Subproduct systems, Gysin sequences and $SU(2)$-symmetries

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Gysin sequences in K-theory (Sphere bundles)

Theorem (Karoubi?)

Let M be a **compact Hausdorff space** *and let $V \to M$ be a* **hermitian complex vector bundle** *of rank n.*

- *Then there exists a* **six term exact sequence** *of K-groups:*

$$\begin{array}{ccccc} K_0(C(M)) & \xrightarrow[\chi(V)]{} & K_0(C(M)) & \longrightarrow & K_0(C(S(V))) \\ \uparrow & & & & \downarrow \\ K_1(C(S(V))) & \longleftarrow & K_1(C(M)) & \xleftarrow{\chi(V)} & K_1(C(M)) \end{array}$$

- *where $S(V) \to M$ is the* **sphere bundle** *and*

$$\chi(V) = \sum_{i=0}^{n} (-1)^i \big[\Gamma(\Lambda^i(V))\big] \in KK_0\big(C(M), C(M)\big)$$

is the **Euler characteristic***.*

Gysin sequences in K-theory (Pimsner algebras)

Theorem (Pimsner)

Let X be a countably generated and full C^*-correspondence from a separable C^*-algebra A to itself such that

- *the left action $\phi : A \to \mathbb{L}(X)$ is injective and $\phi(A) \subseteq \mathbb{K}(X)$.*
- *Then there exists a* **six term exact sequence** *of K-groups:*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow[1-[X]]{} & K_0(A) & \longrightarrow & K_0(\mathbb{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{O}_X) & \longleftarrow & K_1(A) & \xleftarrow{1-[X]} & K_1(A) \end{array}$$

- *where $\mathbb{O}_X$ is the* **Cuntz-Pimsner algebra** *and*

$$1 - [X] \in KK_0(A, A)$$

is the **Euler characteristic***.*

Subproduct systems

Definition

A **subproduct system** *is a sequence* $\{X(m)\}_{m=0}^{\infty}$ *of* C^**-correspondences from* C^**-algebra* A *to itself together with* **adjointable isometries** $\iota_{k,m} : X(k+m) \to X(k)\widehat{\otimes}_A X(m)$ *for all* $k, m \in \mathbb{N}_0$ *such that*

- *all left actions are injective and non-degenerate;*
- $X(0) = A$*;*
- *the adjointable isometries satisfy* **unitality** *and* **associativity** *constraints.*

Fock spaces and creation operators

Definition

Let (X, ι) be a subproduct system.

- *The* **Fock space** *is the Hilbert C^*-module direct sum*

$$F(X) := \bigoplus_{m=0}^{\infty} X(m).$$

- *For each $\xi \in X(k)$ the* **creation operator** $T_\xi \in \mathbb{L}\big(F(X)\big)$ *is defined by*

$$T_\xi(\eta) := \iota_{k,m}^*(\xi \otimes \eta) \qquad \eta \in X(m).$$

Toeplitz and Cuntz-Pimsner algebras

Definition

Let (X, ι) be a subproduct system.

- *The* **Toeplitz algebra** $\mathbb{T}_X \subseteq \mathbb{L}\big(F(X)\big)$ *is the smallest unital C^*-subalgebra containing all the creation operators.*
- *The* **Cuntz-Pimsner algebra** $\mathbb{O}_X$ *is the quotient of* $\mathbb{T}_X$ *by the* **ideal** $\mathbb{I}_X$ *consisting of those Toeplitz operators that "vanish at infinity".*

Subproduct systems from representations of $SU(2)$

Definition

Let $\rho : SU(2) \to U(H)$ be a non-trivial irreducible representation. The **determinant** *of ρ is the 1-dimensional subspace*

$$\det(\rho) := \left\{ \xi \in H \otimes H \mid \rho^{\otimes 2}(g)(\xi) = \xi \ , \ \ \forall g \in SU(2) \right\} \subseteq H \otimes H .$$

Lemma

- *For each $m \geq 2$ define the subspaces*

$$K_\rho(m) := \sum_{i=0}^{m-2} H^{\otimes i} \otimes \det(\rho) \otimes H^{\otimes (m-2-i)} \subseteq H^{\otimes m} \qquad \text{and}$$

$$H_\rho(m) := K_\rho(m)^\perp \ , \ \ H_\rho(0) = \mathbb{C} \ \ \text{and} \ \ H_\rho(1) := H .$$

- *Then the sequence $\{H_\rho(m)\}_{m=0}^{\infty}$ and the inclusions $\iota_{k,m} : H_\rho(k+m) \subseteq H_\rho(k) \otimes H_\rho(m)$ form a* **subproduct system** *equipped with an $SU(2)$-***action**.

Toeplitz and Cuntz-Pimsner algebras from $SU(2)$-actions

Definition

Let $\rho : SU(2) \to U(H)$ be a non-trivial irreducible representation.

- *The* **Toeplitz algebra** $\mathbb{T}_\rho$ *is the Toeplitz algebra coming from the subproduct system* (H_ρ, ι)*.*
- *The* **Cuntz-Pimsner algebra** $\mathbb{O}_\rho$ *agrees with the quotient* C^**-algebra* $\mathbb{T}_\rho / \mathbb{K}\big(F(H_\rho)\big)$*.*

Remark

The Toeplitz algebra $\mathbb{T}_\rho$ *and the Cuntz-Pimsner algebra* $\mathbb{O}_\rho$ *admit a* **gauge action of** $SU(2)$ *coming from the* $SU(2)$*-action on the underlying subproduct system.*

Fusion rules

Theorem

*Let $\rho : SU(2) \to U(H)$ be a non-trivial irreducible representation. For each $k, m \in \mathbb{N}_0$ there exists an explicit $SU(2)$-**equivariant unitary isomorphism***

$$W_{k,m} : H_\rho(k) \otimes H_\rho(m) \to H_\rho(k+m) \oplus H_\rho(k+m-2) \oplus \ldots \oplus H_\rho(|k-m|).$$

Remark

*For $\dim(H) = 2$ we recover the usual **fusion rules** for the irreducible representations of $SU(2)$. For $\dim(H) > 2$ this is no longer the case since $H_\rho(m)$ is **not irreducible** for $m \geq 2$.*

KK-equivalences

Theorem (Arici and K.)

Let $\rho : SU(2) \to U(H)$ be a non-trivial irreducible representation.

- *The Toeplitz algebra $\mathbb{T}_\rho$ is KK-***equivalent*** to $\mathbb{C}$.*
- *In particular, we have an* **isomorphism of K-groups***:*

$$K_*(\mathbb{T}_\rho) \cong K_*(\mathbb{C}).$$

The Gysin sequence

Theorem (Arici and K.)

Let $\rho : SU(2) \to U(H)$ be a non-trivial irreducible representation. Then there exists a **six term exact sequence** *of K-groups:*

$$\begin{array}{ccccc} K_0(\mathbb{C}) & \xrightarrow[1-[H]+[\det(\rho)]]{} & K_0(\mathbb{C}) & \longrightarrow & K_0(\mathbb{O}_\rho) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{O}_\rho) & \longleftarrow & K_1(\mathbb{C}) & \xleftarrow{1-[H]+[\det(\rho)]} & K_1(\mathbb{C}) \end{array}$$

Computation of K-groups

Corollary (Arici and K.)

Let $\rho : SU(2) \to U(H)$ *be an irreducible representation with* $\dim(H) = n \geq 2$. *Then*

$$K_0(\mathbb{O}_\rho) \cong \mathbb{Z}/(n-2)\mathbb{Z} \qquad \textit{and} \qquad K_1(\mathbb{O}_\rho) \cong \begin{cases} \mathbb{Z} & \text{for} \quad n = 2 \\ \{0\} & \text{for} \quad n > 2 \end{cases} .$$