Strong Birkhoff-James orthogonality in Hilbert C^*-modules

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(joint work with A. Guterman, B. Kuzma, R. Rajić, S. Zhilina)

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Orthogonality in normed spaces

If X is an inner product space then $x, y \in X$ are orthogonal if $(x, y) = 0$.
Let $(X, \|\cdot\|)$ be a normed linear space, $x, y \in X$. We say that x is Birkhoff–James orthogonal to y, $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\|, \quad \forall \lambda \in \mathbb{F}.$$

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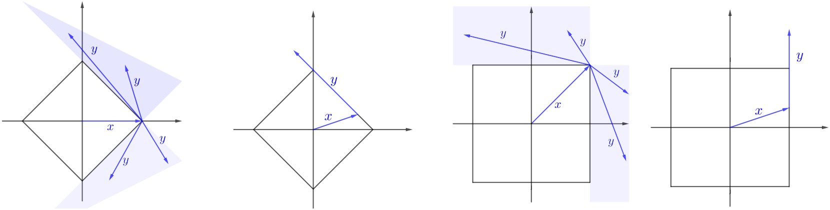

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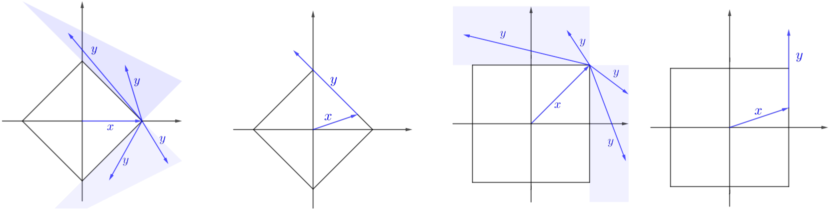

- $\perp_B$ is homogeneous, not symmetric, not additive
- $x \perp_B y \Leftrightarrow$ there is $f \in X^*, \|f\| = 1$ such that $|f(x)| = \|x\|, f(y) = 0$.
- For every $x, y \in X$ there is $\lambda \in \mathbb{C}$ such that $x \perp_B (y + \lambda x)$.

BJ orthogonality in $\mathbb{B}(H)$ with the operator norm

Let $A, B \in \mathbb{B}(H)$. Then for all λ and a unit vector ξ we have

$$\|A + \lambda B\|^2 \geq \|(A + \lambda B)\xi\|^2 = \|A\xi\|^2 + 2\mathrm{Re}\,(\overline{\lambda}(A\xi, B\xi)) + |\lambda|^2\|B\xi\|^2.$$

If there is $\xi \in H, \|\xi\| = 1$ s.t. $\|A\xi\| = \|A\|$ and $(A\xi, B\xi) = 0$ then $A \perp_B B$.

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Bhatia, Šemrl, 1999., Magajna, 1993

- $A \perp_B B \Leftrightarrow$ there is (ξ_n) in $H, \|\xi_n\| = 1$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty}(A\xi_n, B\xi_n) = 0$.
- If $\dim H < \infty$ then $A \perp_B B \Leftrightarrow$ there is $\xi \in H, \|\xi\| = 1$ such that $\|A\xi\| = \|A\|$ and $(A\xi, B\xi) = 0$.

BJ-orthogonality in $C(K)$

K a compact Hausdorff space, $C(K)$ with $\|f\| = \max\{|f(x)| : x \in K\}$.

- $f, g \in C(K)$ such that $|f(x_0)| = \|f\|$ and $g(x_0) = 0$ for some $x_0 \in K$ $\Rightarrow \|f + \lambda g\| \geq |f(x_0) + \lambda g(x_0)| = \|f\|, \forall \lambda \in \mathbb{C} \Rightarrow f \perp_B g$.
- $f, g \in C([0,1])$ defined by $f(t) = 1$ and $g(t) = e^{2\pi i t}$ then $f \perp_B g$.

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Kečkić, 2012.

Let $f, g \in C(K)$. Let $E := \{x \in K : |f(x)| = \|f\|\}$.
Then $f \perp_B g$ if and only if the set $(f\bar{g})(E)$ is not contained in an open half plane in $\mathbb{C}$ with boundary that contains the origin.
In particular, if $E = \{x_0\}$, then $f \perp_B g$ if and only if $g(x_0) = 0$.

This gives a characterization of BJ orthogonality in $\mathbb{C}^n$ with $\|\cdot\|_\infty$.

BJ orthogonality in Hilbert C^*-modules

Let X be a right Hilbert C^*-module over a C^*-algebra $\mathcal{A}$.
We say that x and y are orthogonal (with respect to C^*-valued inner product) if $\langle x, y\rangle = 0$. We write $x \perp y$.
It holds: $x \perp y \Rightarrow x \perp_B y$.

A., Rajić, LAA, 2012, Bhattacharyya, Grover, JMAA, 2013

Let X be a Hilbert $\mathcal{A}$-module, and $x, y \in X$. Then $x \perp_B y$ if and only if there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$.

- $x \perp_B y \Leftrightarrow \langle x, x\rangle \perp_B \langle x, y\rangle \Leftrightarrow \langle x, x\rangle \perp_B \langle y, x\rangle$;
- $x \perp_B y \Rightarrow (x \perp_B x\langle x, y\rangle$ and $x \perp_B x\langle y, x\rangle)$.
- If $\dim X \geq 2$, then $\forall x \in X$ there is $y \neq 0$, such that $x \perp_B x\langle x, y\rangle$.
- $x \perp_B \|x\|^2 y - y\langle x, x\rangle$ for all $x, y \in X$.

Generalizations of BJ-orthogonality?

$$x \perp_B y \overset{def}{\iff} \|x + \lambda y\| \geq \|x\|, \forall \lambda \in \mathbb{C}.$$

In Hilbert C^*-modules the role of scalars is played by the elements of the underlying C^*-algebra. Also, we have the C^*-valued "norm".

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4. $|x + \lambda y| \geq |x|$ for all $\lambda \in \mathbb{C}$ - ?
5. $\|x + ya\| \geq \|x\|, \forall a \in \mathcal{A} \not\Rightarrow \langle x, y \rangle = 0$

Strong BJ-orthogonality

An element x of a Hilbert $\mathcal{A}$-module X is strongly BJ-orthogonal to $y \in X$, in short $x \perp_B^s y$, if $\|x + ya\| \geq \|x\|, \forall a \in \mathcal{A}$.

$x \perp y \Rightarrow x \perp_B^s y \Rightarrow x \perp_B y$.

A., Rajić, AFA, 2014

Let X be a Hilbert $\mathcal{A}$-module, $x, y \in X$. Then

$$\begin{aligned} x \perp_B^s y &\Leftrightarrow x \perp_B ya, \forall a \in \mathcal{A} \\ &\Leftrightarrow x \perp_B y\langle y, x\rangle \\ &\Leftrightarrow \exists \varphi \in S(\mathcal{A}) : \varphi(\langle x, x\rangle) = \|x\|^2 \text{ and } \varphi(\langle x, y\rangle\langle y, x\rangle) = 0. \end{aligned}$$

- $x \perp_B^s y \Leftrightarrow \langle x, x\rangle \perp_B^s \langle x, y\rangle$.
- If $\langle x, y\rangle \geq 0$, then $x \perp_B y$ if and only if $x \perp_B^s y$.
- $x \perp_B^s (\|x\|^2 x - x\langle x, x\rangle)$ for all $x \in X$.

Strong BJ in $\mathbb{B}(H)$ and $C(K)$

Strong BJ-orthogonality in $\mathbb{B}(H)$

For every $A, B \in \mathbb{B}(H)$ the following statements hold.

1. $A \perp_B^s B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ s.t. $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty} B^* A\xi_n = 0$.
2. If $\dim H < \infty$, then $A \perp_B^s B$ if and only if there is a unit vector $\xi \in H$ s.t. $\|A\xi\| = \|A\|$ and $B^* A\xi = 0$.

Strong BJ-orthogonality in $C(K)$

$f, g \in C(K)$. Then $f \perp_B^s g$ if and only if there is $x_0 \in K$ such that $|f(x_0)| = \|f\|$ and $g(x_0) = 0$.

The (strong) BJ-orthogonality in $\mathbb{B}(H)$

We can characterize certain classes of operators in $\mathbb{B}(H)$ in terms of the (strong) Birkhoff–James orthogonality.
Let $A \in \mathbb{B}(H)$.

- A is a scalar multiple of isometry or coisometry if and only if whenever $B \perp_B A$ then $A \perp_B B$.
- A is a rank-one operator if and only if whenever $A \perp_B^s B$ then $B \perp_B^s A$.
- A is a scalar multiple of coisometry if and only if $A \perp_B^s B$ for all $B \in \mathbb{B}(H)$ such that BB^* is noninvertible.

When are $\perp$, $\perp_B$, $\perp_B^s$ the same?

It follows from characterizations of $\perp_B$ and $\perp_B^s$:

- $x \perp_B (\|x\|^2 y - y\langle x, x\rangle)$ for all $x, y \in X$.
- $x \perp_B^s (\|x\|^2 x - x\langle x, x\rangle)$ for all $x \in X$.
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A., Rajić, LAMA, 2015

Let $X \neq \{0\}$ be a full Hilbert $\mathcal{A}$-module (and a left Hilbert $\mathbb{K}(X)$-module).

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3. $(\perp_B^s = \perp) \Leftrightarrow \mathcal{A}$ or $\mathbb{K}(X)$ is isomorphic to $\mathbb{C}$.

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Symmetrized strong BJ orthogonality in C^*-algebras

A., Rajić, AFA, 2016

Let $X \neq \{0\}$ be a full Hilbert $\mathcal{A}$-module. Then:
$\perp_B$ is symmetric $\Leftrightarrow$ $\perp_B^s$ is symmetric $\Leftrightarrow \mathcal{A}$ or $\mathbb{K}(X)$ is isomorphic to $\mathbb{C}$.

In particular, the only C^*-algebra in which $\perp_B^s$ is symmetric is $\mathbb{C}$.
If $a^*b = 0$ then $a \perp_B^s b$ and $b \perp_B^s a$, but there are other cases of such $a, b \in \mathcal{A}$.

Definition

We say that elements $a, b \in \mathcal{A}$ are mutually strongly B-J orthogonal, and we write $a \perp\!\!\!\perp_B^s b$, if $a \perp_B^s b$ and $b \perp_B^s a$.

- Let $a \in \mathcal{A}$. Is there a nonzero $b \in \mathcal{A}$ such that $a \perp\!\!\!\perp_B^s b$?
- Let $a, b \in \mathcal{A}$ such that $a \not\perp\!\!\!\perp_B^s b$. Is there a nonzero $c \in \mathcal{A}$ such that $a \perp\!\!\!\perp_B^s c \perp\!\!\!\perp_B^s b$?
- Is there $n \in \mathbb{N}$ such that for arbitrary $a, b \in \mathcal{A}$ there are nonzero $c_1, \ldots, c_k \in \mathcal{A}$ (for $k \leq n$) such that $a \perp\!\!\!\perp_B^s c_1 \perp\!\!\!\perp_B^s \ldots \perp\!\!\!\perp_B^s c_k \perp\!\!\!\perp_B^s b$?

In the language of graphs

Let $\Gamma = \Gamma(\mathcal{A})$ be the graph with the vertex set

$$V(\Gamma(\mathcal{A})) = \{[a] = \mathbb{C}a : a \in \mathcal{A} \setminus \{0\}\}$$

and with vertices $[a], [b] \in V(\Gamma(\mathcal{A}))$ adjacent if $a \perp\!\!\!\perp_B^s b$. We identify a vertex $[a]$ with its representative a.

A vertex in a graph is isolated if there is no path between this vertex and any other vertex in the graph.

The distance between two distinct vertices is the length of the shortest path between them.

A graph is said to be connected if there exists a path from any vertex to any other vertex of the graph.

A connected component of a graph is a maximal (in the sense of inclusion) connected subgraph.

The diameter $\operatorname{diam}(\Gamma)$ of a graph Γ is the maximum of distances between vertices for all pairs of vertices in the graph; in the same way we define the diameter of a connected component of a graph.

In the language of graphs

We would like to answer the following questions:

- Let $a \in \mathcal{A}$. Is there a nonzero $b \in \mathcal{A}$ such that $a \perp_B^s b$?
- Let $a, b \in \mathcal{A}$ such that $a \not\perp_B^s b$. Is there a nonzero $c \in \mathcal{A}$ such that $a \perp_B^s c \perp_B^s b$?
- Is there $n \in \mathbb{N}$ such that, for arbitrary $a, b \in \mathcal{A}$ there are $c_1, \ldots, c_k \in \mathcal{A}$ for some $k \leq n$, such that $a \perp_B^s c_1 \perp_B^s \ldots \perp_B^s c_k \perp_B^s b$?

In terms of orthograph:

- Are there isolated points in $\Gamma(\mathcal{A})$?
- What are connected components of $\Gamma(\mathcal{A})$?
- What are diameters of connected components of $\Gamma(\mathcal{A})$?

We shall discuss two classes of C^*-algebras:

- the commutative C^*-algebras
- the C^*-algebra $\mathbb{B}(H)$

Commutative unital C^*-algebras

1. $a \perp\!\!\!\perp_B^s b \Rightarrow a$ and b are right noninvertible
   (Right invertible elements of $\mathcal{A}$ are isolated vertices in $\Gamma(\mathcal{A})$).
2. $f \perp_B^s g$ if and only if there is $x_0 \in K$ such that $|f(x_0)| = \|f\|$ and $g(x_0) = 0$.

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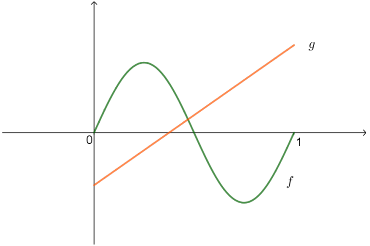

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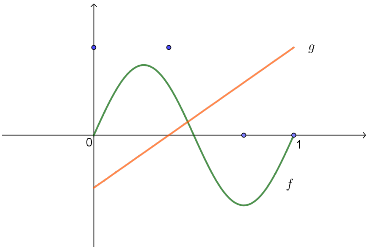

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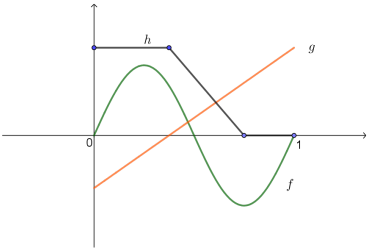

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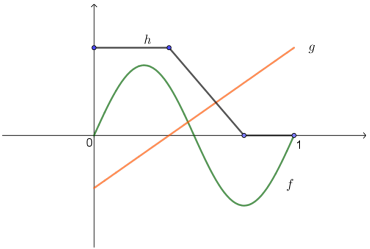

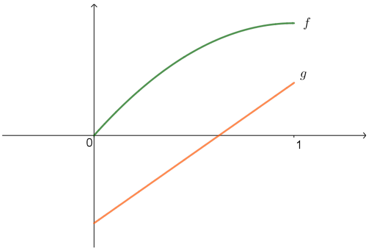

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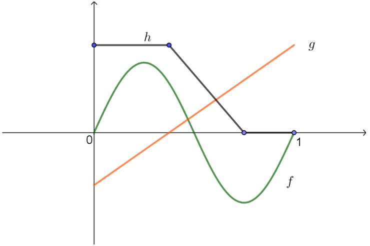

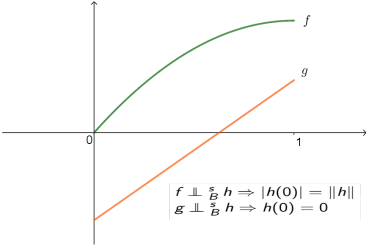

Commutative unital C^*-algebras

A., Guterman, Kuzma, Rajić, Zhilina, BJMA, 2020

Let K be a compact Hausdorff space, $|K| \geq 3$.

- There is $f \in C(K)$ with unique zero point t_0 if and only if t_0 has a countable local basis in K.
- Suppose no point in K has a countable local basis. If $f, g \in C(K)$ are noninvertible and such that $f \not\perp_B^s g$, then there is a nonzero $h \in C(K)$ such that $f \perp_B^s h \perp_B^s g$.
- Suppose there is a point in K with a countable local basis. There are noninvertible functions $f, g \in C(K)$ such that $f \not\perp_B^s g$, and the only $h \in C(K)$ which satisfies $f \perp_B^s h \perp_B^s g$ is $h = 0$. For such $f, g \in C(K)$ there are nonzero $h_1, h_2 \in C(K)$ such that $f \perp_B^s h_1 \perp_B^s h_2 \perp_B^s g$.

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- $f \in C(K)$ is isolated point in $\Gamma(C(K)) \Leftrightarrow f(t) \neq 0, \forall t \in K$.
- The set of all $f \in C(K)$ with a zero point in K is a connected component of the orthograph $\Gamma(C(K))$. Its diameter is 3 if at least one point of K has a countable local basis, otherwise its diameter is 2.

C^*-algebras $\mathbb{B}(H)$

A., Guterman, Kuzma, Rajić, Zhilina, BJMA, 2020

1. $A \in \mathbb{B}(H)$ is an isolated vertex of $\Gamma(\mathbb{B}(H))$ if and only if A is right invertible.
2. If $\dim H = 2$, then the connected components of the orthograph $\Gamma(\mathbb{B}(H))$ are either isolated vertices or the sets of the form $\mathcal{S}_\xi = \{A \in \mathbb{B}(H) : \operatorname{Im} A = \operatorname{span}\{\xi\} \text{ or } \operatorname{Im} A = \operatorname{span}\{\xi\}^\perp\}$ where $\xi \in H$ is nonzero. The diameter of each $\mathcal{S}_\xi$ is 2.
3. If $\dim H = 3$, then the set of all (right) noninvertible operators is a connected component whose diameter is 4.
4. If $\dim H \geq 4$, then the set of all right noninvertible operators is a connected component whose diameter is 3.

C^*-algebras $\mathbb{B}(H)$

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Thank you very much for your attention!

Спасибо! Danke!