Pre-Hilbert modules, normed modules and the parallelogram law

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Definition

Let A be a C*-algebra.
A **pre-Hilbert A-module** is a right module H over A which is equipped with a generalized inner product, that is with an A-valued map $\langle\, .\, ,\, .\,\rangle$ on $H \times H$, having the following properties:

(H1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in H$,

(H2) $\langle x, ya\rangle = \langle x, y\rangle a$ for all $x, y \in H$ and $a \in A$,

(H3) $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in H$,

(H4) $\langle x, x\rangle \geq 0$ for every $x \in H$ and $\langle x, x\rangle = 0$ implies $x = 0$.

Each pre-Hilbert A-module $(H, \langle\, .\, ,\, .\,\rangle)$ is a metric space with respect to a metric $d\colon H \times H \to \mathbb{C}$ given by $d(x, y) = \|\langle x - y,\, x - y\rangle\|^{1/2}$ (where $\|\, .\, \|$ denotes the norm in A). If this metric space is complete, then $(H, \langle\, .\, ,\, .\,\rangle)$ is called a **Hilbert A-module**.

Definition

Let A be an H*-algebra.
A **pre-Hilbert A-module** is a right module H over A which is equipped with a generalized inner product, that is with an $\tau(A)$-valued map $\langle\,.\,,\,.\,\rangle$ on $H \times H$, having the following properties:

(H1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in H$,

(H2) $\langle x, ya\rangle = \langle x, y\rangle a$ for all $x, y \in H$ and $a \in A$,

(H3) $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in H$,

(H4) $\langle x, x\rangle \geq 0$ for every $x \in H$ and $\langle x, x\rangle = 0$ implies $x = 0$.

An **H*algebra** is a complex Banach algebra A whose norm is induced by an inner product $\langle\,.\,,\,.\,\rangle$ such that for each $a \in A$ there exists a unique $a^* \in A$ satisfying

$$\langle ab, c\rangle = \langle b, a^* c\rangle \text{ and } \langle ba, c\rangle = \langle b, ca^*\rangle, \quad b, c \in A.$$

The **trace-class** for A is $\tau(A) = \{ab : a, b \in A\}$ equipped with the norm $\tau(\,.\,)$ such that $(\tau(A), \tau(\,.\,))$ is a Banach $*$-algebra.

Example

Let A be a C*-algebra or an H*-algebra such that there exists $x \in A$ with $xv \neq x$ for every $v \in A$. The set $A \times \mathbb{R}$, endowed with coordinatewise addition, is a right A-module under its action on A given by

$$(a, \lambda)b = (ab + \lambda b, 0), \quad a, b \in A, \ \lambda \in \mathbb{R}.$$

Note that

$$M = \{(a, \lambda) \in A \times \mathbb{R} \ : \ xa + \lambda x = 0\}$$

is a submodule of $A \times \mathbb{R}$. Let us put $H = (A \times \mathbb{R})/M$ and define

$$\big((a, \lambda) + M\big) + \big((b, \mu) + M\big) = (a + b, \lambda + \mu) + M, \quad a, b \in A, \ \lambda, \mu \in \mathbb{R},$$

$$\big((a, \lambda) + M\big)b = (ab + \lambda b, 0) + M \quad a, b \in A, \ \lambda \in \mathbb{R}.$$

Under these operations, H is a right module over A. We can equip H with the structure of a pre-Hilbert A-module via

$$\big\langle (a, \lambda) + M, (b, \mu) + M \big\rangle = (xa + \lambda x)^*(xb + \mu x), \quad a, b \in A, \ \lambda, \mu \in \mathbb{R}.$$

Example (continuation)

Assume that H has the structure of a complex vector space compatible with the structure of A. Let $\{e_\lambda : \lambda \in \Lambda\}$ be an approximate identity for A. We have

$$\big(i \cdot ((0,1) + M)\big)e_\lambda = \big((0,1) + M\big)(ie_\lambda), \quad \text{that is,}$$

$$\big(i \cdot ((0,1) + M)\big)e_\lambda = (ie_\lambda, 0) + M.$$

If $i \cdot \big((0,1) + M\big) = (u, \eta) + M \in H$, then

$$\big(i \cdot ((0,1) + M)\big)e_\lambda = \big((u,\eta) + M\big)e_\lambda = (ue_\lambda + \eta e_\lambda, 0) + M.$$

Therefore,

$$(ue_\lambda + \eta e_\lambda - ie_\lambda, 0) \in M, \quad \lambda \in \Lambda, \qquad \text{so}$$

$$x(ue_\lambda + \eta e_\lambda - ie_\lambda) = 0, \quad \lambda \in \Lambda.$$

After taking limits, we get $xu = (i - \eta)x$. Since $\eta \in \mathbb{R}$, we have $\eta \neq i$. If we define $v = \frac{1}{i-\eta}u \in A$, then $xv = x$, contradicting our assumption.

Let A be a C*-algebra or an H*-algebra and let $(H, \langle\,.\,,\,.\,\rangle)$ be a pre-Hilbert A-module.
For $x \in H$ set $N(x) = \langle x, x\rangle^{1/2} \in A$. Then we have:

(1) $N(x) \geq 0$ for every $x \in H$,

(2) $N(x) = 0$ implies $x = 0$,

(3) $N(xa) = |N(x)a|$ for all $x \in H$ and $a \in A$,

(4) $\|N(x+y)\| \leq \|N(x)\| + \|N(y)\|$ for all $x, y \in H$.

Definition

Let $(A, \|\cdot\|)$ be a C*-algebra or an H*-algebra.
A **normed** A**-module** is a right module H over A together with a map $N\colon H \to A$ having the following properties:

(N1) $N(x) \geq 0$ for every $x \in H$,

(N2) $N(x) = 0$ implies $x = 0$,

(N3) $N(xa) = |N(x)a|$ for all $x \in H$ and $a \in A$,

(N4) $\|N(x+y)\| \leq \|N(x)\| + \|N(y)\|$ for all $x, y \in H$.

Every pre-Hilbert A-module is a normed A-module.

Note that (N3) and (N1) imply

$$N(xa)^2 = |N(x)a|^2 = a^* N(x)^2 a, \qquad x \in H,\ a \in A.$$

A generalized normed space

Definition (B. Zalar, Acta Math. Hungar., 1995)

Let A be an H*-algebra and let H be a right A-module.
Let $N\colon H \to A$ be a map with the following properties:

(1) $N(x) \geq 0$ for every $x \in H$,

(2) $N(x) = 0$ implies $x = 0$,

(3) $N(xa) = |N(x)a|$ for all $x \in H$ and $a \in A$,

(4) $\|N(x+y)\| \leq \|N(x)\| + \|N(y)\|$ for all $x, y \in H$,

(5) if $\{x_\lambda\}$ is a generalized sequence in H such that for all $\varepsilon > 0$ there exists λ_0 such that for all $\lambda, \mu \geq \lambda_0$ we have $\|N(x_\lambda - x_\mu)\| < \varepsilon$ then $\{N(x_\lambda)\}$ is a generalized Cauchy sequence in A.

Then N is called an A-valued generalized norm (A-norm) and (H, N) is called a **generalized normed space** or a **normed A-module**.

Definition (N.C. Phillips and N. Weaver, Pacific J. Math., 1998)

Let A be a C*-algebra and let A_+ denote the set of positive elements in A. A **Finsler module** is a right A-module H which is equipped with a map $N\colon H \to A_+$ such that

(1) the map $\|\cdot\|_H\colon x \mapsto \|N(x)\|$ is a Banach space norm on H,

(2) $N(xa)^2 = aN(x)^2a$ for all $x \in H$, $a \in A$.

When do these structures arise from (pre-)Hilbert modules?

Theorem (B. Zalar, Acta Math. Hungar., 1995)

Let $(A, \|\cdot\|)$ be an H^*-algebra. A normed A-module (H, N) satisfies the parallelogram law

$$N(x+y)^2 + N(x-y)^2 = 2N(x)^2 + 2N(y)^2, \quad x, y \in H,$$

if and only if H is a pre-Hilbert A-module with respect to the generalized inner product $\langle \cdot, \cdot \rangle$ such that $N(x)^2 = \langle x, x \rangle$ holds for all $x \in H$.

Theorem (N.C. Phillips and N. Weaver, Pacific J. Math., 1998)

Every Finsler A-module, when A is a C^*-algebra without nonzero commutative closed two-sided ideals, arises from a unique Hilbert A-module.

Let M be a right module over a $*$-ring R. A map $Q\colon M \to R$ is called a **quadratic functional** if it satisfies the parallelogram law

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y), \quad x,y \in M$$

and the homogeneity equation

$$Q(xa)=a^*Q(x)a, \quad x \in M,\ a \in R.$$

A biadditive form $S\colon M \times M \to R$ is called **sesquilinear** if

$$S(xa,y)=a^*S(x,y) \text{ and } S(x,ya)=S(x,y)a, \quad x,y \in M,\ a \in R.$$

For every sesquilinear form $S\colon M \times M \to R$, the map $Q : M \to R$ defined by $Q(x)=S(x,x)$ is a quadratic functional.

What about the converse?

Let $(A, \|\,.\,\|)$ be a complex Banach $*$-algebra with an approximate identity $\{e_\lambda\}$. If the generalized sequence $\{L_{{e_\lambda}^*}\}$ is bounded, then

$$\begin{aligned} \|{e_\lambda}^* a e_\lambda - a\| &\leq \|{e_\lambda}^* a e_\lambda - {e_\lambda}^* a\| + \|{e_\lambda}^* a - a\| = \\ &= \|L_{e_\lambda^*}(a e_\lambda - a)\| + \|a^* e_\lambda - a^*\| \leq \\ &\leq \|L_{e_\lambda^*}\| \cdot \|a e_\lambda - a\| + \|a^* e_\lambda - a^*\| \longrightarrow 0. \end{aligned}$$

Examples of such algebras and approximate identities:

(I) A is a C*-algebra and $\{e_\lambda\}$ is the canonical approximate identity for A.

(II) A is an H*-algebra and $\{e_\lambda\}$ is an approximate identity consisting of projections.

(III) A is the trace-class associated with an H*-algebra and $\{e_\lambda\}$ is an approximate identity for that H*-algebra, consisting of projections.

Theorem

Let $(A, \|\,.\,\|)$ be a complex Banach $*$-algebra with an approximate identity $\{e_\lambda\}$ such that

$$\lim_\lambda \|{e_\lambda}^* a e_\lambda - a\| = 0, \quad a \in A.$$

Let M be a right module over A and let $Q\colon M \to A$ be a quadratic functional. There exists a unique sesquilinear form $S\colon M \times M \to A$ satisfying $S(x,x) = Q(x)$ for every $x \in M$ if and only if for all $x, y \in M$ the limit $\lim_\lambda Q(x(ie_\lambda) + y) \in A$ exists. In that case,

$$\begin{aligned} S(x,y) &= \tfrac{1}{4}\big(Q(x+y) - Q(x-y)\big) \\ &\quad + \tfrac{i}{4}\lim_\lambda \big(Q(x(ie_\lambda) + y) - Q(x(ie_\lambda) - y)\big). \end{aligned}$$

If $Q(x)^* = Q(x)$ for every $x \in M$, then $S(x,y)^* = S(y,x)$ for all $x, y \in M$.

Theorem

Let $(A, \| \, . \, \|)$ be a complex Banach $*$-algebra with an approximate identity $\{e_\lambda\}$ such that

$$\lim_{\lambda} \|{e_\lambda}^* a e_\lambda - a\| = 0, \quad a \in A.$$

Let the right module M over A be a complex vector space compatible with the structure of A. For each quadratic functional $Q\colon M \to A$ there exists a unique sesquilinear form $S\colon M \times M \to A$ such that $S(x,x) = Q(x)$ for every $x \in M$. Furthermore,

$$S(x,y) = \tfrac{1}{4}\big(Q(x+y) - Q(x-y)\big) + \tfrac{i}{4}\big(Q(ix+y) - Q(ix-y)\big).$$

If $Q(x)^* = Q(x)$ for every $x \in M$, then $S(x,y)^* = S(y,x)$ for all $x, y \in M$.

Let $\mathcal{H}$ be a (real or complex) Hilbert space. A **standard operator algebra** is a (not necessarily self-adjoint) subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the ideal $\mathcal{F}(\mathcal{H})$ of finite rank operators.

Example

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Let us choose $U \in \mathcal{B}(\mathcal{H})$ such that the kernel of U^* is infinite-dimensional and such that $U^*U = I$ (concretely, $U = \sum_n e_{2n} \otimes e_n$, where (e_n) is an orthonormal basis for $\mathcal{H}$).
The algebra $A = \mathcal{F}(\mathcal{H}) + U \cdot \mathcal{B}(\mathcal{H})$ is a standard operator algebra and $U = 0 + U \cdot I \in A$.
If we assume $U^* \in A$, then $U^* = F + UB$ for some $F \in \mathcal{F}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$, hence $U = F^* + B^*U^*$. This first implies $I = U^*F^* + U^*B^*U^*$ and then $I - U^*B^*U^* = U^*F^* \in \mathcal{F}(\mathcal{H})$.
However, $I - U^*B^*U^*$ is the identity operator on the kernel of U^*, hence it is not a finite rank operator. Therefore, $U^* \notin A$.

Theorem

Let $A \subseteq \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} > 1$, be a standard operator algebra and let M be a right module over A. Then for each quadratic functional $Q\colon M \to A$ there exists a sesquilinear form $S\colon M \times M \to \mathcal{B}(\mathcal{H})$ such that $S(x,x) = Q(x)$ for every $x \in M$.

One of the key ingredients in the proof is the fact that

$$\lim_{\lambda} |p_\lambda a p_\lambda \xi - a\xi| = 0, \quad a \in A,\ \xi \in \mathcal{H}$$

for a generalized sequence $\{p_\lambda\}$ in $\mathcal{H}$ consisting of finite rank projections such that $\lim_\lambda |p_\lambda \xi - \xi| = 0$ for every $\xi \in \mathcal{H}$.

Let $A \subseteq \mathcal{B}(\mathcal{H})$, $\dim\mathcal{H} > 1$, be a standard operator algebra.
Let $\xi \in \mathcal{H}$ and let p be the orthogonal projection on the subspace of $\mathcal{H}$ generated by ξ. Take any η orthogonal to ξ having the same norm as ξ and define $q : \mathcal{H} \to \mathcal{H}$ by $q(\lambda\eta + \varrho) = \lambda\xi$ for every scalar λ and every ϱ orthogonal to η. Then $p, q \in A$ are such that

$$p = p^*,\ p\xi = \xi,\ qp = 0,\ qq^* = p.$$

If $Q\colon M \to A$ satisfies $Q(xa) = a^*Q(x)a$ for all $x \in M$ and $a \in A$ then we have

$$\langle Q(x \pm y)\xi, \xi\rangle = \langle (p \pm q)Q(xp + yq)(p \pm q^*)\xi, \xi\rangle\,,\ \xi \in \mathcal{H},\ x, y \in M.$$

Adding yields the parallelogram law.

Lemma

Let $(A, \|\cdot\|)$ be a C-algebra or an H*-algebra without nonzero commutative closed two-sided ideals.*
Let (H, N) be a normed A-module.
If $\{e_\lambda\}$ is an approximate identity for A, then, for all $x, y \in H$, a generalized sequence $\{N(x(ie_\lambda) + y)^2\}$ converges in $(A, \|\cdot\|)$ if A is a C-algebra or in $(\tau(A), \tau(\cdot))$ if A is an H*-algebra.*

The proof is based on the following inequalities:

$$\|N(x)^2 - N(y)^2\| \leq \|N(x-y)\|(\|N(x)\| + \|N(y)\|), \quad x, y \in H$$

in case of C*-algebras, and

$$\tau(N(x)^2 - N(y)^2) \leq \|N(x-y)\|(\|N(x)\| + \|N(y)\|), \quad x, y \in H$$

in case of H*-algebras.

Theorem

Let A be a C-algebra or an H*-algebra without nonzero commutative closed two-sided ideals.*
Then the class of normed A-modules coincides with the class of pre-Hilbert A-modules.

The A-valued inner product can be recovered from the A-valued norm via

$$\begin{aligned}\langle x, y\rangle &= \tfrac{1}{4}\big(N(x+y)^2 - N(x-y)^2\big)\\ &\quad + \tfrac{i}{4}\lim_{\lambda}\big(N(x(ie_\lambda)+y)^2 - N(x(ie_\lambda)-y)^2\big).\end{aligned}$$

D.I., *Quadratic functionals on modules over complex Banach $*$-algebras with an approximate identity*, Studia Math. 171 (2005)

Homogeneity equation: $Q(xa) = a^* Q(x)a$ for all x and a, that is, for A-valued norm: $N(xa) = |N(x)a|$ for all x and a.

In normed spaces: $\|\lambda x\| = |\lambda|\|x\|$ for all λ and x.
Note that $\|\lambda x\| \leq |\lambda|\|x\|$ implies $\|\lambda x\| = |\lambda|\|x\|$.

Let us also note that $\|N(xa)\| \leq \|N(x)\| \, \|a\|$.

Example

Let H be a pre-Hilbert A-module and let $x_0 \in H$. Let $M = \{x \in H \,:\, \langle x_0, x\rangle = 0\}$; it is a submodule of H. On H/M we define an A-valued map N by $N(x) = (\langle x_0, x\rangle\langle x, x_0\rangle)^{1/2}$. Then N is positive definite, and satisfies both the homogeneity and the triangle inequality for the composition of the norm in A and N.

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Let $(A, \|\cdot\|)$ be a C*-algebra or an H*-algebra.
A **normed** A**-module** is a right module H over A together with a map $\|\cdot\|_H \colon H \to \mathbb{C}$ having the following properties:

(N1) $\|x\|_H \geq 0$ for every $x \in H$,

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(N3) $\|xa\|_H \leq \|x\|_H \|a\|$ for all $x \in H$ and $a \in A$,

(N4) $\|x + y\|_H \leq \|x\|_H + \|y\|_H$ for all $x, y \in H$.

The map $\|\cdot\|_H \colon H \to \mathbb{C}$ defined by $\|x\|_H = \|\langle x, x\rangle\|^{1/2}$ is a norm on H and $\|xa\|_H \leq \|x\|_H \|a\|$ holds for all $x \in H$ and $a \in A$.

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(N4) $\|N(x+y)\| \leq \|N(x)\| + \|N(y)\|$ for all $x, y \in H$.

- Birkhoff-James orthogonality: $\|x\| \leq \|x + \lambda y\|, \forall \lambda$.
- Isosceles orthogonality: $\|x + y\| = \|x - y\|$.
- Roberts' orthogonality: $\|x + \lambda y\| = \|x - \lambda y\|, \forall \lambda$.
- Pythagorean orthogonality: $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- etc.

There are many known results of the following type:

If a given orthogonality is equivalent to (or implies) another one, then a normed space is an inner product space.

Corresponding results for normed modules?