BURES DISTANCE FOR COMPLETELY POSITIVE MAPS

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International Workshop
HILBERT C^*-MODULES ONLINE WEEKEND
In memory of William L. Paschke

Acknowledgements

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- Minimality: $\mathcal{H} = \overline{\text{span}}\{\pi(a)z : a \in \mathcal{A}\}$.

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- Then (H, π, z_1, z_2) is a common representation for ϕ_1, ϕ_2.

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- The infimum is attained in every common representation.
- The result has found many applications.

Completely positive (CP) maps

- A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be completely positive (CP) if,

$$\sum_{i,j} b_i^* \phi(a_i^* a_j) b_j \geq 0$$

for $a_i \in \mathcal{A}, b_i \in \mathcal{B}$.

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- CP maps are very important for understanding C^*-algebras and from applications point of view.

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- Then β is a metric.
- The infimum is attained in some representation and one has lower and upper bounds for β:

$$\frac{\|\phi_1 - \phi_2\|_{cb}}{\sqrt{\|\phi_1\|_{cb}} + \sqrt{\|\phi_2\|_{cb}}} \leq \beta(\phi_1, \phi_2) \leq \sqrt{\|\phi_1 - \phi_2\|_{cb}}$$

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- In good situations, such as when $\mathcal{B}$ is a von Neumann algebra, or an injective C^*-algebra, β is a metric and has similar bounds.
- Remark: The infimum is not attained in all common representations and in general it is not a metric (triangle inequality fails).

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- $\phi_i(a) = \langle z, \pi_i(a)z \rangle$, $i = 1, 2$.
- It can be proved that such joint representations $(\mathcal{E}, \pi_1, \pi_2, z)$ always exist.

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- It suffices to consider minimal joint representations.
- Theorem 1: If the range algebra $\mathcal{B}$ is a von Neumann algebra or injective C^*-algebra then γ is a metric.
- Theorem 2: γ is invariant under ampliations:

$$\gamma(\phi_1^{(n)}, \phi_2^{(n)}) = \gamma(\phi_1, \phi_2)$$

  for all $n \geq 1$.

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This is a C^* norm. Completion of $\mathcal{C} \circ \mathcal{D}$ under this norm is called the full free product of $\mathcal{C}$ and $\mathcal{D}$ and is denoted by $\mathcal{C} * \mathcal{D}$.

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- There are canonical injections $\rho_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} * \mathcal{D}$, $\rho_{\mathcal{D}} : \mathcal{D} \to \mathcal{C} * \mathcal{D}$. This way, $\mathcal{C}, \mathcal{D}$ are considered as sub-algebras of $\mathcal{C} * \mathcal{D}$.

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- There is a 1-1 correspondence between the $*$-representations of $\mathcal{C} * \mathcal{D}$ and pairs of $*$-representations of $\mathcal{C}$ and $\mathcal{D}$ on a common Hilbert space $\mathcal{H}$.

Joint representation module and free product

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- The joint representation module is minimal if and only if $\overline{\mathcal{A} * \mathcal{A} x \mathcal{B}} = \mathcal{E}$.

Representation metric through free products

- Theorem 3: Let $\mathcal{A} * \mathcal{A}$ be the free product of $\mathcal{A}$ with itself.

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- $(\mathcal{E}, \sigma, x)_\phi$ is the minimal Stinespring dilation of ϕ.
- Remark: A CP map in $K(\phi_1, \phi_2)$ is like a bivariate distribution with given marginals. This shows that the metric γ is somewhat like the Wasserstein metric for probability measures.

Consequences

- Theorem 4: Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be C^*-algebras. Let ϕ_1, ϕ_2 be UCP maps from $\mathcal{A}$ to $\mathcal{B}$ and ψ is a UCP map from $\mathcal{B}$ to $\mathcal{C}$. Then
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- Theorem 5 (Attainability of the metric): There is a $\phi \in K(\phi_1, \phi_2)$ for which the infimum is attained for $\gamma(\phi_1, \phi_2)$, that is,
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Main result for states: Relationship with Bures metric

- Theorem 6. For states ϕ_1, ϕ_2,

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- Only the negative sign is permissible, as $0 \leq \beta^2(\phi_1, \phi_2), \gamma(\phi_1, \phi_2) \leq 2$ is trivially true for unital CP maps.

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- It follows that

$$\gamma(\phi_1, \phi_2) = \inf_{\{\mathcal{K}, \pi, U, x, y\}} \|\pi - U^* \pi U\|_{cb}.$$

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- Proposition: Let T be a strict contraction on a Hilbert space $\mathcal{K}$. Then any unitary dilation V of T on $\mathcal{K} \oplus \mathcal{K}$ is up to unitary equivalence of the form

$$V = \begin{pmatrix} T & -(I - TT^*)^{\frac{1}{2}} W \\ (I - T^*T)^{\frac{1}{2}} & T^*W \end{pmatrix}$$

for some unitary W on $\mathcal{K}$.

Example

- Example: Let H be a separable infinite dimensional Hilbert space. Let $\mathcal{K}$ denote the set of all compact operators on H. Set $\mathcal{K}_+ =$ span $\{\mathcal{K}, \mathbb{C}I_H\}$. Let

$$\mathcal{A} = \begin{pmatrix} \mathcal{K}_+ & \mathcal{K} \\ \mathcal{K} & \mathcal{K}_+ \end{pmatrix} \subset \mathcal{B}(H \oplus H), \quad \mathcal{B} = \mathcal{K}_+.$$

Let p be a projection on H such that range of p and $1-p$ are both infinite dimensional subspaces of H.
Let $0 < \theta < \frac{\pi}{2}$. Set

$$u := e^{i\theta}p + e^{-i\theta}(1-p).$$

Then u is a unitary and $u \notin \mathcal{K}_+$. Let

$$z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I \\ I \end{pmatrix}, \quad z_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ I \end{pmatrix}.$$

Define unital CP maps $\phi_i : \mathcal{A} \to \mathcal{B}$, by
$\phi_i(a) = z_i^* a z_i, a \in \mathcal{A}, i = 1, 2.$

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- Qpen Question: Does the formula for β in terms of γ hold when the range algebra is a general von Neumann algebra?

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THANK YOU FOR YOUR PATIENCE