Gabor duality theory for Morita equivalent C*-algebras

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- **Gabor duality theory for Morita equivalent C*-algebras,** International Journal of Mathematics, 2020.
- **Duality theory of Gabor frames and Heisenberg modules,** Journal of Noncommutative Geometry, accepted 2020.

Main theme

Study of equivalence bimodules between two C^*-algebras from the perspective of frame theory. Motivation comes from the duality principle for Gabor frames.

Equivalence bimodules

Morita equivalence

Let A and B be C^*-algebras. An A-B-**equivalence bimodule** between A and B is a Hilbert C^*-module E satisfying the following conditions:

- $\overline{{}_\bullet\langle E, E\rangle} = A$ and $\overline{\langle E, E\rangle_\bullet} = B$, where ${}_\bullet\langle E, E\rangle = \mathrm{span}\{{}_\bullet\langle f, g\rangle \mid f, g \in E\}$ and likewise for $\langle E, E\rangle_\bullet$.
- For all $f, g \in E$, $a \in A$ and $b \in B$,
$$\langle af, g\rangle_\bullet = \langle f, a^*g\rangle_\bullet \text{ and } {}_\bullet\langle fb, g\rangle = {}_\bullet\langle f, gb^*\rangle.$$
- For all $f, g, h \in E$,
$${}_\bullet\langle f, g\rangle h = f\langle g, h\rangle_\bullet .$$

Notation

$\Theta_{f,g}$ denotes the **rank-one module operator** $\Theta_{f,g} : h \mapsto {}_\bullet\langle h, f\rangle g$ on E and $\mathbb{K}_A(E)$ denotes the C^*-algebra of compact module operators on E.

Observation

It is a well-known result that if E is an A-B-equivalence bimodule, then $B \cong \mathbb{K}_A(E)$ through the identification $\Theta_{f,g} \mapsto \langle f, g\rangle_\bullet$.

Fact

Let E be an A-B-equivalence bimodule. Then E is a finitely generated projective A-module if and only if B is unital.

Localization

Suppose ϕ is a faithful positive linear functional on a C^*-algebra B, and that E is a right Hilbert B-module. We define an inner product

$$\langle \cdot, \cdot \rangle_\phi : E \times E \to \mathbb{C}, \quad (f, g) \mapsto \phi(\langle g, f \rangle_B),$$

where $\langle \cdot, \cdot \rangle_B$ is the B-valued inner product and complete E with respect to $\langle \cdot, \cdot \rangle_\phi$, denoted by H_E, and called the **localization of** E in ϕ
There is a natural map $\rho_\phi : E \to H_E$ which induces an injective $*$-homomorphism $\rho_\phi : \mathrm{End}_B(E) \to \mathbb{B}(H_E)$.

Localization of equivalence bimodules

For unital Morita equivalent C^*-algebras A and B a result by Rieffel says that there is a bijection between normalized finite traces on A and non-normalized finite traces on B under which to a trace tr_B on B there is an associated trace tr_A on A satisfying

$$\mathrm{tr}_A({}_\bullet\langle f, g\rangle) = \mathrm{tr}_B(\langle g, f\rangle_\bullet)$$

for all f, g in the equivalence bimodule E.

Localization

Let E be an A-B-equivalence bimodule, tr_B a faithful finite trace on B.

i) There is a finite trace on A, denoted tr_A, for which $\mathrm{tr}_A({}_\bullet\langle f, g\rangle) = \mathrm{tr}_B(\langle g, f\rangle_\bullet)$, for all $f, g \in E$.
   For $f, g \in E$ we set

$$\langle f, g\rangle_{\mathrm{tr}_A} = \mathrm{tr}_A({}_\bullet\langle f, g\rangle), \quad \langle f, g\rangle_{\mathrm{tr}_B} = \mathrm{tr}_B(\langle g, f\rangle_\bullet),$$

   with $\langle f, g\rangle_{\mathrm{tr}_A} = \langle f, g\rangle_{\mathrm{tr}_B}$ for all $f, g \in E$.

Frames for Hilbert C^*-modules

Definitions

Let E be an A-B-equivalence bimodule. For $g \in E$ we define the **analysis operator** by

$$\begin{aligned} \Phi_g : E &\to A \\ f &\mapsto {}_\bullet\langle f, g\rangle \,, \end{aligned}$$

and the **synthesis operator**:

$$\begin{aligned} \Psi_g : A &\to E \\ a &\mapsto a \cdot g \,. \end{aligned}$$

Frames for Hilbert C^*-modules

Remark

Since E is an A-B-bimodule, we also have defined the analysis operator and the synthesis operator with respect to the B-valued inner product. If we do so, then it will be indicated by writing Φ_g^B.

Definitions-ctd

Let E be an A-B-equivalence bimodule. We define the **frame-like operator** $\Theta_{g,h}$ to be

$$\begin{aligned} \Theta_{g,h} : E &\to E \\ f &\mapsto {}_{\bullet}\langle f, g\rangle \cdot h. \end{aligned}$$

$\Theta_{g,h} = \Psi_h \Phi_g = \Phi_h^* \Phi_g$. The **frame operator of** g is the operator $\Theta_g := \Theta_{g,g} = \Phi_g^* \Phi_g$.

Frames for Hilbert C^*-modules

Module frames

Suppose we have $g_1, \ldots, g_k \in E$, such that $\sum_{i=1}^k \Theta_{g_i}$ is invertible E. Then we call $\{g_1, \ldots, g_k\}$ a **module frame** for E.
This is equivalent to existence of constants $C, D > 0$ such that

$$C {}_\bullet\langle f, f\rangle \leq \sum_{i=1}^k {}_\bullet\langle f, g_i\rangle \, {}_\bullet\langle g_i, f\rangle \leq D {}_\bullet\langle f, f\rangle$$

for all $f \in E$.

Frame condition

Let E be an A-B-equivalence bimodule. Then $f = \sum_{i=1}^k \Theta_{g_i, h_i} f$ for all $f \in E$ if and only if B is unital and $\sum_{i=1}^k \langle g_i, h_i \rangle_\bullet = 1_B$.

Lemma

Let E be an A-B-equivalence bimodule. Suppose $g, h \in E$ satisfy ${}_\bullet\langle f, h\rangle\, g = f$ for all $f \in E$. Then

$$f = h\,\langle g, f\rangle_\bullet \quad \text{for all } f \in \overline{h \cdot B}.$$

Definition

Let E be an A-B-equivalence bimodule. If $g \in E$ is such that Θ_g is invertible on E, then $h = \Theta_g^{-1} g$ is called the **canonical dual atom** of g.

Module Duality Principle

Theorem (Duality principle)

Let E be an A-B-equivalence bimodule, and let $g \in E$. The following are equivalent:

1. $\Theta_g : E \to E$ *is invertible.*
2. $\Phi_g^B(\Phi_g^B)^* : B \to B$ *is an isomorphism.*

Density theorems

Let E be an A-B-equivalence bimodule where both A and B are unital.

- If $g \in E$ is such that $\Theta_g : E \to E$ is invertible, then $\mathrm{tr}_B(1_B) \leq \mathrm{tr}_A(1_A)$.
- If $g \in E$ is such that $\Phi_g \Phi_g^* : A \to A$ is an isomorphism, then $\mathrm{tr}_B(1_B) \geq \mathrm{tr}_A(1_A)$.

Localization of module frames

Let E be an A-B-equivalence bimodule, where B is unital and equipped with a faithful finite trace tr_B and induced trace tr_A on A. We denote by H_E the localization of E in tr_A, and by $(-,-)_E$ its inner product, i.e. $(f_1, f_2)_E = \mathrm{tr}_A({}_\bullet\langle f_1, f_2\rangle)$ for all $f_1, f_2 \in E$.

Proposition

For $g \in E$ there exists an $h \in E$ such that we have ${}_\bullet\langle f, g\rangle\, h = f$ for all $f \in E$ if and only if there exist constants $C, D > 0$ such that

$$C(f, f)_E \leq (f\langle g, g\rangle_\bullet, f)_E \leq D(f, f)_E$$

for all $f \in H_E$.

Riesz sequences

Proposition

Let E be an A-B-equivalence bimodule where A is unital and equipped with a faithful finite trace tr_A. We localize E and A as described above. For $g \in E$ we have $\Phi_g \Phi_g^* : A \to A$ is an isomorphism if and only if there exist $C, D > 0$ such that for all $a \in A$:

$$C(a, a)_A \leq (ag, ag)_E \leq D(a, a)_A.$$

We say g generates a **module Riesz sequence** for E with respect to A.

Heisenberg modules

modulation operator and **translation operator**

$$E_\beta f(t) = e^{2\pi i\beta t} f(t),\ \ T_\alpha f(t) = f(t-\alpha),\ \ \ \alpha,\beta\in\mathbb{R}\backslash\{0\},\ \ f\in L^2(\mathbb{R}),$$

$$\mathcal{A} = \Big\{ \mathrm{a} \in \mathrm{B}(L^2(\mathbb{R}))\ :\ \mathrm{a} = \sum_{m,n\in\mathbb{Z}} a(m,n)\, E_{m\beta} T_{n\alpha}\,,\ \ a\in\ell^1(\mathbb{Z}^2)\Big\}.$$

$\mathcal{A}$ is a faithful representation of the twisted group algebra $\ell^1(\alpha\mathbb{Z}\times\beta\mathbb{Z}, c)$:

$$a_1 \natural\, a_2(m,n) = \sum_{m',n'\in\mathbb{Z}} a_1(m',n')\, a_2(m-m',n-n')\, e^{2\pi i\beta\alpha(m-m')n'}, \quad (1)$$

$$a^*(m,n) = e^{2\pi i\alpha\beta mn}\,\overline{a(-m,-n)}. \quad (2)$$

The enveloping C^*-algebra is the twisted group C^*-algebra $C^*(\mathbb{Z}^2, c)$

Heisenberg modules

The left-action that $a \in \mathcal{A}$ has on functions $f \in L^2(\mathbb{R})$ is given by $a \cdot f = \sum_{m,n\in\mathbb{Z}} a(m,n) E_{m\beta} T_{n\alpha} f$.
For functions in $\mathbf{S}_0(\mathbb{R})$ we define an $\mathcal{A}$-valued inner-product in the following way: ${}_\bullet\langle \cdot\,,\,\cdot \rangle : \mathbf{S}_0(\mathbb{R}) \times \mathbf{S}_0(\mathbb{R}) \to \mathcal{A}$,

$$ {}_\bullet\langle f, g \rangle = \sum_{m,n\in\mathbb{Z}} \langle f, E_{m\beta} T_{n\alpha} g \rangle \, E_{m\beta} T_{n\alpha}. $$

Here $\mathbf{S}_0(\mathbb{R})$ is **Feichtinger's algebra**: a suitable Banach space of test-functions, which is widely used in time-frequency analysis. Let g be a Gaussian function. Then $f \in L^2(\mathbb{R})$ is in $\mathbf{S}_0(\mathbb{R})$ if and only if

$$ \|f\|_{\mathbf{S}_0} = \iint_{\mathbb{R}^2} |\langle f, E_\omega T_x g \rangle| dx d\omega < \infty. $$

Let $\mathcal{B}$ be the twisted group algebra $\ell^1(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ and $C^*(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ its enveloping C^*-algebra.
We define a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_\bullet : \mathbf{S}_0(\mathbb{R}) \times \mathbf{S}_0(\mathbb{R}) \to \mathcal{B}$: $\langle f, g \rangle_\bullet = \frac{1}{|\alpha\beta|} \sum_{m,n\in\mathbb{Z}} \langle g, \left(E_{m/\alpha} T_{n/\beta}\right)^* f \rangle \left(E_{m/\alpha} T_{n/\beta}\right)^*$ that has a right-action on functions $g \in L^2(\mathbb{R})$ given by

$$g \cdot b = \frac{1}{|\alpha\beta|} \sum_{m,n\in\mathbb{Z}} b(m,n) \left(E_{m/\alpha} T_{n/\beta}\right)^* g.$$

Associativity condition:

$${}_\bullet\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_\bullet \quad \text{for all} \quad f, g, h \in \mathbf{S}_0(\mathbb{R}).$$

The completion of $\mathbf{S}_0(\mathbb{R})$ with respect to $\|f\|_E = \|{}_\bullet\langle f, f \rangle\|^{1/2}$ (or $\|\langle f, f \rangle_\bullet\|^{1/2}$) is the **Heisenberg module**, E.

Heisenberg modules

Bessel family

Denote by $B_{\alpha\beta}$ the subspace of $L^2(\mathbb{R})$ consisting of those $g \in L^2(\mathbb{R})$ such that

$$\sum_{k,l \in \mathbb{Z}} |\langle f, E_{\beta l} T_{\alpha k} g \rangle|^2 \leq \infty,$$

for all $f \in L^2(\mathbb{R})$. This is a Banach space with respect to

$$\|g\|_{B_{\alpha\beta}} = \sup_{\|f\|_2 = 1} \Big(\sum_{k,l \in \mathbb{Z}} |\langle f, E_{\beta l} T_{\alpha k} g \rangle|^2 \Big)^{1/2}.$$

Gabor system $\{E_{\beta l} T_{\alpha k} g\}_{k,l \in \mathbb{Z}}$

$$f = {}_\bullet\langle f, g\rangle\, h = \sum_{m,n\in\mathbb{Z}} \langle f, E_{\beta n} T_{\alpha m} g\rangle E_{\beta n} T_{\alpha m} h$$

Austad-Enstad

- For $g \in \mathbf{S}_0(\mathbb{R})$ we have $\|g\|_{B_{\alpha\beta}} = \|g\|_E$, i.e. Heisenberg modules consists of the Bessel vectors of the Gabor system $\{E_{\beta l} T_{\alpha k} g\}_{k,l\in\mathbb{Z}}$.
- The set $\{g_1, ..., g_k\}$ generates E as a left $C^*(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$-module if and only if $\{E_{\beta n} T_{\alpha m} g_i : i = 1, .., k\}_{m,n\in\mathbb{Z}}$ is a (multi-window) frame for $L^2(\mathbb{R})$, i.e. there exist constant $A, B > 0$ such that

$$A\,\|f\|_2 \le \sum_{i=1}^{k} \sum_{m,n\in\mathbb{Z}} |\langle f, E_{m\beta} T_{n\alpha} g_i\rangle|^2 \le B\,\|f\|_2 \quad \text{for all} \quad f \in L^2(\mathbb{R}).$$

Gabor duality principle

Definitions

(i) The Gabor system $\{E_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ is a **frame** for $L^2(\mathbb{R})$, with bounds $A, B > 0$, i.e.,

$$A\,\|f\|_2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{m\beta}T_{n\alpha}g\rangle|^2 \leq B\,\|f\|_2 \quad \text{for all} \quad f \in L^2(\mathbb{R}).$$

(ii) The Gabor system $\{(E_{m/\alpha}T_{n/\beta})^*g\}_{m,n\in\mathbb{Z}}$ is a **Riesz sequence** for $L^2(\mathbb{R})$ with bounds $|\alpha\beta|A, |\alpha\beta|B$, i.e.,

$$|\alpha\beta|\,A\,\|c\|_2^2 \leq \Big\| \sum_{m,n\in\mathbb{Z}} c(m,n)\big(E_{m\alpha^{-1}}T_{m\beta^{-1}}\big)^*g\,\Big\|_2^2 \leq |\alpha\beta|\,B\,\|c\|_2^2,$$

for all $c \in \ell^2(\mathbb{Z}^2)$.

Gabor duality principle

Theorem (Duality principle)

$\{E_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ *is a* **frame** *for* $L^2(\mathbb{R})$ *if and only if* $\{(E_{m/\alpha}T_{n/\beta})^*g\}_{m,n\in\mathbb{Z}}$ *is a* **Riesz sequence** *for* $\overline{\mathrm{span}\{E_{m/\alpha}T_{n/\beta})^*g}_{m,n\in\mathbb{Z}}\}$.

We have the standard trace on $\mathcal{A}$: $\mathrm{tr}_{\mathcal{A}}(a) = a(0,0)$. The localization of E is a subspace of $L^2(\mathbb{R})$.
Consequently, the module duality principle for the Heisenberg module E reduces to the Gabor duality principle.

Extensions

Multi-super Gabor frames or more generally matrix Gabor frames.