On operators on Hilbert C^*–modules

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Introduction

In this presentation we let $\mathcal{A}$ be a unital C^*-algebra, $H_{\mathcal{A}}$ be the standard module over $\mathcal{A}$ (this is $H_{\mathcal{A}} = l_2(\mathcal{A})$) and $B^a(H_{\mathcal{A}})$ be the set of all $\mathcal{A}$-linear, bounded adjointable operators on $H_{\mathcal{A}}$.
We wish to solve the equations of the form $Fx = y$, where $F \in B^a(H_{\mathcal{A}})$ and $x, y \in H_{\mathcal{A}}$. Even if F is not invertible, we can still handle this equation if F is regular i.e. if F admits generalized inverse. This happens if ImF is closed and in this case F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, w.r.t. the decomposition

$$H_{\mathcal{A}} = \ker F^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus ImF^{\perp} = H_{\mathcal{A}},$$

where F_1 is an isomorphism and the generalized inverse of F has the matrix $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ w.r.t. the decomposition

$$H_{\mathcal{A}} = ImF \oplus ImF^{\perp} \longrightarrow \ker F^{\perp} \oplus \ker F = H_{\mathcal{A}}.$$

If in addition $ImF^{\perp}$ is finitely generated, then it is easy to check whether the equation $Fx = y$ has a solution. On the other hand, if F is regular and in addition $\ker F$ is finitely generated, then we have an explicit formula for the solutions of the equation $Fx = y$ in the case when the solution exists. This motivates to study the following classes of operators on H_A.

Semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$

Inspired by definition of $\mathcal{A}$-Fredholm operator given in [**MF**], we give now the following definition.

Definition

Let $F \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\mathcal{M}\Phi_+(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm } \},$$

$$\mathcal{M}\Phi_-(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm } \},$$

$\mathcal{M}\Phi(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F$ is $\mathcal{A}$-Fredholm operator on $H_\mathcal{A}\}$. Then obviously $\mathcal{M}\Phi(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A})$. We are going to show later in this section that actually "$=$" holds.
Notice that if M, N are two arbitrary Hilbert modules C^*-modules, the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M, N)$ and $\mathcal{M}\Phi_-(M, N)$.

We let now $K^*(H_{\mathcal{A}})$ denote the closed, two sided ideal of adjointable compact operators in $B^a(H_{\mathcal{A}})$, see [**MT**].

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent
1) $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$
2) There exists $D \in B^a(H_{\mathcal{A}})$ such that $DF = I + K$ for some $K \in K^*(H_{\mathcal{A}})$

Theorem

Let $D \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:
1) $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$
2) There exist $F \in B^a(H_{\mathcal{A}}), K \in K^*(H_{\mathcal{A}})$ s.t. $DF = I + K$

Corollary

$\mathcal{M}\Phi(H_{\mathcal{A}}) = \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$

Corollary

$\mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}(H_{\mathcal{A}})$ are semigroups under multiplication.

Corollary

Let $F \in B^{a}(M, N)$. Then $F \in \mathcal{M}\Phi_{+}(M, N)$ if and only if $F^{*} \in \mathcal{M}\Phi_{-}(N, M)$. Moreover, if $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, then $F^{*} \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\mathrm{index} F = -\mathrm{index} F^{*}$.

Lemma

Let M be a closed submodule of $H_{\mathcal{A}}$ s.t. $H_{\mathcal{A}} = M\tilde{\oplus}N$ for some finitely generated submodule N. Let $F \in B^a(H_{\mathcal{A}})$, J_M be the inclusion map from M into $H_{\mathcal{A}}$ and suppose that $FJ_M \in \mathcal{M}\Phi_+(M, H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$.

Lemma

Suppose that $D, F \in B^a(H_{\mathcal{A}})$ $DF \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $\mathrm{Im}F$ is closed. Then $DJ_{\mathrm{Im}F} \in \mathcal{M}\Phi_+(\mathrm{Im}F, H_{\mathcal{A}})$.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and suppose that there are two decompositions

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

with respect to which F has matrices

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}, \begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix},$$

respectively, where $F_1, {F_1}'$ are isomorphisms, $N_1, {N_1}', N_2$ are closed, finitely generated and ${N_2}'$ is just closed. Then ${N_2}'$ is finitely generated also.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ *and let*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be a decomposition with respect to which F *has the matrix*

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 *is an isomorphism,* N_2 *is finitely generated and* N_1 *is just closed. Then* N_1 *is finitely generated.*

Lemma

Let $F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ *and suppose that* $\mathrm{Im}F$ *is closed. If*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

are two $\mathcal{M}\Phi_{+}$ *decomposition for* F *then* $F(N_1), F(N_1')$ *are closed finitely generated projective modules and*

$$[N_1] - [F(N_1)] = [N_1'] - [F(N_1')]$$

in $K(A)$.

Lemma

Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. *Then there is no sequence of unit vectors* $\{x_n\}$ *in* $H_{\mathcal{A}}$ *such that* $\varphi(x_n) \to 0$ *in* $\mathcal{A}$ *for all* $\varphi \in H'_{\mathcal{A}}$ *and* $\lim_{n\to\infty} \| Fx_n \| = 0$.

Generalized Schechter characterization of $\mathcal{M}\Phi_{+}$ operators on $H_{\mathcal{A}}$

Lemma

Let $F \in B^{a}(M,N)$ Then $F \in \mathcal{M}\Phi_{+}(M,N)$ if and only if there exists a closed, orthogonally complementable submodule $M' \subseteq M$ such that $F_{|_{M'}}$ is bounded below and $M'^{\perp}$ is finitely generated.

Lemma

Let $F \in B^{a}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$. Then there exists a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ s.t.

$$x_k \in L_{n_k} \setminus L_{n_{k-1}} \text{ for all } k \in \mathbb{N}, \| x_k \| \leq 1 \text{ for all } k \in \mathbb{N}$$

and

$$\| Fx_k \| \leq 2^{1-2k} \text{ for all } k \in \mathbb{N}.$$

Openness of the set of semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$

Theorem

The sets $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ *and* $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ *are open in* $B^{a}(H_{\mathcal{A}})$, *where* $B^{a}(H_{\mathcal{A}})$ *is equipped with the norm topology.*

Corollary

If $F \in B^{a}(H_{\mathcal{A}})$ *belongs to the boundary of* $\mathcal{M}\Phi(H_{\mathcal{A}})$ *in* $B^{a}(H_{\mathcal{A}})$ *then* $F \notin \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$.

Corollary

Let $f : [0,1] \to B^{a}(H_{\mathcal{A}})$ *be continuous and assume that* $f([0,1]) \subseteq \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})$. *Then the following statments hold:*
1) If $f(0) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, *then* $f(1) \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$
2) If $f(0) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$, *then* $f(1) \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$
3) If $f(0) \in \mathcal{M}\Phi(H_{\mathcal{A}})$, *then* $f(1) \in \mathcal{M}\Phi(H_{\mathcal{A}})$ *and* $\mathrm{index} f(0) = \mathrm{index} f(1)$.

Definition

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$. We say that $F \in \tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1, N_2 are closed, finitely generated and $N_1 \preceq N_2$, that is N_1 is isomorphic to a closed submodule of N_2. We define similarly the class $\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})$, the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi_+^-(H_{\mathcal{A}}) = (\tilde{\mathcal{M}}\Phi_+^-(H_{\mathcal{A}})) \cup (\mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}))$$

and

$$\mathcal{M}\Phi_-^+(H_{\mathcal{A}}) = (\tilde{\mathcal{M}}\Phi_-^+(H_{\mathcal{A}})) \cup (\mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}))$$

Further, we define $\mathcal{M}\Phi_0(H_\mathcal{A})$ to be the set of all $F \in \mathcal{M}\Phi(H_\mathcal{A})$ for which there exists an $\mathcal{M}\Phi$-decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A},$$

where $N_1 \cong N_2$.

Lemma

Suppose that $K(\mathcal{A})$ satisfies "the cancellation property". If $F \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$, then for any decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_\mathcal{A}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix},$$

where F_1' is an isomorphism, N_1', N_2' are finitely generated, we have $N_1' \preceq N_2'$. Similarly $N_1' \preceq N_2'$ if $F \in \tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$.

Proposition

Let $K \in K^*(H_{\mathcal{A}})$ and $T \in B^a(H_{\mathcal{A}})$. Suppose that T is invertible and that $K(\mathcal{A})$ satisfies the cancellation property.Then the equation $(T+K)x = y$ has a solution for every $y \in H_{\mathcal{A}}$ if and only if $T+K$ is bounded below. In this case the solution of the equation above is unique.

Lemma

$\tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}})$ are semigroups under multiplication.

Lemma

$\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})$ are semigroups under multiplication.

Lemma

$\tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$ and $\tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}})$ are open.

Definition

Let $F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$. We say that $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$, only in this case $F \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$, N_2 is finitely generated and $N_2 \preceq N_1$.

Proposition

$$\tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}}) = \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}), \tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}}) = \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}}).$$

Lemma

The sets $\mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$ *and* $\mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ *are open. Moreover, if* $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ *and* $K \in K^{*}(H_{\mathcal{A}})$, *then*

$$(F+K) \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}}).$$

If $F \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$ *and* $K \in K^{*}(H_{\mathcal{A}})$, *then*

$$(F+K) \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}}).$$

Lemma

The sets $\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$, $\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$ *and* $\mathcal{M}\Phi(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_{0}(H_{\mathcal{A}})$ *are open.*

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent

1) $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$

2) There exist $D \in B^a(H_{\mathcal{A}}), K \in K^*(H_{\mathcal{A}})$ such that D is bounded below and $F = D + K$

Proposition

1) $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$

2) $F \in \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \tilde{\mathcal{M}}\Phi_{-}^{+}(H_{\mathcal{A}})$

3) $F \in \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \Leftrightarrow F^* \in \mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})$

Definition

We set $M^a(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is bounded below}\}$ and $Q^a(H_{\mathcal{A}}) = \{D \in B^a(H_{\mathcal{A}}) \mid D \text{ is surjective } \}$.

Lemma

Let $B^a(H_{\mathcal{A}})$. *Then* $F \in M^a(H_{\mathcal{A}})$ *if and only if* $F^* \in Q^a(H_{\mathcal{A}})$.

Corollary

Let $D \in B^a(H_{\mathcal{A}})$. *The following statements are equivalent:*
1) $D \in \mathcal{M}\Phi_{-}^{+'}(H_{\mathcal{A}})$
2) There exist $Q \in Q^a(H_{\mathcal{A}}), K \in K^*(H_{\mathcal{A}})$ *s.t.* $D = Q + K$.

Theorem

Let $B^a(H_{\mathcal{A}})$. *Then the following statements are equivalent:*
1) $F \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$
2) There exist an invertible $D \in B^a(H_{\mathcal{A}})$ *and* $K \in K^*(H_{\mathcal{A}})$ *such that* $F = D + K$.

Non adjointable semi-$\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$

Definition

Let $F \in B(H_{\mathcal{A}})$, where $B(H_{\mathcal{A}})$ is the set of all bounded, (not necessarily adjointable) $\mathcal{A}$-linear operators on $H_{\mathcal{A}}$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array} \right],$$

where F_1 is an isomorphism M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm }\},$$

$\widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) = \{F \in B(H_{\mathcal{A}}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_{\mathcal{A}}\}$.
Then, by definition we have

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}),$$

$$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$$

and

$$\mathcal{M}\Phi(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}).$$

Definition

[**IM**] An $\mathcal{A}$-operator $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called a finitely generated $\mathcal{A}$-operator if it can be represented as a composition of bounded $\mathcal{A}$-operators f_1 and f_2:

$$K : H_{\mathcal{A}} \xrightarrow{f_1} M \xrightarrow{f_2} H_{\mathcal{A}},$$

where M is a finitely generated Hilbert C^*-module. The set $FG(\mathcal{A}) \subset B(H_{\mathcal{A}})$ of all finitely generated $\mathcal{A}$-operators forms a two sided ideal. By definition, an $\mathcal{A}$-operator K is called compact if it belongs to the closure

$$K(H_{\mathcal{A}}) = \overline{FG(\mathcal{A})} \subset B(H_{\mathcal{A}}),$$

which also forms two sided ideal.

Clearly, any operator $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ is also left invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, whereas any operator $G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ is right invertible $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$. The converse also holds:

Proposition

If F is left invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. If F is right invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.

Corollary

The sets $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ are closed under multiplication.

Inspired by definition of externel (Noether) decomposition given in [**IM**], we give the following definition.

Definition

We say that F has an upper external (Noether) decomposition if there exist two closed C^*-modules X_1, X_2 and two bounded $\mathcal{A}$-operators E_2, E_3, where X_2 finitely generated, the operator F_0 given by the operator matrix $\begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix}$ with respect to the decomposition $H_{\mathcal{A}} \oplus X_1 \xrightarrow{F_0} H_{\mathcal{A}} \oplus X_2$ is invertible and ImE_2 is complementable in $H_{\mathcal{A}}$. Similarly, we say that F has a lower external (Noether) decomposition if the above decomposition exists and F_0 is invertible, only in this case we assume that X_1 is finitely generated and that $\ker E_3$ is complementable in $H_{\mathcal{A}}$.

Proposition

A bounded $\mathcal{A}$-linear operator $F : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ belongs to $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ if and only if it admits an upper external (Noether) decomposition. Similarly, F belongs to $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ if and only if F admits a lower external (Noether) decomposition.

Lemma

Let $F, G \in B(H_{\mathcal{A}})$ and suppose that $GF \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$. Then there exist decompositions

$$H_{\mathcal{A}} = M_1 \oplus N_1 \xrightarrow{F} H_{\mathcal{A}} = M_3 \oplus N_3 \xrightarrow{G} H_{\mathcal{A}} = M_2 \oplus N_2$$

with respect to which F, G have matrices $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}, \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix}$, respectively, where F_1, G_1 are isomorphisms and N_1, N_2 are finitely generated.

Lemma

Let V be a finitely generated Hilbert submodule of $H_{\mathcal{A}}$, $F \in B(H_{\mathcal{A}})$ and suppose that $P_{V^\perp} F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}, V^\perp)$, where $P_{V^\perp}$ denotes the orthogonal projection onto $V^\perp$ along V. Then $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.

Lemma

Let $G, F \in B(H_{\mathcal{A}})$, suppose that ImG is closed. Assume in addition that $\ker G$ *and ImG are complementable in $H_{\mathcal{A}}$. If $GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, then*

$$\sqcap F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}, N),$$

where $\ker G \tilde{\oplus} N = H_{\mathcal{A}}$ *and $\sqcap$ denotes the projection onto N along* $\ker G$.

Lemma

Let $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ *and suppose that*

$$H_{\mathcal{A}} = M_1'\tilde{\oplus}N_1{}' \xrightarrow{F} M_2'\tilde{\oplus}N_2' = H_{\mathcal{A}}$$

is a decomposition with respect to which F *has the matrix* $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$, *where* F_1' *is an isomorphism,* N_2' *is finitely generated and* N_1' *is just closed. Then* N_1' *is finitely generated.*

Lemma

Let $F \in B(H_{\mathcal{A}})$. *Then* F *admits an upper external (Noether) decomposition with the property that* $X_2 \preceq X_1$ *if and only if* $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$. *Similarly,* F *admits a lower external (Noether) decomposition with the property that* $X_1 \preceq X_2$ *if and only if* $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$.

Recall now the definition of the closses $\mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$. We are going to keep this notion in the next results, but without assuming the adjointability of operators.

Lemma

Let $F \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$. Then $F + K \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$ for all $K \in K(H_{\mathcal{A}})$.

Lemma

Let $F \in B(H_{\mathcal{A}})$ and suppose that

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is a decomposition w.r.t. which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Then $N_1 = F^{-1}(N_2)$.

Lemma

Let $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ and $K \in K(H_{\mathcal{A}})$. Then $F + K \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$.

Semi-Fredholm operators over W^*-algebras

Proposition

Let $F \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$ or $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. Then there exists a decomposition.

$$H_\mathcal{A} = M_0 \tilde{\oplus} M_1' \tilde{\oplus} \ker F \xrightarrow{F} N_0 \tilde{\oplus} N_1' \tilde{\oplus} {N_1'}' = H_\mathcal{A}$$

w.r.t. which F has the matrix

$$\begin{bmatrix} F_0 & 0 & 0 \\ 0 & F_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where F_0 is an isomorphism, M_1' and $\ker F$ are finitely generated. Moreover $M_1' \cong N_1'$ If $F \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$ and ImF is closed, then ImF is complementable in $H_\mathcal{A}$.

In this case F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, w.r.t. the decomposition

$$H_{\mathcal{A}} = \ker F^0 \tilde{\oplus} \ker F \xrightarrow{F} ImF \tilde{\oplus} ImF^0 = H_{\mathcal{A}}$$

where F_1 is an isomorphism and $\ker F^0$, ImF^0 denote the complements of $\ker F$, ImF respectively.

Proposition

If $D \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and ImD is closed and complementable in $H_{\mathcal{A}}$, then the decomposition given above exists for the operator D. In this case, instead of $\ker D$, we have that N_1'' is finitely generated and N_1'' is the complement of ImD.

Lemma

If $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$, ImF is closed and complementable, then the complement of ImF is not finitely generated.

Theorem

Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ if and only if $\ker(F - K)$ *is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.*

Moreover, $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ if and only if $Im(F-K)^{\perp}$ is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.

Definition

Let $F \in B(H_{\mathcal{A}})$. We say that $F \in \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$ if there exist a closed submodule M and a finitely generated submodule N s.t. $H_{\mathcal{A}} = M \tilde{\oplus} N$ and $F_{|_M}$ is bounded below.

Lemma

Let $F \in B(H_{\mathcal{A}})$. Then $F \in \widehat{\mathcal{M}\Phi}_{+}(H_{\mathcal{A}})$ iff $\ker(F-K)$ is finitely generated for all $K \in K^{*}(H_{\mathcal{A}})$.

Set $\widehat{\mathcal{M}\Phi}_{-}(H_{\mathcal{A}}) = \{G \in B(H_{\mathcal{A}} \mid$ there exists closed submodules M, N, M' of $H_{\mathcal{A}}$ s.t. $H_{\mathcal{A}} = M \tilde{\oplus} N$, N is finitely generated and $G_{|_{M'}}$, is an isomorphism onto $M\}$.

Proposition

Let $G \in \widehat{\mathcal{M}\Phi}_{-}(H_{\mathcal{A}})$. Then for every $K \in K(H_{\mathcal{A}})$ there exists an inner product equivalent to the initial one and such that the orthogonal complement of $\overline{Im(G+K)}$ w.r.t this new inner product is finitely generated.

Lemma

$\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_{+}(H_{\mathcal{A}}) \cap B^{a}(H_{\mathcal{A}})$,
$\mathcal{M}\Phi_{-}(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_{-}(H_{\mathcal{A}}) \cap B^{a}(H_{\mathcal{A}})$.

Proposition

Let $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ with closed images and suppose that $ImGF$ is closed. Then ImF, ImG and $ImGF$ are complementable in $H_{\mathcal{A}}$. Moreover, if $ImF^0, ImG^0, ImGF^0$ denote the complements of $ImF, ImG, ImGF$, respectively, then

$$ImGF^0 \preceq ImF^0 \oplus ImG^0,$$

$$\ker GF \preceq \ker G \oplus \ker F.$$

If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and $ImF, ImG, ImGF$ are closed, then the statement above holds under additional assumption that $ImF, ImG, ImGF$ are complementable in $H_{\mathcal{A}}$.

Lemma

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that ImF, ImD and $ImDF$ are closed. Then

$$ImDF^{\perp} \preceq ImF^{\perp} \oplus ImD^{\perp}$$

$$\ker DF \preceq \ker D \oplus \ker F$$

Lemma

Let $F \in \mathcal{M}\Phi(M)$ be such that ImF is closed, where M is a Hilbert W^*-module. Then there exists an $\epsilon > 0$ such that for every $D \in B^a(M)$ with $\| D \| < \epsilon$, we have

$$\ker(F+D) \preceq \ker F \ , \ Im(F+D)^{\perp} \preceq ImF^{\perp}.$$

Definition

Let M be a countably generated Hilbert W^*- module. For $F \in \mathcal{M}\Phi(M)$, we say that F satisfies the condition (*) if the following holds:
1) ImF^n is closed for all n
2) $F(\bigcap_{n=1}^{\infty} Im(F^n)) = \bigcap_{n=1}^{\infty} Im(F^n)$

Theorem

Let $F \in \mathcal{M}\Phi(\tilde{M})$ where $\tilde{M}$ is countably generated Hilbert $\mathcal{A}$-module and suppose that F satisfies (*). Then there exists an $\epsilon > 0$ such that, if $\alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$ and $\| \alpha \| < \epsilon$, then $[\ker(F - \alpha I)] + [N_1] = [\ker F]$ and $[Im(F - \alpha I)^{\perp}] + [N_1] = [Im(F)^{\perp}]$ for some fixed, finitely generated closed submodule N_1.

Theorem

Let $\tilde{M}$ be a Hilbert module over a C^*-algebra $\mathcal{A}$, $\alpha \in \mathbb{C}$ and $F \in B^a(H_{\mathcal{A}})$. Suppose that $\alpha \in iso\ \sigma(F)$ and assume either that $R(F - \alpha I)$ is closed or that $R(P_0)$ is self dual and that $\mathcal{A}$ is a W^*-algebra, where P_0 denotes the spectral projection corresponding to α of the operator F. Then the following conditions are equivalent:

a) $(F - \alpha I) \in \mathcal{M}\Phi_{\pm}(\tilde{M})$

b) There exist closed submodules $M, N \subseteq \tilde{M}$ such that. $(F - \alpha I)$ has the matrix

$$\begin{bmatrix} (F - \alpha I)_1 & 0 \\ 0 & (F - \alpha I)_4 \end{bmatrix}$$

w.r.t. the decomposition $\tilde{M} = M \tilde{\oplus} N \xrightarrow{F - \alpha I} M \tilde{\oplus} N = \tilde{M}$, where $(F - \alpha I)_1$ is an isomorphism and N is finitely generated. Moreover, if $(F - \alpha I)$ is not invertible in $B(\tilde{M})$, then $N(F - \alpha I) \neq \{0\}$.

On generalized $\mathcal{A}$-Fredholm and $\mathcal{A}$-Weyl operators

Definition

Let $F \in B^a(H_{\mathcal{A}})$.
1) We say that $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ if ImF is closed and in addition $\ker F$ and $ImF^{\perp}$ are self-dual.
2) We say that $F \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ if ImF is closed and $kerF \cong ImF^{\perp}$ (here we do not require the self-duality of $kerF$, $ImF^{\perp}$).

Proposition

Let $F, D \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.

Definition

Let $M_1, ..., M_n$ be Hilbert submodules of $H_{\mathcal{A}}$. We say that the sequence $0 \to M_1 \to M_2 \to ... \to M_n \to 0$ is exact if for each $k \in \{2, ..., n-1\}$ there exist closed submodules M_k' and M_k'' such that the following holds:
1) $M_k = M_k' \tilde{\oplus} M_k''$ for all $k \in \{2, ..., n-1\}$;
2) $M_2' \cong M_1$ and $M_{n-1}'' \cong M_n$;
3) $M_k'' \cong M_{k+1}'$ for all $k \in \{2, ..., n-2\}$.

Lemma

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that $ImF, ImD, ImDF$ are closed. Then the sequence

$$0 \to \ker F \to \ker DF \to \ker D \to ImF^{\perp} \to ImDF^{\perp} \to ImD^{\perp} \to 0$$

is exact.

Lemma

Let $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$*. Then* $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$ *if and only if* $F^* \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$*.*

Proposition

Let $F, D \in B^a(H_{\mathcal{A}})$, suppose that ImF, ImD are closed and $DF \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$. Then the following statements hold:
a) $D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}}) \Leftrightarrow F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$;
b) if $\ker D$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$;
c) if $ImF^{\perp}$ is self-dual, then $F, D \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$ *and suppose that* ImF *is closed. Moreover, assume that there exist operators* $D, D' \in B^a(H_{\mathcal{A}})$ *with closed images such that* $D'F, FD \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$*. Then* $F \in \mathcal{M}\Phi^{gc}(H_{\mathcal{A}})$*.*

Definition

Let X, Y be Banach spaces and $T \in B(X, Y)$. Then T is called a regular operator if $T(X)$ is closed in Y and in addition $T^{-1}(0)$ and $T(X)$ are complementable in X and Y, respectively.

Definition

[**DDj2**] Let X, Y be Banach spaces and $T \in B(X, Y)$. Then we say that T is generalized Weyl, if $T(X)$ is closed in Y, and $T^{-1}(0)$ and $Y/_{T(X)}$ are mutually isomorphic Banach spaces.

Proposition

Let X, Y, Z be Banach spaces and let $T \in B(X, Y), S \in B(Y, Z)$. Suppose that T, S, ST are regular, that is $T(X), S(Y), ST(X)$ are closed and T, S, ST admit generalized inverse. If T and S are generalized Weyl operators, then ST is a generalized Weyl operator.

Definition

Let X, Y be Banach spaces and $T \in B(X, Y)$ be a regular operator. Then T is said to be a generalized upper semi-Weyl operator if $\ker T \preceq Y \setminus R(T)$. Similarly T is said to be a generalized lower semi-Weyl operator if $Y \setminus R(T) \preceq \ker T$.

Lemma

Let $T \in B(X, Y)$ $S \in B(Y, Z)$ and suppose that S, T, ST are regular. If S and T are upper (or lower) generalized semi-Weyl operators, then ST is an upper (or respectively lower) generalized semi-Weyl operator.

Definition

For two Hilbert C^*- modules M and M', We set $\tilde{\mathcal{M}}\Phi_0^{gc}(M, M')$ to be the class of all closed range operators $F \in B^a(M, M')$ such that there exist finitely generated Hilbert submodules $N, \tilde{N}$ with the property that $N \oplus \ker F \cong \tilde{N} \oplus ImF^{\perp}$.

Lemma

Let $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}})$ *and* $F \in B^a(H_{\mathcal{A}})$ *s.t.* ImF *is closed, finitely generated. Suppose that* $Im(T+F), T(\ker F), P(\ker T), P(\ker(T+F))$ *are closed, where* P *denotes the orthogonal projection onto* $\ker F^{\perp}$*. Then* $T+F \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}})$.

Corollary

Let $T \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$ *and suppose that* $\ker T \cong ImT^{\perp} \cong H_{\mathcal{A}}$*. If* $F \in B^a(H_{\mathcal{A}})$ *satisfies the assumptions of Lemma 64, then* $\ker(T+F) \cong Im(T+F)^{\perp} \cong H_{\mathcal{A}}$*. In particular,* $T+F \in \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})$.

Lemma

Let $F \in B^a(M)$ *where* M *is a Hilbert* C^**-module and suppose that* ImF *is closed. Then the following statements hold:*
a) $F \in \mathcal{M}\Phi_+(M)$*, if and only if* $\ker F$ *is finitely generated;*
b) $F \in \mathcal{M}\Phi_-(M)$*, if and only if* $ImF^{\perp}$ *is finitely generated.*

Lemma

Let $T \in \mathcal{M}\Phi(H_{\mathcal{A}})$ *and suppose that* ImT *is closed. Then* $T \in \tilde{\mathcal{M}}\Phi_0^{gc}(H_{\mathcal{A}})$.

On semi-$\mathcal{A}$-B-Fredholm operators

Definition

Let $F \in B^a(H_{\mathcal{A}})$. Then F is said to be an upper semi-$\mathcal{A}$-B-Fredhom operator if there exists some $n \in \mathbb{N}$ such that ImF^m is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an upper semi-$\mathcal{A}$-Fredhom operator.

Similarly, F is said to be a lower semi-$\mathcal{A}$-B-Fredholm operator if the conditions above hold except that in this case we assume that $F_{|_{ImF^n}}$ is a lower semi-$\mathcal{A}$-Fredhom operator and not an upper semi-$\mathcal{A}$-Fredhom operator.

Proposition

If F is an upper semi-$\mathcal{A}$-B-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-B-Fredholm operator) and $n \in \mathbb{N}$ is such that ImF^m is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an upper semi-$\mathcal{A}$-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-Fredholm operator), then $F_{|_{ImF^m}}$ is an upper semi-$\mathcal{A}$-Fredholm operator (respectively, a lower semi-$\mathcal{A}$-Fredholm operator) for all $m \geq n$. Moreover, if F is an $\mathcal{A}$-B-Fredholm operator and $n \in \mathbb{N}$ is such that $ImF^n \cong H_{\mathcal{A}}, ImF^m$ is closed for all $m \geq n$ and $F_{|_{ImF^n}}$ is an $\mathcal{A}$-Fredholm operator, then $ImF^m \cong H_{\mathcal{A}}$ $F_{|_{ImF^m}}$ is an $\mathcal{A}$ Fredholm operator and index $F_{|_{ImF^m}} = index F_{|_{ImF^n}}$ for all $m \geq n$.

Lemma

Let $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$, let $P \in B(H_{\mathcal{A}})$ be a projection such that $N(P)$ is finitely generated. Then $PF_{|_{R(P)}} \in \mathcal{M}\Phi(R(P))$ and $\text{index} PF_{|_{R(P)}} = \text{index} F$.

Theorem

Let T be an $\mathcal{A}$-B-Fredholm operator on $H_{\mathcal{A}}$ and suppose that $m \in \mathbb{N}$ is such that $T_{|_{ImT^m}}$ is an $\mathcal{A}$-Fredholm operator and ImT^n is closed for all $n \geq m$. Let F be in the linear span of elementary operators and suppose that $Im(T+F)^n$ is closed for all $n \geq m$. Finally, assume that $ImT^m \cong H_{\mathcal{A}}$ and that $Im(\tilde{F})$, $T^m(\ker \tilde{F})$ are closed, where $\tilde{F} = (T+F)^m - T^m$. Then $T+F$ is an $\mathcal{A}$-B-Fredholm operator and $\text{index} T+F = \text{index} T$.

Proposition

Let $F \in B(H_{\mathcal{A}})$. If $n \in \mathbb{N}$ is s.t. ImF^n closed, $ImF^n \cong H_{\mathcal{A}}$, $F_{|_{ImF^n}}$ is upper semi-$\mathcal{A}$-Fredholm and ImF^m is closed for all $m \geqslant n$, then $F_{|_{ImF^m}}$ is upper semi-$\mathcal{A}$- Fredholm and $ImF^m \cong H_{\mathcal{A}}$ for all $m \geqslant n$. If $n \in \mathbb{N}$ is s.t. ImF^n is closed, $ImF^n \cong H_{\mathcal{A}}, ImF^m$ is closed and complementable in ImF^n for all $m \geqslant n$ and $F_{|_{ImF^n}}$ is lower semi-$\mathcal{A}$-Fredholm, then $F_{|_{ImF^m}}$ is lower semi-$\mathcal{A}$-Fredholm for all $m \geqslant n$ and $ImF^m \cong H_{\mathcal{A}}$ for all $m \geqslant n$.

On closed range operators over C^*-algebras.

Lemma

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that ImF, ImD are closed. If $ImF + \ker D$ is closed, then $ImF + \ker D$ is orthogonally complementable.

Corollary

Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that ImF, ImD are closed. Then $ImDF$ is closed if and only if $ImF + \ker D$ is orthogonally complementable.

Definition

Given two closed submodules M, N of $H_{\mathcal{A}}$, we set

$$c_0(M,N) = \sup\{\| < x, y > \| \mid x \in M, y \in N, \| x \|, \| y \| \leq 1\}.$$

We say then that the Dixmier angle between M and N is positive if $c_0(M,N) < 1$.

Lemma

Let M, N be two closed, submodules of $H_{\mathcal{A}}$, assume that M orthogonally complementable and suppose that $M \cap N = \{0\}$. Then $M + N$ is closed if the Dixmier angle between M and N is positive.

Corollary

Let $F, D \in B^a(H_{\mathcal{A}})$ *and suppose that* ImF, ImD *are closed. Set* $M = ImF \cap (\ker D \cap ImF)^{\perp}$, $M' = \ker D \cap (\ker D \cap ImF)^{\perp}$. *Assume that* $\ker D \cap ImF$ *is orthogonally complementable. Then* $ImDF$ *is closed if the Dixmier angle betwen* M' *and* ImF, *or equivalently the Dixmier angle between* M *and* $\ker D$ *is positive.*

Lemma

Let M *and* N *be two closed submodules of* $H_{\mathcal{A}}$. *Suppose that* M *and* N *are orthogonally complementable in* $H_{\mathcal{A}}$ *and that* $M \cap N = \{0\}$. *Then* $M + N$ *is closed if and only if* $P_{|_N}$ *is bounded below, where* P *denotes the orthogonal projection onto* $M^{\perp}$.

Corollary

Let $F, D \in B^a(H_{\mathcal{A}})$ *and suppose that* ImF, ImD *are closed. Then* $ImDF$ *is closed if and only if* $\ker D \cap ImF$ *is orthogonally complementable and* $P_{|_{ImF \cap (\ker D \cap ImF)^{\perp}}}$ *is bounded below, or equivalently* $Q_{|_{\ker D \cap (\ker D \cap ImF)^{\perp}}}$ *is bounded below, where* P *and* Q *denote the orthogonal projections onto* $\ker D^{\perp}$ *and* $ImF^{\perp}$*, respectively.*

Lemma

Let $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ *and suppose that* ImG *and* ImF *are closed. Then* $ImGF$ *is closed if and only if* $ImF + \ker G$ *is closed and complementable. If* $F, G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ *and* ImG, ImF *are closed, then the statment above holds under additional assumtion that* ImG, ImF *are complementable. Moreover, if* $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ *and* ImF, ImG *are closed and if the Dixmier angle between* $\ker G$ *and* $ImF \cap (\ker G \cap ImF)^0$ *is positive, or equivalently the Dixmier angle berween* ImF *and* $\ker G \cap (\ker G \cap ImF)^0$ *is positive, where* $(\ker G \cap ImF)^0$ *denotes the complement of* $\ker G \cap ImF$*, then* $ImGF$ *is closed.*

Proposition

Let $F \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:
1) ImF is closed in $H_{\mathcal{A}}$
2) ImL_F is closed in $B^a(H_{\mathcal{A}})$
3) ImR_F is closed in $B^a(H_{\mathcal{A}})$.

Lemma

Let $F \in M^a(H_{\mathcal{A}})$. If there exists a sequence $\{F_n\} \subseteq \mathcal{M}\Phi(H_{\mathcal{A}}))$ of constant index such that $F_n \to F$, then $F \subset \mathcal{M}\Phi(H_{\mathcal{A}})$ and $indexF = indexF_n$ for all n.

Lemma

Let $F \in B(H_{\mathcal{A}})$ and suppose that ImF is closed. Then F is a regular operator with the property that $ImF^0, \ker F$ are finitely generated if and only if $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$.

Proposition

Let $F \in B(H_{\mathcal{A}})$ be bounded below and suppose that there exists a sequence $\{F_n\} \subseteq \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ of constant index and such that $F_n \to F$. Suppose also that for each n there exists an $\widehat{\mathcal{M}\Phi}-$ decomposition

$$H_{\mathcal{A}} = M_1^{(n)} \tilde{\oplus} N_1^{(n)} \xrightarrow{F_n} M_2^{(n)} \tilde{\oplus} N_2^{(n)} = H_{\mathcal{A}}$$

such that the sequence of projections $\{\sqcap_n\}$ is uniformly bounded, where $\sqcap_n$ denotes the projection onto $N_2^{(n)}$ along $M_2^{(n)}$ for each n. Then $F \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ and $index F_n = index F$ for all n.

Lemma

Let X, Y be Banach spaces and $F \in M(X, Y)$. Suppose that there exists a sequence $\{F_n\}$ of regular operators in $B(X, Y)$ such that $F_n \to F$. Moreover, assume that there exists a sequence of projections $\{\sqcap_n\}$ in $B(Y)$ which is uniformly bounded in the norm and such that $Im(I - \sqcap_n) = Im F_n$ for all n. Then, F is a regular operator, i.e. $Im F$ is complementable in Y.

On generalized spectra of operators over C^*-algebras

Question: If $\mathcal{A}$ is a C^*-algebra, then for $\alpha \in \mathcal{A}$ could we define in a suitable way the operator αI on $H_{\mathcal{A}}$ and the generalized spectra in $\mathcal{A}$ of operators in $B^a(H_{\mathcal{A}})$ by setting for every $F \in B^a(H_{\mathcal{A}})$
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in} B^a(H_{\mathcal{A}})\}$?
Answer: For $a \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by $\alpha I(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots)$. It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\| \alpha I \| = \| \alpha \|$.
Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.
We introduce then the following notion:
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in} B^a(H_{\mathcal{A}})\}$;
$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \ker(F - \alpha I) \neq \{0\}\}$;
$\sigma_{rl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is bounded below, but not surjective on } H_{\mathcal{A}})\}$;
$\sigma_{cl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid Im(F - \alpha I) \text{ is } \underline{\text{not}} \text{ closed } \}$. (where $F \in B^a(H_{\mathcal{A}})$)).

Proposition

Let $\mathcal{A}$ be a unital C^*-algebra, $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is unilateral shift and $S^* e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1))$ or if $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leq 1\}$, (where in the case when $\mathcal{A} = L^\infty((0,1))$, we set $\inf|\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}[0,C]) > 0\} = \sup\{K > 0 \mid |\alpha| > K)$ a.e. on $(0,1)\}$). Moreover, $\sigma_p^{\mathcal{A}}(S) = \varnothing$ in both cases.

Corollary

Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then $\sigma^{\mathcal{A}}(S) = \mathcal{A} \setminus G(\mathcal{A}) \cup \{\alpha \in G(\mathcal{A}) | (\alpha^{-1}, \alpha^{-2}, \cdots, \alpha^{-k}, \cdots) \notin H_{\mathcal{A}}\}$.

Proposition

Let $\alpha \in \mathcal{A}$. We have

1. If $\alpha I - F$ is bounded below, and $F \in B^a(H_{\mathcal{A}})$ then $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$.
2. If $F, D \in B^a(H_{\mathcal{A}})$ and $D = U^* F U$ for some unitary operator U, then $\sigma^{\mathcal{A}}(F) = \sigma^{\mathcal{A}}(D), \sigma_p^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(D), \sigma_{cl}^{\mathcal{A}}(F) = \sigma_{cl}^{\mathcal{A}}(D)$ and $\sigma_{rl}^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(D)$.

Proposition

Let $U \in B^a(H_{\mathcal{A}})$ be unitary. Then $\sigma^{\mathcal{A}}(U) \subseteq \{\alpha \in \mathcal{A} \mid \| \alpha \| \geq 1\}$ and $\sigma^{\mathcal{A}}(U) \cap G(\mathcal{A}) \subseteq \{\alpha \in G(\mathcal{A}) \mid \| \alpha^{-1} \|, \| \alpha \| \geq 1\}$.

Consider again the orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider bilateral shift operator V w.r.t. this basis i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^* e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition

Let V be bilateral shift operator. Then the following holds
1) If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \emptyset\}$
2) If $\mathcal{A} = L^\infty([0,1])$, then
$\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon)) > 0 \ \forall \epsilon > 0\}$. In both cases $\sigma_p^{\mathcal{A}}(V) = \emptyset$.

Lemma

If F is a self-adjoint operator on $H_{\mathcal{A}}$, then $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$, that is $\alpha \in \sigma_p^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F)$ in the case when $\mathcal{A}$ is a commutative C^*-algebra.

Lemma

Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, then $\overline{R(F-\alpha I)}^{\perp} = \{0\}$. Hence, if $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$ and in addition $F - \alpha I$ is bounded below, then $\alpha \in \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.

Lemma

Let $\mathcal{A}$ be a commutative unital C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$, that is $FF^* = F^*F$. If $\alpha_1, \alpha_2 \in \sigma_p^{\mathcal{A}}(F)$ and $\alpha_1 - \alpha_2$ is invertible in $\mathcal{A}$, then $\ker(F - \alpha_1 I) \perp \ker(F - \alpha_2 I)$.

Lemma

Let $\mathcal{A}$ be a commutative C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$. Then $\sigma_{rl}^{\mathcal{A}}(F) = \varnothing$, hence $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent:
a) $\alpha \in \mathcal{A} \setminus \sigma_a(F)$
b) $\alpha \in \mathcal{A} \setminus \sigma_l(F)$
c) $\alpha^* \in \mathcal{A} \setminus \sigma_r(F^*)$
d) $Im(\alpha^* I - F^*) = H_{\mathcal{A}}$.

Next, for $F \in B^a(H_{\mathcal{A}})$, set $\sigma_a^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I$ is not bounded below $\}$.

Proposition

For $F \in B^a(H_{\mathcal{A}})$, we have that $\sigma_a^{\mathcal{A}}(F)$ is a closed subset of $\mathcal{A}$ in the norm topology and $\sigma^{\mathcal{A}}(F) = \sigma_a^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F)$.

Proposition

If $F \in B^a(H_{\mathcal{A}})$, then $\partial\sigma^{\mathcal{A}}(F) \subseteq \sigma_a^{\mathcal{A}}(F)$. Moreover, if M is a closed submodule of $H_{\mathcal{A}}$ and invariant with respect to F, and $F_0 = F_{|_M}$, then we have $\partial\sigma^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F)$, $\sigma^{\mathcal{A}}(F_0) \cap \sigma^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(F_0)$.

Definition

Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. We set

$$\sigma_{ew}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in \mathcal{A} \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \mathcal{M}\Phi_0(H_{\mathcal{A}})\},$$

$$\sigma_{e\alpha}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in \mathcal{A} \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$

$$\sigma_{e\beta}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in \mathcal{A} \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{ek}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in \mathcal{A} \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}}) \cup \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$

$$\sigma_{ef}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in \mathcal{A} \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \mathcal{M}\Phi(H_{\mathcal{A}})\}.$$

Definition

We set $ms_\Phi(F) = \inf\{\| \alpha \| | \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi(H_\mathcal{A})\}$,

$$ms(F) = \inf\{\| \alpha \| | \alpha \in \mathcal{A}, F - \alpha I \notin (\mathcal{M}\Phi_+(H_\mathcal{A}) \cup \mathcal{M}\Phi_-(H_\mathcal{A}))\},$$

$$ms_+(F) = \inf\{\| \alpha \| | \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_+(H_\mathcal{A})\},$$

$$ms_-(F) = \inf\{\| \alpha \| | \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_-(H_\mathcal{A})\}.$$

It follows that $ms_\Phi(F) = \max\{\epsilon \geq 0 \, | \, \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi(H_\mathcal{A})\}$,

$$ms_+(F) = \max\{\epsilon \geq 0 \, | \, \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_+(H_\mathcal{A})\},$$

$$ms_-(F) = \max\{\epsilon \geq 0 \, | \, \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_-(H_\mathcal{A})\},$$

$$ms(F) = \max\{\epsilon \geq 0 \, | \, \| \alpha \| < \epsilon \Rightarrow F - \alpha I \in (\mathcal{M}\Phi_+(H_\mathcal{A}) \cup \mathcal{M}\Phi_-(H_\mathcal{A}))\},$$

it follows that $ms_\Phi(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi(H_\mathcal{A})$,

$$ms_+(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_+(H_\mathcal{A}), ms_-(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_-(H_\mathcal{A}),$$

$$ms(F) > 0 \Leftrightarrow F \in (\mathcal{M}\Phi_+(H_\mathcal{A}) \cup \mathcal{M}\Phi_-(H_\mathcal{A})), \text{ it follows that}$$

$$ms_+(F) = ms_-(F^*), ms_\Phi(F) = ms_\Phi(F^*), ms(F) = ms(F^*).$$

Lemma

Let $F \in B(H_{\mathcal{A}})$. *If* $ms_{+}(F) > 0$ *and* $ms_{-}(F) > 0$, *then* $ms_{+}(F) = ms_{-}(F)$.

Lemma

Let $F \in B(H_{\mathcal{A}})$. *Then*
1) $ms_{\Phi}(F) = \min\{ms_{+}(F), ms_{-}(F)\}$
2) $ms(F) = \max\{ms_{+}(F), ms_{-}(F)\}$.

Lemma

Let $F \in B(H_{\mathcal{A}})$, *where* $\mathcal{A}$ *be a* W^{*}*-algebra and suppose that* $K(\mathcal{A})$ *satisfies the cancellation property. Then* $\sigma^{\mathcal{A}}(F) = \sigma_{ew}^{\mathcal{A}}(F) \cup \sigma_{p}^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Lemma

Let now $\mathcal{A}$ be an arbitrary C^*-algebra. For $F \in B^a(H_{\mathcal{A}})$ set $\sigma_{ewgc}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_0^{gc}(H_{\mathcal{A}})\}$. Then $\sigma^{\mathcal{A}}(F) = \sigma_{ewgc}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F)$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$ and supppose $K(\mathcal{A})$ satisfies the cancellation property. Then $\sigma^{\mathcal{A}}(F) = \sigma_{ew}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.

Proposition

If $F \in B^a(H_{\mathcal{A}})$ then the components of $\mathcal{A} \setminus (\sigma_{e\alpha}^{\mathcal{A}}(F) \cap \sigma_{e\beta}^{\mathcal{A}}(F))$ are either completely contained in $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or in $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or index $(F - \alpha I)$ is constant on them.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. If $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma_{e\alpha}^{\mathcal{A}}(F) \cap \sigma_{e\beta}^{\mathcal{A}}(F))$, then $\alpha \in \mathcal{M}\Phi_0(F)$.

Let now $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$ be the set of all $\mathrm{F} \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

w.r.t. which F has the matrix $\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1, N_2 are finitely generated and

$$N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for some closed submodule $N \subseteq H_{\mathcal{A}}$.
Notice that this implies that $\mathrm{F} \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $N_1 \cong N_2$, so that index $\mathrm{F} = [N_1] - [N_2] = 0$. Hence $\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_0(H_{\mathcal{A}})$.
Let $\mathrm{P}(H_{\mathcal{A}}) = \{\mathrm{P} \in B(H_{\mathcal{A}}) \mid \mathrm{P}$ is a projection and $\mathrm{N}(\mathrm{P})$ is finitely generated$\}$
and let

$$\sigma_{e\mathrm{W}}^{\mathcal{A}}(\mathrm{F}) = \{\alpha \in Z(\mathcal{A}) \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})\}$$

for $\mathrm{F} \in B^a(H_{\mathcal{A}})$.

Theorem

Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. *Then*

$$\sigma_{eW}^{\mathcal{A}}(\mathrm{F}) = \cap\{\sigma^{\mathcal{A}}(\mathrm{PF}_{|_{\mathrm{R(P)}}}) \mid \mathrm{P} \in \mathrm{P}(H_{\mathcal{A}})\}$$

where
$\sigma^{\mathcal{A}}(\mathrm{PF}_{|_{\mathrm{R(P)}}}) = \{\alpha \in Z(\mathcal{A}) \mid (\mathrm{PF} - \alpha \mathrm{I})_{|_{\mathrm{R(P)}}}$ *is not invertible in* $B(\mathrm{R(P)})\}$.

Lemma

$\tilde{\mathcal{M}}\Phi_0(H_{\mathcal{A}})$ *is open in* $B^a(H_{\mathcal{A}})$.

We let now $\widehat{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$ be the space of all $\mathrm{F} \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

w.r.t. which F has the matrix $\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}$, where F_1 is an isomorphism, N_1 is finitely generated and such that there exist closed submodules N_2', N where $N_2' \subseteq N_2, N_2' \cong N_1$, $H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2'$ and the projection onto N along N_2' is adjointable.

Then we set

$$\sigma_{e\tilde{a}}^{\mathcal{A}}(\mathrm{F}) := \{\alpha \in Z(\mathcal{A}) \mid (\mathrm{F} - \alpha \mathrm{I}) \notin \widehat{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})\}.$$

Theorem

Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. *Then* $\sigma_{e\tilde{a}}^{\mathcal{A}}(\mathrm{F}) = \cap\{\sigma_a^{\mathcal{A}}(\mathrm{PF}_{|_{\mathrm{R(P)}}}) \mid \mathrm{P} \in \mathrm{P}^a(H_{\mathcal{A}})\}$ *where* $\sigma_a^{\mathcal{A}}(\mathrm{PF}_{|_{\mathrm{R(P)}}})$ *is the set of all* $\alpha \in Z(\mathcal{A})$ *s.t.* $(\mathrm{PF} - \alpha \mathrm{I})_{|_{\mathrm{R(P)}}}$ *is not bounded below on* $\mathrm{R(P)}$ *and* $\mathrm{P}^a(H_{\mathcal{A}}) = \mathrm{P}(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$.

Definition

We set $\widehat{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})$ to be the set of all $\mathrm{D} \in B^a(H_{\mathcal{A}})$ such that there exists a decomposition

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

w.r.t. which D has the matrix $\begin{bmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_4 \end{bmatrix}$, where D_1 is an isomorphism, N_2' is finitely generated and such that $H_{\mathcal{A}} = M_1' \tilde{\oplus} N \tilde{\oplus} N_2'$ for some closed submodule N, where the projection onto $M_1' \tilde{\oplus} N$ along N_2' is adjointable. Then we set

$$\sigma_{e\tilde{d}}^{\mathcal{A}}(\mathrm{D}) = \{\alpha \in Z(\mathcal{A})) \mid (\mathrm{D} - \alpha \mathrm{I}) \notin \widehat{\mathcal{M}\Phi}_{-}^{+}(H_{\mathcal{A}})\}$$

and for $\mathrm{P} \in \mathrm{P}^a(H_{\mathcal{A}})$ we set

$$\sigma_{d}^{\mathcal{A}}(\mathrm{PD}_{|_{\mathrm{R(P)}}}) = \{\alpha \in Z(\mathcal{A})) \mid (\mathrm{PD} - \alpha \mathrm{I})_{|_{\mathrm{R(P)}}} \text{ is not onto } \mathrm{R(P)}\}.$$

Theorem

Let $\mathrm{D} \in B^a(H_{\mathcal{A}})$. *Then*

$$\sigma_{e\tilde{d}}^{\mathcal{A}}(\mathrm{D}) = \bigcap \{\sigma_{d}^{\mathcal{A}}(\mathrm{PD}_{|_{\mathrm{R(P)}}}) \mid \mathrm{P} \in \mathrm{P}^a(H_{\mathcal{A}})\}$$

Definition

We let $\widehat{\mathcal{M}\Phi}_{+}^{-}(H_{\mathcal{A}})$ be the set of all $F \in B(H_{\mathcal{A}})$ such that there exists an $\mathcal{M}\Phi_{+}$-decomposition for F

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

and closed submodules N, N_2' with the property that N_1 is isomorphic to N_2', $N_2' \subseteq N_2$ and

$$H_{\mathcal{A}} = N \tilde{\oplus} N_1 = N \tilde{\oplus} N_2'.$$

Theorem

For $F \in B(H_{\mathcal{A}})$ we have

$$\sigma_{e\tilde{a}0}^{\mathcal{A}}(F) = \cap\{\sigma_{a0}^{\mathcal{A}}(PF_{|_{R(P)}}) \mid P \in P(H_{\mathcal{A}})\},$$

where $\sigma_{a0}^{\mathcal{A}}(PF_{|_{R(P)}}) = \{\alpha \in Z(\mathcal{A}) \mid (PF - \alpha I)_{|R(P)}$ is not bounded below on $R(P)$ or $R(PF - \alpha P)$ is not complementable in $R(P)\}$.

Definition

We set $\widehat{\widehat{\mathcal{M}\Phi}}_{-}^{+}(H_{\mathcal{A}})$ to be the set of all $G \in B(H_{\mathcal{A}})$ such that there exists an $\mathcal{M}\Phi_{-}$-decomposition for G

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1{}' \xrightarrow{G} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

and a closed submodule N with the property that $H_{\mathcal{A}} = M_1' \tilde{\oplus} N \tilde{\oplus} N_2{}'$.

Theorem

For $G \in B(H_{\mathcal{A}})\}$ we have

$$\sigma_{e\tilde{d}0}^{\mathcal{A}}(G) = \cap\{\sigma_{d0}^{\mathcal{A}}(PG_{|_{R(P)}}) \mid P \in P(H_{\mathcal{A}})\},$$

where $\sigma_{d0}^{\mathcal{A}}(PG_{|_{R(P)}}) = \{\alpha \in Z(\mathcal{A}) \mid R(P)$ does not split into the decomposition $R(P) = \tilde{N} \tilde{\oplus} \tilde{\tilde{N}}$ with the property that $PG_{|_{\tilde{N}}}$ is an isomorphism onto $R(P)\}$.

The boundary of several kinds of Fredholm spectra in $\mathcal{A}$

Theorem

Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. *Then the following inclusions hold:*

$$\partial\sigma_{ew}^{\mathcal{A}}(\mathrm{F}) \subseteq \partial\sigma_{ef}^{\mathcal{A}}(\mathrm{F}) \subseteq \begin{matrix} \partial\sigma_{e\beta}^{\mathcal{A}}(\mathrm{F}) \\ \partial\sigma_{e\alpha}^{\mathcal{A}}(\mathrm{F}) \end{matrix} \subseteq \partial\sigma_{ek}^{\mathcal{A}}(\mathrm{F}).$$

Theorem

Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. *Then*

$$\partial\sigma_{ew}^{\mathcal{A}}(\mathrm{F}) \subseteq \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(\mathrm{F}) \subseteq \partial\sigma_{ea}^{\mathcal{A}}(\mathrm{F})$$

Moreover, $\partial\sigma_{ea}^{\mathcal{A}}(\mathrm{F}) \subseteq \partial\sigma_{e\alpha}^{\mathcal{A}}(\mathrm{F})$ *if* $K(\mathcal{A})$ *satisfies the cancellation property.*

Lemma

$\mathcal{M}I(H_{\mathcal{A}})$ is a closed two sided ideal in $B^a(H_{\mathcal{A}})$ and

$$\mathcal{M}I(H_{\mathcal{A}}) = \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in B^a(H_{\mathcal{A}})\} =$$

$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + DF \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in \mathcal{M}\Phi(H_{\mathcal{A}})\} =$$

$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in B^a(H_{\mathcal{A}})\} =$$

$$= \{D \in B^a(H_{\mathcal{A}}) \mid I + FD \in \mathcal{M}\Phi(H_{\mathcal{A}})\ \forall F \in F \in \mathcal{M}\Phi(H_{\mathcal{A}})\}.$$

Lemma

a) If $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F + D \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
b) If $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $F + D \in \mathcal{M}\Phi_-(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})$.
c) If $\mathcal{M}\Phi(H_{\mathcal{A}})$ and $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$, then $D + F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and index $D + F =$ index F.

Lemma

We have $P(\mathcal{M}\Phi_0(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Proposition

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\sigma_{ew}^{\mathcal{A}}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D) = \bigcap_{D \in \mathcal{M}I(H_{\mathcal{A}})} \sigma^{\mathcal{A}}(F + D).$$

Theorem

The operator $D \in B^a(H_{\mathcal{A}})$ *satisfies the condition* $\sigma_{ek}^{\mathcal{A}}(F + D) = \sigma_{ek}^{\mathcal{A}}(F)$ *for every* $F \in B^a(H_{\mathcal{A}})$ *if and only if*
$D \in P(\mathcal{M}\Phi_+(H_{\mathcal{A}})) \cap P(\mathcal{M}\Phi_-(H_{\mathcal{A}})) = P(\mathcal{M}\Phi(H_{\mathcal{A}}))$.

Lemma

The operator $D \in B^a(H_{\mathcal{A}})$ *satisfies the condition* $\sigma^{\mathcal{A}}_{e\alpha}(F + D) = \sigma^{\mathcal{A}}_{e\alpha}(F)$ *for every* $F \in B^a(H_{\mathcal{A}})$ *if and only if* $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$*.*

Lemma

The operator $D \in B^a(H_{\mathcal{A}})$ *satisfies the condition* $\sigma^{\mathcal{A}}_{e\beta}(F + D) = \sigma^{\mathcal{A}}_{e\beta}(F)$ *for every* $F \in B^a(H_{\mathcal{A}})$ *if and only if* $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$*.*

Lemma

The operator $D \in B^a(H_{\mathcal{A}})$ *satisfies the condition* $\sigma^{\mathcal{A}}_{ef}(F + D) = \sigma^{\mathcal{A}}_{ef}(F)$ *for every* $F \in B^a(H_{\mathcal{A}})$ *if and only if* $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$*.*

Lemma

The operator $D \in B^a(H_{\mathcal{A}})$ *satisfies the condition* $\sigma^{\mathcal{A}}_{ew}(F + D) = \sigma^{\mathcal{A}}_{ew}(F)$ *for every* $F \in B^a(H_{\mathcal{A}})$ *if and only if* $D \in P(\mathcal{M}\Phi(H_{\mathcal{A}}))$*.*

Definition

For $F \in B^a(H_{\mathcal{A}})$ we set $\sigma^{\mathcal{A}}_{e\alpha'}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \notin \mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}})$ and $\sigma^{\mathcal{A}}_{e\beta'}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \notin \mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}})\}$.

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then

$$\sigma^{\mathcal{A}}_{e\alpha'}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma^{\mathcal{A}}_{a}(F+D) = \bigcap_{D \in P(\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}))} \sigma^{\mathcal{A}}_{a}(F+D),$$

$$\sigma^{\mathcal{A}}_{e\beta'}(F) = \bigcap_{D \in K^*(H_{\mathcal{A}})} \sigma^{\mathcal{A}}_{d}(F+D) = \bigcap_{D \in P(\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}))} \sigma^{\mathcal{A}}_{d}(F+D),$$

Lemma

Let $F \in B^a(H_{\mathcal{A}})$. Then

1) We have $\sigma^{\mathcal{A}}_{e\alpha'}(F+D) = \sigma^{\mathcal{A}}_{e\alpha'}(D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only if $F \in P(\mathcal{M}\Phi_+^{-\prime}(H_{\mathcal{A}}))$.

2) We have $\sigma^{\mathcal{A}}_{e\beta'}(D) = \sigma^{\mathcal{A}}_{e\beta'}(F+D)$ for every $D \in B^a(H_{\mathcal{A}})$ if and only $F \in P(\mathcal{M}\Phi_-^{+\prime}(H_{\mathcal{A}}))$.

On operator 2×2 matrices over C^*-algebras

We will consider the operator $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}(\mathrm{F}, \mathrm{D}) : H_{\mathcal{A}} \oplus H_{\mathcal{A}} \to H_{\mathcal{A}} \oplus H_{\mathcal{A}}$ given as 2×2 operator matrix

$$\begin{bmatrix} \mathrm{F} & \mathrm{C} \\ 0 & \mathrm{D} \end{bmatrix},$$

where $\mathrm{F}, \mathrm{C}, \mathrm{D} \in B^a(H_{\mathcal{A}})$.
To simplify notation we will only write $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}$ instead of $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}(\mathrm{F}, \mathrm{D})$ when $\mathrm{F}, \mathrm{D} \in B^a(H_{\mathcal{A}})$ are given.

Proposition

For given $\mathrm{F}, \mathrm{C}, \mathrm{D} \in B^a(H_{\mathcal{A}})$, one has

$$\sigma_e^{\mathcal{A}}(\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}) \subset (\sigma_e^{\mathcal{A}}(\mathrm{F}) \cup \sigma_e^{\mathcal{A}}(\mathrm{D})).$$

Theorem

Let $\mathrm{F}, \mathrm{D} \in B^a(H_{\mathcal{A}})$. *If* $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ *for some* $\mathrm{C} \in B^a(H_{\mathcal{A}})$, *then* $\mathrm{F} \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), \mathrm{D} \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ *and for all decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

w.r.t. which F, D *have matrices* $\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_4 \end{bmatrix}$, *respectively, where* $\mathrm{F}_1, \mathrm{D}_1$ *are isomorphisms, and* N_1, N_2' *are finitely generated, there exist closed submodules* $\tilde{N}_1', \tilde{\tilde{N}}_1', \tilde{N}_2, \tilde{\tilde{N}}_2$ *such that* $N_2 \cong \tilde{N}_2, N_1' \cong \tilde{N}_1'$, $\tilde{\tilde{N}}_2$ *and* $\tilde{\tilde{N}}_1'$ *are finitely generated and*

$$\tilde{N}_2 \tilde{\oplus} \tilde{\tilde{N}}_2 \cong \tilde{N}_1' \tilde{\oplus} \tilde{\tilde{N}}_1'.$$

Proposition

Suppose that there exists some $\mathrm{C} \in B^a(H_{\mathcal{A}})$ such that the inclusion $\sigma_e^{\mathcal{A}}(\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}) \subset \sigma_e^{\mathcal{A}}(\mathrm{F}) \cup \sigma_e^{\mathcal{A}}(\mathrm{D})$ is proper. Then for any

$$\alpha \in [\sigma_e^{\mathcal{A}}(\mathrm{F}) \cup \sigma_e^{\mathcal{A}}(\mathrm{D})] \setminus \sigma_e^{\mathcal{A}}(\mathbf{M}_{\mathrm{C}}^{\mathcal{A}})$$

we have

$$\alpha \in \sigma_e^{\mathcal{A}}(\mathrm{F}) \cap \sigma_e^{\mathcal{A}}(\mathrm{D}).$$

Next, we define the following classes of operators on $H_{\mathcal{A}}$:

$$\mathcal{M}S_{+}(H_{\mathcal{A}}) = \{\mathrm{F} \in B^a(H_{\mathcal{A}}) \mid (\mathrm{F} - \alpha 1) \in \mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})$$
$$\text{whenever } \alpha \in \mathcal{A} \text{ and } (\mathrm{F} - \alpha 1) \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})\},$$
$$\mathcal{M}S_{-}(H_{\mathcal{A}}) = \{\mathrm{F} \in B^a(H_{\mathcal{A}}) \mid (\mathrm{F} - \alpha 1) \in \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}})$$
$$\text{whenever } \alpha \in \mathcal{A} \text{ and } (\mathrm{F} - \alpha 1) \in \mathcal{M}\Phi_{\pm}(H_{\mathcal{A}})\}.$$

Proposition

If $\mathrm{F} \in \mathcal{M}S_{+}(H_{\mathcal{A}})$ or $\mathrm{D} \in \mathcal{M}S_{-}(H_{\mathcal{A}})$, then for all $\mathrm{C} \in B^a(H_{\mathcal{A}})$, we have

$$\sigma_e^{\mathcal{A}}(\mathbf{M}_{\mathrm{C}}^{\mathcal{A}}) = \sigma_e^{\mathcal{A}}(\mathrm{F}) \cup \sigma_e^{\mathcal{A}}(\mathrm{D})$$

Theorem

Let $\mathrm{F} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}}), \mathrm{D} \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$ *and suppose that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} N_2^{\perp} \oplus N_2 = H_{\mathcal{A}}$$

$$H_{\mathcal{A}} = {N_1'}^{\perp} \oplus N_1' \xrightarrow{\mathrm{D}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

w.r.t. which F, D *have matrices*

$$\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_4 \end{bmatrix},$$

respectively, where $\mathrm{F}_1, \mathrm{D}_1$ *are isomorphims,* N_1, N_2' *are finitely generated and assume also that one of the following statements hold:*
a) There exists some $\mathrm{J} \in B^a(N_2, N_1')$ *such that* $N_2 \cong \mathrm{ImJ}$ *and* $\mathrm{ImJ}^{\perp}$ *is finitely generated.*
b) There exists some $\mathrm{J}' \in B^a(N_1', N_2)$ *such that* $N_1' \cong \mathrm{ImJ}', (\mathrm{ImJ}')^{\perp}$ *is finitely generated.*
Then $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ *for some* $\mathrm{C} \in B^a(H_{\mathcal{A}})$.

Theorem

Suppose $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ *for some* $\mathrm{C} \in B^{a}(H_{\mathcal{A}})$*. Then* $\mathrm{D} \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$ *and in addition the following statement holds: Either* $\mathrm{F} \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$ *or there exists decompositions*

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}'} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}'} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

w.r.t. which F', D' *have the matrices* $\begin{bmatrix} \mathrm{F}_1' & 0 \\ 0 & \mathrm{F}_4' \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1' & 0 \\ 0 & \mathrm{D}_4' \end{bmatrix}$*, where* $\mathrm{F}_1', \mathrm{D}_1'$ *are isomorphisms,* N_2' *is finitely generated,* N_1, N_2, N_1' *are closed, but* <u>*not*</u> *finitely generated, and* $M_2 \cong M_1', N_2 \cong N_1'$.

Theorem

Let $\mathrm{F}, \mathrm{D} \in B^a(H_{\mathcal{A}})$ *and suppose that* $\mathrm{D} \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ *and either* $\mathrm{F} \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ *or that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} N_2^{\perp} \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = N_1'^{\perp} \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}},$$

w.r.t. which F, D *have the matrices* $\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_4 \end{bmatrix}$, *respectively, where* $\mathrm{F}_1, \mathrm{D}_1$ *are isomorphisms* N_2', *is finitely generated and that there exists some* $\iota \in B^a(N_2, N_1')$ *such that* ι *is an isomorphism onto its image in* N_1' . *Then* $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi_-(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ *for some* $\mathrm{C} \in B^a(H_{\mathcal{A}})$.

Theorem

Let $\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$. *Then* $\mathrm{F}' \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}} \oplus H_{\mathcal{A}})$ *and either* $\mathrm{D} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ *or there exist decompositions*

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}'} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

$$H_{\mathcal{A}} \oplus H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}'} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}} \oplus H_{\mathcal{A}},$$

w.r.t. which F', D' *have matrices* $\begin{bmatrix} \mathrm{F}_1' & 0 \\ 0 & \mathrm{F}_4' \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1' & 0 \\ 0 & \mathrm{D}_4' \end{bmatrix}$, *respectively, where* $\mathrm{F}_1', \mathrm{D}_1'$ *are isomorphisms,* $M_2 \cong M_1'$ *and* $N_2 \cong N_1'$, N_1 *is finitely generated and* N_2, N_1' *are closed, but not finitely generated.*

Theorem

Let $\mathrm{F} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ *and suppose that either* $\mathrm{D} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ *or that there exist decompositions*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} N_2^{\perp} \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = {N_1'}^{\perp} \tilde{\oplus} N_1' \xrightarrow{\mathrm{D}} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

w.r.t. which F, D *have matrices* $\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix}, \begin{bmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_4 \end{bmatrix}$, *respectively, where* $\mathrm{F}_1, \mathrm{D}_1$ *are isomorphisms,* N_1' *is finitely generated and in addition there exists some*
$\iota \in B^a(N_1', N_2)$ *such that* ι *is an isomorphism onto its image. Then*

$$\mathbf{M}_{\mathrm{C}}^{\mathcal{A}} \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}} \oplus H_{\mathcal{A}}),$$

for some $\mathrm{C} \in B^a(H_{\mathcal{A}})$.

Definition

Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *topologically transitive* if for each non-empty open subsets U, V of $\mathcal{X}$, $T_n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$. If $T_n(U) \cap V \neq \varnothing$ holds from some n onwards, then $(T_n)_{n\in\mathbb{N}_0}$ is called *topologically mixing*.

Definition

Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *hypercyclic* if there is an element $x \in \mathcal{X}$ (called *hypercyclic vector*) such that the orbit $\mathcal{O}_x := \{T_n x : n \in \mathbb{N}_0\}$ is dense in $\mathcal{X}$. The set of all hypercyclic vectors of a sequence $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $HC((T_n)_{n\in\mathbb{N}_0})$. If $HC((T_n)_{n\in\mathbb{N}_0})$ is dense in $\mathcal{X}$, the sequence $(T_n)_{n\in\mathbb{N}_0}$ is called *densely hypercyclic*. An operator $T \in B(\mathcal{X})$ is called *hypercyclic* if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is hypercyclic.

Definition

Let $\mathcal{X}$ be a Banach space, and $(T_n)_{n\in\mathbb{N}_0}$ be a sequence of operators in $B(\mathcal{X})$. A vector $x \in \mathcal{X}$ is called a *periodic element* of $(T_n)_{n\in\mathbb{N}_0}$ if there exists a constant $N \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $T_{kN}x = x$. The set of all periodic elements of $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$. The sequence $(T_n)_{n\in\mathbb{N}_0}$ is called *chaotic* if $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive and $\mathcal{P}((T_n)_{n\in\mathbb{N}_0})$ is dense in $\mathcal{X}$. An operator $T \in B(\mathcal{X})$ is called *chaotic* if the sequence $\{T^n\}_{n\in\mathbb{N}_0}$ is chaotic.

Definition

Let $U, W \in B(\mathcal{H})$. We define the operator $T_{U,W} : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$T_{U,W}(F) := WFU \tag{1}$$

for all $F \in B(\mathcal{H})$.

Then the operator $T_{U,W}$ is invertible and its inverse is given by $T_{U^*,W^{-1}}$, i.e. $(T_{U,W})^{-1} = T_{U^*,W^{-1}}$.
We will denote this inverse by $S_{U,W}$ and for each $n \in \mathbb{N}$ we set

$$C_{U,W}^n = \frac{1}{2}(T_{U,W}^n + S_{U,W}^n).$$

Theorem

Let $\mathcal{H}$ be a separable Hilbert space. Let $W \in B(\mathcal{H})$ be invertible and $U \in B(\mathcal{H})$ be unitary such that for each $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ with

$$U^n(L_k) \perp L_k \quad \text{for all } n \geq N_k. \tag{2}$$

Then, the following statements are equivalent.

(i) $T_{U,W}$ *is hypercyclic on* $B_0(\mathcal{H})$, *where* $B_0(\mathcal{H})$ *is equipped with the operator norm* $\|\cdot\|$.

(ii) *For each* $m \in \mathbb{N}$ *there exist a strictly increasing sequence* $\{n_k\}$ *in* $\mathbb{N}$ *and the sequences* $\{D_k\}$ *and* $\{G_k\}$ *of operators in* $B_0(\mathcal{H})$ *such that*

$$\lim_{k\to\infty} \|D_k - P_m\| = \lim_{k\to\infty} \|G_k - P_m\| = 0, \tag{3}$$

and

$$\lim_{k\to\infty} \|W^{n_k} G_k\| = \lim_{k\to\infty} \left\|W^{-n_k} D_k\right\| = 0, \tag{4}$$

where P_m *denotes the orthogonal projection onto* L_m.

Definition

Let $\mathcal{X}$ be a Banach space, $a \in \mathcal{X}$, and $T \in B(\mathcal{X})$. We say that T is *a-transitive* if for each two non-empty open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathcal{X}$ with $a \in \mathcal{O}_1$, there are $m, n \in \mathbb{N}$ such that

$$T^n(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \varnothing, \qquad T^m(\mathcal{O}_2) \cap \mathcal{O}_1 \neq \varnothing.$$

Theorem

Let $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Then, the following statements are equivalent.

(i) *$T_{U,W}$ and $S_{U,W}$ are 0-transitive on $B_0(\mathcal{H})$.*

(ii) *For every finite dimensional subspace K of $\mathcal{H}$ there are strictly increasing sequences $\{n_j\}$ and $\{m_j\}$ in $\mathbb{N}$ and sequences of operators $\{G_j\}$ and $\{D_j\}$ in $B_0(\mathcal{H})$ such that*

$$\lim_{j\to\infty} \|G_j - P_K\| = \lim_{j\to\infty} \|D_j - P_K\| = 0, \tag{5}$$

and

$$\lim_{j\to\infty} \|W^{-m_j} G_j\| = \lim_{j\to\infty} \|W^{n_j} D_j\| = 0. \tag{6}$$

Theorem

Let $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. If $T_{U,W}$ is hypercyclic on $B_0(\mathcal{H})$, then $m(W) < 1 < \|W\|$.

Theorem

Let $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. Suppose that there is a finite dimensional subspace K of $\mathcal{H}$ such that for a constant $N > 0$, $U^n(K) \perp K$ for all $n \geq N$. Then, we have (i) $\Rightarrow$ (ii):

(i) *P_K belongs to the closure of $\mathcal{P}(\{S^n_{U,W}\}_{n\in\mathbb{N}_0})$ in $B_0(\mathcal{H})$.*

(ii) *There exists an increasing sequence (n_k) in $\mathbb{N}$ such that $m(W^{-n_k}) \to 0$ as $k \to \infty$.*

Theorem

Let $\mathcal{H}$ be a separable Hilbert space and $U, W \in B(\mathcal{H})$ such that W be invertible and U be unitary. Then, we have (ii) $\Rightarrow$ (i)*:*

(i) *the operators $T_{U,W}$ and $S_{U,W}$ are chaotic on $B_0(\mathcal{H})$.*

(ii) *For each $m \in \mathbb{N}$ there is a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sum_{l=1}^{\infty} \|W^{ln_k} P_m\| = \lim_{k\to\infty} \sum_{l=1}^{\infty} \|W^{-ln_k} P_m\| = 0,$$

where the corresponding series are convergent for each k.

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Then, we have (ii) $\Rightarrow$ (i)*:*

(i) *The sequence $(C_{U,W}^{(n)})_{n\in\mathbb{N}_0}$ is topologically transitive on $B_0(\mathcal{H})$.*

(ii) *For each $m \in \mathbb{N}$, there are sequences (E_k) and (R_k) of subspaces of L_m and an strictly increasing sequence (n_k) of positive integers such that $L_m = E_k \oplus R_k$ and*

$$\lim_{k\to\infty} \|W^{n_k} P_m\| = \lim_{k\to\infty} \left\|W^{-n_k} P_m\right\| = 0, \tag{7}$$

$$\lim_{k\to\infty} \left\|W^{2n_k} P_{E_k}\right\| = \lim_{k\to\infty} \left\|W^{-2n_k} P_{R_k}\right\| = 0. \tag{8}$$

Theorem

Suppose that $U, W \in B(\mathcal{H})$ *such that* W *is invertible and* U *is unitary. Let there exist a closed subspace* K *of* $\mathcal{H}$ *such that* $U^n(K) \perp K$ *for all* $n \geq N$. *Then,* (i) $\Rightarrow$ (ii).

(i) $\mathcal{P}(C_{U,W}^{(n)})$ *is dense in* $B_0(\mathcal{H})$, *and for each* $F \in B_0(\mathcal{H})$, $\lim_{n\to\infty} S_{U,W}^n(F) = 0$ *in* $B_0(\mathcal{H})$.

(ii) $m(W) < 1$.

Theorem

Suppose that $U, W \in B(\mathcal{H})$ *such that* W *is invertible and* U *is unitary. Assume that there exists a closed subspace* K *of* $\mathcal{H}$ *such that* $U^n(K) \perp K$ *for all* $n \geq N$. *We have* (i) $\Rightarrow$ (ii).

(i) $\mathcal{P}(C_{U,W}^{(n)})$ *is dense in* $B_0(\mathcal{H})$, *and* $\lim_{n\to\infty} T_{U,W}^n F = F$ *for all* $F \in B_0(\mathcal{H})$.

(ii) $m(W^{-1}) < 1$.

Theorem

Let $\mathcal{H}$ be a separable Hilbert space. We have (ii) $\Rightarrow$ (i)*:*

(i) *The sequence* $\{C_{U,W}^{(n)}\}$ *is chaotic on* $B_0(\mathcal{H})$.

(ii) *For each* $m \in \mathbb{N}$, *there exists a strictly increasing sequence* $\{n_k\} \subseteq \mathbb{N}$ *such that*

$$\lim_{k\to\infty} \sum_{l=1}^{\infty} \|W^{ln_k} P_m\| = \lim_{k\to\infty} \sum_{l=1}^{\infty} \|W^{-ln_k} P_m\| = 0, \qquad (9)$$

where the corresponding series are convergent for each k.

Remark

Our sufficient conditions for topological transitivity in the norm topology of $B_0(\mathcal{H})$ are also sufficient conditions for topological transitivity in the strong topology of $B(\mathcal{H})$. Indeed, since $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$, it is easily seen that the set $\{P_nF : F \in B(\mathcal{H}), n \in \mathbb{N}\}$ is dense in $B(\mathcal{H})$ in the strong operator topology. Moreover, in this case the conditions (3)-(4) in Theorem 128 can even be relaxed by considering the strong limits instead of the limit in norm and by dropping the requirement that the sequences $\{D_k\}$ and $\{G_k\}$ should belong to $B_0(\mathcal{H})$. Hence, also in the case of strong operator topology on $B(\mathcal{H})$, the operator W in Example satisfies the sufficient conditions for topological transitivity of $T_{U,W}$ and $\{C_{U,W}^{(n)}\}_n$.

Remark

Except from the implication (i) $\Rightarrow$ (ii) in Theorem 128, all our results about sufficient conditions for topological transitivity, easily generalize to the case where $B_0(\mathcal{H})$ is replaced by an arbitrary non-unital C^*-algebra $\mathcal{A}$, and the set of all finite rank orthogonal projections on $\mathcal{H}$ is replaced by the canonical approximate unit in $\mathcal{A}$. Indeed, if $\mathcal{A}$ is a non-unital C^*-algebra, then it can be isometrically embedded into a unital C^*-algebra $\mathcal{A}_1$ such that $\mathcal{A}$ becomes an ideal in $\mathcal{A}_1$. If u and w are invertible elements in $\mathcal{A}_1$ and u is unitary (i.e. $uu^* = u^*u = 1_{\mathcal{A}_1}$), then we can define the operator $T_{u,w}$ on $\mathcal{A}$ by $T_{u,w}(a) := wau$ for all $a \in \mathcal{A}$. Therefore, all our results regarding the sufficient conditions for $T_{u,w}$ to be topologically transitive or chaotic can be generalized in this setting.

Moreover, if $\mathcal{A}$ is a unital C^*-algebra and $H_{\mathcal{A}}$ denotes the standard Hilbert module over $\mathcal{A}$, then all our results so far can be transferred directly to the case where $B_0(\mathcal{H})$ and $B(\mathcal{H})$ are replaced by $K(H_{\mathcal{A}})$ and $B(H_{\mathcal{A}})$, respectively. Here, $K(H_{\mathcal{A}})$ and $B(H_{\mathcal{A}})$ stand for the set of all compact and all bounded $\mathcal{A}$-linear operators on $H_{\mathcal{A}}$, respectively.

Theorem

Let $w \in \mathcal{A}_1$ be invertible and u be a unitary element of $\mathcal{A}_1$. Suppose that there exist an element $a \in \mathcal{A}^+$ and an $N \in \mathbb{N}$ such that $au^n a = 0$ for all $n \geq N$. Then, (i) $\Rightarrow$ (ii).

(i) $\mathcal{P}((C_{u,w}^{(n)})_n)$ *is dense in* $\mathcal{A}$.

(ii) $m(\varphi(w)) < 1 < \| \varphi(w) \|$, *where* $(\varphi, \mathcal{H})$ *is the universal representation of* $\mathcal{A}_1$.

Theorem

Suppose that for every $m \in \mathbb{N}$ there exist sequences (E_k) and (R_k) of subspaces of L_m and an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that for each k, $L_m = E_k \oplus R_k$ and

$$\lim_{k\to\infty} \|W^{n_k} P_m\| = \lim_{k\to\infty} \left\|W^{-n_k} P_m\right\| = 0, \tag{10}$$

$$\lim_{k\to\infty} \left\|W^{2n_k} P_{E_k}\right\| = \lim_{k\to\infty} \left\|W^{-2n_k} P_{R_k}\right\| = 0. \tag{11}$$

Then, $\{C_{U,W}^{*(n)}\}$ is topologically transitive on $B_1(\mathcal{H})$.

Theorem

Suppose that $U, W \in B(\mathcal{H})$ such that W is invertible and U is unitary. Assume that there exists a finite dimensional subspace K of $\mathcal{H}$ such that $U^n(K) \perp K$ for all $n \geq N$. Then, (i) $\Rightarrow$ (ii).

(i) $\mathcal{P}(C_{U,W}^{(n)^*})$ *is dense in* $B_1(\mathcal{H})$, *and for each* $F \in B_1(\mathcal{H})$, $\lim_{n\to\infty} {S^*}_{U,W}^{n}(F) = 0$ *in* $B(\mathcal{H})$.

(ii) $m(W) < 1$.

Theorem

Let $U, W \in B(\mathcal{H})$ be invertible such that U is unitary. Suppose that there exists a finite dimensional subspace K of $\mathcal{H}$ and $N \in \mathbb{N}$ such that $U^n(K) \perp K$ for all $n \geq N$. Then, (i) $\Rightarrow$ (ii)*:*

(i) $\mathcal{P}\{(C^*_{U,W})^n\}$ *is dense in* $B(\mathcal{H})'$ *and* $\lim_{n\to\infty}(S^*_{U,W})^n\varphi = 0$ *for all* $\varphi \in B(\mathcal{H})'$.

(ii) $m(W) < 1$.

Theorem

We have (ii) $\Rightarrow$ (i)*:*

(i) $(C^{(n)*}_{U,W})$ *is topologically transitive in* $B(\mathcal{H})'$.

(ii) *For every $m \in \mathbb{N}$ there exist sequences (E_k) and (R_k) of subspaces of L_m and an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that for each k, $L_m = E_k \oplus R_k$ and*

$$\lim_{k\to\infty} \left\|P_m W^{n_k}\right\| = \lim_{k\to\infty} \left\|P_m W^{-n_k}\right\| = 0, \tag{12}$$

$$\lim_{k\to\infty} \left\|P_{E_k} W^{2n_k}\right\| = \lim_{k\to\infty} \left\|P_{R_k} W^{-2n_k}\right\| = 0. \tag{13}$$

Theorem

We have (i) $\Rightarrow$ (ii)*:*

(i) $P(T_{U,W}^{*^n})$ *is dense in* $B(\mathcal{H})'$.

(ii) $m(W) < 1$.

Theorem

We have (i) $\Rightarrow$ (ii)*:*

(i) $P(S_{U,W}^{*^n})$ *is dense in* $B(\mathcal{H})'$.

(ii) $m(W^{-1}) = \|W\|^{-1} < 1$, *that is* $\|W\| > 1$.

Theorem

Let $B(\mathcal{H})$ be equipped with the strong topology, and $B(\mathcal{H})'$ be equipped with the w^*-topology, where $B(\mathcal{H})'$ is the dual of $B(\mathcal{H})$. Then we have (ii) $\Rightarrow$ (i)*:*

(i) $\{T^{*^n}_{U,W}\}$ *and* $\{S^{*^n}_{U,W}\}$ *are topologically transitive on* $B(\mathcal{H})'$.

(ii) *for every* $n \in \mathbb{N}$ *there exist an increasing sequence* $\{n_k\} \subseteq \mathbb{N}$ *and sequences of operators* $\{G_k\}$ *and* $\{D_k\}$ *in* $B(\mathcal{H})$ *such that same as theorem 3.2 in the draft with*

$$\lim_{k\to\infty} \|G_k W^{n_k}\| = \lim_{k\to\infty} \|D_k W^{-n_k}\| = 0,$$

and

$$\mathrm{s-}\lim_{k\to\infty} G_k = \mathrm{s-}\lim_{k\to\infty} D_k = P_n,$$

where $\mathrm{s-}\lim$ *denotes the limit in the strong operator topology.*

Example

Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define $W \in B(\mathcal{H})$ by

$$W(e_j) := \begin{cases} \frac{1}{2}\, e_{j+2}, & \text{if } j \text{ is odd,} \\ \\ 2\, e_{j-2}, & \text{if } j \text{ is even and } j > 2, \\ \\ e_1, & \text{if } j = 2. \end{cases}$$

Then, W is invertible and $\|W\| = 2$. For each fixed $k \in \mathbb{N}$ it is easily checked that $\|W^{2k-1+m}P_{2k}\| = \frac{1}{2^m}$ for all $m \in \mathbb{N}$. Consequently, $\|W^{2k-1+m}P_{2k-1}\| \leq \frac{1}{2^m}$. Further, it is also easily verified that for each $k, m \in \mathbb{N}$ we have $\|W^{-2k-m}P_{2k+1}\| = \frac{1}{2^{m-1}}$, and this gives that $\|W^{-2k-m}P_{2k}\| \leq \frac{1}{2^{m-1}}$. As above, P_n denotes the orthogonal projection onto span$\{e_1, \dots, e_n\}$.
It follows that

$$\|P_{2k}(W^*)^{2k-1+m}\| = \frac{1}{2^m}, \qquad \|P_{2k+1}(W^*)^{-2k-m}\| = \frac{1}{2^{m-1}},$$

for all $k, m \in \mathbb{N}$.

Then W and W^* satisfy the sufficient condition in various results above on topological transitivity. If we instead of H consider $H_{\mathcal{A}}$ and let $\{e_j\}_{j\in\mathbb{N}}$ denote the standard basis, then the same arguments applies in this case also.

Example

Let $F(e_k) = e_{2k}$ for all k.
Then $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$

Example

Let $D(e_{2k-1}) = 0, D(e_{2k}) = e_k$.
Then $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$

Example

In general, let $\iota : \mathbb{N} \to \iota(\mathbb{N})$ be a bijection such that $\iota(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathbb{N} \setminus \iota(\mathbb{N})$ infinite. Moreover we may define ι in a such way s.t. $\iota(1) < \iota(2) < \iota(3) < \ldots$. Then, if we define an $\mathcal{A}$-linear operator F as $F(e_k) = e_{\iota(k)}$ for all k, we get that $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. Moreover, if we define an $\mathcal{A}$-linear operator D as

$$D(e_k) = \begin{cases} e_{\iota^{-1}(k)}, & \text{for } k \in \iota(\mathbb{N}), \\ 0, & \text{else} \end{cases}$$

then $D \in \mathcal{M}\Phi_-(H_\mathcal{A})$.

Those examples are also valid in the case when $\mathcal{A} = \mathbb{C}$, that is when $H_{\mathcal{A}} = H$ is a Hilbert space. We will now introduce examples where we use the structure of $\mathcal{A}$ itself in the case when $\mathcal{A} \neq \mathbb{C}$:

Example

Let $\mathcal{A} = (L^{\infty}([0,1]), \mu)$, where μ is a Borel probability measure. Set

$$F(f_1, f_2, f_3, ...) = (\mathcal{X}_{[0,\frac{1}{2}]} f_1, \mathcal{X}_{[\frac{1}{2},1]} f_1, \mathcal{X}_{[0,\frac{1}{2}]} f_2, \mathcal{X}_{[\frac{1}{2},1]} f_2, ...) \ .$$

Then F is bounded $\mathcal{A}-$ linear operator, $\ker F = \{0\}$,

$$ImF = Span_{\mathcal{A}}\{\mathcal{X}_{[0,\frac{1}{2}]} e_1, \mathcal{X}_{[\frac{1}{2},1]} e_2, \mathcal{X}_{[0,\frac{1}{2}]} e_3, \mathcal{X}_{[\frac{1}{2},1]} e_4, ...\},$$

and clearly $F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$.

Example

Let again $\mathcal{A} = (L^{\infty}([0,1]), \mu)$. Set

$$D(g_1, g_2, g_3, ...) = (\mathcal{X}_{[0,\frac{1}{2}]} g_1 + \mathcal{X}_{[\frac{1}{2},1]} g_2, \mathcal{X}_{[0,\frac{1}{2}]} g_3 + \mathcal{X}_{[\frac{1}{2},1]} g_4, ...) \ .$$

Then $\ker D = ImF$, D is an $\mathcal{A}$-linear, bounded operator and $ImD = H_{\mathcal{A}}$. Thus $D \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$. Indeed, $D = F^{*}$.

Example

Let $\mathcal{A} = B(H)$, where H is a Hilbert space and let P be an orthogonal projection on H. Set

$$F(T_1, T_2, ...) = (PT_1, (I - P)T_1, PT_2, (I - P)T_2, ...),$$

$$D(S_1, S_2, ...) = (PS_1 + (I - P)S_2, PS_3 + (I - P)S_4, ...),$$

then by similar arguments $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Example

In general, supose that $\{p_j^i\}_{j,i\in\mathbb{N}}$ is a family of projections in $\mathcal{A}$ s.t. $p_{j_1}^i p_{j_2}^i = 0$ for all i, whenever $j_1 \neq j_2$ and $\sum_{j=1}^{k} p_j^i = 1$ for some $k \in \mathbb{N}$.

Set

$$F'(\alpha_1, ..., \alpha_n, ...) = (p_1^1\alpha_1, p_2^1\alpha_1, ...p_k^1\alpha_1, p_2^1\alpha_2, p_2^2\alpha_2, ...p_k^2\alpha_2, ...),$$

$$D'(\beta_1, ..., \beta_n, ...) = (\sum_{i=1}^{k} p_i^1\beta_i, \sum_{i=1}^{k} p_i^2\beta_{i+k}, ...).$$

Then $F' \in \mathcal{M}\Phi_+(H_{\mathcal{A}}), D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.
Recalling now that a composition of two $\mathcal{M}\Phi_+$ operators is again an $\mathcal{M}\Phi_+$ operator and that the same is true for $\mathcal{M}\Phi_-$ operators, we may take suitable comprositions of operators from these examples in order to construct more $\mathcal{M}\Phi_\pm$ operators.
Even more $\mathcal{M}\Phi_\pm$ operators can be obtained by composing these operators with isomorphisms of $H_{\mathcal{A}}$. We will present here also some isomorphisms of $H_{\mathcal{A}}$.

Example

Let $j : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the operator U given by $U(e_k) = e_{j(k)}$ for all k is an isomorphism of $H_{\mathcal{A}}$. This is a classical well known example of an isomorphism.

Example

Let $(\alpha_1, ..., \alpha_n, ...) \in \mathcal{A}^{\mathbb{N}}$ be a sequence of invertible elements in $\mathcal{A}$ s.t. $\| \alpha_k \| \leq M$ for all $k \in \mathbb{N}$ and some $M > 0$. If the operator V is given by $V(e_k) = e_k \cdot \alpha_k$ for all k, then V is an isomorphism of $H_{\mathcal{A}}$. Moreover, if $(\alpha_1, \cdots, \alpha_n, \cdots)$ is the sequence from above, we may let $\tilde{V}$ be the operator on $H_{\mathcal{A}}$ given by $\tilde{V}(x_1, \cdots, x_n) = (\alpha_1 x_1, \cdots, \alpha_n x_n, \cdots)$. Then $\tilde{V}$ is also an isomorphism of $H_{\mathcal{A}}$.

Thank you for attention!

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