HILBERT C^*-MODULES WITH COMPLEMENTING PROPERTY

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Content

1. **Introduction and preliminaries**
2. **The main results**
   - Modules over algebra with compact like ideal
   - Modules with complementing property
   - Example

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{X}$ equipped with an $\mathcal{A}$-valued inner product $\langle \cdot | \cdot \rangle$ which is $\mathcal{A}$-linear in the second and $*$-conjugate linear in the first variable such that $\mathcal{X}$ is a Banach space with the norm $\|x\| = \|\langle x|x\rangle\|^{\frac{1}{2}}$. $\mathcal{X}$ is a full Hilbert $\mathcal{A}$-module if $\mathcal{A} = \langle \mathcal{X}|\mathcal{X}\rangle$ where $\langle \mathcal{X}|\mathcal{X}\rangle$ is the closed linear span of all elements in the underlying C^*-algebra $\mathcal{A}$ of the form $\langle x|y\rangle, x, y \in \mathcal{X}$.

The main objects and tools in this presentation are related to the extensions of Hilbert modules on which Prof. Damir Bakić and I worked about fifteen years ago.

First of all let's note that throughout $\mathcal{A}$ is a C^*-algebra with an essential closed two-sided ideal $\mathcal{I}$ and $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module. We prefer that the ideal is the proper ideal because otherwise our results coincide with the already known results on the characterization of Hilbert modules over C^*-algebras of compact operators.

- The first object ideal submodule $\mathcal{X}_{\mathcal{I}}$ of $\mathcal{X}$ associated to $\mathcal{I}$ is $\mathcal{X}_{\mathcal{I}} = \mathcal{X}\mathcal{I} = \{x \in \mathcal{X} : \langle x|x \rangle \in \mathcal{I}\} = \{x \in \mathcal{X}; \langle x|y \rangle \in \mathcal{I}, \forall y \in \mathcal{X}\}$. If $\mathcal{X}$ is full module, then $\mathcal{X}_{\mathcal{I}}$ is full as a Hilbert $\mathcal{I}$-module.

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- Also we have so called multiplier module $M(\mathcal{X}_{\mathcal{I}})$ of $\mathcal{X}_{\mathcal{I}}$ that is (not necessarily full) Hilbert C^*-module over the multiplier algebra $M(\mathcal{I})$ and contains $\mathcal{X}$.

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- Also we have so called multiplier module $M(\mathcal{X}_{\mathcal{I}})$ of $\mathcal{X}_{\mathcal{I}}$ that is (not necessarily full) Hilbert C^*-module over the multiplier algebra $M(\mathcal{I})$ and contains $\mathcal{X}$.
- Besides the norm topology, $M(\mathcal{X}_{\mathcal{I}})$ is also endowed with the strict topology induced by $\mathcal{X}_{\mathcal{I}}$. This is the topology induced by two families of seminorms: $v \mapsto \|\langle v|y\rangle\|$, $(y \in \mathcal{X}_{\mathcal{I}})$, and $v \mapsto \|vb\|$, $(b \in \mathcal{I})$. The strict topology is Hausdorff since $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.

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A net (v_λ) in $M(\mathcal{X}_{\mathcal{I}})$ converges strictly to $v \in M(\mathcal{X}_{\mathcal{I}})$, which is denoted by $v = \text{st-}\lim_\lambda v_\lambda$, if and only if $\langle v|y\rangle = \lim_\lambda \langle v_\lambda|y\rangle$, $\forall y \in \mathcal{X}_{\mathcal{I}}$, and $vb = \lim_\lambda v_\lambda b$, $\forall b \in \mathcal{I}$.

It is known that $\mathcal{X}_{\mathcal{I}}$ is strictly dense in $M(\mathcal{X}_{\mathcal{I}})$; moreover, it turns out that $M(\mathcal{X}_{\mathcal{I}})$ is the strict completion of $\mathcal{X}_{\mathcal{I}}$. Also, if $\mathcal{X}_{\mathcal{I}}$ is a full $\mathcal{I}$-module, we can look at $M(\mathcal{X}_{\mathcal{I}})$ as a largest Hilbert C^*-module over C^*-algebra containing $\mathcal{I}$ as an essential ideal such that $\mathcal{X}_{\mathcal{I}}$ is its ideal submodule with respect to $\mathcal{I}$.

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- For a submodule $\mathcal{F}$ of $M(\mathcal{X}_{\mathcal{I}})$, $\mathcal{X}$ and $\mathcal{X}_{\mathcal{I}}$ we denote by $\mathcal{F}^{\perp}$, $\mathcal{F}^{\perp_{\mathcal{X}}} = \mathcal{F}^{\perp} \cap \mathcal{X}$ and $\mathcal{F}^{\perp_{\mathcal{X}_{\mathcal{I}}}} = \mathcal{F}^{\perp} \cap \mathcal{X}_{\mathcal{I}}$ the orthogonal complement of $\mathcal{F}$ in $M(\mathcal{X}_{\mathcal{I}})$, $\mathcal{X}$ and $\mathcal{X}_{\mathcal{I}}$, respectively.

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- For a submodule $\mathcal{F}$ of $M(\mathcal{X}_\mathcal{I})$, $\mathcal{X}$ and $\mathcal{X}_\mathcal{I}$ we denote by $\mathcal{F}^\perp$, $\mathcal{F}^{\perp_\mathcal{X}} = \mathcal{F}^\perp \cap \mathcal{X}$ and $\mathcal{F}^{\perp_{\mathcal{X}_\mathcal{I}}} = \mathcal{F}^\perp \cap \mathcal{X}_\mathcal{I}$ the orthogonal complement of $\mathcal{F}$ in $M(\mathcal{X}_\mathcal{I})$, $\mathcal{X}$ and $\mathcal{X}_\mathcal{I}$, respectively.
- We denote by $c\ell(\mathcal{F})$ the closure of $\mathcal{F}$ in $M(\mathcal{X}_\mathcal{I})$ with respect to the norm topology.

For a submodule $\mathcal{F}$ of $M(\mathcal{X}_\mathcal{I})$, $\mathcal{X}$ and $\mathcal{X}_\mathcal{I}$ we denote by $c\ell^{st}(\mathcal{F})$, $c\ell^{st}_\mathcal{X}(\mathcal{F}) = c\ell^{st}(\mathcal{F}) \cap \mathcal{X}$ and $c\ell^{st}_{\mathcal{X}_\mathcal{I}}(\mathcal{F}) = c\ell^{st}(\mathcal{F}) \cap \mathcal{X}_\mathcal{I}$ the (relative) strict closure of $\mathcal{F}$ in $M(\mathcal{X}_\mathcal{I})$, $\mathcal{X}$ and $\mathcal{X}_\mathcal{I}$, respectively.

- The hereditary c^*-subalgebras of some c^*-algebra $\mathcal{B}$ is a c^*-subalgebra $\mathcal{A}$ having the property that if for $0 \leq b \in \mathcal{B}$ there exists $0 \leq a \in \mathcal{A}$ such that $b \leq a$ then $b \in \mathcal{A}$. The useful characterization of hereditary c^*-subalgebras $\mathcal{A}$ is $\mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$. By $\mathfrak{ha}(\mathcal{B})$ we denote the set of all such algebras. For nonempty set $S \subset \mathcal{B}$ we denote by $\mathfrak{ha}_S(\mathcal{B})$ the set of all c^*-algebras from $\mathfrak{ha}(\mathcal{B})$ containing S.

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- The hereditary $\mathcal{A}$-module of a full Hilbert $\mathcal{B}$-module $\mathcal{X}$, where $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$, is $\mathcal{X}_\mathcal{A} = \mathcal{X}\mathcal{A} = \{x \in \mathcal{X} : \langle x|x\rangle \in \mathcal{A}\} = \{x \in \mathcal{X}; |\langle y|x\rangle| \in \mathcal{A}, \forall y \in \mathcal{X}\}$. We note that hereditary $\mathcal{A}$-module of a Hilbert $\mathcal{B}$-module is its submodule iff it is a full module over some ideal in $\mathcal{B}$ what is generally not the case with its hereditarry subalgebras. We denote by $\mathfrak{hm}(\mathcal{X})$ the set of all hereditary C^*-modules of the Hilbert C^*-module $\mathcal{X}$ and for nonempty set $S \subset \mathcal{X}$ we denote by $\mathfrak{hm}_S(\mathcal{X})$ the set of all C^*-modules in $\mathfrak{hm}(\mathcal{X})$ containing S.

- If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, we denote by $\mathbf{B}(\mathcal{X},\mathcal{Y})$, $\mathbf{B}_a(\mathcal{X},\mathcal{Y})$ and $\mathbf{K}(\mathcal{X},\mathcal{Y})$ the Banach space of all bounded, adjointable and "compact" operators from $\mathcal{X}$ to $\mathcal{Y}$, respectively, and $\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X},\mathcal{X})$. The Banach space of all "compact" operators is generated by elementary "compact" operators $\Theta_{y,x}$, for all $x \in \mathcal{X}, y \in \mathcal{Y}$ acting as $\Theta_{y,x}z = y\langle x|z\rangle$, for all $z \in \mathcal{X}$.

- Let $(\mathcal{B}_j, \|\cdot\|_j)_{j\in\mathcal{J}}$ be a family of Banach spaces. For any closed ideal $\mathcal{C}$ of $\mathcal{C}_\infty(\mathcal{J})$ containing $\mathcal{C}_0$ we denote the outer direct sum

$$\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathcal{B}_j = \{x = (x_j)_{j\in\mathcal{J}} \in \Pi_{j\in\mathcal{J}}\mathcal{B}_j ; (\|x_j\|_j)_{j\in\mathcal{J}} \in \mathcal{C}\}. \qquad (1)$$

The set $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathcal{B}_j$ is a Banach space with the norm $\|x\|_\infty = \sup_{j\in\mathcal{J}} \|x_j\|_j$ and componentwise operations.

In what follows, for simplicity, we call *compact like C^*-algebra* any C^*-algebra which is isomorphic to a C^*-algebra of not necessarily all compact operators on a Hilbert space.

The main results

First we give characterizations and description of a class of full Hilbert C^*-modules over C^*-algebras containing an essential compact like ideal.

Theorem 1

Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(*i*) $\mathcal{I}$ *is a compact like C^*-algebra.*

Theorem 1

Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(i) *$\mathcal{I}$ is a compact like C^*-algebra.*

(ii) *There is a strict orthogonal bases for $\mathcal{X}$.*

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(ii) *There is a strict orthogonal bases for $\mathcal{X}$.*

(iii) *For every relatively strictly closed submodule $\mathcal{F}$ in $\mathcal{X}$ submodule $\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}}$ is relatively strictly dense in $\mathcal{X}$, i.e. $\mathcal{X} = c\ell^{st}_{\mathcal{X}}(\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}})$.*

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(iv) *Each relatively strictly closed submodule in $\mathcal{X}$ is orthogonally closed.*

Theorem 1

Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(v) *There are families of Hilbert spaces $(H_j)_{j\in\mathcal{J}}$, $(G_j)_{j\in\mathcal{J}}$, a family of C^*-algebras $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$,*

$$\mathbf{K}(H_j) \subseteq \mathbf{A}_j \subseteq \mathbf{B}(H_j),\ j \in \mathcal{J}, \tag{1}$$

and a family of Banach spaces of bounded linear operators $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$,

$$\mathbf{K}(H_j, G_j) \subseteq \mathbf{X}_j \subseteq \mathbf{B}(H_j, G_j),\ j \in \mathcal{J}, \tag{2}$$

such that $\mathcal{I}$ is isomorphic to c_0-$\oplus_j \mathbf{K}(H_j)$, ideal submodule $\mathcal{X}_\mathcal{I}$ is isomorphic to c_0-$\oplus_j \mathbf{K}(H_j, G_j)$ and $\mathcal{A}$-module $\mathcal{X}$ is isomorphic to $\mathbf{A}$-module $\mathbf{X}$.

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In what follows we say that Hilbert's C^*-module has the complementing property if each of its relatively strictly closed submodules is complemented.

Theorem 2

Let $\mathcal{A}$ be a C^*-algebra with an essential compact like ideal $\mathcal{K}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(*i*) *Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*

Theorem 2

Let $\mathcal{A}$ be a C^*-algebra with an essential compact like ideal $\mathcal{K}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(*i*) *Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*

(*ii*) *For each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the orthogonal sum $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$.*

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(ii) *For each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the orthogonal sum $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$.*

(iii) *For every relatively strictly closed submodules $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}, \mathcal{F} \perp \mathcal{G}$, the orthogonal sum $\mathcal{F} \oplus \mathcal{G}$ is relatively strictly closed in $\mathcal{X}$.*

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(ii) *For each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the orthogonal sum $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$.*

(iii) *For every relatively strictly closed submodules $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}, \mathcal{F} \perp \mathcal{G}$, the orthogonal sum $\mathcal{F} \oplus \mathcal{G}$ is relatively strictly closed in $\mathcal{X}$.*

(iv) *C^*-algebras $\mathbf{B}_a(\mathcal{X})$ and $\mathbf{B}_a(\mathcal{X}_{\mathcal{K}})$ of all adjointable operators on $\mathcal{X}$ and $\mathcal{X}_{\mathcal{K}}$ are isomorphic by isomorphism acting as restriction.*

In the following corollary we characterize one class of C^*-algebras by generic categorical property of some class of Hilbert C^*-modules over them.

Corollary 3

Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$. Then $\mathcal{A}$ is a hereditary subalgebra of $M(\mathcal{I})$ if and only if for any C^*-Hilbert module of the form $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A}$ the mapping $\beta : \mathbf{B}_a(\mathcal{X}) \to \mathbf{B}_a(\mathcal{X}_{\mathcal{I}})$ which acts as restriction is an isomorphism.

In the following theorem we characterize and describe the class of all full Hilbert C^*-modules which have the complementing property.

Theorem 3

Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:

(*i*) *Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*

Theorem 3

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(i) *Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*

(ii) *$\mathcal{I}$ is compact like C^*-algebra, $\mathcal{A} \in \mathfrak{ha}_{\mathcal{I}}(M(\mathcal{I}))$ and $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A} \in \mathfrak{hm}_{\mathcal{X}_{\mathcal{I}}}(M(\mathcal{X}_{\mathcal{I}}))$.*

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(iii) *There are families of Hilbert spaces $(H_j)_{j\in\mathcal{J}}$, $(G_j)_{j\in\mathcal{J}}$, C^*-algebra $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}} \in \mathfrak{ha}_{\mathbf{K}}(M(\mathbf{K}))$ and $\mathbf{A}$-module $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}} \in \mathfrak{hm}_{\mathbf{X}_{\mathbf{K}}}(M(\mathbf{X}_{\mathbf{K}}))$, such that $\mathbf{K} = c_0\text{-}\bigoplus_j \mathbf{K}(H_j)$, ideal submodule $\mathbf{X}_{\mathbf{K}} = c_0\text{-}\bigoplus_j \mathbf{K}(H_j, G_j)$, $\mathcal{I}$ is isomorphic to $\mathbf{K}$ and $\mathcal{A}$-module $\mathcal{X}$ is isomorphic to $\mathbf{A}$-module $\mathbf{X}$.*

Finally, we give an important hereditary property of Hilbert C^*-modules having the complementing property. Namely we claim that any relatively strictly closed submodule of such module possess the complementing property.

Proposition 4

Let $\mathcal{X} = (\mathcal{X}_j)_{j\in\mathcal{J}}$ be a full hereditary Hilbert $\mathcal{A}$-module of $M(\mathcal{X}_{\mathcal{K}})$ and $\mathcal{A} = (\mathcal{A}_j)_{j\in\mathcal{J}} \in \mathfrak{ha}_{\mathcal{K}}(M(\mathcal{K}))$, were $\mathcal{K} = c_0\text{-}\bigoplus_{j\in\mathcal{J}} \mathcal{K}_j$ and $\mathcal{K}_j$ is isomorphic to C^*-algebra of all compact operators on some Hilbert space H_j, $j \in \mathcal{J}$. Then every relatively strictly closed submodule $\mathcal{Y}$ in $\mathcal{X}$ is a hereditary (not necessarily full) $\mathcal{A}$-submodule of $M(\mathcal{Y}_{\mathcal{K}})$.

Proposition 4

Let $\mathcal{X} = (\mathcal{X}_j)_{j\in\mathcal{J}}$ be a full hereditary Hilbert $\mathcal{A}$-module of $M(\mathcal{X}_{\mathcal{K}})$ and $\mathcal{A} = (\mathcal{A}_j)_{j\in\mathcal{J}} \in \mathfrak{ha}_{\mathcal{K}}(M(\mathcal{K}))$, were $\mathcal{K} = c_0\text{-}\bigoplus_{j\in\mathcal{J}} \mathcal{K}_j$ and $\mathcal{K}_j$ is isomorphic to C^*-algebra of all compact operators on some Hilbert space H_j, $j \in \mathcal{J}$. Then every relatively strictly closed submodule $\mathcal{Y}$ in $\mathcal{X}$ is a hereditary (not necessarily full) $\mathcal{A}$-submodule of $M(\mathcal{Y}_{\mathcal{K}})$.
If $p = (p_j)_{j\in\mathcal{J}} \in \mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$ is a projection such that $\mathcal{Y} = p\mathcal{X}$, then for all $j \in \mathcal{J}$ we have $\langle\mathcal{Y}_j|\mathcal{Y}_j\rangle_j = \mathcal{I}_j$, where $\mathcal{I}_j$ is an ideal of $\mathcal{A}_j$ containing $\mathcal{K}_j$, if and only if $\mathcal{Y}_j = \mathcal{Y}_{j\mathcal{I}_j}$ if and only if $p_j \in \mathbf{K}_{\mathcal{I}_j}(\mathcal{X}_j) = c\ell(\mathrm{span}(\{\Theta_{x,y}; x, y \in \mathcal{X}_{j\mathcal{I}_j}\})) = \{T \in \mathbf{B}_a(\mathcal{X}_j); T\mathcal{X}_j \subseteq \mathcal{X}_{j\mathcal{I}_j}\} \simeq \mathbf{K}(\mathcal{X}_{j\mathcal{I}_j})$. If H_j is separable Hilbert space then $\langle\mathcal{Y}_j|\mathcal{Y}_j\rangle_j = \mathcal{A}_j$ or $\langle\mathcal{Y}_j|\mathcal{Y}_j\rangle_j = \mathcal{K}_j$, for all $j \in \mathcal{J}$.

Example

In the following example we discuss some basic C^*-algebras and Hilbert C^*-modules with the complementing property and those without it.

Example 5

Let H be infinite-dimensional separable Hilbert space, let $\mathbf{B}$ be a c^*-algebra of all bounded operators on H and let $\mathbf{K}$ be a c^*-algebra of all compact operators on H.
Then for c^*-algebra $\mathbf{A}$, where $\mathbf{K} \subset \mathbf{A} \subseteq \mathbf{B}$, $\mathbf{K}$ is an unique essential ideal in $\mathbf{A}$ and $\mathbf{B}$. In $\mathbf{B}$ we define inner product $\forall x, y \in \mathbf{B}$, $\langle x|y\rangle = x^*y$ with which $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ are full right Hilbert c^*-modules over $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ respectively. Strict topology in this modules is standard strict topology in $\mathbf{B}$ generated by $\mathbf{K}$. Submodules in this Hilbert modules are right ideals in corresponding algebras and ideal submodule of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{K}$.

Example 5

As for Hilbert modules in which each relatively strictly closed submodule is orthogonally complemented in this example we have basic cases $\mathbf{A} = \mathbf{K}$, $\mathbf{X} = \mathbf{K}$ and $\mathbf{A} = \mathbf{B}$, $\mathbf{X} = \mathbf{B}$. Also, for any projection $p \in \mathbf{B}$ with infinite rank and kernel $\mathbf{A} = p\mathbf{B}p + \mathbf{K}$ is hereditary c^*-subalgebra of $\mathbf{B}$ containing $\mathbf{K}$, and associated hereditary module is $\mathbf{X} = \mathbf{BA} = \mathbf{B}p + \mathbf{K}$. By the way, the multiplier algebra of $\mathbf{A}$ is $M(\mathbf{A}) = p\mathbf{B}p + (e-p)\mathbf{B}(e-p) + \mathbf{K}$, where e is unit in $\mathbf{B}$. Note that the c^*-algebra $\mathbf{Y} = \mathbf{A}$ is also a full Hilbert c^*-module over $\mathbf{A}$, but $\mathbf{Y} = p\mathbf{B}p + \mathbf{K} \subsetneqq \mathbf{X}$. This implies that some relatively strictly closed submodules in $\mathbf{Y}$ are not complemented.

Example 5

In order to determine which submodules are complemented we write Hilbert space $H = R \oplus L$, where $R = pH$, $L = (e - p)H$. Then projection p has a 2×2 matrix form $p = \begin{bmatrix} e_1 & 0_3 \\ 0_3^* & 0_2 \end{bmatrix}$ where $e_1 \in \mathbf{B}(R)$ is unit, $0_3 \in \mathbf{B}(L, R)$, $0_2 \in \mathbf{B}(R)$, and therefore $\mathbf{Y} = \begin{bmatrix} \mathbf{B}(R) & \mathbf{K}(L, R) \\ \mathbf{K}(R, L) & \mathbf{K}(L) \end{bmatrix}$. Any projection q from $\mathbf{B}(H)$ can be identified with the matrix $q = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix}$, where $a \in \mathbf{B}(R)$, $b, e_2 \in \mathbf{B}(L)$, e_2 is unit, and $c \in \mathbf{B}(L, R)$ such that $0 \leq a \leq e_1$, $0 \leq b \leq e_2$, $cc^* = a(e_1 - a)$, $c^*c = b(e_2 - b)$ and $ac = c(e_2 - b)$, i.e. $a = \frac{1}{2}(e_1 \pm (e_1 - 4cc^*)^{\frac{1}{2}})$ and $b = \frac{1}{2}(e_2 \mp (e_2 - 4c^*c)^{\frac{1}{2}})$.

Example 5

Then $q\mathbf{Y}$ is of the form

$$q\mathbf{Y} = \begin{bmatrix} a\mathbf{B}(R) + c\mathbf{K}(R,L) & a\mathbf{K}(L,R) + c\mathbf{K}(L) \\ c^*\mathbf{B}(R) + b\mathbf{K}(R,L) & c^*\mathbf{K}(L,R) + b\mathbf{K}(L) \end{bmatrix}.$$

From that it follows $q\mathbf{Y} \subseteq \mathbf{Y} \Leftrightarrow c^*\mathbf{B}(R) \subseteq \mathbf{K}(R,L) \Leftrightarrow c \in \mathbf{K}(L,R)$. Thus complemented are those and only those submodules in $\mathbf{Y}$ of the form $q\mathbf{B} \cap \mathbf{Y} = q\mathbf{Y}$ for which the c component of the projection q is a compact operator. Other submodules of the form $q\mathbf{B} \cap \mathbf{Y}$ are not complemented in $\mathbf{Y}$, but they are relatively strictly closed in $\mathbf{Y}$ and hence orthogonally closed in $\mathbf{Y}$ by Theorem 1.

Example 5

Let's consider now $\mathcal{A} = \mathbb{C}e + \mathbf{K}$, a minimal unitization of **K**, which is not a hereditary c^*-subalgebra of **B**. We know that every submodule in $\mathcal{A}$ which is closed in relatively strict topology, which is also orthogonally closed in $\mathcal{A}$, is of the form $\mathcal{G} = (e - p)\mathbf{B} \cap \mathbf{A}$ for some projection $p \in \mathbf{B}$. Then every $t \in \mathcal{G}$ is of the form $t = (e - p)b = \alpha e + k$ for some $b \in \mathbf{B}$, $k \in \mathbf{K}$ and $\alpha \in \mathbb{C}$. This implies that compact operator $k = (e - p)b - \alpha e$ for some $b \in \mathbf{B}$ and $\alpha \in \mathbb{C}$. Then $pk = -\alpha p$, and this is possible if and only if $\alpha = 0$ or the dimension of the range of p is finite.

Example 5

Relatively strictly closed submodules in $\mathcal{A}$ defined by projections with infinite-dimensional range and kernel are closed submodules of $\mathbf{K}$ (case $\alpha = 0$), i.e. $\mathcal{G} = (e - p)\mathbf{K}$ and $\mathcal{G}^{\perp_{\mathcal{A}}} = \mathcal{G}^{\perp_{\mathbf{K}}} = p\mathbf{K}$. They are not complemented in $\mathcal{A}$, but they are complemented in $\mathbf{K}$, i.e. $\mathcal{G} \oplus \mathcal{G}^{\perp_{\mathcal{A}}} = \mathbf{K}$, so the orthogonal sum is not relatively strictly closed in $\mathcal{A}$, but it is relatively strictly dense in $\mathcal{A}$. This submodule $\mathcal{G}$ is also an example of a submodule that is orthogonally closed in $\mathcal{A}$ but not complemented in $\mathcal{A}$.

Example 5

Consider a relatively strictly closed submodule in $\mathcal{A}$ that is defined by a projection with a finite-dimensional range or kernel. Then exactly one of the projections p or $e - p$ is in **K**. If this is p, then $p\mathcal{A} = \mathbb{C}p + p\mathbf{K} \subset \mathbf{K} \subset \mathcal{A}$, and then we have $(e - p)\mathcal{A} \subseteq \mathcal{A} - p\mathcal{A} \subseteq \mathcal{A}$, which gives $\mathcal{A} = (e - p)\mathcal{A} \oplus p\mathcal{A} = \mathcal{G} \oplus \mathcal{G}^{\perp_{\mathcal{A}}}$. Thus, submodules are orthogonally complemented in $\mathcal{A}$ if and only if the associated projection has a finite-dimensional range or kernel. $\diamondsuit$

THANK YOU FOR YOUR ATTENTION