Boltzmann Ergodicity Hypothesis

(Project 2018-1 proposed by Lykov A.A. and Malyshev V.A.)

There is no generally accepted formulation of Boltzmann's hypothesis. Here we propose our own formulation, consisting of three parts:

1. Most closed particle systems are not ergodic;
2. For most particle systems minimal contact with external world is sufficient to imply ergodicity.
3. For convergence to Gibbs invariant measure the external influence should not have memory.

Here we want to **formulate exactly** and justify the second and third part.

Many mathematicians wanted to prove various types of ergodicity for N-particle systems "in finite volume" for various classes of potentials U_{jk}. It is useful to define more exactly what means "in finite volume".

For N-particle system with Hamiltonian H denote

$$M_h = \{\psi : H(\psi) = h\}$$

the energy surface in the phase space $R^{2dN} = \{\psi = (q_1, p_1), \ldots, q_N, p_N), q_k, p_k = m_k v_k \in R^d\}$. It is compact if something does not allow particles to escape to infinity: either some common external potential $U(x) \to \infty$ as $x \to \infty$ (for example, the walls of the finite volume), or if the same is true for external potential acting on one particle but also for all interaction potentials. Liouville measure λ_h on M_h can be roughly understood as the restriction of Lebesgue measure λ on the energy surface. Gibbs measure (Liouville measure with random energy, that is if h is random) is

$$d\mu_\beta = Z_\beta^{-1} e^{-\beta H} d\lambda$$

where Z_β is the normalization factor (partition function), β – inverse temperature.

Both measures are invariant with respect to the Hamiltonian dynamics. Mathematicians formulated convergence to Liouville measure as follows: for Hamiltonian systems the following should hold: for any initial state $\psi(0) \in M_h$ and any function $f(\psi)$ on M_h (for example, for functions f from $L_2(M_h, \lambda_h)$.)

$$\frac{1}{T} \int_0^T f(\psi(t)) dt \to_{T \to \infty} \int_{M_h} f(\psi) d\lambda_h$$

"Ergodic theory of dynamical systems" created many deep theories, but for Hamiltonian systems of N particles still there are no any examples of such ergodicity. On the contrary, the examples of nonergodic systems are numerous and their number increases.

Moreover, some general considerations suggest completely different picture: any closed physical system, even very complex, tends to "obsession".

Note that to get convergence to Gibbs measure one should either choose random initial conditions or have random contact with external world. Thus, one could expect the following result: even minimal contact with external world implies ergodicity. For example, for the N-particle system minimality could mean that only one fixed particle of N has contact with the external world. We have proved exactly this for **"most nonergodic"** system. Namely, we have chosen the "**worst**" system in the sense of ergodicity: the system with quadratic interaction

$$H = \sum_{k=1}^{N} \frac{mv_k^2}{2} + \sum_{j<k} V_{jk} q_j q_k$$

where the matrix $V = (V_{jk})$ is positive definite. For such system continuum (for example, N-parametric) of invariant tori provides evident non-ergodicity. And nevertheless, we have proved the following

Theorem If we fixed a particle and in random time moments change the sign of its velocity, then for almost all (in Lebesgue measure) V there is convergence to Liouville measure for any initial $\psi(0)$, see [5, 6].

Some adherents of "nonlinear science" claimed that this result is not interesting due to linearity (for example, we have some referee reports). But firstly, the proof of this theorem is sufficiently long and very "nonlinear". And secondly, simple logics lead us to the following conclusion. We follow the folklore of "nonlinear" adepts in mathematics and physics, that nonlinear systems have better mixing properties than linear systems. Thus, the same results should take place for almost all nonlinear hamiltonians as well. Should one loose time for such labor ? The answer is – of course yes. but one should also avoid narrow down oneself for a long time – there are even more interesting problems. Unfortunately, nonlinear adherents avoid open scientific discussion.

For the Gibbs measure we have proved [1, 2, 3, 4] similar statements. If there is random external force (stationary random process) acting on one fixed particle, then the system is also ergodic. But there are very important restrictions:

1) to converge to Gibbs distribution this random process SHOULD NOT HAVE MEMORY (like white noise) and have dissipation to avoid energy pumping.

2) If there is memory, then convergence also takes place but not to Gibbs measure, see [3, 4].

3) In [8] we have proved the same results for collision-like forces.

4) Note that in [7] some results are obtained for quantum case, but we do not want to discuss it here.

Examples of other problems

1. Rates of convergence to equilibrium depending on the number of particles having contact with external world.

2. BBGKY hierarchy for such convergence processes.

3. What will be the limit if for different particles the parameters of external force differ.

4. Control of chaos in large systems, that is what simple means could stop global distribution of chaos (from one particle only) through all system.

5. Use KAM perturbation techniques to prove the assertions of references below for weakly perturbed linear systems.

References

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