An Open Problem—or Two ...

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Hilbert C^*–Modules Online Weekend
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- Mention another suspect statement that, first, I thought was related, but that, now, I think is unrelated.

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Under the passage $E \rightsquigarrow \overline{E}$, S loses the property $S^\perp = \{0\}$.

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- (Guljas's talk: Essential ideal $F \subset E$. Then $\mathcal{B}^r(E) \subset \mathcal{B}^r(F)$.
  If $a \in \mathcal{B}^r(E)$ is in $\mathcal{B}^a(F)$, then $\ker a^{\perp\perp} = \ker a$.)

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- Suppose v is an isometry on E, so that $E_u := \bigcap_{n \in \mathbb{N}_0} v^n E$ is the unique maximal invariant submodule on which v restricts to a unitary.

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- Suppose v is an isometry on E, so that $E_u := \bigcap_{n \in \mathbb{N}_0} v^n E$ is the unique maximal invariant submodule on which v restricts to a unitary. If **the statement** is true, then also $E_u^{\perp\perp}$ is invariant for v.

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  (Note: By maximality of E_u, if $E_u^{\perp\perp} \neq E_u$, then the restriction of v to $E_u^{\perp\perp}$ cannot be a unitary.)

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Violating any of the four statements will, thus, disprove **the statement**.

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However, embedding E into a von Neumann (or W^*) module can be done better, making that proof-idea work at least for ideals.

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  $\mathcal{C} = \overline{\operatorname{span}}\, \mathcal{C}\mathscr{K}(E)\mathcal{C} = \mathscr{K}(F)$.
  Translate hypotheses on E and F into those of $\overline{\operatorname{span}}\, \mathscr{K}(E)\mathscr{K}(F) = \mathscr{K}(F, E)$ and $\mathscr{K}(F)$.]

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$$\mathcal{B} = (p\mathcal{A}''p) \cap \mathcal{A}$$
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If $\mathcal{A}''$ acts on H and $G = pH$, then $\Phi \in \mathcal{B}(H, G)$. I did not figure out yet, if this helps.

Another open problem. (Appendix C in Shalit-ms 2020.)
(Though, I did not really research for an existing answer.)

- Vector spaces: $V \supset V_i, W \supset W_i$

$$(V_1 \otimes W_1) \cap (V_2 \otimes W_2) \;=\; (V_1 \cap V_2) \otimes (W_1 \cap W_2).$$

  (Elementary linear algebra of tensor products.)

- Hilbert spaces: $G \supset G_i, H \supset H_i$

$$(G_1 \otimes H_1) \cap (G_2 \otimes H_2) \;=\; (G_1 \cap G_2) \otimes (H_1 \cap H_2).$$

  (Quite different proof. Generalizes to von Neumann modules.)

Question: Is it true for C^*–correspondences $E \supset E_i, F \supset F_i$ that

$$(E_1 \odot F_1) \cap (E_2 \odot F_2) \;=\; (E_1 \cap E_2) \odot (F_1 \cap F_2)?$$

Relevance: Is the intersection of two product subsystems a product subsystem?

The obvious inclusion of RHS in LHS $\rightsquigarrow$ intersection of superproduct subsystems is a superproduct subsystem.

For the intersection of subproduct subsystems we don't know.

Thank you!

Bibliography I

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