A closer look at the B-spline interpolation problem in the setting of Hilbert C^*-modules

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(Joint work with R. Eskandari, M. Frank, and V. M. Manuilov)

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Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, let Λ be a family of continuous linear forms over $\mathscr{H}$, and let $B(x, y)$ be a bounded bilinear form on $\mathscr{H} \times \mathscr{H}$ such that

$$B(x,x) \geqslant 0 \ \text{ for all } x \in N(\Lambda) = \{x \in \mathscr{H} : \lambda(x) = 0 \text{ for all } \lambda \in \Lambda\}.$$

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A vector $s \in \mathscr{H}$ is called a B-spline if $B(s, y) = 0$ for all $y \in N(\Lambda)$.

The closed linear space of all B-splines is denoted by $Sp(B, \Lambda)$. For $x \in \mathscr{H}$, an element $s \in \mathscr{H}$ is said to be a $Sp(B, \Lambda)$-interpolate of x if s is a B-spline and $s - x \in N(\Lambda)$.

Lucas [1] gives conditions that insure the existence of a $Sp(B,\Lambda)$-interpolate of any element in $\mathscr{H}$.

[1] T. R. Lucas, *M-splines*, J. Approximation Theory **5** (1972), 1–14. (see also R. Arcangéli, M. C. López de Silanes, and J. C. Torrens, *Multidimensional minimizing splines. Theory and applications*, Grenoble Sciences. Kluwer Academic Publishers, Boston, MA, 2004.)

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One of the conditions is that the system $(\mathscr{H}, \Lambda, B, N(\Lambda))$ is well-posed in the sense that if $N_1 := \{x \in N(\Lambda) : B(x,x) = 0\}$, then $B(x,y) = 0$ for all $x \in \mathscr{H}$ and all $y \in N_1$.

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Moreover, if N_2 is the orthogonal complement of N_1 in $N(\Lambda)$, then B is definite on N_2, that is, $B(x,x) \geqslant c\|x\|^2$ for some $c > 0$ and all $x \in N_2$.

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Inspired by the theory of B-splines in the setting of Hilbert spaces, we investigate the B-spline interpolation problem in the framework of Hilbert modules over C^*-algebras and W^*-algebras. This talk is based on a recent paper. [2]

[2] M. Frank, V. Manuilov, and M. S. Moslehian, *B-spline interpolation problem in Hilbert C^*-modules*, J. Oper. Theory (2021), arXiv:2004.01444.

The notion of a pre-Hilbert C^*-module $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ is a natural generalization of that of an inner product space in which we allow the inner product to take its values in a C^*-algebra $\mathscr{A}$ instead of the field of complex numbers.

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If $\mathscr{X}$ together with the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is complete, then it is called an $\mathscr{A}$-Hilbert C^*-module. The positive square root of $\langle x, x\rangle$ is denoted by $|x|$ for $x \in \mathscr{X}$. We say that a closed submodule $\mathscr{Y}$ of a Hilbert C^*-module $\mathscr{X}$ is orthogonally complemented if $\mathscr{X} = \mathscr{Y} \oplus \mathscr{Y}^\perp$, where

$$\mathscr{Y}^\perp = \{x \in \mathscr{X} : \langle x, y\rangle = 0 \text{ for all } y \in \mathscr{Y}\}.$$

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STANDING NOTATION: Throughout this talk, let $\mathscr{A}$ be a C^*-algebra (W^*-algebra if we explicitly state it) whose pure state space is denoted by $\mathcal{PS}(\mathscr{A})$, and let $\mathscr{X}$ denote a Hilbert $\mathscr{A}$-module.

If $\mathscr{X}'$ denotes the set of all bounded $\mathscr{A}$-linear maps from $\mathscr{X}$ into $\mathscr{A}$, named as the dual of $\mathscr{X}$, then $\mathscr{X}'$ becomes a right $\mathscr{A}$-module equipped with the following actions:

$$(\rho+\lambda\tau)(x)=\rho(x)+\overline{\lambda}\tau(x) \quad \text{and} \quad (\tau b)(x)=b^*\tau(x)$$

for $\rho, \tau \in \mathscr{X}'$, $b \in \mathscr{A}$, $\lambda \in \mathbb{C}$ and $x \in \mathscr{X}$.

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$$(\rho + \lambda\tau)(x) = \rho(x) + \overline{\lambda}\tau(x) \quad \text{and} \quad (\tau b)(x) = b^*\tau(x)$$

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Trivially, to every bounded $\mathscr{A}$-linear map $T : \mathscr{X} \to \mathscr{Y}$ one can associate a bounded $\mathscr{A}$-linear map $T' : \mathscr{Y}' \to \mathscr{X}'$ defined by

$$T'(g)(x) = g(T(x)) \quad g \in \mathscr{Y}'.$$

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It is easy to verify that the map $x \mapsto \hat{x}$ is isometric and $\mathscr{A}$-linear. Hence one can identify $\mathscr{X}$ with

$$\widehat{\mathscr{X}} := \{\hat{x} : x \in \mathscr{X}\}$$

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A module $\mathscr{X}$ is called self-dual if $\widehat{\mathscr{X}} = \mathscr{X}'$. For example, a unital C^*-algebra $\mathscr{A}$ is self-dual as a Hilbert $\mathscr{A}$-module via $\langle a, b \rangle = a^* b$.

Given $x \in \mathscr{X}$, one can define

$$\dot{x} \in \mathscr{X}''$$

by

$$\dot{x}(f) = f(x)^* \quad (f \in \mathscr{X}').$$

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There is an $\mathscr{A}$-valued inner product on the second dual $\mathscr{X}''$ defined by

$$\langle F, G \rangle = F(\dot{G}), \text{ where } \dot{G}(x) := G(\widehat{x}) \ (x \in \mathscr{X}).$$

It is an extension of the inner product on $\mathscr{X}$. In addition, the map $F \mapsto \dot{F}$ is an isometric inclusion, and $\dot{\dot{x}} = \widehat{x}$ because of

$$\dot{\dot{x}}(y) = \dot{x}(\widehat{y}) = \widehat{y}(x)^* = \langle x, y \rangle = \widehat{x}(y).$$

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Thus we have the chain of inclusions as $\mathscr{X} \subseteq \mathscr{X}'' \subseteq \mathscr{X}'$, and every self-dual Hilbert C^*-module is reflexive, too.

A comprehensive result of Paschke reads as follows.[3]

[3] W. L. PASCHKE, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1972), 443–468.

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Theorem

Let $\mathscr{X}$ be a pre-Hilbert C^*-module over a W^*-algebra $\mathscr{A}$. The $\mathscr{A}$-valued inner product on $\mathscr{X} \times \mathscr{X}$ can be extended to $\mathscr{X}' \times \mathscr{X}'$ in such a way as to make $\mathscr{X}'$ into a self-dual Hilbert $\mathscr{A}$-module.

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By an $\mathscr{A}$-sesquilinear form on a Hilbert $\mathscr{A}$-module $\mathscr{X}$ we mean a bounded map $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that it is anti-$\mathscr{A}$-linear in the first variable and $\mathscr{A}$-linear in the second one. We say that it is positive on a set $\mathscr{Y}$ if $B(y, y) \geqslant 0$ for all $y \in \mathscr{Y}$.

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It is elliptic on a set $\mathscr{Y}$ if $B(y, y) \geqslant c\langle y, y\rangle$ for any $y \in \mathscr{Y}$ and some positive real constant c.

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The **B-spline interpolation problem** asks whether for each $x \in \mathscr{X}$ there exists a B-spline element s in the coset $x + \mathscr{Y}$.

Example

Suppose that P is a non-trivial projection on a Hilbert C^*-module $\mathscr{X}$ and set

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Then there is an orthogonal decomposition $\mathscr{X} = \mathrm{ran}(P) \oplus \ker(P)$. Let $\mathscr{Z} \subseteq \ker(P)$ be a closed submodule and set $\mathscr{Y} := \mathrm{ran}(P) \oplus \mathscr{Z} \subseteq \mathscr{X}$. Given $x \in \mathscr{X}$ the element $s \in x + \mathscr{Y}$ can be selected as $s = (1 - P)(x)$, i.e. the B-spline interpolation has a solution.

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It might be not unique when $\mathscr{Z} \neq \{0\}$. Indeed, let $z \in \mathscr{Z}$, and let $s = (1-P)(x) + z$. Then $s - x = -P(x) + z \in \mathscr{Y}$ and $B(s,y) = 0$ for any $y \in \mathscr{X}$.

Example

Consider a Hilbert space $\mathscr{H}$ as a Hilbert C^*-module over the C^*-algebra $\mathbb{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$ under the C^*-inner product

$$[x, y] := x \otimes y,$$

where $x \otimes y$ is defined by $(x \otimes y)(z) = \langle z, y\rangle x$, and the actions

$$\lambda \cdot x = \bar{\lambda} x \;\text{ and }\; x \cdot T = T^*(x).$$

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$$\lambda \cdot x = \bar{\lambda} x \quad \text{and} \quad x \cdot T = T^*(x).$$

Then the B-spline interpolation problem has no solution for any given nontrivial closed subspace $\mathscr{Y}$ of $\mathscr{X}$.

Example

Let $\mathscr{H}$ be an infinite-dimensional Hilbert space, $\mathscr{A} = \mathbb{B}(\mathscr{H})$, and let $\mathbb{K}(\mathscr{H})$ be the norm-closed two-sided ideal of $\mathbb{B}(\mathscr{H})$ of all compact operators on $\mathscr{H}$. Let $\mathscr{X}$ be $\mathscr{A}$ with the Hilbert $\mathscr{A}$-module operations inherited from the algebraic operations in $\mathscr{A}$, in particular,

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$$\langle T, S\rangle = T^*S.$$

For

$$B(\cdot,\cdot) = \langle\cdot,\cdot\rangle$$

and $\mathscr{Y} = \mathbb{K}(\mathscr{H})$, the B-spline interpolation problem has no solution.

It is known that in the setting of Hilbert spaces if σ is a bounded sesquilinear form on $\mathscr{H}$, then there is a unique bounded linear operator U on $\mathscr{H}$ such that

$$\sigma(x, y) = \langle U(x), y \rangle.$$

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Next, we show that the above representation is valid in a self-dual Hilbert C^*-module. To achieve it we need a lemma.

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Next, we show that the above representation is valid in a self-dual Hilbert C^*-module. To achieve it we need a lemma.

Theorem

Let $\mathscr{X}$ be a self-dual Hilbert $\mathscr{A}$-module. Let $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be an $\mathscr{A}$-sesquilinear form on $\mathscr{A}$. Then there is a unique operator $T \in \mathscr{L}(\mathscr{X})$ such that

$$B(x, y) = \langle T(x), y \rangle \qquad (x, y \in \mathscr{X}).$$

Let B be an $\mathscr{A}$-sesquilinear form on a Hilbert $\mathscr{A}$-module $\mathscr{X}$. Let $\mathscr{Y}$ be a closed submodule of $\mathscr{X}$.

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Set

$$\breve{\mathscr{Y}} = \{\breve{y} \in \mathscr{Y} : B(\breve{y}, y) = 0 \text{ for all } y \in \mathscr{Y}\},$$
$$\widetilde{\mathscr{Y}} = \{\tilde{y} \in \mathscr{Y} : B(y, \tilde{y}) = 0 \text{ for all } y \in \mathscr{Y}\},$$

and

$$\mathscr{Y}_1 = \{y \in \mathscr{Y} : B(y, y) = 0\}.$$

Clearly, $\breve{\mathscr{Y}} \subseteq \mathscr{Y}_1$ and $\widetilde{\mathscr{Y}} \subseteq \mathscr{Y}_1$.

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If $B(\cdot,\cdot)$ is skew-symmetric, i.e. $B(x,y) = -B(y,x)$ on $\mathscr{X}$, then always $\mathscr{Y}_1 = \mathscr{Y}$, but the other two sets are most often smaller. Moreover, for $\mathscr{A}$-sesquilinear forms both $\breve{\mathscr{Y}}$ and $\widetilde{\mathscr{Y}}$ are norm-complete $\mathscr{A}$-submodules of $\mathscr{Y}$.

Theorem

Let B be positive on $\mathscr{Y}$, i.e. $B(y,y) \geqslant 0$ for any $y \in \mathscr{Y}$. Then

$$\breve{\mathscr{Y}} = \mathscr{Y}_1 = \widetilde{\mathscr{Y}}.$$

Theorem

Let $\mathscr{Y}$ be a closed submodule of $\mathscr{X}$, and B be an $\mathscr{A}$-sesquilinear form on $\mathscr{X}$. Suppose that the B-spline interpolation problem has a solution for $\mathscr{Y}$ for an element $x \in \mathscr{X}$.

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Then the following two conditions are equivalent:

1. *The solution of the B-spline problem for x is unique.*
2. $\breve{\mathscr{Y}} = \{0\}$.

Theorem

Let $T : \mathscr{X} \to \mathscr{X}$ be adjointable. Let $B_1 : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be the $\mathscr{A}$-sesquilinear form defined by

$$B_1(x, y) = \langle T(x), y \rangle \quad (x, y \in \mathscr{X}).$$

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Let $B_2 : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be the $\mathscr{A}$-sesquilinear form defined by

$$B_2(x, y) = \langle T^*(x), y \rangle \qquad (x, y \in \mathscr{X}).$$

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Then $\widetilde{\mathscr{X}_{B_1}}=\breve{\mathscr{X}}_{B_2}$ and $\widetilde{\mathscr{X}_{B_2}}=\breve{\mathscr{X}}_{B_1}$.

The following result is a generalization of a result of Arambašić and Rajić [4].

[4] Lj. Arambašić and R. Rajić, *Operator version of the best approximation problem in Hilbert C^*-modules*, J. Math. Anal. Appl. **413** (2014), no. 1, 311–320.

The following result is a generalization of a result of Arambašić and Rajić [4].

Theorem

Let $\mathscr{Y}$ be a closed submodule of $\mathscr{X}$. Let B be an $\mathscr{A}$-sesquilinear form on $\mathscr{X}$ and $T : \mathscr{X} \to \mathscr{Z}'$ be such that

$$B(x, z) = T(x)(z) \qquad (x \in \mathscr{X}, z \in \mathscr{Z}).$$

and $\breve{\mathscr{Y}} = \{0\}$. Then B-spline interpolation problem has a solution for $\mathscr{Y}$ if and only if

$$\{Tx|_{\mathscr{Y}} : x \in \mathscr{X}\} \subseteq \{Ty|_{\mathscr{Y}} : y \in \mathscr{Y}\}.$$

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Then a necessary condition for the B-spline interpolation problem for $\mathscr{Y}$ to have a solution is

$$B(x, \check{y}) = 0 \quad \text{for all } x \in \mathscr{X}, \check{y} \in \check{\mathscr{Y}}$$

The next two results give some properties inherited from a module to its second dual.

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Let $\mathscr{X}$ and $\mathscr{Z}$ be Hilbert $\mathscr{A}$-modules over a C^*-algebra $\mathscr{A}$. Let $B : \mathscr{X} \times \mathscr{Z} \to \mathscr{A}$ be an $\mathscr{A}$-sesquilinear form. Then B is uniquely extended to an $\mathscr{A}$-sesquilinear form on $\mathscr{X}'' \times \mathscr{Z}''$.

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If $\mathscr{X}$ is a Hilbert C^*-module over a W^*-algebra, then $\mathscr{X}'$ is self-dual, and so $\mathscr{X}'' = \mathscr{X}'$.

The next two results give some properties inherited from a module to its second dual.

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Corollary

Let $\mathscr{X}$ be a Hilbert C^*-module over a W^*-algebra $\mathscr{A}$. Let B be an $\mathscr{A}$-sesquilinear form on $\mathscr{X}$. Then B is uniquely extended to an $\mathscr{A}$-sesquilinear form on $\mathscr{X}'$.

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Let $\mathscr{Y}$ be an orthogonally complemented submodule of $\mathscr{X}$. Then $\mathscr{Y}''$ is an orthogonally complemented submodule of $\mathscr{X}''$.

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Let $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be an $\mathscr{A}$-sesquilinear form and positive on $\mathscr{Y}$. Assume there exists $c > 0$ and $k > 0$ such that for every $f \in \mathcal{PS}(\mathscr{A})$ and every $x \in \mathscr{Y} \backslash \breve{\mathscr{Y}}$ there exists $y \in \mathscr{Y}$ with $\|y\| = 1$ such that $f(|y|^2) \geqslant k$ and

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Then a necessary condition for the B-spline interpolation problem for $\mathscr{Y}$ to have a solution is

$$B(x, \check{y}) = 0 \qquad (x \in \mathscr{X}, \check{y} \in \breve{\mathscr{Y}}). \tag{1}$$

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If $\dot{\mathscr{Y}}$ is orthogonally complemented in $\mathscr{X}''$, then (1) is sufficient.

Theorem

Let $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module over a W^*-algebra $\mathscr{A}$ and $\mathscr{Y}$ be an orthogonally complemented submodule of $\mathscr{X}$. Let $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be an $\mathscr{A}$-sesquilinear form on $\mathscr{X}$. Let $\widetilde{B}$ be the extension of B on $\mathscr{X}'$. If B is positive on $\mathscr{Y}$, then $\widetilde{B}$ is positive on $\mathscr{Y}'$

Our second main result reads as follows. [5]

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Theorem

Let $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module over a W^*-algebra $\mathscr{A}$ and $\mathscr{Y}$ be a nontrivial orthogonally complemented submodule of $\mathscr{X}$.

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Theorem

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$$|fB(x, y)|^2 \geqslant cf(|x|^2)f(|y|^2). \qquad (2)$$

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$$|fB(x,y)|^2 \geqslant cf(|x|^2)f(|y|^2). \tag{2}$$

Then, the $\widetilde{B}$-spline interpolation problem has a solution for $\mathscr{Y}'$ if and only if

$$B(x, \breve{y}) = 0 \qquad (x \in \mathscr{X}, \breve{y} \in \breve{\mathscr{Y}}).$$

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Now we give an example in which the conditions of our second main Theorem simultaneously occur.

Example

Let $\mathscr{B}$ be an abelian von Neumann algebra of operators acting on a Hilbert space $\mathscr{H}$. Let $\mathscr{A} = \mathscr{B} \oplus \mathscr{B}$ be the von Neumann algebra of all $u \in \mathbb{B}(\mathscr{H})$ having the representation

$$u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \qquad (u_1, u_2 \in \mathscr{B}). \tag{3}$$

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Let $\mathscr{X} := \mathscr{A}$ and $\mathscr{Y} := \mathscr{B} \oplus 0$. Define $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ by

$$B(u, v) = \begin{bmatrix} u_1^* v_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where u and v have the representations as presented in (3).

Example (continued from the previous slide)

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Then B is an $\mathscr{A}$-sesquilinear form on $\mathscr{X}$ and positive on $\mathscr{Y}$. Clearly, $\undersim{\mathscr{Y}} = \{0\}$. In addition, any pure state of $\mathscr{A}$ is multiplicative on $\mathscr{A}$.
Hence,

$$|fB(u,v)|^2 = \left| f\left(\begin{bmatrix} u_1^* v_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right|^2 = f(|u|^2) f(|v|^2)$$

for all $u, v \in \mathscr{Y}$. In addition, $\mathscr{X}$ is self-dual. Hence, our second main Theorem ensures that B-spline interpolation has a solution for $\mathscr{Y}$.

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Let $B : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ be an $\mathscr{A}$-inner product on $\mathscr{X}$. Assume there exist $c > 0$ and $k > 0$ such that for every $f \in \mathcal{PS}(\mathscr{A})$ and for every $x \in \mathscr{Y}$ there exists a unit vector $y \in \mathscr{Y}$ such that $f(|y|^2) \geqslant k$ and

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Then $\mathscr{Y}'$ is an orthogonally complemented submodule of $\mathscr{X}'$ with respect to the inner product $\widetilde{B}$.

As a consequence, we show when an orthogonally complemented submodule of a self-dual Hilbert W^*-module $\mathscr{X}$ is orthogonally complemented with respect to another C^*-inner product on $\mathscr{X}$.

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Then $\mathscr{Y}$ is an orthogonally complemented submodule of $\mathscr{X}$ with respect to the inner product B.

Thank you very much for your attention.

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