The initial condition for equations (5.10), (5.11) is generated by conditions (5.3):

$$z\big|_{t=0} = \tilde{v}_0, \qquad r\big|_{t=0} = \tilde{\theta}_0. \tag{5.12}$$

We replace the approximate controllability condition (5.4) by the exact controllability condition

$$z(T,\,\cdot\,) = \tilde{v}_1, \qquad r(T,\,\cdot\,) = \tilde{\theta}_1. \tag{5.13}$$

Furthermore, $\tilde{v}_j \in V^0(\Pi) \cap (C^\infty(\Pi))^n$ and $\tilde{\theta}_j \in C^\infty(\Pi)$, $j \in \{0,1\}$, will be chosen close to v_j and θ_j, respectively, $j \in \{0,1\}$.

The coefficients of system (5.10), (5.11) are determined by the vector field m constructed in the following lemma.

Lemma 5.1. *There exist a vector field $m(t,x) = (m_1,\dots,m_n) \in (C^\infty(Q))^n$ satisfying conditions (5.5) and a time $T > 0$ such that*

$$m(0,x) \equiv m(T,x) \equiv 0, \quad \frac{\partial^k m(t,x)}{\partial t^k}\bigg|_{t=0} \equiv \frac{\partial^k m(t,x)}{\partial t^k}\bigg|_{t=T} \equiv 0, \ \ k \in \mathbb{N} \tag{5.14}$$

(k is an arbitrary positive integer) and the relation

$$\big\{(t, x(t,x_0)),\ t \in (0,T)\big\} \cap Q^\omega \neq \varnothing \tag{5.15}$$

is valid for every $x_0 \in \Pi$, where $x(t,x_0)$ is the solution of the Cauchy problem

$$\frac{d}{dt}x(t,x_0) = m\big(t, x(t,x_0)\big), \qquad x(t,x_0)\big|_{t=0} = x_0. \tag{5.15$'$}$$

Moreover, $x(T,x_0) = x_0$ for each $x_0 \in \Pi$. Furthermore, there exist a finite cover $\{\mathcal{O}_i,\ i = 1,\dots,k\}$ of the torus Π by open sets $\mathcal{O}_i$ and a number $\Delta > 0$ such that for each i all the curves $x(t,x_0)$, $x_0 \in \mathcal{O}_i$, simultaneously lie in ω for some time interval of length Δ.

Theorem 5.1. *Let $m(t,x)$ satisfy all the hypotheses of Lemma 5.1, and let $\tilde{v}_i \in (C^\infty(\Pi))^n \cap V^1(\Pi)$ and $\tilde{\theta}_i \in C^\infty(\Pi)$, $i = 0,1$, be given. Then there is a solution*

$$\begin{aligned}(z,\nabla q, r, u) \in{} &\big((C^\infty(Q))^n \cap V^{1,2}(Q)\big)\\ &\times (C^\infty(Q))^n \times C^\infty(Q) \times \big(U(\omega;0,T) \cap (C^\infty(Q))^{n+1}\big)\end{aligned}$$

of problem (5.10)–(5.13) satisfying the inequality

$$\begin{aligned}&\|z\|^2_{C^1(0,T;(C^{k,\alpha}(\Pi))^n)} + \|r\|^2_{C^1(0,T;C^{k,\alpha}(\Pi))}\\ &\quad + \|\nabla q\|^2_{C(0,T;(C^{k,\alpha}(\Pi))^n)} + \|u\|^2_{C(0,T;(C^{k,\alpha}(\Pi))^{n+1})}\\ &\qquad \leqslant c_k\bigg(\sum_{j=0}^{1}\big(\|\tilde{v}_j\|^2_{(C^{k,\alpha}(\Pi))^n} + \|\tilde{\theta}_j\|^2_{C^{k,\alpha}(\Pi)}\big)\bigg)\end{aligned} \tag{5.16}$$

for any $k \geqslant 2$ *and* $\alpha \in (0,1)$, *where the constant* c_k *depends only on the norm of the vector field* $m(t,x)$ *in the space* $(C^{k,\alpha}(Q))^n$.

The proofs of Lemma 5.1 and Theorem 5.1 will be given in the next section. Now we derive the approximate controllability of the Boussinesq system from Theorem 5.1.

5.2. Approximate controllability of the Boussinesq system.

Proof of Theorem 1.6. Suppose that

$$(v_0,\theta_0) \in \bigl(V^0(\Pi) \cap (C^{2,\alpha}(\Pi))^n\bigr) \times C^{2,\alpha}(\Pi),$$
$$(v_1,\theta_1) \in \bigl(V^0(\Pi) \cap (C^{2,\alpha}(\Pi))^n\bigr) \times C^{2,\alpha}(\Pi),$$

and $\varepsilon > 0$ are given. We must find a control $u \in U(\omega;0,T)$ such that inequality (5.4) is valid for the solution $(v,\nabla p,\theta)$ of problem (5.1)–(5.3). By Theorem 5.1, there is a control $u \in U(\omega;0,T)$ such that equations (5.13) hold for the solution $(z,\nabla q,r)$ of problem (5.10)–(5.12). Using the functions z, ∇q, r, m, and u, we construct the functions

$$\begin{gathered} z_\delta(t,x) = z\left(\frac{t}{\delta},x\right), \quad r_\delta(t,x) = r\left(\frac{t}{\delta},x\right), \quad m_\delta(t,x) = \frac{1}{\delta} m\left(\frac{t}{\delta},x\right), \\ \nabla q_\delta(t,x) = \frac{1}{\delta}\nabla q\left(\frac{t}{\delta},x\right), \quad u_\delta(t,x) = \frac{1}{\delta} u\left(\frac{t}{\delta},x\right), \end{gathered} \tag{5.17}$$

where $\delta > 0$ is some parameter. Let us substitute the functions (5.17) into system (5.10), (5.11) for the corresponding functions z, r, m, ∇q, and u. As a result, we find that the functions (5.17) on the cylinder $Q_{\delta T} \equiv (0,\delta T) \times \Pi$ satisfy the system

$$\partial_t z_\delta + (m_\delta,\nabla) z_\delta + (z_\delta,\nabla) m_\delta - \nabla q_\delta = u'_\delta, \quad \operatorname{div} z_\delta = 0, \tag{5.18}$$
$$\partial_t r_\delta + (m_\delta,\nabla r_\delta) = (u_{n+1})_\delta. \tag{5.19}$$

Moreover, relations (5.12) and (5.13) become

$$z_\delta\big|_{t=0} = \tilde v_0, \qquad r_\delta\big|_{t=0} = \tilde\theta_0, \tag{5.20}$$
$$z_\delta(\delta T,\,\cdot\,) = \tilde v_1, \qquad r_\delta(\delta T,\,\cdot\,) = \tilde\theta_1. \tag{5.21}$$

We recall that after representing the solution of system (5.1), (5.2) in the form (5.6), we reduced this system to system (5.10), (5.11) by discarding some terms, declared to be small. Since, obviously, $z_\delta + m_\delta$, ∇q_δ, r_δ, u_δ cannot be an exact solution of system (5.1), (5.2), we seek the exact solution in the form

$$v = z_\delta + m_\delta + y, \qquad \theta = r_\delta + \tau, \qquad u = u_\delta - \chi_\omega \Delta m_\delta. \tag{5.22}$$

Let us substitute (5.22) into (5.1), (5.2) and, using (5.18), (5.19), and (5.9), rewrite the resulting relations as equations for the functions (y,τ) defined for $(t,x) \in Q_{\delta T}$:

$$\begin{aligned} \partial_t y - \Delta y + (y,\nabla)(y + z_\delta + m_\delta) \qquad & \\ + (z_\delta + m_\delta,\nabla) y + \tau\vec e - \nabla q_2 = f_1, \quad \operatorname{div} y = 0, & \end{aligned} \tag{5.23}$$
$$\partial_t \tau - \Delta\tau + (m_\delta + z_\delta + y,\nabla\tau) + (y,\nabla r_\delta) = g_1, \tag{5.24}$$

where

$$f_1 = f + \Delta z_\delta - (z_\delta, \nabla)z_\delta - z_\delta \vec{e}, \qquad g_1 = g + \Delta r_\delta - (z_\delta, \nabla r_\delta), \tag{5.25}$$

$\nabla q_2 = \nabla p - \frac{1}{\delta}\nabla q_1\left(\frac{t}{\delta}, x\right)$, and ∇q_1 is defined in (5.9).

We choose the initial and terminal conditions $(\tilde{v}_j, \tilde{\theta}_j) \in (V^1(\Pi) \cap (C^\infty(\Pi))^n) \times C^\infty(\Pi)$, $j \in \{0, 1\}$, in problem (5.10)–(5.12) such that the estimates

$$\|\tilde{v}_0 - v_0\|^2_{V^1(\Pi)\cap C^{2,\alpha}(\Pi)} + \|\tilde{\theta}_0 - \theta_0\|^2_{C^{2,\alpha}(\Pi)} < \delta^2, \tag{5.26}$$

$$\|\tilde{v}_1 - v_1\|^2_{V^1(\Pi)\cap C^{2,\alpha}(\Pi)} + \|\tilde{\theta}_1 - \theta_1\|^2_{C^{2,\alpha}(\Pi)} < (\varepsilon/2)^2 \tag{5.27}$$

are satisfied, where (v_0, θ_0) and (v_1, θ_1) are the initial and terminal conditions, respectively, of problem (5.1)–(5.4).[4] It follows from (5.22), (5.3), (5.20), and (5.14) that (y, τ) satisfy the initial conditions

$$y\big|_{t=0} = v_0 - \tilde{v}_0, \qquad \tau\big|_{t=0} = \theta_0 - \tilde{\theta}_0. \tag{5.28}$$

In view of (5.22), (5.14), (5.21), and (5.27), to prove (5.4) it suffices to establish the inequality

$$\|y(T\delta, \cdot)\|^2_{V^1(\Pi)} + \|\tau(T\delta, \cdot)\|^2_{H^1(\Pi)} < (\varepsilon/2)^2. \tag{5.29}$$

Let us show that the L_2-norms of the right-hand sides f_1 and g_1 in problem (5.23), (5.24), (5.28) are small for small δ. Indeed, by virtue of (5.25), (5.17), and (5.16), we obtain with the help of obvious estimates and changes of variables that

$$\begin{aligned}\|g_1\|^2_{L_2(Q_{T\delta})} &\leqslant \int_0^{T\delta}\int_\Pi |g(t,x)|^2\,dx\,dt + \delta\int_0^{T\delta}\left\|r\left(\frac{t}{\delta}, \cdot\right)\right\|^2_{H^2(\Pi)} d\,\frac{t}{\delta} \\ &\quad + c\delta\int_0^{T\delta}\left\|z\left(\frac{t}{\delta}, \cdot\right)\right\|^2_{V^1(\Pi)}\left\|r\left(\frac{t}{\delta}, \cdot\right)\right\|^2_{H^2(\Pi)} d\,\frac{t}{\delta} \\ &\leqslant \int_0^{T\delta}\|g(t, \cdot)\|^2_{L_2(\Pi)}\,dt \\ &\quad + c\delta\sum_{j=0}^{1}\left(\|\tilde{v}_j\|^2_{V^2(\Pi)} + \|\tilde{\theta}_j\|^2_{H^2(\Pi)}\right) \to 0 \quad \text{as} \quad \delta \to 0.\end{aligned} \tag{5.30}$$

In just the same way, we can derive the inequality

$$\|f_1\|^2_{L_2(Q_{T\delta})} \leqslant \int_0^{T\delta}\|f(t, \cdot)\|^2_{L_2(\Pi)}\,dt + c\delta\sum_{j=0}^{1}\left(\|\tilde{v}_j\|^2_{V^2(\Pi)} + \|\tilde{\theta}_j\|^2_{H^2(\Pi)}\right) \to 0 \quad \text{as} \quad \delta \to 0. \tag{5.31}$$

[4] We note that the $\tilde{v}_0$ and $\tilde{\tau}_0$ defined in (5.26) depend on δ, of course. Consequently, the functions z and r constructed in Theorem 5.1 also depend on δ. However, by virtue of (5.16), the norms of z and r in the space $C^1(0, T; C^{2,\alpha}(\Pi))$ are bounded by a constant independent of δ. This remark is used below in the estimates (5.30)–(5.38).

Since the right-hand sides and initial conditions in problem (5.23), (5.24), (5.26) are small, we see that to prove the existence and uniqueness of the solution it suffices to estimate the solution of the linearized system

$$\begin{aligned}\partial_t y - \Delta y + (y, \nabla)(z_\delta + m_\delta) \\ + (z_\delta + m_\delta, \nabla) y + \tau \vec{e} - \nabla q_2 = f_1, \quad \operatorname{div} y = 0, \qquad (5.32)\\ \partial_t \tau - \Delta \tau + (m_\delta + z_\delta, \nabla \tau) + (y, \nabla r_\delta) = g_1. \qquad (5.33)\end{aligned}$$

Let us take the inner products of the first equation in (5.32) by y in $V^0(\Pi)$ and of (5.33) by τ in $L_2(\Pi)$. By summing the resulting inequalities and arguing as in the derivation of (2.13), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|y(t,\cdot)\|^2_{V^0(\Pi)} + \|\tau(t,\cdot)\|^2_{H^0(\Pi)}\big) + \big(\|\nabla y(t,\cdot)\|^2_{V^0(\Pi)} + \|\nabla\tau(t,\cdot)\|^2_{H^0(\Pi)}\big) \\ \leqslant \big(\|f_1(t,\cdot)\|^2_{(L_2(\Pi))^n} + \|g_1(t,\cdot)\|^2_{L_2(\Pi)}\big) + c\big(\|y(t,\cdot)\|^2_{V^0(\Pi)} + \|\tau(t,\cdot)\|^2_{H^0(\Pi)}\big) \\ \times \big(1 + \|z_\delta(t,\cdot)\|_{V^1_\infty(\Pi)} + \|r_\delta(t,\cdot)\|_{W^1_\infty(\Pi)} + \|m_\delta(t,\cdot)\|_{(C^1(\Pi))^n}\big). \qquad (5.34)\end{aligned}$$

We set

$$a(t) = 1 + \|z_\delta(t,\cdot)\|_{V^0(\Pi)\cap(C^{2,\alpha}(\Pi))^n} + \|r_\delta(t,\cdot)\|_{C^{2,\alpha}(\Pi)} + \|m_\delta(t,\cdot)\|_{(C^{2,\alpha}(\Pi))^n},$$

where $0 < \alpha < 1$. Then we obtain the following estimate from (5.34) with the help of Gronwall's lemma:

$$\begin{aligned}\|y(t,\cdot)\|^2_{V^0(\Pi)} + \|\tau(t,\cdot)\|^2_{H^0(\Pi)} \\ \leqslant c\Big[e^{\int_0^t a(s)\,ds}\big(\|\tilde v_0 - v_0\|^2_{V^0(\Pi)} + \|\tilde\theta_0 - \theta_0\|^2_{H^0(\Pi)}\big) \\ + \int_0^t e^{\int_0^s a(s_1)\,ds_1}\big(\|f_1(s,\cdot)\|^2_{(L_2(\Pi))^n} + \|g_1(s,\cdot)\|^2_{L_2(\Pi)}\big)\,ds\Big]. \qquad (5.35)\end{aligned}$$

We note that

$$\begin{aligned}\int_0^{T\delta} \|m_\delta(t,\cdot)\|_{(C^{2,\alpha}(\Pi))^n}\,dt = \int_0^{T\delta}\Big\|m\Big(\frac{t}{\delta},\cdot\Big)\Big\|_{(C^{2,\alpha}(\Pi))^n}\,d\,\frac{t}{\delta} \\ = \int_0^T \|m(t,\cdot)\|_{(C^{2,\alpha}(\Pi))^n}\,dt.\end{aligned}$$

In a similar way we obtain the estimates

$$\begin{aligned}\int_0^{T\delta}\big(\|r_\delta(t,\cdot)\|_{C^{2,\alpha}(\Pi)} + \|z_\delta(t,\cdot)\|_{(C^{2,\alpha}(\Pi))^n}\big)\,dt \\ \leqslant \delta\int_0^T\big(\|r(t,\cdot)\|_{C^{2,\alpha}(\Pi)} + \|z(t,\cdot)\|_{(C^{2,\alpha}(\Pi))^n}\big)\,dt \\ \leqslant c\delta\sum_{j=0}^1\big(\|\tilde v_j\|_{(C^{2,\alpha}(\Pi))^n} + \|\tilde\theta_j\|_{C^{2,\alpha}(\Pi)}\big).\end{aligned}$$

It follows from these estimates that

$$\int_0^t a(s)\,ds \leqslant c\biggl(1+\delta\sum_{j=0}^{1}\bigl(\|\tilde v_j\|_{(C^{2,\alpha}(\Pi))^n}+\|\tilde\theta_j\|_{C^{2,\alpha}(\Pi)}\bigr)\biggr) \tag{5.36}$$

for any $t\in(0,T\delta)$. Now inequalities (5.34)–(5.36) imply the energy inequality ($t\in(0,T\delta)$)

$$\begin{aligned}\|y(t,\,\cdot\,)\|^2_{V^0(\Pi)}&+\|\tau(t,\,\cdot\,)\|^2_{H^0(\Pi)}+\int_0^t\bigl(\|\nabla y(s,\,\cdot\,)\|^2_{V^0(\Pi)}+\|\nabla\tau(s,\,\cdot\,)\|^2_{H^0(\Pi)}\bigr)\,ds\\ &\leqslant\gamma_K\biggl(\|\tilde v_0-v_0\|^2_{V^0(\Pi)}+\|\tilde\theta_0-\theta_0\|^2_{H^0(\Pi)}\\ &\qquad+\int_0^t\bigl(\|f_1(s,\,\cdot\,)\|^2_{L_2(\Pi)}+\|g_1(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds\biggr),\end{aligned} \tag{5.37}$$

where γ_K is a monotone continuous function of K, K being the constant in (1.32). We apply the operator ∇ to (5.32) and (5.33), take the L_2 inner products of the first of the resulting equations by ∇y and of the second by $\nabla\tau$, argue as in the derivation of (5.34)–(5.37), and use (5.36) to obtain the following analogue of the estimate (5.37):

$$\begin{aligned}\|y(t,\,\cdot\,)\|^2_{V^1(\Pi)}&+\|\tau(t,\,\cdot\,)\|^2_{H^1(\Pi)}+\int_0^t\bigl(\|y(s,\,\cdot\,)\|^2_{V^2(\Pi)}+\|\tau(s,\,\cdot\,)\|^2_{H^2(\Pi)}\bigr)\,ds\\ &\leqslant\gamma_K\biggl(\|\tilde v_0-v_0\|^2_{V^1(\Pi)}+\|\tilde\theta_0-\theta_0\|^2_{H^1(\Pi)}\\ &\qquad+\int_0^t\bigl(\|f_1(s,\,\cdot\,)\|^2_{L_2(\Pi)}+\|g_1(s,\,\cdot\,)\|^2_{L_2(\Pi)}\bigr)\,ds\biggr).\end{aligned} \tag{5.38}$$

It follows from the estimate (5.38) by simple iteration that there is a solution (y,τ) of the non-linear problem (5.23), (5.24), (5.28), and moreover, this solution satisfies the estimate (5.38) in which the constant γ_K is replaced by $2\gamma_K$. By (5.38), (5.26), (5.30), and (5.31), there is a sufficiently small δ such that inequality (5.29) holds. Furthermore, we set $T_{\varepsilon,K}=T_\delta$.

§ 6. Exact controllability of a linear system

In this section we prove Lemma 5.1 and Theorem 5.1.

6.1. Proof of Lemma 5.1. First, let us prove the following assertion.

Lemma 6.1. *For each* $x_0\in\Pi$ *there exist a time* $T=T_{x_0}$ *and a vector field* $m=m_{x_0}(t,x)\in(C^\infty((0,T_{x_0})\times\Pi))^n$ *satisfying conditions* (5.5), (5.14), *and* (5.15) *with* $x(t,x_0)$ *defined in* (5.15′).

Proof. Suppose the contrary. Let M be the set of initial conditions x_0 for which there are no vector fields m_{x_0} with the desired property. The set $\Pi\setminus M$ is open, since the solutions of ordinary differential equations depend continuously on the initial data. Hence, M is closed. Let $x_0 \in M$ and $y_0 \in \overline{\omega}$ be points such that

$$|x_0 - y_0| \;=\; \min_{x \in M,\ y \in \overline{\omega}} |x - y|. \tag{6.1}$$

Let us show that for some $\xi \in C^\infty(\partial\omega)$ with $\int_{\partial\omega} \xi \, d\sigma = 0$ the solution $q(x)$ of the Neumann problem

$$\Delta q(x) = 0, \quad x \in \Pi \setminus \omega, \qquad \frac{\partial q}{\partial n}\Big|_{\partial\omega} = \xi \tag{6.2}$$

satisfies the condition

$$\big(\nabla q(x_0), y_0 - x_0\big) > 0, \tag{6.3}$$

where $(\,\cdot\,,\cdot\,)$ is the inner product in $\mathbb{R}^n$ and x_0, y_0 are the points from (6.1). Suppose the contrary: for each $\xi \in C^\infty(\partial\omega)$ with $\int_{\partial\omega} \xi \, ds = 0$ the solution q of problem (6.2) satisfies

$$\big(\nabla q(x_0), y_0 - x_0\big) = 0. \tag{6.4}$$

Let us consider the Neumann problem

$$\Delta z(x) = (\nabla \delta(x - x_0), y_0 - x_0), \quad x \in \Pi \setminus \omega, \qquad \frac{\partial z(x)}{\partial n}\Big|_{\partial\omega} = 0, \tag{6.5}$$

where $\delta(x - x_0)$ is the Dirac δ function at the point x_0. A necessary condition for the solubility of problem (6.5) is given by

$$\int_{\Pi\setminus\omega} \big(\nabla \delta(x - x_0), y_0 - x_0\big) \mathbf{1}(x) \, dx = 0, \tag{6.6}$$

where $\mathbf{1}(x)$ is the function identically equal to 1 and the integral is treated as the pairing between distributions and test functions. Equation (6.6) obviously holds. Hence (see [99], Theorem 6.6) there is a solution $z(x)$ of problem (6.5) which is infinitely differentiable at all points $x \in \Pi \setminus \omega$ except for $x = x_0$. Let us take the $L_2(\Pi \setminus \omega)$ inner product of the first equation in (6.5) by the solution q of problem (6.2) and integrate by parts with regard to (6.2) and the boundary condition in (6.5). Then we obtain the relations

$$\begin{aligned} -\int_{\partial\omega} z(x)\xi(x) \, d\sigma &= \int_{\Pi\setminus\omega} \big(\nabla \delta(x - x_0), y_0 - x_0\big) q(x) \, dx \\ &= -\big(\nabla q(x_0), y_0 - x_0\big) = 0, \end{aligned} \tag{6.7}$$

where the last equality in (6.7) is valid by virtue of (6.4). Since $\xi(x)$ in (6.7) is an arbitrary smooth function with zero mean, we see that

$$z|_{\partial\omega} \equiv \text{const}\,.$$

Since the solution z of problem (6.5) is determined up to a constant, we can assume that

$$z\big|_{\partial\omega} = 0. \tag{6.8}$$

It follows from Holmgren's uniqueness theorem for the Cauchy problem for the Laplace operator and from relations (6.5) and (6.8) that $z(x) = 0$ for all $x \in (\Pi \setminus \omega) \setminus x_0$, and hence $z(x)$ is a distribution supported at x_0. Consequently,

$$z(x) = \sum_{|\alpha| \leqslant N} C_\alpha D^\alpha \delta(x - x_0).$$

By substituting this into (6.5), we obtain

$$\sum_{|\alpha| \leqslant N} C_\alpha D^\alpha \Delta\delta(x - x_0) = \big(\nabla\delta(x - x_0), y_0 - x_0\big).$$

But this equation cannot be valid for any N and C_α, since the right-hand side contains a sum of first derivatives of the δ function, while the derivatives on the left-hand side are of order $\geqslant 2$. Thus we have proved condition (6.3).

Let $q(x)$ be a solution of problem (6.2) satisfying (6.3). We extend $\nabla q(x)$ from $\Pi \setminus \omega$ to Π as a smooth divergence-free vector field, which will be denoted by $r(x)$. This extension is possible (see [40]), since

$$\int_{\partial\omega} (\nu, \nabla q)\, d\sigma = \int_{\partial\omega} \xi\, d\sigma = 0$$

by virtue of (6.2). Clearly, the solution $x(t)$ of the problem

$$\frac{d}{dt}x(t) = r\big(x(t)\big), \qquad x\big|_{t=0} = x_0$$

belongs to $\Pi \setminus M$ for all $t \in (0, \varepsilon)$ provided that ε is sufficiently small: $x(t) \in \Pi \setminus M$, $t \in (0, \varepsilon)$. If we take

$$\varphi(t) \in C^\infty(0, \varepsilon), \qquad 0 \leqslant \varphi(t) \leqslant 1,$$
$$\varphi(0) = \varphi(\varepsilon) = \varphi^{(k)}(0) = \varphi^{(k)}(\varepsilon) = 0 \qquad \forall\, k = 1, 2, \ldots,$$

then the solution of the problem

$$\frac{d}{dt}x(t) = \varphi(t) r\big(x(t)\big), \qquad x\big|_{t=0} = x_0$$

also satisfies $x(\varepsilon) \in \Pi \setminus M$. The definition of the set $\Pi \setminus M$ implies the existence of $T_{x(\varepsilon)}$ and of a vector field $m_{x(\varepsilon)}(t, x) \in C^\infty$ satisfying conditions (5.5) and (5.14) and relation (5.15) in which x_0 is replaced by $x(\varepsilon)$. Let us define a vector field $m_{x_0}(t, x)$ by the formula

$$m_{x_0}(t, x) = \begin{cases} \varphi(t) r(x) & \text{for } t \in (0, \varepsilon), \\ m_{x(\varepsilon)}(t + \varepsilon, x) & \text{for } t \in (\varepsilon, T_{x(\varepsilon)} + \varepsilon). \end{cases}$$

Obviously, relation (5.15) holds, where $x(t,x_0)$ is the solution of the problem

$$\frac{d}{dt}x(t,x_0)=m_{x_0}\big(t,x(t,x_0)\big),\qquad x\big|_{t=0}=x_0.$$

But this contradicts the inclusion $x_0\in M$. Consequently, $M=\varnothing$.

Proof of Lemma 5.1. Let $x_0\in\Pi$, and let T_{x_0} be the time and $m=m_{x_0}(t,x)$ the vector field whose existence is stated in Lemma 6.1. Since solutions of differential equations depend continuously on the initial data, it follows that each point $x_0\in\Pi$ has a neighbourhood $\mathcal{O}(x_0)$ such that the solution $x(t,z)$ of the problem

$$\frac{d}{dt}x(t,z)=m_{x_0}\big(t,x(t,z)\big),\quad x(t,z)\big|_{t=0}=z,\quad z\in\overline{\mathcal{O}(x_0)},$$

satisfies the relation $\{(t,x(t,z)),\,t\in(0,T_{x_0})\}\cap Q^{\omega}_{T_{x_0}}\neq\varnothing$, where $Q^{\omega}_{T_{x_0}}=(0,T_{x_0})\times\omega$. Moreover, there is a finite time interval on which all the curves $x(t,z)$, $z\in\mathcal{O}(x_0)$, simultaneously lie in ω. From the cover $\{\mathcal{O}(x_0),\,x_0\in\Pi\}$ we extract a finite subcover $\mathcal{O}_1,\dots,\mathcal{O}_K$. By T_i and $m_i(t,x)$ we denote a time and a vector field satisfying the assertion of Lemma 6.1 with $x_0=z\in\mathcal{O}_i$. Using $m_i(t,x)$, we construct the vector field

$$\widehat{m}_i(t,x)=\begin{cases} m_i(t,x) & \text{for } t\in(0,T_i),\\ -m_i(2T_i-t,x) & \text{for } t\in(T_i,2T_i).\end{cases}\tag{6.9}$$

This vector field obviously has the following properties: all solutions $x(t,z)$, $z\in\mathcal{O}_i$, of the Cauchy problem

$$\frac{d}{dt}x(t,z)=\widehat{m}_i\big(t,x(t,z)\big),\qquad x(t,z)\big|_{t=0}=z\tag{6.10}$$

simultaneously lie in ω for $t\in\Delta_i$. Moreover, the trajectory $x(t,x_0)$ issuing at $t=0$ from an arbitrary point $x_0\in\Pi$ returns at $t=2T_i$ to the same point x_0. Hence, if we define T and $m(t,x)$ by the formulae

$$T=2\sum_{i=1}^{K}T_i,$$

$$m(t,x)=\left\{\widehat{m}_i\left(t-2\sum_{j=0}^{i-1}T_j,x\right)\text{ for } t\in\left(2\sum_{j=0}^{i-1}T_j,2\sum_{j=0}^{i}T_j\right),\ i=1,\dots,K\right\},$$

where the $\widehat{m}_i$ are defined in (6.9) and $T_0=0$, then this pair satisfies all assertions of Lemma 5.1.

6.2. Proof of Theorem 5.1. Prior to studying the exact controllability problem (5.10)–(5.12), let us consider the problem

$$\partial_t y+(m,\nabla)y+(y,\nabla)m=u',\tag{6.11}$$

$$\partial_t r+(m,\nabla r)=u_{n+1},\tag{6.12}$$

$$y\big|_{t=0}=\tilde v_0,\qquad r\big|_{t=0}=\tilde\theta_0,\tag{6.13}$$

$$y\big|_{t=T}=\tilde v_1,\qquad r\big|_{t=T}=\tilde\theta_1,\tag{6.14}$$

obtained by omitting the unknown function ∇q and the divergence-free condition $\operatorname{div} z=0$ in (5.10).

Theorem 6.1. *Let* $(0,T)$ *be a time interval and* $m(t,x) \in (C^\infty(Q))^n$ *the vector field constructed in Lemma* 5.1. *Then for any* $\tilde{v}_i \in (C^\infty(\Pi))^n$ *and* $\tilde{\theta}_i \in C^\infty(\Pi)$ *there exist a control* $u = (u', u_{n+1}) \in U(\omega; 0, T) \cap (C^\infty(Q))^{n+1}$ *and a pair* $(y, r) \in (C^\infty(Q))^n \times C^\infty(Q)$ *satisfying relations* (6.11)–(6.14). *Moreover, for each positive integer* p *there is a constant* C_p *depending only on the vector field* $m(t,x)$ *and its derivatives of order* $\leqslant p+1$ *such that*

$$\|y\|^2_{C^1(0,T;(C^{p,\alpha}(\Pi))^n)} + \|r\|^2_{C^1(0,T;C^{p,\alpha}(\Pi))} + \|u\|^2_{U(\omega;0,T)\cap C(0,T;(C^{p,\alpha}(\Pi))^{n+1})}$$
$$\leqslant c_p\left(\sum_{j=0}^{1}\left(\|\tilde{v}_j\|^2_{(C^{p,\alpha}(\Pi))^n} + \|\tilde{\theta}_j\|^2_{C^{p,\alpha}(\Pi)}\right)\right). \tag{6.15}$$

Proof. Let $\{\mathcal{O}_i,\ i = 1, \dots, k\}$ be the finite cover of Π constructed in Lemma 5.1, and let $\{\varphi_i\}$ be a partition of unity subordinate to this cover.

First, we construct a solution of the exact controllability problem with conditions (6.13) and (6.14) replaced by the conditions

$$y\big|_{t=0} = \varphi_i \tilde{v}_0, \qquad r\big|_{t=0} = \varphi_i \tilde{\theta}_0, \tag{6.16}$$
$$y\big|_{t=T} = \varphi_i \tilde{v}_1, \qquad r\big|_{t=T} = \varphi_i \tilde{\theta}_1. \tag{6.17}$$

We solve the system

$$\partial_t \rho(t,x) + (m, \nabla)\rho + (\rho, \nabla)m = 0, \qquad \partial_t \beta(t,x) + (m, \nabla\beta) = 0, \tag{6.18}$$

equipped with the initial conditions (6.16) with $y = \rho$ and $r = \beta$, by the method of characteristics. The characteristics of system (6.18) are the solutions of the Cauchy problem

$$\frac{d}{dt}x(t,z) = m\big(t, x(t,z)\big), \tag{6.19}$$
$$x(t,z)\big|_{t=0} = z, \tag{6.20}$$

where z ranges over Π. By substituting $x(t,z)$ for x in (6.18), we obtain the Cauchy problem for a linear system of ordinary differential equations. By solving this Cauchy problem, we obtain

$$\tilde{\rho}_i\big(t, x(t,z)\big) = e^{\int_0^t M(\tau)\,d\tau}\varphi_i(z)\tilde{v}_0(z), \qquad \tilde{\beta}_i\big(t, x(t,z)\big) \equiv \varphi_i(z)\tilde{\theta}_0(z), \tag{6.21}$$

where the matrix $M(t)$ is the adjoint of $\nabla_x m\,(t, x(t,z))$, and the exponential in (6.21) gives a formal notation of the solving operator for the problem $\dot{\rho} + M\rho = 0$, $\rho\,\big|_{t=0} = \varphi_i(z_0)v_0(z_0)$. Obviously, these formulae uniquely determine $\tilde{\rho}_i(t,x)$, $\tilde{\beta}_i(t,x)$.

By Lemma 5.1, there is a time interval $\Delta_i = (\tau_{i,0}, \tau_{i,1})$ such that the inclusion $x(t,z) \in \omega$ holds for any $t \in \Delta$ and $z \in \mathcal{O}_i$, where the $x(t,z)$ are the characteristics defined in (6.19), (6.20). Let

$$\tilde{\chi}_i(t) \in C^\infty(0,T), \qquad \tilde{\chi}_i(t) = \begin{cases} 1 & \text{for } t \in (0, \tau_{i,0}), \\ 0 & \text{for } t \in \left(\dfrac{\tau_{i,0} + \tau_{i,1}}{2}, T\right). \end{cases} \tag{6.22}$$

We set

$$\tilde{y}_i(t,x) = \tilde{\chi}_i(t)\tilde{\rho}(t,x), \qquad \tilde{r}_i(t,x) = \tilde{\chi}_i(t)\tilde{\beta}(t,x). \tag{6.23}$$

By virtue of (6.21)–(6.23), we have $(\tilde{y}_i, \tilde{r}_i) \in (C^\infty(Q))^{n+1}$, and the supports of these functions lie in the curvilinear tube formed by the characteristics (6.19), (6.20) issuing from the set $\mathcal{O}_i$. We find $\widehat{y}_i$ and $\widehat{r}_i$ in a similar way by solving problem (6.18), (6.17). Here the characteristics are obviously specified by (6.19) with the following initial condition at time T:

$$x(t,z)\big|_{t=T} = z. \tag{6.24}$$

It follows from the method in Lemma 5.1 for constructing the vector field $m(t,x)$ that the set of characteristics defined by (6.19) and (6.20) with $z \in \mathcal{O}_i$ coincides with the set of characteristics defined by (6.19) and (6.24) with $z \in \mathcal{O}_i$.

By solving the Cauchy problem (6.18), (6.17) with $y = \alpha$ and $r = \beta$ with the help of the characteristics (6.19), (6.24) by analogy with (6.21)–(6.23), we obtain

$$\widehat{\rho}_i(t,x(t,z)) = e^{-\int_t^T M(\tau)\,d\tau}\varphi_i(z)\tilde{v}_1(z), \quad \widehat{\beta}_i(t,x(t,z)) = \varphi_i(z)\tilde{\theta}_1(z), \tag{6.25}$$

$$\widehat{y}_i(t,x) = \widehat{\chi}_i(t)\rho_i(t,x), \quad \widehat{r}_i(t,x) = \widehat{\chi}_i(t)\beta_i(t,x), \tag{6.26}$$

where

$$\widehat{\chi}_i(t) \in C^\infty(0,T), \qquad \widehat{\chi}_i(t) = \begin{cases} 1 & \text{for } t \in (\tau_{i,1}, T), \\ 0 & \text{for } t \in \left(0, \dfrac{\tau_{i,0}+\tau_{i,1}}{2}\right). \end{cases} \tag{6.27}$$

Finally, we set

$$y(t,x) = \sum_{i=1}^K \bigl(\tilde{y}_i(t,x) + \widehat{y}_i(t,x)\bigr), \quad r(t,x) = \sum_{i=1}^K \bigl(\tilde{r}_i(t,x) + \widehat{r}_i(t,x)\bigr). \tag{6.28}$$

Obviously, the pair $(y(t,x), r(t,x)) \in (C^\infty(Q))^{n+1}$ satisfies the boundary-value problem (6.11)–(6.14), and moreover,

$$u'(t,x) = \sum_{i=1}^K \left(\frac{d\tilde{\chi}_i(t)}{dt}\,\tilde{\rho}_i(t,x) + \frac{d\widehat{\chi}_i(t)}{dt}\,\widehat{\alpha}_i(t,x)\right), \tag{6.29}$$

$$u_{n+1}(t,x) = \sum_{i=1}^K \left(\frac{d\tilde{\chi}_i(t)}{dt}\,\tilde{\beta}_i(t,x) + \frac{d\widehat{\chi}_i(t)}{dt}\,\widehat{\beta}_i(t,x)\right). \tag{6.30}$$

It follows from the construction of the functions (6.29) and (6.30) that

$$\operatorname{supp} u' \subset Q^\omega, \qquad \operatorname{supp} u_{n+1} \subset Q^\omega.$$

Moreover, using formulae (6.21)–(6.23) and (6.25)–(6.30), one can readily obtain the estimate (6.15).

Proof of Theorem 5.1. Let (y, r, u) be the solution of problem (6.11)–(6.14) constructed in Theorem 6.1. For the vector field y, we write out the Weyl decomposition

$$y(t,x) = z(t,x) + \nabla p(t,x), \tag{6.31}$$

where $z(t,x)$ is a solenoidal vector field: $\operatorname{div} z = 0$. We recall that the construction of the decomposition (6.31) can be reduced by taking the divergence of both sides to the solution of the equation

$$\Delta p(t,x) = \operatorname{div} y(t,x). \tag{6.32}$$

By substituting (6.31) into (6.11), we obtain

$$\partial_t z + (m,\nabla)z + (z,\nabla)m + \nabla\partial_t p + (m,\nabla)\nabla p + (\nabla p,\nabla)m = u'. \tag{6.33}$$

Since $m = \nabla\gamma$ for $x \in \Pi\setminus\omega$, we have

$$(m,\nabla)\nabla p + (\nabla p,\nabla)m = \sum_{i,k}(\partial_i\gamma\,\partial_i\partial_k p + \partial_i p\,\partial_i\partial_k\gamma) = \nabla(\nabla\gamma,\nabla p).$$

Consequently,

$$\nabla\partial_t p + (m,\nabla)\nabla p + (\nabla p,\nabla)m = \nabla\big(\partial_t p + (\nabla\gamma,\nabla p)\big) = \nabla q \tag{6.34}$$

for $x \in \Pi\setminus\omega$, where $q = \partial_t p + (\nabla\gamma,\nabla p)$. Extending q from $\Pi\setminus\omega$ to a smooth function $q(t,x)$ defined on Π with the help of Whitney's extension operator and substituting q in (6.33), we obtain

$$\partial_t z + (m,\nabla)z + (z,\nabla)m + \nabla q = u' + u''(t,x),$$

where $u''(t,x) \in C^\infty(Q)$ and $\operatorname{supp} u'' \subset Q^\omega$. Using inequality (6.15) and the Schauder estimates for the solutions of the elliptic equation (6.32) (see [78]), one can readily derive the estimate (5.16).

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Moscow State University
Iowa State University
E-mail address: fursikov@dial01.msu.ru, vika@iastate.edu

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