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MATHEMATICAL PHYSICS

Self-Interaction Model of Classical Point Particle in One-Dimension[1]

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Abstract—We consider a hamiltonian system on the real line, consisting of real scalar field $\phi(x, t)$ and point particle with trajectory $y(t)$. The dynamics of this system is defined by the system of two equations: wave equation for the field, "radiated: by the point particle, and Newton's equation for the particle in its own field. We find the solution where the particle is strongly damped, but the kinetic and interaction energies of the field change linearly in time, in despite of the full energy conservation.

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INTRODUCTION

In classical physics the matter (for example, point particles) and continuous fields are linked with two types of equations: 1) fields (forces) move the particles, 2) particles generate fields. The first type are the Newton equations, a particular case is the Lorentz equation (Lorentz force). The second are based on the Maxwell equations with fixed trajectories of the point charges. The problem of joining these two systems together always, starting possibly with [6], drew much attention of physicists, but still rests terra incognita. Possible approaches to this problem can differ both globally and in small details. For example, one can introduce additional forces, keeping smeared charge inside balls, as in the Abraham model (see bibliography in the book [4]). Note that the considered model of particle-field interaction is a non-relativistic analog of the scalar gravity theory by G. Nordstrom [7]. We want also to mention another mathematical model of particle-field interaction [2], however it is sufficiently far away from our model.

In this paper we study seemingly the simplest self-interaction model without introducing additional forces. The main interest of this model is not only that it is hamiltonian, allows rigorous analysis and appears to be almost explieitely solvable, but rather that the energy of the field and of the interaction energy, taken separately, are not bounded (in despite of the energy conservation). This fact seems to be important but demands further comprehension.

THE MODEL

We consider a system on the real line, consisting of the real scalar field $\phi(x, t)$, $x \in R$, $t \in R_+$, and point particle with trajectory $y(t) \in R$. The dynamics of this system is defined by two equations: wave equation for the field, "radiated" by the particle,

$$\frac{\partial^2 \phi(x, t)}{\partial t^2} = c^2 \frac{\partial^2 \phi(x, t)}{\partial x^2} + \beta f(x - y(t)) \tag{1}$$

with the initial conditions

$$\phi(x, 0) = 0, \quad \phi'_t(x, 0) = 0 \tag{2}$$

and the Newton equation for the particle, driven by its own field,

$$m\frac{d^2 y(t)}{dt^2} = \beta \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \phi(x, t) f(x - y) dx \tag{3}$$

with the initial conditions

$$y(0) = 0, \quad \frac{dy}{dt}(0) = v(0) = v_0. \tag{4}$$

It is well known, see for example [3], that for any locally integrable f, and given smooth $y(t)$ the unique solution $\phi(x, t)$ of the linear inhomogeneous equation (1) with initial conditions (2) is also locally integrable and can be written as

$$\phi(x, t) = \frac{\beta}{2c} \int_0^t \int_{x - c(t-\tau)}^{x + c(t-\tau)} f(x_1 - y(\tau)) dx_1 d\tau. \tag{5}$$

However the joint system of these equations is nonlinear, and we do not know general rigorous results concerning the structure of its solutions.

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Lemma 1. *If the function f is smooth and bounded, then the solution of the system* (1)–(4) *exists and is unique on all time interval* $[0, \infty)$.

The goal of this paper is to give exact sense and get complete picture of the dynamics for the ultra-local interaction, that is for the case when f is the δ-function.

Theorem 1. *If* $f = \delta$ *and* $|v_0| < c$, *then there exists a solution* $(\phi(x, t), y(t))$ *of the equations (*1)–(4) *in the domain* $x \in R$, $t \in [0, \infty)$, *such that* $v(t) = y'(t)$ *is a smooth, monotone function on* $[0, \infty)$. *For this solution*

$$\sup_{0 \le t < \infty} |v(t)| < c \tag{6}$$

and $v(t) \to 0$ *exponentially fast as* $t \to \infty$. *Moreover, this solution is unique in the class of smooth solutions, satisfying condition* (6).

ENERGY

The equations (1) and (3) can be written in the hamiltonian form

$$\frac{\partial^2 \phi(x, t)}{\partial t^2} = -\frac{\delta U}{\delta \phi(U)}, \quad m\frac{d^2 y(t)}{dt^2} = -\frac{\partial U}{\partial y}, \tag{7}$$

$$U = U_{ff} + U_{fp}$$

with the formal hamiltonian $H = T_f + T_p + U_{ff} + U_{fp}$, where

$$T_f = \int \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 dx, \quad T_p = \frac{m}{2}v^2 \tag{8}$$

are the kinetic energies of the field and of the particle, and

$$U_{ff} = \int \frac{c^2}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 dx, \; U_{fp} = -\beta \int \phi(x) f(x - y) dx$$

$$= -\beta\phi(y)$$

where U_{ff} is the self-interaction energy of the field, U_{fp} is the particle-field interaction.

Theorem 2. *Let* $f = \delta$. *Then for any fixed t, the supports of the derivatives* $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t}$ *are bounded in x, and all energy constituents are finite and have the following asymptotics as* $t \to \infty$

$$T_p(t) \to 0, \quad T_f(t) \sim \frac{\beta^2}{4c}, \quad U_{ff}(t) \sim \frac{\beta^2}{4c}t,$$

$$U_{fp}(t) = -\frac{\beta^2}{2c}t.$$

The energy conservation is proved by the standard calculation

$$\frac{dH}{dt} = \int\Big[c^2 \frac{\partial \phi}{\partial x}\frac{d}{dt}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial t}\frac{d}{dt}\left(\frac{\partial \phi}{\partial t}\right)$$

$$- \beta\frac{\partial \phi}{\partial t}f(x - y(t)) - \beta\phi(x, t)\frac{\partial f}{\partial y}v\Big]dx + mv\frac{dv}{dt}$$

$$= \int\Big[-c^2\frac{\partial^2 \phi}{\partial x^2}\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2} - \beta\frac{\partial \phi}{\partial t}f(x - y(t))$$

$$- \beta v\phi(x, t)\frac{\partial f(x - y)}{\partial y} + \beta v\phi(x, t)\frac{\partial f(x - y)}{\partial y}\Big]dx = 0,$$

where the last two terms mutually cancel, and the first three terms give zero due to equation (1).

Proof of Theorem 1. The plan of the proof is the following. By explicit formula (5), one can forget about equation (1) and, substituting (5) to (3), solve the obtained integro-differential equation. It is not clear how to calculate integrals for arbitrary $y(t)$, but if one assumes in advance the condition (6) on $y(t)$, then one gets the solution of equations (1)–(4), which miraculously appears to satisfy this assumption.

Proof of Lemma 1. Local (in time) existence and uniqueness of the solution can be proved in the standard way. To prove that the solution exists on all time interval one needs more accurate estimates. From (5) we have

$$\frac{\partial}{\partial x}\phi(x, t) = \frac{\beta}{2c}\int_0^t [f(x + c(t - \tau) - y(\tau))$$

$$- f(x - c(t - \tau) - y(\tau))]d\tau. \tag{9}$$

From (5) it is clear that $\frac{d\phi}{dx}$ for given t and all x does not exceed Bt, where

$$B = \frac{\beta}{c}\sup|f|.$$

This means that the absolute value of the particle acceleration does not exceed $\frac{B}{m}t$, that is $y(t)$ does not exceed constt^3. Thus there cannot be vertical asymptote for finite t. Lemma is proved.

Lemma 2. *Let* $y(t)$ *be sufficiently smooth and let the condition* (6) *be satisfied. Then*

$$\frac{\partial \phi(x, t)}{\partial x} = \begin{cases} 0, & x \notin [-ct, ct], \\ \frac{\beta}{2c}\frac{1}{c + y'(\tau(x))}, & x \in [-ct, y(t)), \\ -\frac{\beta}{2c}\frac{1}{c - y'(\tau(x))}, & x \in (y(t), ct], \\ -\frac{\beta}{2c}\frac{y'(t)}{c^2 - (y'(t))^2}, & x = y(t). \end{cases} \tag{10}$$

Proof. Instead of directly using (rather arbitrary) substitution techniques for δ-function (as for example in [5]), it is more convenient to use the gaussian approximation for the δ-function

$$f_\sigma(x,t) = \delta_\sigma(x-y(t)),$$

$$\delta_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} \xrightarrow[\sigma\to 0]{} \delta(x).$$

In our case

$$\frac{\partial\phi_\sigma(x,t)}{\partial x} = \frac{\beta}{2c\sigma\sqrt{2\pi}}\int_0^t\left(e^{\frac{h_+}{\sigma^2}} - e^{\frac{h_-}{\sigma^2}}\right)d\tau, \qquad (11)$$

where for given t

$$h_\pm = h_\pm(x,\tau) = -\frac{1}{2}(x-y(\tau)\pm c(t-\tau))^2.$$

For given t define the functions $x_\pm(\delta) = y(\tau) \mp c(t-\tau)$. In other words we choose them so that

$$h_\pm(x_\pm(\tau),\tau) = 0.$$

When τ runs along the interval $[0, t]$, $x_\pm(\tau)$ runs inside the interval $[\mp ct, y(t)]$ correspondingly. Define the functions $\tau_\pm(x)$ on the intervals $[-ct, y(t)]$ and $[y(t), ct]$ correspondingly by the condition

$$h_\pm(x,\tau_\pm(x)) = 0.$$

These functions are uniquely defined because the functions $y(\tau) \pm c(t - \tau)$ are strictly monotone by our assumption. These functions coincide at the point $y(t)$ where they equal to t. Thus they can be glued in one function on the interval $[-ct, ct]$, we denote this function $\tau(x)$. Outside this interval we put $\tau(x) = 0$.

Then

$$h'_\pm = \frac{\partial h_\pm}{\partial\tau}(x,\tau) = -(x-y(\tau)\pm c(t-\tau))(\mp c - y'(\tau)),$$

$$h''_\pm = \frac{\partial^2 h_\pm}{\partial\tau^2}(x,\tau)$$

$$= (x-y(\tau)\pm c(t-\tau)y''(\tau)) - (\mp c - y'(\tau))^2,$$

and at the point $(x, \tau_\pm(x))$ we have

$$\frac{\partial h_\pm}{\partial\tau}(x,\tau_\pm(x))$$

$$= -(x-y(\tau_\pm(x))\pm c(t-\tau_\pm(x)))(\mp c - y'(\tau_\pm(x))) = 0,$$

$$h''(x,\tau_\pm(x)) = \frac{\partial^2 h_\pm}{\partial\tau^2}(x,\tau_\pm(x))$$

$$= -(\mp c - y'(\tau_\pm(x)))^2.$$

Note that if $x \neq x_\pm(\tau)$ for any $\tau \in [0, t]$, then

$$\int_0^t e^{\frac{h_\pm}{\sigma^2}}d\tau \xrightarrow[\sigma\to 0]{} 0.$$

In particular, this takes place for any $x \notin [-ct, ct]$. Moreover, $x \neq x_+(\tau)$ for all $\tau \in [0, t]$, if $x \notin [-ct, y(t)]$. Similarly, $x \neq x_-(\tau)$ for all $\tau \in [0, t]$, if $x \notin [y(t), ct]$. For other x we will apply to any integral (11) the Laplace method, using the following result

$$\int_a^b e^{nh(\tau)}d\tau \sim e^{nh(a)}\left[-\frac{\pi}{2nh''(a)}\right]^{\frac{1}{2}}, \qquad (12)$$

if $h(\tau)$ has its maximum at the point a, $h'(a) = 0$ and $h''(a) < 0$ (similarly for b). If the maximum is reached at some point u, lying inside the interval (a, b), and also $h'(u) = 0$ and $h''(u) < 0$, then the right hand side in (12) is multiplied by 2.

In our case $h'(x, \tau_\pm(x)) = 0$, $h''(x, \tau_\pm(x)) \neq 0$, and

$$\frac{\beta}{2c\sigma\sqrt{2\pi}}\int_0^t e^{\frac{h_+}{\sigma^2}}d\tau \sim \frac{\beta}{c\sigma\sqrt{2\pi}}\sigma\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{c+y'(\tau(x))},$$

$$x \in [-ct, y(t)],$$

$$\frac{\beta}{2c\sigma\sqrt{2\pi}}\int_0^t e^{\frac{h_-}{\sigma^2}}d\tau \sim \frac{\beta}{c\sigma\sqrt{2\pi}}\sigma\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{c-y'(\tau(x))},$$

$$x \in [y(t), ct],$$

and for $x = y(t)$ as $\sigma \to 0$

$$\frac{\partial\phi_\sigma(y(t),t)}{\partial x} \to \frac{\beta}{2c\beta\sqrt{2\pi}}\sigma\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{1}{c+y'(t)} - \frac{1}{c-y'(t)}\right) = -\frac{\beta}{2c}\frac{y'(t)}{c^2-(y'(t))^2}.$$

Lemma is proved.

Let us prove now the first assertion of the theorem 1. By lemma 2, the equation (3) for the particle becomes

$$m\frac{d^2y(t)}{dt^2} = \beta\frac{\partial}{\partial y}\phi(y(t),t) = -\frac{\beta^2}{2c}\frac{y'(t)}{c^2-(y'(t))^2}$$

or

$$m\frac{dv}{dt} = -\frac{\beta^2}{2c}\frac{v}{c^2-v^2}, \qquad (13)$$

This means that $v(t)$ tends to the fixed point $v = 0$ exponentially fast.

Corollary 1. *Under the conditions of theorem* 1, *the solution* $(\phi_\sigma(x, t), y_\sigma(t))$, *which accordingly to Lemma* 1 *exists and is unique, converges* (*as* $\sigma \to 0$) *to the solution obtained in Theorem* 1.

Proof of Theorem 2. If $f_\sigma \to \delta$, then

$$\phi_\sigma(x,t) \to \frac{\beta}{2c}\int_0^t \mathbf{1}\{x: y(\tau)-c(t-\tau) < x < c(t-\tau)$$

$$+ y(\tau)\}d\tau = \frac{\beta}{2c}\tau(x),$$

and, by $\tau(y(t)) = t$, we get

$$U_{fp} = -\beta\phi(y(t),t) = -\frac{\beta^2}{2c}t.$$

Further on, using Lemma 2 and taking into account that $v(t) \to 0$ as $t \to \infty$, we have

$$U_{ff} = \frac{c^2}{2}\int\left(\frac{\partial\phi}{\partial x}\right)^2 dx$$

$$= \frac{c^2}{2}\left[\int_{-ct}^{y(t)}\left(\frac{\beta}{2c}\frac{1}{c + y'(\tau(x))}\right)^2 dx\right.$$

$$\left.+ \int_{y(t)}^{ct}\left(\frac{\beta}{2c}\frac{1}{c - y'(\tau(x))}\right)^2 dx\right] \sim \frac{\beta^2}{4c}t.$$

Finally

$$\frac{\partial}{\partial t}\phi_\sigma(x, t) = \frac{\beta}{2}\int_0^t (f(x + c(t-\tau) - y(\tau))$$

$$+ f(x - c(t-\tau) - y(\tau)))d\tau \quad (14)$$

and in the limit $f = \delta$

$$\frac{\partial}{\partial t}\phi_\sigma(x, t) = c\frac{\partial\phi_\sigma(x, t)}{\partial x}\operatorname{sgn}(y(t) - x).$$

That is why

$$T_f = U_{ff} \sim \frac{\beta^2}{4c}t,$$

that also follows (as we know asymptotics of other energy constituents) from the energy conservation.

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