φ-maps on Hilbert C^*-mosules, φ-module domains and ternary domains

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Completely positive linear maps

Let A and B be two C^*-algebras. For each positive integer n, $M_n(A)$ denotes the C^*-algebra of all $n \times n$ matrices with elements in A.

Definition

Let $\varphi : A \to B$ be a linear map.

- φ is *positive* if $\varphi(a^* a) \geq 0$ for all $a \in A$.
- $\varphi : A \to B$ is a cp map (*completely positive linear map)* if for each positive integer n, the linear map

$$\varphi_n : M_n(A) \to M_n(B), \varphi_n\left([a_{ij}]_{i,j=1}^n\right) = [\varphi(a_{ij})]_{i,j=1}^n$$

is positive.

- A cp map $\varphi : A \to B$ is continuous, and it is contractive if $\|\varphi\| \leq 1$.
- Any C^*-morphism is a cp map.
- Not all cp maps are C^*-morphisms.

Multiplicative domains for completely positive linear maps

Theorem (M.D. Choi, 1974)

Let $\varphi : A \to B$ be a cp map and
$M_\varphi = \{a \in A : \varphi(ab) = \varphi(a)\varphi(b)$ *and* $\varphi(ba) = \varphi(b)\varphi(a), (\forall)$
$b \in A\}$.

1. *The set M_φ is a C^*-subalgebra of A and $\varphi|_{M_\varphi}$(the restriction of φ to M_φ) is a C^*-morphism.*
2. *If φ is a contractive cp map, then*
   $M_\varphi = \{a \in A : \varphi(aa^*) = \varphi(a)\varphi(a)^*$ *and* $\varphi(a^*a) = \varphi(a)^*\varphi(a)\}$
   and M_φ is the largest C^*-subalgebra C of A such that the map $\varphi|_C$ (the restriction of φ to C) is a C^*-morphism.

Definition

M_φ is called the multiplicative domain of φ.

If φ is unital, then 1_A, the unit of A, is an element in M_φ.

Hilbert C*-modules

Definition

A *Hilbert C^*-module* X over a C^*-algebra A is a linear space X that is a right A-module, together with an A-valued inner product $\langle \cdot, \cdot \rangle$ that is A-linear in the second variable, with the following properties:

1. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
2. $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in X$

   such that X is a Banach space with the norm induced by the inner product, $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

- $L(\mathcal{H}, \mathcal{K})$, the vector space of all bounded linear operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, has a canonical structure of Hilbert C^*-module over $L(\mathcal{H})$ with:
  - $(T, S) \in L(\mathcal{H}, \mathcal{K}) \times L(\mathcal{H}) \to TS \in L(\mathcal{H}, \mathcal{K})$
  - $(T_1, T_2) \in L(\mathcal{H}, \mathcal{K}) \times L(\mathcal{H}, \mathcal{K}) \to \langle T_1, T_2\rangle = T_1^* T_2 \in L(\mathcal{H})$.
- A is a Hilbert C^*-module over A with the inner product given by

$$\langle a, b\rangle = a^* b.$$

Hilbert C*-modules

Definition

Let X be a Hilbert C^*-module over A and $\varphi : A \to L(\mathcal{H})$ a linear map. A map $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ is called a *φ-map* if

$$\Phi(x)^* \Phi(y) = \varphi(\langle x, y \rangle) \text{ for all } x, y \in X.$$

- If φ is a $*$-representation of A on the Hilbert space $\mathcal{H}$, we say that Φ is a φ-representation of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.
- If φ is a cp map, we say that Φ is a cp φ-map.

Remark

1. *If Φ is a φ-map, then Φ is linear.*
2. *If Φ is a φ-completely positive map, then Φ is continuous.*
3. *If Φ is a φ-representation, then Φ is a φ-module map,* $\Phi(xa) = \Phi(x)\varphi(a)$ *for all $x \in X$ and $a \in A$.*

Ternary maps

Definition

The *ternary product* on a Hilbert C^*-module X is the map $[\cdot,\cdot,\cdot] : X \times X \times X \to X$ defined by

$$[x, y, z] = x\langle y, z\rangle \text{ for all } x, y, z \in X.$$

A map between two Hilbert C^*-modules is called a *ternary map*, if it preserves the ternary product.

A Hilbert C^*-module X over A is *full* if A coincides with the closed two-sided $*$-ideal $\langle X, X\rangle$ generated by $\{\langle x, y\rangle : x, y \in X\}$.

Theorem (M. Skeide, K. Sumesh, 2014)

Let $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a map. Then:

1. *If Φ is a φ-representation, then Φ is a linear ternary map.*
2. *If X is full and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ is a linear ternary map, then there is a $*$-homomorphism $\varphi : A \to L(\mathcal{H})$ such that Φ is a φ-map.*

Module domains for cp maps on Hilbert C*-modules

Theorem

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ a cp map, $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ be a φ-map and

$$X_\Phi = \{x \in X : \Phi(xb) = \Phi(x)\varphi(b), (\forall)\, b \in A\}.$$

Then:

1. *X_Φ is a Hilbert C^*-module over M_φ;*
2. *$\Phi|_{X_\Phi} : X_\Phi \to L(\mathcal{H}, \mathcal{K})$ is a $\varphi|_{M_\varphi}$-representation.*

Definition

For a a φ-map, $\Phi : X \to L(\mathcal{H}, \mathcal{K})$,

$$X_\Phi = \{x \in X : \Phi(xb) = \Phi(x)\varphi(b), (\forall)\, b \in A\}$$

is called the *φ-module domain* of Φ.

Example

Example

The map $\Phi : M_2(\mathbb{C}) \to L(\mathbb{C}^2, \mathbb{C}^2)$ defined by

$$\Phi\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right)(\xi, \eta) = (x_{11}\xi, x_{21}\xi)\,; (\xi, \eta) \in \mathbb{C}^2$$

is a cp φ-map, where $\varphi : M_2(\mathbb{C}) \to L(\mathbb{C}^2)$ is given by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)(\xi, \eta) = (a_{11}\xi, 0)\,; (\xi, \eta) \in \mathbb{C}^2.$$

It is easy to check that φ ia a cp map,
$M_\varphi = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C}\right\}$ and $X_\Phi = \left\{\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} : x, y \in \mathbb{C}\right\}$.

Module domains for cp maps on Hilbert C*-modules

Let $\Phi : A \to L(H, K)$ be a cp φ-map.

- $A_\Phi \subseteq \{a \in A; \varphi(ab) = \varphi(a)\varphi(b)\ (\forall)\ b \in A\}$

For $(u_i)_{i \in I}$ - an approximate unit for A,

$$\varphi(ab) = \lim_i \varphi(u_i ab) = \lim_i \Phi(u_i)^* \Phi(ab) = \lim_i \Phi(u_i)^* \Phi(a)\varphi(b) \\ = \lim_i \varphi(u_i a)\varphi(b) = \varphi(a)\varphi(b), (\forall)\ b \in A.$$

- If A is unital and $\Phi(1_A)$ is onto, then $A_\Phi = \{a \in A; \varphi(ab) = \varphi(a)\varphi(b)\ (\forall)\ b \in A\}$

If $\Phi(1_A)$ is onto, then $\Phi(1_A)^*$ has left inverse, S.
$(\forall)\ a \in A$ s. t. $\varphi(ab) = \varphi(a)\varphi(b)\ (\forall)\ b \in A$,

$$\Phi(ab) - \Phi(a)\varphi(b) = S\Phi(1_A)^*(\Phi(ab) - \Phi(a)\varphi(b)) \\ = S(\varphi(ab) - \varphi(a)\varphi(b)) = 0, (\forall)\ b \in A.$$

- If A is unital and $\Phi(1_A)$ is a coisometry, then φ is a $*$-homomorphism and $A_\Phi = A$.
  $\varphi(ab) = \Phi(a^*)^* \Phi(1_A)\Phi(1_A)^* \Phi(b) = \varphi(a)\varphi(b), (\forall)\ a, b \in A.$

Module domains for cp maps on Hilbert C*-modules

Theorem

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ be a cp map, $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ be a φ-map and $x_0 \in X$. Then

$$x_0 \in X_\Phi \Leftrightarrow \Phi\left(x_0 \langle y, z\rangle\right) = \Phi\left(x_0\right) \varphi\left(\langle y, z\rangle\right), (\forall)\, y, z \in X.$$

Proof.

" $\Rightarrow$ " It is clear.
" $\Leftarrow$ "

Lemma

Let I be a closed two-sided $*$-ideal of A and $\varphi : A \to L(\mathcal{H})$ a cp map. Then $M_{\varphi|_I} \subseteq M_\varphi$, where $\varphi|_I$ is the restriction of φ to I.

□

Proof.

$\Phi(x_0\langle y,z\rangle)=\Phi(x_0)\varphi(\langle y,z\rangle)\,(\forall)\,y,z\in X$

$\Rightarrow \varphi(\langle x_0,x_0\rangle\langle y,z\rangle)=\varphi(\langle x_0,x_0\rangle)\varphi(\langle y,z\rangle),\,(\forall)\,y,z\in X$

$\Rightarrow \langle x_0,x_0\rangle\in M_{\varphi|_I}\subseteq M_\varphi$, I - the closed two sided $*$-ideal $\langle X,X\rangle$ of A.

$\Rightarrow \varphi(\langle y,z\rangle\langle x_0,x_0\rangle b)=\varphi(\langle y,z\rangle)\varphi(\langle x_0,x_0\rangle)\varphi(b),\,(\forall)\,y,z\in X,b\in A.$

Let $b\in A$, and $(u_\lambda)_{\lambda\in\Lambda}$ an approximate unit for $\langle X,X\rangle$.

$$\begin{aligned}\langle\varphi(b^*\langle x_0,x_0\rangle b)\xi,\eta\rangle&=\left\langle\lim_\lambda\varphi(u_\lambda b^*u_\lambda\langle x_0,x_0\rangle b)\xi,\eta\right\rangle\\&=\lim_\lambda\langle\varphi(u_\lambda b^*u_\lambda)\varphi(\langle x_0,x_0\rangle)\varphi(b)\xi,\eta\rangle\\&=\langle\varphi(b^*)\varphi(\langle x_0,x_0\rangle)\varphi(b)\xi,\eta\rangle,\,(\forall)\,\xi,\eta\in\mathcal{H}\end{aligned}$$

$\Rightarrow \varphi(b^*\langle x_0,x_0\rangle b)=\varphi(b^*)\varphi(\langle x_0,x_0\rangle)\varphi(b)$

$\Rightarrow (\Phi(x_0b)-\Phi(x_0)\varphi(b))^*(\Phi(x_0b)-\Phi(x_0)\varphi(b))=0$

$\Rightarrow \Phi(x_0b)=\Phi(x_0)\varphi(b).$ □

Remark

$\Phi : X \to L(\mathcal{H}, \mathcal{K})$ *is a ternary map, if*

$$\Phi(x\langle y, z\rangle) = \Phi(x)\langle \Phi(y), \Phi(z)\rangle = \Phi(x)\Phi(y)^*\Phi(z) \text{ for all}$$

$x, y, z \in X$.

Definition

Let $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ be a map. The set

$$T_\Phi = \{y \in X : \Phi(x\langle y, z\rangle) = \Phi(x)\Phi(y)^*\Phi(z) \text{ for all } x, z \in X\}$$

is called the *ternary domain* of Φ.

Module domains and ternary domains for cp maps on Hilbert C*-modules

Proposition

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ be a cp map, $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ be a φ-map and $x_0 \in X$. Then the following statements are equivalent:

1. $x_0 \in X_\Phi$;
2. $\Phi\left(x_0 \langle y, z\rangle\right) = \Phi\left(x_0\right)\Phi\left(y\right)^*\Phi\left(z\right)\ y, z \in X$;
3. $\Phi\left(y \langle x_0, z\rangle\right) = \Phi\left(y\right)\Phi\left(x_0\right)^*\Phi\left(z\right)\ y, z \in X$;
4. $x_0 \in T_\Phi$.

Ternary domains for cp maps

Definition

Fie $\varphi : A \longrightarrow L(\mathcal{H})$ be a cp map

$$T_\varphi = \{a \in A; \varphi(ba^*c) = \varphi(ba^*)\varphi(c) = \varphi(b)\varphi(a^*c) = \varphi(b)\varphi(a^*)\varphi(c), (\forall)\ b, c \in A\}$$

is called the ternary domain of φ.

Example

The map $\varphi : M_2(\mathbb{C}) \longrightarrow L(\mathbb{C}^2)$ is given by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)(\xi, \eta) = (a_{11}\xi, 0)\,;(\xi, \eta) \in \mathbb{C}^2.$$

is a cp map. It is easy to check that

$$M_\varphi = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C}\right\} \text{ and } T_\varphi = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{C}\right\}.$$

Ternary domains for cp maps

Proposition

Let $\varphi : A \to L(\mathcal{H})$ be a cp map. Then:

1. $T_\varphi = \{a \in M_\varphi : \varphi(ba^*c) = \varphi(b)\varphi(a)^*\varphi(c)$ *for all* $b, c \in A\}$;
2. T_φ *is a closed two-sided* $*$*-ideal in* M_φ;
3. $T_\varphi A T_\varphi = T_\varphi$;

   If A and φ are unital, then:
4. $T_\varphi = \{a \in A : \varphi(ba^*c) = \varphi(b)\varphi(a^*)\varphi(c), (\forall)\ b, c \in A\}$;
5. $1_A \in T_\varphi \Leftrightarrow \varphi$ *is a* $*$ *-morphism.*

Module domains for cp maps on Hilbert C*-modules

Proposition

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ be a cp map, $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ be a φ-map and $x_0 \in X$. Then the following statements are equivalent:

1. $x_0 \in X_\Phi$;
2. $\langle x_0, x_0 \rangle \in M_\varphi$ and $\varphi\left(a \langle x_0, x_0 \rangle b\right) = \varphi\left(a\right) \varphi\left(\langle x_0, x_0 \rangle\right) \varphi\left(b\right), (\forall)$ $a, b \in A$;
3. $\langle x_0, x_0 \rangle \in T_\varphi$.

Module domains for cp maps on Hilbert C*-modules

Corollary

Let $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ *be a* φ*-map. Then:*

1. X_Φ *is a Hilbert* C^**-module over the* C^**-algebra* T_φ;
2. $XT_\varphi = X_\Phi$.

Corollary

If $\Phi_1 : X \to L(\mathcal{H}, \mathcal{K}_1)$ *and* $\Phi_2 : X \to L(\mathcal{H}, \mathcal{K}_2)$ *are two cp* φ*-maps, then* $X_{\Phi_1} = X_{\Phi_2}$.

Ternary domains of cp maps on Hilbert C*-modules

Theorem (Asadi, Behmani, Medghalchi, Nikpey, 2017)

Let X be a Hilbert A-module and $\varphi : A \to L(\mathcal{H})$ be a cp map. If $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)$ is a minimal Stinespring representation associated to φ, then there two Hilbert spaces $\mathcal{K}_{\pi_\varphi}$ and $\mathcal{K}_\varphi$, a π_φ-representation $\Pi_{\pi_\varphi} : X \to L(\mathcal{H}_\varphi, \mathcal{K}_{\pi_\varphi})$ of X on the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{K}_{\pi_\varphi}$, and a unitary operator $W_\varphi : \mathcal{K}_\varphi \to \mathcal{K}_{\pi_\varphi}$ such that the map $\Phi_{\varphi,X} : X \to L(\mathcal{H}, \mathcal{K}_\varphi)$ given by

$$\Phi_{\varphi,X}(x) = W_\varphi^* \Pi_{\pi_\varphi}(x) V_\varphi$$

is a φ-map. Moreover, if $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ is a φ-map, then there is a unique isometry $S_\Phi : \mathcal{K}_\varphi \to \mathcal{K}$ such that

$$\Phi(\cdot) = S_\Phi \Phi_{\varphi,X}(\cdot)$$

Ternary domains of cp maps on Hilbert C*-modules

Definition

Let X be a Hilbert A-module and $\varphi : A \to L(\mathcal{H})$ a cp map. We denote the φ-module domain of each φ-map on X by X_φ and call it the **ternary domain of φ on X**.

Remark

1. *X_φ is a Hilbert C^*-module over the C^*-algebra T_φ;*
2. *If X is full, then X_φ is a full Hilbert C^*-module over the C^*-algebra T_φ;*
3. *Every cp φ-map $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ is a ternary map on X_φ.*
4. $$X_\varphi = \overline{span}\{x \in X; \langle x, x\rangle \in M_\varphi,\ \varphi(a\langle x, x\rangle b) = \varphi(a)\varphi(\langle x, x\rangle)\varphi(b),\ (\forall)\ a, b \in A\}$$
$$= \overline{span}\{x \in X; \langle x, x\rangle \in M_\varphi,\ \varphi(b^*\langle x, x\rangle b) = \varphi(b)^*\varphi(\langle x, x\rangle)\varphi(b),\ (\forall)\ b \in A\}.$$

Linking algebra of a Hilbert C*-module

Let X and Y be two Hilbert C^*-modules over A.

- $K(X,Y)$ denotes the space of all 'compact' operators from X to Y.
- $K(X,Y)$ is generated by $\{\theta_{y,x} : x \in X, y \in Y\}$, where $\theta_{y,x} : X \to Y, \theta_{y,x}(z) = y \langle x, z \rangle$.
- The linking C^*-algebra $\mathcal{L}_A(X)$ of X is the C^*-algebra of all *'compact'* operators on the Hilbert C^*-module $X \oplus A$ over A.

Linking algebra of a Hilbert C*-module

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  - The map $x \mapsto r_x$, where $r_x : A \to X, r_x(a) = xa$ $(r_x^* = l_x)$, is an isometric linear isomorphism from X to $K(A, X)$, and we denote $K(A, X)$ by X and r_x by x.
  - The map $y \mapsto l_y$, where $l_y : X \to A, l_y(z) = \langle y, z \rangle$, is an isometric conjugate linear isomorphism from X to $K(X, A)$, and we denote $K(X, A)$ by X^* and l_x by x^*.
  - The map $a \mapsto T_a$, where $T_a : A \to A$, $T_a(b) = ab$, is an isometric linear $*$-isomorphism from A to $K(A)$, and we denote $K(A)$ by A and T_a by a.

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$$\begin{aligned} \mathcal{L}_A(X) &= \begin{bmatrix} K(X) & K(A, X) \\ K(X, A) & K(A) \end{bmatrix} = \begin{bmatrix} K(X) & X \\ X^* & A \end{bmatrix} \\ &= \left\{ \begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : T \in K(X), x, y \in X, a \in A \right\}. \end{aligned}$$

Stinespring type theorem for cp on Hilbert C*-modules

Theorem (Bhat, Ramesh,Sumesh, 2012)

Let X be a Hilbert C^*-module over the C^*-algebra A, $\varphi : A \to L(\mathcal{H})$ a cp and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a φ-map. Then there is a triple of pairs $\left(\left(\Pi_\Phi, \pi_\varphi\right), (W_\Phi, V_\varphi), (\mathcal{H}_\varphi, \mathcal{K}_\Phi)\right)$ consisting of the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{K}_\Phi$, a bounded linear operator $V_\varphi : \mathcal{H} \to \mathcal{H}_\varphi$, a coisometry $W_\Phi : \mathcal{K} \to \mathcal{K}_\Phi$, a $*$-representation $\pi_\varphi : A \to L(\mathcal{H}_\varphi)$ and a π_φ-representation $\Pi_\Phi : X \to L(\mathcal{H}_\varphi, \mathcal{K}_\Phi)$ such that $\left(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi\right)$ is a minimal Stinespring representation associated to φ and $\Phi(x) = W_\Phi^* \Pi_\Phi(x) V_\varphi$ for all $x \in X$. Moreover, $\left[\Pi_\Phi(X) V_\varphi \mathcal{H}\right] = \mathcal{K}_\Phi$.

The triple of pairs $\left(\left(\Pi_\Phi, \pi_\varphi\right), (W_\Phi, V_\varphi), (\mathcal{H}_\varphi, \mathcal{K}_\Phi)\right)$ is called a minimal Stinespring representation associated to the cp φ- map Φ, which is unique up to unitary equivalence.

Induced completely positive maps on the linking algebra of a Hilbert C*-module

Proposition

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ a cp map and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a φ-map. Then there is a unique $*$-representation $\psi_{\Phi,\varphi} : K(X) \to L(\mathcal{K})$ such that

$$\widetilde{\varphi}_{\Phi} = \begin{bmatrix} \psi_{\Phi,\varphi} & \Phi \\ \Phi^* & \varphi \end{bmatrix} : \mathcal{L}_A(X) \to L(\mathcal{K} \oplus \mathcal{H})$$

is a cp map and for every minimal Stinespring representation associated to the φ-map Φ such as $((\Pi, \pi), (W, V), (\mathcal{K}', \mathcal{H}'))$, there is a $*$-representation $\Gamma : K(X) \to L(\mathcal{K}')$ such that $\Gamma(\theta_{x,y}) = \Pi(x)\Pi(y)^*$ for all $x, y \in X$ and

$$\widetilde{\varphi}_{\Phi} = \begin{bmatrix} \psi_{\Phi,\varphi} & \Phi \\ \Phi^* & \varphi \end{bmatrix} = \begin{bmatrix} W^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} \Gamma(\cdot) & \Pi(\cdot) \\ \Pi^*(\cdot) & \pi(\cdot) \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}.$$

Moreover, if φ is contractive, then $\widetilde{\varphi}_{\Phi}$ is contractive.

Induced completely positive maps on the linking algebra of a Hilbert C*-module

Lemma

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ a cp map and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a φ-map. Then:

1. $\Phi(T(z)) = \psi_{\Phi,\varphi}(T)\,\Phi(z), \forall T \in K(X), \forall z \in X$;
2. $K(X)X_\varphi \subseteq X_\varphi$.

   If φ is contractive, then:
3. $x \in X_\varphi \Leftrightarrow \psi_{\Phi,\varphi}(\theta_{x,x}) = \Phi(x)\,\Phi(x)^*$;
4. $x \in X_\varphi \Leftrightarrow \psi_{\Phi,\varphi}(\theta_{x,y}) = \Phi(x)\,\Phi(y)^*, (\forall)\, y \in X$;
5. $\psi_{\Phi,\varphi}(\theta_{xa,y}) = \Phi(x)\,\varphi(a)\,\Phi(y)^*, (\forall)\, x, y \in X, (\forall)\, a \in T_\varphi$.

Multiplicative domains of the induced completely positive maps on the linking algebra

Theorem

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ a contractive cp linear map and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a φ-map. If $\widetilde{\varphi}_\Phi : \mathcal{L}_A(X) \to L(\mathcal{K} \oplus \mathcal{H})$ is the cp linear map on $\mathcal{L}_A(X)$ associated to the φ-map Φ, then:

$$M_{\widetilde{\varphi}_\Phi} = \left\{ \begin{bmatrix} T & y \\ x^* & a \end{bmatrix} : T \in K(X), x, y \in X_\varphi, a \in M_\varphi \right\},$$

and

$$T_{\widetilde{\varphi}_\Phi} = \left\{ \begin{bmatrix} T & y \\ x^* & a \end{bmatrix} : T \in K(X), x, y \in X_\varphi, a \in T_\varphi \right\}.$$

Remark

The above theorem remains valid for all completely positive linear map

Multiplicative domains of the induced completely positive maps on the linking algebra

Corollary

Let X be a Hilbert A-module, $\varphi : A \to L(\mathcal{H})$ a contractive cp linear map and $\Phi : X \to L(\mathcal{H}, \mathcal{K})$ a φ-map. If $\widetilde{\varphi}_{\Phi} : \mathcal{L}_A(X) \to L(\mathcal{K} \oplus \mathcal{H})$ is the cp linear map on $\mathcal{L}_A(X)$ associated to the φ-map Φ, then

$$\mathcal{L}_{M_{\varphi}}(X_{\varphi}) \subseteq M_{\widetilde{\varphi}_{\Phi}} \text{ and } \mathcal{L}_{T_{\varphi}}(X_{\varphi}) \subseteq T_{\widetilde{\varphi}_{\Phi}}.$$

Corollary

Let $\varphi : A \to L(\mathcal{H})$ be a completely positive linear map and Φ and Ψ be two operator-valued φ-maps on a Hilbert A-module X. Then

$$M_{\widetilde{\varphi}_{\Phi}} = M_{\widetilde{\varphi}_{\Psi}} \text{ and } T_{\widetilde{\varphi}_{\Phi}} = T_{\widetilde{\varphi}_{\Psi}}.$$

Multiplicative domains of the induced completely positive maps on the linking algebra

Example

If φ and Φ are as in the previous example, then

$$M_{\widetilde{\varphi}_\Phi} = \left\{ \begin{bmatrix} a & b & x & 0 \\ c & d & y & 0 \\ u & v & s & 0 \\ 0 & 0 & 0 & t \end{bmatrix} ; a, b, c, d, x, y, s, t \in \mathbb{C} \right\}$$

and

$$T_{\widetilde{\varphi}_\Phi} = \left\{ \begin{bmatrix} a & b & x & 0 \\ c & d & y & 0 \\ u & v & s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; a, b, c, d, x, y, s \in \mathbb{C} \right\}.$$

Thank you for your attention!