Representations of $*$-Algebras on Hilbert C^*-Modules

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Contents

Throughout this talk:

- A is a complex unital $*$-algebra with involution $a \mapsto a^+$,
- $\mathfrak{A}$ denotes a C^*-algebra,
- $\mathcal{X}$ is a Hilbert $\mathfrak{A}$-module.

The talk is based on Section 14.6 and Chapter 12 of the recent book

K. Schmüdgen, *An Invitation to Representations of $*$-Algebras on Hilbert Space*, Graduate Texts in Mathematics **285**, Springer-Verlag, 2020.

Other references are

K. Schmüdgen, On well-behaved unbounded representations of $*$-algebras, J. Operator Theory **48**(2002), 487–502.

Y. Savchuk and K. Schmüdgen, Unbounded induced representations of $*$-algebras, Algebras Represent. Theory **16**(2013), 309–376.

R. Meyer, Representations by unbounded operators, Doc. Math. **22**(2017), 1375–1466.

$*$-Representations on Hilbert space

Suppose $(\mathcal{D}, \langle\cdot,\cdot\rangle)$ is a complex inner product space.

Definition 1:

A $*$-**representation** of A on $\mathcal{D}$ is an algebra homomorphism π of A into $L(\mathcal{D})$ such that $\langle\pi(a)\varphi, \psi\rangle = \langle\varphi, \pi(a^+)\psi\rangle$ for $a \in$ A, $\varphi, \psi \in \mathcal{D}$.

Another Definition:

$\mathcal{L}^+(\mathcal{D}) := \{a \in L(\mathcal{D}) : \exists\ b \in L(\mathcal{D})$ s.t. $\langle a\varphi, \psi\rangle = \langle\varphi, b\psi\rangle,\ \varphi, \psi \in \mathcal{D}\}$.

b is uniquely determined by a, denoted a^+, and $\mathcal{L}^+(\mathcal{D})$ is a $*$-algebra with involution $a \mapsto a^+$.

Then a $*$-**representation** is a $*$-homomorphism of A into $\mathcal{L}^+(\mathcal{D})$.

$\mathcal{L}^+(\mathcal{D})$ is the counter-part of $\mathbf{B}(\mathcal{H})$.

An Example: Schrödinger representation of the Weyl algebra

Let A be the **Weyl algebra**, i.e. A is the unital $*$-algebra with generators p, q s. t. $pq - qp = -i$. There is a $*$-representation π of A on $\mathcal{D} := \mathcal{S}(\mathbb{R})$ such that $\pi(q) = t$ and $\pi(p) = -i\frac{d}{dt}$.

Unbounded $*$-representations may have pathological properties!

Example: $\mathsf{A} = \mathbb{C}[x, y]$

Let π be a $*$-representation of A. It may happen that:
- $\overline{\pi(x)}$ and $\overline{\pi(y)}$ are self-adjoint, but their spectral projections do **not** commute.
- $\pi(p)$ is **not** positive even if the polynomial p is positive on $\mathbb{R}^2$.

To avoid such pathologies one has to select classes of "well-behaved" representations. For the Weyl algebra, there is only one irreducible well-behaved representation: the Schrödinger representation.

Fundamental problem in unbounded representation theory:
Describe and characterize "well-behaved" representations of A.

One possible approach:

One takes an appropriate $*$-representation of A on a Hilbert $\mathfrak{A}$-module. Any representation of the C^*-algebra $\mathfrak{A}$ induces a $*$-representation of A. **These induced representations are considered as "well-behaved" representations of A.**

We will discuss this briefly in Section 4.

$\mathcal{B}$-Operators

Suppose $\mathfrak{B}$ is a $*$-subalgebra of $\mathfrak{A}$.

Definition 2: $\mathfrak{B}$-Operators

A $\mathfrak{B}$-**operator** on $\mathcal{X}$ is a $\mathbb{C}$-linear and $\mathfrak{B}$-linear map t of a $\mathfrak{B}$-submodule $\mathcal{D}(t)$ of $\mathcal{X}$, called the *domain* of t, into $\mathcal{X}$, that is,

$$t(\lambda x) = \lambda t(x) \quad \text{and} \quad t(x \cdot b) = t(x) \cdot b \quad \text{for} \quad x \in \mathcal{D}(t),\, \lambda \in \mathbb{C},\, b \in \mathfrak{B}.$$

Example:

Consider the Hilbert $\mathfrak{A}$-module $\mathcal{X} := \mathfrak{A} = C([0,1])$ and $\mathfrak{B} = \mathbb{C}[x]$. The multiplication operator t by the variable x with domain $\mathcal{D}(t) = \mathbb{C}[x]$ is a $\mathfrak{B}$-operator. $\mathcal{D}(t)$ is not an $\mathfrak{A}$-submodule and t is not an $\mathfrak{A}$-operator. The closure of the operator t is an $\mathfrak{A}$-operator.

Adjoint Operator

Definition 3: Adjoint Operator

Let t be a $\mathfrak{B}$-operator s.t. $\mathcal{D}(t)$ is **essential**, i.e. $\mathcal{D}(t)^\perp = \{0\}$. Define

$$\mathcal{D}(t^*) = \big\{ y \in \mathcal{X} : \exists\, z \in \mathcal{X} \text{ such that } \langle tx, y\rangle_{\mathcal{X}} = \langle x, z\rangle_{\mathcal{X}},\ x \in \mathcal{D}(t) \big\}.$$

Since $\mathcal{D}(t)$ is essential, z is uniquely determined by y. Hence $t^*y := z$ gives a well-defined map t^* of $\mathcal{D}(t^*)$ into $\mathcal{X}$. It can be shown that t^* is a $\mathfrak{B}$-operator, called the *adjoint operator* of t. By definition,

$$\langle tx, y\rangle_{\mathcal{X}} = \langle x, t^*y\rangle_{\mathcal{X}} \quad \text{for} \quad x \in \mathcal{D}(t),\ y \in \mathcal{D}(t^*).$$

Note that the adjoint operator t^* is already well defined if the domain $\mathcal{D}(t)$ is essential; it is not needed that $\mathcal{D}(t)$ is dense.

Suppose that $\mathcal{D}$ is a $\mathfrak{B}$-**submodule** of $\mathcal{X}$. The counter-part of the $*$-algebra $\mathcal{L}^+(\mathcal{D})$ is the following.

Definition 4:

Let $\mathcal{L}^+_{\mathfrak{B}}(\mathcal{D})$ denote the set of maps $t : \mathcal{D} \mapsto \mathcal{D}$ for which there exists a map $s : \mathcal{D} \mapsto \mathcal{D}$ such that

$$\langle tx, y\rangle_{\mathcal{X}} = \langle x, sy\rangle_{\mathcal{X}} \quad \text{for} \quad x, y \in \mathcal{D}. \tag{1}$$

If t and s are as in (1), then t and s are $\mathfrak{B}$-operators with domain $\mathcal{D}$. Further, s is uniquely determined by t and will be denoted by t^+.

Lemma 1:

$\mathcal{L}^+_{\mathfrak{B}}(\mathcal{D})$ is a unital complex $*$-algebra with operator multiplication and involution $t \mapsto t^+$.

If $\mathcal{D}$ is essential, then t^+ is the restriction to $\mathcal{D}$ of the adjoint operator t^*.

Definition 5:

A $*$-**representation** of A on a $\mathfrak{B}$-submodule $\mathcal{D}$ of $\mathcal{X}$ is a $*$-homomorphism π of A in the $*$-algebra $\mathcal{L}^+_{\mathfrak{B}}(\mathcal{D})$. We write $\mathcal{D}(\pi) := \mathcal{D}$.

Many facts for Hilbert space representations carry to $*$-representations on $\mathfrak{B}$-submodules.

Definition 6: The graph topology

Let π be a $*$-representation of A on a $\mathfrak{B}$-submodule $\mathcal{D}$. The **graph topology** of π is the l. c. topology on $\mathcal{D}$ defined by the seminorms

$$\|x\|_a := \|\pi(a)x\|_{\mathcal{X}}, \quad a \in \mathsf{A},\ x \in \mathcal{D}.$$

Theorem 1:

Suppose $\mathfrak{B}$ is dense in $\mathfrak{A}$. Let π be a $*$-representation of A on a $\mathfrak{B}$-**submodule** $\mathcal{D}$. Each $\mathfrak{B}$-operator $\pi(a)$ on $\mathcal{X}$ is closable. The completion $\hat{\mathcal{D}}$ of $\mathcal{D}$ in the graph topology is a $\mathfrak{A}$-**submodule** s.t.

$$\hat{\mathcal{D}} = \bigcap_{a \in \mathsf{A}} \mathcal{D}\big(\overline{\pi(a)}\big).$$

Many examples can be derived from the following simple lemma.

Lemma 2:

Let $\mathcal{X}$ be the Hilbert $\mathfrak{A}$-module $\mathfrak{A}$ with $\mathfrak{A}$-valued inner product $\langle x, y\rangle_{\mathcal{X}} = x^+ y$ and let $\mathfrak{B}$ be a $*$-subalgebra of $\mathfrak{A}$.
Suppose $\mathfrak{B}$ is also a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ and there is a $*$-representation ρ of A on the inner product space $\mathcal{D}$ s.t.

$$\rho(a) \cdot b \in \mathfrak{B} \quad \text{for} \quad a \in \mathrm{A}, b \in \mathfrak{B}. \tag{2}$$

Then there is a $*$-representation π of A on the $\mathfrak{B}$-submodule $\mathfrak{B}$ of $\mathcal{X}$ defined by $\pi(a)b := \rho(a) \cdot b$ for $a \in \mathrm{A}, b \in \mathfrak{B}$.

The dot "$\cdot$" in (2) refers to the product of $\rho(a)$ and b in $\mathcal{L}^+(\mathcal{D})$, while $\pi(a)b$ means the action of $a \in \mathrm{A}$ on $b \in \mathfrak{B}$ by the representation π.

Crucial step of proof: Let $a \in \mathrm{A}$, $b, c \in \mathfrak{B}$. By $*$-algebra properties,

$$\langle \pi(a)b, c\rangle_{\mathcal{X}} = (\rho(a) \cdot b)^+ \cdot c = (b^+ \cdot \rho(a)^+) \cdot c = b^+ \cdot (\rho(a^+) \cdot c) = \langle b, \pi(a^+)c\rangle_{\mathcal{X}}.$$

“Ordinary” Hilbert space representation are related a $*$-representation on the Hilbert C^*-module of compacts.

Example 1: Hilbert space representations and C^*-algebras of compacts

Let ρ be a $*$-representation of A on a Hilbert space $\mathcal{H}(\rho)$. Let $\mathfrak{A}$ be the C^*-algebra of compact operators on $\mathcal{H}(\rho)$ and $\mathcal{X}$ the C^*-module $\mathfrak{A}$. Then the $*$-subalgebra $\mathcal{F}$ of all finite rank operators of $\mathcal{L}^+(\mathcal{D}(\rho))$ is also a $*$-subalgebra of $\mathfrak{A}$. Clearly, $\rho(a) \cdot x \in \mathcal{F}$ for $a \in \mathsf{A}, x \in \mathcal{F}$. By Lemma 2:

There is a $*$-representation π of A on the $\mathcal{F}$-submodule $\mathcal{F}$ of $\mathcal{X}$ defined by $\pi(a)x = \rho(a) \cdot x$ for $a \in \mathsf{A}$, $x \in \mathcal{F}$.

Each $*$-representation of A on a Hilbert space arises in this manner.

Example 2: Representations on $\mathfrak{A} = C_0(\mathbb{R}^d)$

Suppose $\mathfrak{A}$ is the C^*-algebra $C_0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ vanishing at infinity and $\mathcal{X} = \mathfrak{A}$.

There is a $*$-representation π of $\mathsf{A} := \mathbb{C}[x_1, \dots, x_d]$ on the $\mathfrak{A}$-submodule $\mathcal{D}(\pi)$ of $\mathcal{X}$ given by

$$\pi(p)f := p \cdot f, \quad \mathcal{D}(\pi) := \left\{ f \in C_0(\mathbb{R}^d) : p \cdot f \in C_0(\mathbb{R}^d),\ p \in \mathsf{A} \right\}. \quad (3)$$

Since the domain $\mathcal{D}(\pi)$ contains $C_c(\mathbb{R}^d)$, $\mathcal{D}(\pi)$ is dense in $\mathcal{X}$.

Now suppose $d = 1$ and K is a nonempty nowhere dense subset of $\mathbb{R}$. Let A be the $*$-algebra of rational functions with poles in K.

Then (3) defines also a $*$-representation of A on $\mathcal{X}$.

Since the functions of $\mathcal{D}(\pi)$ vanish at K, $\mathcal{D}(\pi)$ is not dense in $\mathcal{X}$. But $\mathcal{D}(\pi)$ is essential in $\mathcal{X}$, because K is nowhere dense.

Example 3: Representations of enveloping algebras on group C^*-algebras

Let G be a Lie group and $\mathsf{A} := \mathcal{E}(\mathfrak{g})$ the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Then A is a $*$-algebra with involution determined by $x^+ = -x$ for $x \in \mathfrak{g}$.

$\mathfrak{B} := C_0^\infty(G)$ is a $*$-algebra with multiplication and involution:

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g)\, d\mu_l(h),$$
$$f^+(g) := \Delta_G(g)^{-1}\, \overline{f(g^{-1})}, \quad g \in G.$$

Here μ_l is a left Haar measure and Δ_G is the modular function of G.

$\mathfrak{B}$ is a dense $*$-subalgebra of the **group C^*-algebra** $\mathfrak{A} := C^*(G)$.

Each $a \in \mathsf{A} = \mathcal{E}(\mathfrak{g})$ acts as a **right-invariant differential operator** $\tilde{a}$ on G. For $x \in \mathfrak{g}$ this action is given by

$$(\tilde{x}f)(g) = \frac{d}{dt} f(\exp(-tx)g) \,|_{t=0}, \quad f \in C_0^\infty(G),$$

where $x \mapsto \exp x$ is the exponential map of $\mathfrak{g}$ into G.

There is a $*$-representation ρ of A on the domain $\mathcal{D} := C_0^\infty(G)$ of $L^2(G;\mu_l)$ given by $\rho(a)\varphi := \tilde{a}\varphi$, $\varphi \in C_0^\infty(G)$.

It can be shown that $\mathfrak{B} = C_0^\infty(G)$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ and

$$\rho(a) \cdot f = \tilde{a}f \in \mathfrak{B} \quad \textit{for} \;\; a \in \mathsf{A}, f \in \mathfrak{B}. \tag{4}$$

Hence Lemma 2 applies, so:

There is a $*$-representation π of $\mathsf{A} = \mathcal{E}(\mathfrak{g})$ on the $\mathfrak{B}$-submodule $\mathfrak{B} = C_0^\infty(G)$ of $\mathcal{X} = C^*(G)$ defined by $\pi(a)f = \rho(a) \cdot f$, $a \in \mathsf{A}, f \in \mathfrak{B}$.

Induced Representations

Suppose τ is a fixed $*$-**representation of** A **on a Hilbert** $\mathfrak{A}$**-module** $\mathcal{X}$.

Let π be a $*$-representation of the C^*-algebra $\mathfrak{A}$ on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_0)$.
Let $\mathcal{X} \otimes_{\mathfrak{A}} \mathcal{H}$ is the quotient of the complex tensor product $\mathcal{X} \otimes \mathcal{H}$ by

$$\mathcal{N} := \mathrm{Lin}\{x \cdot b \otimes \varphi - x \otimes \pi(b)\varphi : x \in \mathcal{X},\ \varphi \in \mathcal{H},\ b \in \mathfrak{A}\}.$$

There is an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{X} \otimes_{\mathfrak{A}} \mathcal{H}$ defined by

$$\langle x \otimes \varphi, y \otimes \psi\rangle := \langle\pi([y,x]_{\mathcal{X}})\varphi, \psi\rangle_0, \quad x, y \in \mathcal{X},\ \varphi, \psi \in \mathcal{H}. \tag{5}$$

For $a \in$ A, there is a linear mapping $\rho(a)$ on $\mathcal{D} := \mathcal{D}(\tau) \otimes_{\mathfrak{A}} \mathcal{H}$ s.t.

$$\rho(a)(x \otimes \varphi) = \tau(a)x \otimes \varphi, \quad a \in \mathrm{A},\ \varphi \in \mathcal{H}. \tag{6}$$

Then $a \mapsto \rho(a)$ is a $*$-representation of A on the inner product space $\mathcal{D}$.

The closure of this $*$-representation is called the **induced representation** of π and denoted by $\mathrm{Ind}_{\mathcal{X}}\,\pi$.

Examples of Induced Representations

Example 1 revised: $\mathsf{A} = \mathbb{C}[x_1, \ldots, x_d]$, $\mathfrak{A} = C_0(\mathbb{R}^d)$

Recall that A acts on $\mathcal{X} = \mathfrak{A}$ by $\tau(a)f := a \cdot f$, $a \in \mathsf{A}$, $f \in \mathcal{D}(\tau)$.

If π is a $*$-representation of the C^*-algebra $\mathfrak{A}$ on $\mathcal{H}$, then $\mathrm{Ind}_{\mathcal{X}}\pi$ acts on $\pi(\mathfrak{A})\mathcal{H} := \mathrm{Lin}\,\{\pi(f)\varphi : f \in \mathfrak{A}, \varphi \in \mathcal{H}\}$ by

$$\mathrm{Ind}_{\mathcal{X}}\pi\,(a)(\pi(f)\varphi) = \pi(a \cdot f)\varphi, \quad a \in \mathsf{A}, f \in \mathfrak{A}, \varphi \in \mathcal{H}.$$

These representations $\mathrm{Ind}_{\mathcal{X}}\pi$ are precisely the well-behaved representations of the commutative $*$-algebra $\mathsf{A} = \mathbb{C}[x_1, \ldots, x_d]$.

Example 3 revised: $\mathsf{A} = \mathcal{E}(\mathfrak{g})$, $\mathfrak{A} = C^*(G)$

There is a $*$-representation $\tau(a)f := \tilde{a}f$ of A on $\mathfrak{B} = C_0^\infty(G)$.

Let U be a strongly continuous unitary representation of G. Then there is a nondegenerate $*$-representation $f \mapsto U_f$ of $\mathfrak{B} = C_0^\infty(G)$, where

$$U_f := \int U(g) f(g)\, d\mu_l(g).$$

Then $\mathrm{Ind}_{\mathcal{X}} U$ of A acts on $U(\mathfrak{B})\mathcal{H} := \mathrm{Lin}\,\{U_f\varphi : f \in \mathfrak{B}, \varphi \in \mathcal{H}\}$ by

$$\mathrm{Ind}_{\mathcal{X}} U\,(a)(U_f\varphi) = U_{\tilde{a}f}\varphi, \quad a \in \mathsf{A}, f \in \mathfrak{B}, \varphi \in \mathcal{H}.$$

Hence $U(\mathfrak{B})\mathcal{H}$ is the **Garding domain** and $\mathrm{Ind}_{\mathcal{X}}\, U$ is the **infinitesimal representation** dU of $\mathcal{E}(\mathfrak{g})$.

Again, the representations $\mathrm{Ind}_{\mathcal{X}} U = dU$ are precisely the well-behaved representations of the enveloping algebra $\mathsf{A} = \mathcal{E}(\mathfrak{g})$.

There are many $*$-algebras (for instance, coordinate algebras of non-compact quantum spaces and groups) for which the method of induced representations via Hilbert C^*-modules works.