Coulomb Networks

(Project 2018-4 proposed by Malyshev V.A.)

That is electric networks.

0.1 Introduction on school level

Consider a circle S of length L. That is the interval $[0, L]$ with identified end points 0 and L. Assume there are N point particles (call them electrons) with mass m and negative charge $q < 0$ at the points

$$0 \leq x_1 < x_2 < \ldots < x_N < L$$

We assume newtonian dynamics with repulsive Coulomb interaction but for simplicity (technical assumption) we assume that only nearest neighbors interact. That is the following equations hold

$$m\frac{d^2x_k}{dt} = \frac{q^2}{(x_k - x_{k-1})^2} - \frac{q^2}{(x_k - x_{k+1})^2} + F(x_k(t), v_k(t), t) \tag{1}$$

Let first $F = 0$. Then, if initially for all k and some fixed v

$$\Delta_k(0) = x_{k+1}(0) - x_k(0) = \frac{L}{N}, \quad v_k(0) = v, \tag{2}$$

the particles will stand still if $v = 0$, that is $x_k(t) = x_k(0)$. If however $v > 0$ then all particles will turn round the circle:

$$x_k(t) = x_k(0) + vt$$

It seems that this could provide eternal energy. But everyone understands that it is a false impression – there is always dissipation of energy. Our devices can take energy and external media can grab the energy. Normally such interaction with external media is modeled on macro scale by the dissipation (friction) force

$$-\alpha v_k(t) \tag{3}$$

with $0 < \alpha$. On the micro scale this macro force is a consequence of "collisions" with other particles. But in this case, under the same conditions with the dissipation force (3) we will get $v_k(t) \to 0$ for all k. The simplest possibility to avoid this is to add constant force, say $F = qE > 0$. If we define $E = \frac{F}{q}$ it can be imagined as electric field (tension). Or to add potential $U(x) = -\int_0^x E dx$ so that

$$E = -\frac{d}{dx}U(x)$$

Thus, final equations will be

$$m\frac{d^2x_k}{dt} = \frac{q^2}{(x_k - x_{k-1})^2} - \frac{q^2}{(x_k - x_{k+1})^2} + qE - \alpha v_k \tag{4}$$

with the same initial conditions (2). Then, whatever be initial v, for any k as $t \to \infty$

$$v_k(t) \to w = \frac{qE}{\alpha}$$

This is similar to the famous model of electric current proposed by Drude in 1900. This model entered many text books. Note that in Drude's model the particles do not interact. Here they do not interact due to the chosen initial conditions.

Now we can deduce the famous Ohm's law. This is a macroscopic law and we should define the macro variables as the limit of the corresponding discrete quantities as $N \to \infty$. We imagine not discrete point charges but infinitesimally small charges at all points $x \in S$. We refer to the definition of continuum system of point particles in Project 2. So, in each point $x \in S$ there is a "particle" which has trajectory $y(t,x)$ satisfying the equation

$$m\frac{d^2y(t,x)}{dt^2} = qE - \alpha\frac{dy(t,x)}{dt}, \quad y(0,x) = x, \quad \frac{dy(0,x)}{dt} = 0 \tag{5}$$

Note that the array (m, q, α) is defined up to a common multiplication factor. It is important that the trajectories of $x_k(t)$ converge to trajectory $y(t,x)$ as $N \to \infty$, $\frac{k(N)}{N} \to x$.

Anyway the solution is the same as for finite N

$$\frac{dy(t,x)}{dt} \to w = \frac{Eq}{\alpha}$$

Introduce the mass and charge densities (scaling correspondingly m and q),

$$\mu = \lim_{N\to\infty} \frac{Nm}{L}, \quad \rho = \lim_{N\to\infty} \frac{Nq}{L}$$

Moreover, α should be scaled correspondingly.

As the force on any particle is $F = qE$ we assume also that the force on any "particle" of the continuum charged medium is $F = \rho E$ and define the potential as follows:

$$U(x) = -\int_0^x E dx = -Ex$$

It is a multivalued (periodic) function on S but, as it is defined only up to additive constant, its derivative (the force) is uniquely defined.

Current $I(x,t)$ at point x at time t is the amount of charge crossing the point x for unit time. It is also a bit puzzling and can be defined only for large N. Namely, in our case for time Δt the amount of charge crossing point x is

$$Q(x, \Delta t) = w\Delta t\rho$$

and we define the current (not depending on x, t due to constant velocity and constant charge density

$$I = \frac{Q(x, \Delta t)}{\Delta t} = w\rho$$

There are two formulations of **Ohm's law**. The first one is the local formulation:

$$I = w\rho = \frac{qE}{\alpha}\rho = \sigma E, \quad \sigma = \frac{\rho q}{\alpha}$$

where σ is called **conductivity**. For the second one define also the potential difference on the interval $[a, b)]$

$$U_{a,b} = U(b) - U(a) = (b - a)E,$$

the **resistivity** $r = \sigma^{-1}$ and **resistance** $R_{a,b} = \int_a^b r dx = (b-a)r$ of the interval $[a, b]$.

Then we have Ohm's law

$$U_{a,b} = (b - a)E = (b - a)I\sigma^{-1} = (b - a)rI = R_{a,b}I$$

What is bad with this model ? Let us see first what famous people say:

"The force pushes the electrons along the wire. But why does this move the galvanometer, which is so far from the force ? Because when the electrons which feel the magnetic force try to move, they push – by electric repulsion – the electrons a little farther down the wire; they, in turn, repel the electrons a little farther on, and so on for a long distance. An amazing thing. It was so amazing to Gauss and Weber – who first built a galvanometer – that they tried to see how far the forces in the wire would go. They strung the wire all the way across the city." This is in Feynman lectures [9], Ch. 16-1.

So, one should consider the case when external force F is not constant but is different from zero only on some interval of the circle with length possibly even much smaller than L (possibly several meters length compared to $L = 100$ kilometers). The simplest set of axioms is the following.

0.2 Axioms

1. We assume 1-dim space and time. Time is R and space is any interval of length L, or the circle of length L, that is the same interval but with identified boundary points.
2. We assume Coulomb interaction with identical particles (equal masses and equal negative charge (like electrons)). One more (technical) axiom – nearest-neighbor interaction. Thus between neighboring particles there is repulsion force. It is likely that this assumption does not influence the qualitative picture we have here.
3. External forces are of two kinds – dissipative forces $-\alpha v_k$ and deterministic force $F(x), x \in I$, acting on all particles $k = 1, \ldots, N$.

0.3 Phase diagram for static configurations

First of all, the problem of static configuration was considered. Assume that on the interval $[0, L]$ the particles are enumerated as

$$0 \leq x_1 < x_2 < \ldots < x_N \leq L$$

Then if there is no external force, it is clear that in equilibrium the distances between particles will be

$$x_{k+1} - x_k = \frac{L}{N-1} \tag{6}$$

If the external force $F(x)$ is applied, then the most important fact is that the same equality holds but only asymptotically, as $N \to \infty$. Even more important, the new super-micro scale appears as

$$\left| x_{k+1} - x_k - \frac{L}{N-1} \right| = O\Big(\frac{1}{N^2}\Big) \tag{7}$$

Moreover, a rich phase diagram appears if $F(x)$ depends on N, see [2, 1, 4, 6]. The mentioned results are in some sense for zero temperature. Quite similar phase diagram was also obtained for Gibbs distribution, see [8], see also [7].

0.4 Why the Current Flows

Assume now we are on the circle of length L and that initially the particles are at equilibrium, that is the initial velocities are zero and

$$x_{k+1} - x_k = \frac{L}{N}$$

Then it was proved that:

1. almost immediately the particles attain the some constant velocity,
2. they move with this velocity the time at least of the order N
3. during this flow the density and velocity remain constant, that is Ohm's law holds.

0.5 Further problems

Coulomb networks

1. The circle is the simplest graph with one cycle only. One should prove similar results for more complicated graph. In particular Ohm's law and two Kirchhoff laws.
2. Together with direct current (DC) one could consider micro models of alternative current (AC).
3. Networks with capacities,
4. Hodgkin – Huxley equations etc.

Moreover, in any biological organism the main force is the Coulomb force. Natural question arises: possibly this force is sufficient to explain many important biological phenomena, including neural networks.

References

[1] V. A. Malyshev. Critical states of strongly interacting many-particle systems on a circle. Problems of Information Transmission, 2011, v. 47, No. 2, pp. 117–127.

[2] V. A. Malyshev. Fixed points for one-dimensional particle system with strong interaction. Moscow Math. Journal, 2012, v. 12, No. 1, pp. 139–147.

[3] V. A. Malyshev. Analytic dynamics of one-dimensional particle system with strong interaction. Math. Notes, 2012, v. 92, No. 1–2, pp. 237–248.

[4] V. A. Malyshev. Fine structure of one-dimensional discrete system of points. Problems of Information Transmission, 2012, v. 48, No. 3, pp. 57–71.

[5] V. A. Malyshev. Self-organized circular flow of classical point particles. Journal of Math. Physics, 2013, v. 54, No. 023301.

[6] V. A. Malyshev. Phase transitions in the one-dimensional Coulomb medium. Problems of Information Transmission, 2015, v. 51, No. 1, pp. 31–36.

[7] Malyshev V.A., Zamyatin A.A. One-dimensional Coulomb Multiparticle Systems. Advances in Mathematical Physics, 2015, Article ID 857846, pp. 1–9.

[8] Turova-Schmeling T. Phase Transitions in one-dimensional Coulomb gas ensembles. Annals of Applied Probability, 2018, vol. 28, No. 2, 1249–1291.

[9] Feynman R., Leighton R., Sands M. The Feynman lectures on Physics (mainly electromagnetism and matter). 1964, Addison-Wesley.