Bounded and unbounded Fredholm operators on Hilbert modules

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Overview

1. Bounded and unbounded Fredholm operators on Hilbert C*-modules
   - Bounded and unbounded Fredholm operators
   - Noncommutative versions of Atiyah, Jänich, and Singer theorems
2. Representable K-theory and Atiyah–Jänich Theorem
   - Milnor exact sequence
   - Atiyah–Jänich Theorem for σ-C*-algebras

Construction of $K_0(A)$ and $K_1(A)$

Let A be any (unital) C*-algebra, and let $M_\infty(A) := \cup_n M_n(A)$

- $p \in M_\infty(A)$ is a projection if $p = p^* = p^2$.
- $p \sim q \;\; :\Leftrightarrow \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) = v^*v,\ vv^* = \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for a $v \in M_n(A)$;
  $V(A) = \mathcal{P}(M_\infty(A))/\sim,\ [p]+[q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$ semigroup;
  $\mathrm{K}_0(A) := Grot(V(A)) = \{[p]-[q] :\ [p],[q] \in V(A)\}$ group.
- $\mathrm{K}_1(A) := \mathrm{K}_0(SA)$, where $SA := C_0((0,1),A)$ is again a (non-unital) C*-algebra.

Hilbert modules, Fredholm operators, essentially unitary operators

Hilbert modules are essentially objects like Hilbert spaces by allowing the inner product to take values in a C*-algebra A rather than the field of complex numbers. A Hilbert module over a C*-algebra A gets its complete topology from the norm $\| \cdot \| = \|\langle \cdot, \cdot \rangle\|^{1/2}$. Suppose $H = l^2(A)$ is the standard Hilbert A-module:
Hilbert A-module:

- bounded adjointable operators on H : $\mathcal{L}(H)$,
- compact operators on H : $\mathcal{K}(H)$ and $\mathcal{L}(H) \cong M(\mathcal{K}(H))$,
- Calkin algebra : $\mathcal{Q}(H) = \mathcal{L}(H)/\mathcal{K}(H)$,
- Fredholm operators:
  $\mathcal{F}(H) := \{T \in \mathcal{L}(H); \rho(T) \text{ is invertible in } \mathcal{Q}(H)\}$,
- essentially unitary operators:
  $\mathcal{KC}(H) := \{T \in \mathcal{L}(H); \rho(T) \text{ is a unitary in } \mathcal{Q}(H)\}$,

Here $\rho : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{K}(H)$ is the quotient map.

C*-Fredholm operators

Fredholm operators on Hilbert modules were first introduced in the work of Mishchenko and Fomenko [Izv. Akad. Nauk SSSR Ser. Mat. (1979)] which developed the index theory for elliptic operators over C*-algebras. Mingo modified and reformulated the theory and gave a C*-version of the Atkinson Theorem [Trans. Amer. Math. Soc. (1987)].

Generalized Atkinson Theorem

If A is a unital C*-algebra, then $\mathcal{F}(H)$ and the following set coincide:

$$\mathcal{F}'(H) = \{\, F \in \mathcal{L}(H);\ \text{there is } K \in \mathcal{K}(H) \text{ such that } Ker(F+K) \text{ and } Ker(F+K)^* \text{ are finitely generated and } Ran(F+K) \text{ is closed.}\}$$

The Mishchenko index for $F \in \mathcal{F}(H)$ is defined by the formula

$$\mathrm{index}(F) := [Ker\ \ G] - [Ker\ \ G^*] \in \mathrm{K}_0(A),$$

where G is some compact perturbation of F with closed range and finitely generated kernel and cokernel.

Unbounded operators on Hilbert C*-modules

Unbounded adjointable operators or what are now know as regular operators were first introduced by Baaj and Julg [C. R. Acad. Sc. Paris Sér. I Math. (1983)] where they gave a nice construction of Kasparov bimodules in KK-theory using regular operators. An operator $T : Dom(T) \subseteq H \to H$ is said to be regular if

- T is closed and densely defined,
- its adjoint T^* is also densely defined, and
- range of $1 + T^*T$ is dense in H.

The theory of essentially defined operators and useful examples can be found in the paper of Gebhardt and Schmüdgen [Internat. J. Math. (2015)]

Frank and S. (2010s)

Let A be a C*-algebra. The following conditions are equivalent:

- A is a C*-algebras of compact operators (i.e. a C*-algebra that admits a faithful $*$-representation in the set of all compact operators on a certain Hilbert space).
- For every pair of Hilbert A-modules H, K, every densely defined closed operator $T : Dom(T) \subseteq H \to K$ is regular.
- For every pair of Hilbert A-modules H, K, every densely defined closed operator $T : Dom(T) \subseteq H \to K$ has polar decomposition, i.e. there exists a unique partial isometry V with initial set $\overline{Ran(|T|)}$ and the final set $\overline{Ran(T)}$ such that $T = V|T|$.

Unbounded Fredholm operators

Consider a regular operator $T : Dom(T) \subseteq H \to H$. An adjointable bounded operator $G \in \mathcal{L}(H)$ is called a pseudo left inverse of T if GT is closable and its closure $\overline{GT}$ satisfies $\overline{GT} \in \mathcal{L}(H)$ and $\overline{GT} = 1 \mod \mathcal{K}(H)$. Analogously G is called a pseudo right inverse if TG is closable and its closure $\overline{TG}$ satisfies $\overline{TG} \in \mathcal{L}(H)$ and $\overline{TG} = 1 \mod \mathcal{K}(H)$. The regular operator T is called Fredholm, if it has a pseudo left as well as a pseudo right inverse.

Atiyah, Jänich, and Singer (1960s)

- The space of Fredholm operators on an infinite dimensional complex Hilbert space equipped with the norm topology represents the functor $X \mapsto \mathrm{K}^0(X;\mathbb{C})$ from the category of compact Hausdorff spaces to the category of abelian groups. Indeed, Atiyah and Jänich presented different methods to show that

$$[X, \mathcal{F}] \to \mathrm{K}^0(X;\mathbb{C})$$

  is an isomorphism, where $[X, \mathcal{F}]$ denotes the set of homotopy classes of continuous maps from the compact Hausdorff space X into the space of Fredholm operators $\mathcal{F}$.
- In 1969 Atiyah and Singer proved that the functor $X \mapsto \mathrm{K}^1(X;\mathbb{C})$ can be represented by a specific path component of the space of selfadjoint Fredholm operators.

Unbounded selfadjoint Fredholm operators

Boss-Bovenbek, Lesch and Phillips (2005) showed that the space of unbounded selfadjoint Fredholm operators on an infinite dimensional complex Hilbert space equipped with the gap topology has just one path component. They also conjectured that the space of unbounded selfadjoint Fredholm operators would be classifying space for the functor $X \mapsto \mathrm{K}^1(X;\mathbb{C})$. Their conjecture was answered in the affirmative by Joachim with using C*-Fredholm operators on Hilbert modules.

Noncommutative Atiyah–Jänich Theorem

- Troitsky (1985); the abelian group $\mathrm{K}^0(X; A)$ can be realized as the group $[X, \mathcal{F}(H)]$ of homotopy classes of continuous maps from the compact Hausdorff space X to the space of Fredholm operators on the standard Hilbert A-module $H = l^2(A)$.
- Mingo (1987); the abelian group $\mathrm{K}_0(C(X, A))$ can be realized as the group $[X, \mathcal{F}(H)]$ of homotopy classes of continuous maps from the compact Hausdorff space X to the space of Fredholm operators on the standard Hilbert A-module $H = l^2(A)$.

Joachim (2003)

Let A be a unital C*-algebra and let H be the standard Hilbert A-module $l^2(A)$. Let $\mathcal{FR}(H)$ denote the space of regular Fredholm operators on H, equipped with the gap topology, and let $\mathcal{FSR}(H)$ denote the subspace consisting of the set selfadjoint regular Fredholm operators.

- The space $\mathcal{FR}(H)$ represents the functor which associates to a compact space X the group $\mathrm{K}^0(X;A)$, i.e.

$$[X, \mathcal{FR}(H)] \cong \mathrm{K}^0(X;A).$$

- The space $\mathcal{FSR}(H)$ represents the functor which associates to a compact space X the group $\mathrm{K}^1(X;A)$, i.e.

$$[X, \mathcal{FSR}(H)] \cong \mathrm{K}^1(X;A).$$

Inverse systems

Let $(A_i)_{i\in I}$ be a family of groups and suppose we have a family of homomorphisms $f_{ij} : A_j \to A_i$ for all $i \leq j$, with the following properties:

- f_{ii} is the identity on A_i,
- $f_{ik} = f_{ij} \circ f_{jk}$, for all $i \leq j \leq k$.

Then the pair $\{(A_i)_{i\in I},\ (f_{ij})\ i \leq j \in I\}$ is called an inverse system of groups.

σ-C*-algebra (= Countable inverse limit of C*-algebras)

- A σ-C*-algebra is a complete Hausdorff complex topological $*$-algebra A, whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i \in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for every continuous C*-seminorm p on A. In this talk the topology of a σ-C*-algebra is determined by only countably many C*-seminorms.
- Given any σ-C*-algebra A one can find a confinal subset of $S(A)$ which corresponds to $\mathbb{N}$ and explicitly write A as a countable inverse limit of C*-algebras, i.e. $A \cong \varprojlim_n A_n$. Furthermore, the connecting $*$-homomorphisms of the inverse system $\{A_n;\ \pi_n\}_{n \in \mathbb{N}}$ can be arranged to be surjective without altering the inverse limit of C*-algebras.

Hilbert modules

A Hilbert module over a σ-C*-algebra $A = \varprojlim_n A_n$ gets its complete topology from the family of seminorms $\|\cdot\|_p = p(\langle\cdot,\cdot\rangle)^{1/2}$. Suppose H is Hilbert A-module:

- $H \cong \varprojlim_n H_n$,
- bounded adjointable operators on H; $\mathcal{L}(H) \cong \varprojlim_n \mathcal{L}(H_n)$,
- compact operators on H; $\mathcal{K}(H) \cong \varprojlim_n \mathcal{K}(H_n)$ and $\mathcal{L}(H) \cong M(\mathcal{K}(H))$,
- Calkin algebra: $\mathcal{Q}(H) := \mathcal{L}(H)/\mathcal{K}(H) \cong \varprojlim_n \mathcal{Q}(H_n)$.

A-valued continuous functions

If $A = \varprojlim_n A_n$ is a unital σ-C*-algebra and $X = \varinjlim_n X_n$ is a countably compactly generated space, then $C(X, A)$ is the algebra of all continuous functions from X to A is a σ-C*-algebra with the topology determined by the family of C*-seminorms $\|f\|_{X_n,\ \iota_n,\ p_m} = \sup_{x \in X_n} p_m(f \circ \iota_n(x))$ for canonical morphisms $\iota_n : X_n \to X$ and $p_m \in S(A)$. This topology is equivalent with the topology of uniform convergence on each X_n in each continuous C*-seminorm p_m. We also have:

- $C(X) \otimes A \cong C(X, A)$,
- $C(X, A) \cong \varprojlim_n C(X, A_n)$,
- $C(X, A) \cong \varprojlim_n C(X_n, A)$.

Representable K-Theory

Suppose A is a σ-C*-algebra:

- UA : the group of unitary elements of a unital σ-C*-algebra A,
- U_0A : the path component of the identity in UA.

N. C. Phillips (1990)

$\mathrm{RK}_0(A) = U\mathcal{Q}(H)/U_0\mathcal{Q}(H)$ and $\mathrm{RK}_i(A) = \mathrm{RK}_0(S^iA)$, where $SA = A \otimes C_0(\mathbb{R})$ is the suspension of A and $S^iA = A \otimes C_0(\mathbb{R}^i)$ is the i-th suspension of A.

The abelian group $\mathrm{RK}_0(A)$ is isomorphic with the set of homotopy classes in the set of projections p in the unitization $\mathcal{K}(H \oplus H)^+$ such that $p - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{K}(H \oplus H)$.

K-functors are continuous with respect to inductive limits

$$K_0(\varinjlim) \cong \varinjlim K_0(\cdot)$$

Inverse limits do not commute with K-functors

$$K_0(\varprojlim) \ncong \varprojlim K_0(\cdot)$$

Example

Kawamura (2004)

Kawamura constructed an inverse system of Cuntz algebras $\{\mathcal{O}_n : 2 \leq n < \infty\}$ with non-surjective connecting maps $\mathcal{O}_{n+1} \to \mathcal{O}_n$ whose inverse limit is $*$-isomorphic onto $\mathcal{O}_\infty$. The inverse system obtains the following fact:

$$\mathrm{K}_0(\varprojlim_n \mathcal{O}_n) \cong \mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z} \ncong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \varprojlim_n \mathrm{K}_0(\mathcal{O}_n).$$

Milnor $\varprojlim^1$ exact sequence

Milnor (1962)

Suppose $K_1 \subset K_2 \subset \ldots$ is a sequence of CW-complexes and H^* is an additive cohomology theory. Milnor first showed that

$$0 \to \varprojlim_n{}^1 \mathrm{H}^*(K_n) \to \mathrm{H}^*(\cup K_n) \to \varprojlim_n \mathrm{H}^*(K_n) \to 0$$

is an exact sequence where $\varprojlim_n^1$ is the derived functor of the inverse limit.

Milnor $\varprojlim^1$ sequence for representable K-theory

N. C. Phillips (1990)

Let $\{A_n\}_{n\in\mathbb{N}}$ be an inverse system of σ-C*-algebras with surjective maps $A_{n+1} \to A_n$, which can always be arranged, then we have the following Milnor $\varprojlim^1_n$ sequence

$$0 \to \varprojlim_n{}^1 \mathrm{RK}_{1-i}(A_n) \to \mathrm{RK}_i(\varprojlim_n A_n) \to \varprojlim_n \mathrm{RK}_i(A_n) \to 0.$$

The sequence

$$X_1 \xrightarrow{f_1} \cdots \; X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

where each X_n is homotopy equivalent to the sphere S^1 and each $f_n : X_n \to X_{n+1}$ is the standard map of degree p. Here p is a prime number and X_{n+1} arises from rotating the sphere S^1 around itself p^n times. Then $\varprojlim^1_n \mathrm{K}_0(C(X_n)) = \mathbb{Z}_p/\mathbb{Z}$ which is an uncountable group.

The semigroup of essentially unitary operators

If u is a unitary element in $\mathcal{Q}(H)$, we write $[u]$ for its class in $\mathrm{RK}_0(A)$. We equip the set $\mathcal{KC}(H)$ with the topology which is generated by the operator-seminorm $\|\cdot\|_n$. By $[\mathcal{KC}(H)]$ we refer to the set of path components in $\mathcal{KC}(H)$. The equivalence classes are also denoted by $[\cdot]$ and if $t, s \in \mathcal{KC}(H)$ we define the product $[st] := [s]\,[t]$. The multiplication is well defined and makes the set $[\mathcal{KC}(H)]$ into an abelian semigroup just as $\mathcal{KC}(H)$.

Lemmas

- Any unitary element in $\mathcal{Q}(H)$ with trivial class in $\mathrm{RK}_0(A)$ can be lifted to a unitary in $\mathcal{L}(H)$.
- The set of homotopy classes of $\mathcal{KC}(H)$ with the above operation is isomorphic to the abelian group $\mathrm{RK}_0(A)$.
- Suppose that A and B are unital σ-C*-algebras and $t \in \mathcal{KC}(H_{A\otimes B})$. Then there exists $z \in \mathcal{KC}(H_{A\otimes B}) \cap A \otimes \mathcal{L}(H_B)$ such that $[t] = [z]$.
- Suppose that A and B are unital σ-C*-algebras then

$$\mathrm{RK}_0(M(A \otimes \mathcal{K}) \otimes B) = \mathrm{RK}_1(M(A \otimes \mathcal{K}) \otimes B) = \{0\}.$$

In particular, if X is a countably compactly generated space and $H = l^2(A)$, then the unitary group of $C(X, \mathcal{L}(H))$ is path connected.

Theorem

S.

Let X be a countably compactly generated space and let H be the standard Hilbert module over an arbitrary unital σ-C*-algebra A. Then abelian groups $[X, \mathcal{KC}(H)]$ and $\mathrm{RK}_0(XA)$ are isomorphic.

Theorem

S.

Let X be a countably compactly generated space and let H be the standard Hilbert module over an arbitrary unital σ-C*-algebra A. The inclusion $i : \mathcal{KC}(H) \hookrightarrow \mathcal{F}(H)$ induces an isomorphism $i_* : [X, \mathcal{KC}(H)] \to [X, \mathcal{F}(H)]$. In particular, $[X, \mathcal{F}(H)]$ is isomorphic to the abelian group $\mathrm{RK}_0(XA)$.

Remark

Let A be a unital C*-algebra and let X be a compact Hausdorff space. Then, by a result of Roseneberg, the Grothendieck group of A-vector bundles $\mathrm{K}^0(X;A)$ is naturally isomorphic to $\mathrm{K}_0(XA)$. This fact can also be deduced from the results of Troitsky and Mingo. However, the Grothendieck groups of A-vector bundles over X need not be isomorphic to $\mathrm{RK}_0(XA)$, when A is a σ-C*-algebra.

The proof of the main theorem:

We first prove the following assertions for a compact Hausdorff space X.

- (H_1) Any continuous map $f : X \to \mathcal{F}(H)$ is homotopic to a map whose image is contained in $\mathcal{KC}(H)$.
- (H_2) Any continuous map $h : [0,1] \times X \to \mathcal{F}(H)$ for which the images of h_0 and h_1 are contained in $\mathcal{KC}(H)$, is homotopic to a map whose image is contained in $\mathcal{KC}(H)$.

Proof

To prove the assertion (H_1) we suppose $f : X \to \mathcal{F}(H)$ is an arbitrary continues map. It determines the corresponding bounded adjointable operator $f \in \mathcal{L}(XH)$ by $(f(\psi))(x) = f(x)\psi(x)$. Since $f \in \mathcal{F}(XH)$, there exists a compact perturbation of $g \in \mathcal{L}(XH)$ of f which satisfies in the fixed conditions of the polar decomposition. Via the isomorphism $X\mathcal{K}(H) \cong \mathcal{K}(XH)$ the operator $g - f$ corresponds to a continuous map $k : X \to \mathcal{K}(H)$. Then for every $\theta \in [0, 1]$ we can define $f_\theta = f + \theta k : X \to \mathcal{F}(H)$, which is a homotopy between the maps f and g.

Proof

The compact projections $p, q \in \mathcal{K}(XH)$ onto the kernels of g and g^* stratify $qg = 0 = gp$ and $pg^* = 0 = g^*q$. The projections p and q via the isomorphism $X\mathcal{K}(H) \cong \mathcal{K}(XH)$ can be identified with continuous maps $p, q : X \to \mathcal{K}(H)$, which enable us to define $a, b : X \to \mathcal{F}(H)$ by

$$a := g^*g + p \quad \text{and} \quad b := gg^* + q.$$

$$w = b^{-1/2}g : X \to \mathcal{KC}(H).$$

Then for every $\theta \in [0, 1]$ we can define the continuous map

$$g_\theta = (\theta + (1 - \theta)b^{-1})^{1/2}g : X \to \mathcal{F}(H),$$

which is a path joining g and w.

Proof

The operators $g_\theta(x)$ are Fredholm. To see this suppose that $s_1(x)$ is a left generalized inverse for $g(x)$ then $s_1(x)(\theta + (1-\theta)b(x)^{-1})^{-1/2}$ is a left generalized inverse of $g_\theta(x)$. If $s_2(x)$ is a right generalized inverse of $g(x)$, then $s_2(x)(\theta + (1-\theta)b(x)^{-1})^{-1/2}$ is a right generalized inverse of $g_\theta(x)$. Then the continuous map $\mathfrak{H} : [0,1] \times X \to \mathcal{F}(H)$ given by

$$\mathfrak{H}(\theta, x) = \begin{cases} f_{2\theta}(x) & \text{if } 0 \leq \theta \leq 1/2 \\ g_{2-2\theta}(x) & \text{if } 1/2 \leq \theta \leq 1 \end{cases}$$

is a homotopy between the maps $f : X \to \mathcal{F}(H)$ and $\omega : X \to \mathcal{KC}(H)$, which completes the proof of (H_1).

Proof

Surjectivity and injectivity of the map $i_* : [X, \mathcal{KC}(H)] \to [X, \mathcal{F}(H)]$ immediately follow from the properties (H_1) and (H_2).
We now suppose $X = \varinjlim_n X_n$ when each X_n is a compact Hausdorff space. We obtain the following commutative diagram with exact rows

$$\begin{array}{ccccc} \varprojlim^1_n [SX_n, \mathcal{KC}(H)] & \rightarrowtail & [X, \mathcal{KC}(H)] & \twoheadrightarrow & \varprojlim_n [X_n, \mathcal{KC}(H)] \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \varprojlim^1_n [SX_n, \mathcal{F}(H)] & \rightarrowtail & [X, \mathcal{F}(H)] & \twoheadrightarrow & \varprojlim_n [X_n, \mathcal{F}(H)] \end{array} .$$

In particular, the abelian groups $[X, \mathcal{F}(H)]$ and $\mathrm{RK}_0(XA)$ are isomorphic.

Milnor $\varprojlim^1$ sequence for homotopy groups

Let H be the standard Hilbert module over an arbitrary unital σ-C*-algebra A. Then $\pi_i(\mathcal{F}(H))$ is isomorphic to $\mathrm{RK}_i(A)$ and satisfies in the following short exact sequence of abelian groups

$$0 \to \varprojlim_n{}^1 \pi_{1-i}(\mathcal{F}(H_n)) \to \pi_i(\mathcal{F}(H)) \to \varprojlim_n \pi_i(\mathcal{F}(H_n)) \to 0.$$

In particular, we have $\pi_i(\mathcal{F}(H)) \cong \varprojlim_n \pi_i(\mathcal{F}(H_n))$ when the morphisms $\mathrm{K}_i(A_{n+1}) \to \mathrm{K}_i(A_n)$ are surjective, or the abelian groups $\mathrm{K}_i(A_n)$ are finite.

Problem; lifting property

Characterize those σ-C*-algebras A for which

$$\cdots \to \mathcal{F}(H_{n+1}) \to \mathcal{F}(H_n) \cdots \cdot \to \mathcal{F}(H_1)$$

is a sequence of maps having homotopy lifting property.

Problem; deformation retract

Characterize those σ-C*-algebras A for which $\mathcal{KC}(H) \subseteq \mathcal{F}(H)$ is a deformation retract, that is, there is a map $r : \mathcal{F}(H) \to \mathcal{KC}(H)$ with $r\,i = 1_{\mathcal{KC}(H)}$ and the map $i\,r$ is homotopy equivalent to $1_{\mathcal{F}(H)}$.
If A is a unital C*-algebra and $H = l^2(A)$, one can apply functional calculus to show that $\mathcal{KC}(H) \subseteq \mathcal{F}(H)$ is a deformation retract.

Thank you