# On the orthogonal complementarity of closed submodules of Hilbert $C^*$-modules

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## The Douglas theorem

**Theorem (Range inclusion-Factorization-Majorization)**

*Let $H$ be a Hilbert space, and $A, B \in \mathbb{B}(H)$. Then the following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;

(ii) $A = BC$ *for some* $C \in \mathbb{B}(H)$;

(iii) $AA^* \leq k^2 BB^*$ *(or equivalently* $\|A^*\| \leq k\|B^*\|$*) for some* $k \geq 0$.

*Moreover, if* (i)–(iii) *are valid, then there exists a unique operator* $C \in \mathbb{B}(H)$ *(known as the reduced solution) so that*

(a) $\|C\|^2 = \inf \left\{\mu | AA^* \leq \mu BB^*\right\}$;

(b) $\mathcal{N}(A) = \mathcal{N}(C)$;

(c) $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$.

## The orthogonal complementarity

Let $\mathfrak{A}$ be a $C^*$-algebra, $H$ and $K$ be (right) Hilbert $\mathfrak{A}$-modules. The set of adjointable operators from $H$ into $K$ is denoted by $\mathcal{L}(H,K)$.

For every $T \in \mathcal{L}(H,K)$, let $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and the null space of $T$, respectively.

A closed submodule $M$ of $H$ is said to be orthogonally complemented in $H$ if $H = M \dotplus M^\perp$, where

$$M^\perp = \{x \in H : \langle x, y\rangle = 0, \forall\, y \in M\}.$$

In this case, the projection from $H$ onto $M$ is denoted by $P_M$.

## An observation

### Proposition

*Let $M$ and $N$ be closed submodules of $H$ such that $N \subseteq M$ and $M$ is orthogonally complemented in $H$. Then the following statements are equivalent:*

(i) *$N$ is orthogonally complemented in $H$;*

(ii) *$N$ is orthogonally complemented in $M$.*

### Proof.

(i)$\Longrightarrow$(ii): Let $P_N$ denote the projection from $H$ onto $N$. Since $N \subseteq M$, $P_M - P_N$ is a projection. Therefore, $M$ can be decomposed orthogonally as $M = N \dot{+} \mathcal{R}(P_M - P_N)$.

(ii)$\Longrightarrow$(i): Let $X$ denote the orthogonal part of $N$ in $M$. Then clearly, $H = N \dot{+} N^{\perp}$, where $N^{\perp} = X \dot{+} M^{\perp}$. □

## An asymmetry

Let $H$ and $K$ be Hilbert $\mathfrak{A}$-modules, and $T \in \mathcal{L}(H,K)$. It is known that $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed.

### Problem

*Suppose that $\overline{\mathcal{R}(T)}$ is orthogonally complemented. Whether $\overline{\mathcal{R}(T^*)}$ is also orthogonally complemented?*

The answer can be negative.

### Proposition (Xu-Fang, 2017)

*There exist $C^*$-algebra $\mathfrak{A}$, Hilbert $C^*$-modules $H$ and $K$ overt $\mathfrak{A}$, and an operator $T \in \mathcal{L}(H,K)$ such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented in $H$, whereas $\overline{\mathcal{R}(T)}$ fails to be orthogonally complemented in $K$.*

# A classical majorization result

A well-known result reads as follows.

**Proposition**

*Let $A$ and $B$ be bounded linear operators on a Hilbert space $H$. If $AA^* \leq BB^*$, then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

**Proof.**

For every $x \in H$, $\|A^*x\| \leq \|B^*x\|$. So there exists an operator $V : \overline{\mathcal{R}(B^*)} \to \overline{\mathcal{R}(A^*)}$ such that $VB^* = A^*$. An extension of $V$ can be given naturally as $U = VP_{\overline{\mathcal{R}(B^*)}}$. Then $U \in \mathbb{B}(H)$ satisfying $UB^* = A^*$, hence $A = BU^*$, which gives $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. □

Suppose that $H$ is a Hilbert space. Then

- Every closed subspace is orthogonally complemented in $H$;
- Every bounded linear operator on $H$ is adjopintable.

## Range inclusion $\Longrightarrow$ Factorization?

### Proposition (Fang-Moslehian-Xu, 2018)

*There exist $T \in \mathcal{L}(H,K)$ and $T' \in \mathcal{L}(K)$ such that $\mathcal{R}(T') \subsetneq \mathcal{R}(T)$, whereas $TX = T', X \in \mathcal{L}(K,H)$ has no solution.*

### Proof.

Let $E$ be any countably infinite-dimensional Hilbert space, $\mathbb{B}(E)$ (resp. $\mathbb{K}(E)$) be the set of all bounded (resp. compact) linear operators on $E$. Let $\mathfrak{A} = \mathbb{B}(E)$, $H = \mathbb{B}(E)$ and $K = \mathbb{K}(E)$. Then $H$ and $K$ are Hilbert $\mathfrak{A}$-modules in the standard way.

Choose any element $s$ in $K_+$ such that $\overline{s\,K} = K$. Let $T \in \mathcal{L}(H,K)$ be defined by

$$T(x) = sx, \quad \text{for every } x \in H.$$

Then such an operator $T$ will meet the demanding. □

## Majorization $\Longrightarrow$ Factorization?

Let $A$ and $B$ be positive. Then clearly,

$$A^{\frac{1}{2}}(A^{\frac{1}{2}})^* \leq (A+B)^{\frac{1}{2}}\left[(A+B)^{\frac{1}{2}}\right]^*.$$

**Proposition (Luo-Song-Xu, 2019)**

*There exist a Hilbert $C^*$-module $H$ and $A, B \in \mathcal{L}(H)_+$ such that the operator equation $A^{1/2} = (A+B)^{1/2}X$ with $X \in \mathcal{L}(H)$ has no solution.*

Note that neither $A$ nor $B$ constructed in [Luo-Song-Xu, 2019] is a projection, so it is interesting to solve the following problem:

**Problem**

*Find a Hilbert $C^*$-module $H$ and two projections $P, Q \in \mathcal{L}(H)$ such that the equation $P = (P+Q)^{1/2}X$ with $X \in \mathcal{L}(H)$ has no solution.*

## A solution

- V.M. Manuilov, M.S. Moslehian, Q. Xu.
  Proc. Amer. Math. Soc. 148 (2020), no. 3, 1139-1151.

The construction of $P$ and $Q$:

Let $M_2$ be the set of all $2 \times 2$ matrices. Set

$$\mathfrak{A} = \big\{ f \in C([0,1]; M_2) : f(0), f(1) \text{ are diagonal} \big\}, \tag{1.1}$$

and put $H = \mathfrak{A}$. Then $\mathcal{L}(H) = \mathfrak{A}$, since $\mathfrak{A}$ is unital.

Let $P, Q \in \mathcal{L}(H)$ be projections determined by

$$P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q(t) = \begin{pmatrix} c_t^2 & c_t s_t \\ c_t s_t & s_t^2 \end{pmatrix},$$

where $c_t = \cos \frac{\pi}{2} t$ and $s_t = \sin \frac{\pi}{2} t$ for $t \in [0,1]$.

## Extremely discomplementable projections

### Definition

Two projections $P$ and $Q$ are said to be extremely discomplementable if

$$\overline{\mathcal{R}(P+Q)},\ \overline{\mathcal{R}(P+I-Q)},\ \overline{\mathcal{R}(I-P+Q)},\ \overline{\mathcal{R}(I-P+I-Q)}$$

are all not orthogonally complemented in $H$.

Based on the modification of (1.1) by deleting the term "$f(0), f(1)$ are diagonal", two extremely discomplementable projections can then be constructed.

## The reduced solution

### Definition

*Let $E, H$ and $K$ be Hilbert $\mathfrak{A}$-modules, and let $A \in \mathcal{L}(H,K)$ and $C \in \mathcal{L}(E,K)$. An operator $D \in \mathcal{L}(E,H)$ is said to be the reduced solution to the system*

$$AX = C, \quad X \in \mathcal{L}(E,H), \tag{1.2}$$

*if $AD = C$ and $\mathcal{R}(D) \subseteq \overline{\mathcal{R}(A^*)}$.*

Note that $\overline{\mathcal{R}(A^*)} \subseteq \mathcal{N}(A)^\perp$, so the reduced solution (if it exists) is unique.

## The generalized Douglas theorem

**Theorem (Fang-Moslehian-Xu, 2018)**

*For every $A \in \mathcal{L}(H, K)$, the following statements are equivalent:*

(i) $\overline{\mathcal{R}(A^*)}$ *is orthogonally complemented in* $H$*;*

(ii) *Given every* $C \in \mathcal{L}(E, K)$ *with* $\mathcal{R}(C) \subseteq \mathcal{R}(A)$*, system* (1.2) *has the reduced solution.*

**Problem**

Whether 'the reduced solution" in item (ii) can be replaced with "a solution"?

Up to now, we have no answer.

Now we move to the second topic: the polar decomposition.

## Hilbert space case

Let $H, K$ be Hilbert spaces. An element $U \in \mathbb{B}(H,K)$ is called a partial isometry if $U^*U$ is a projection. Let $|T| = (T^*T)^{\frac{1}{2}}$ and $|T^*| = (TT^*)^{\frac{1}{2}}$ for every $T \in \mathbb{B}(H)$.

It is well-known that every $T \in \mathbb{B}(H,K)$ can be decomposed as

$$T = U|T| \quad \text{with} \quad \mathcal{N}(U) = \mathcal{N}(T), \tag{2.1}$$

where $U$ is a partial isometry. Such a decomposition is unique, and simultaneously $T^*$ can be decomposed as

$$T^* = U^*|T^*| \quad \text{with} \quad \mathcal{N}(U^*) = \mathcal{N}(|T^*|).$$


- P. R. Halmos, A Hilbert Space Problem Book (2nd Edn). New York: Springer-Verlag, 1982

## Hilbert $C^*$-module case

**Theorem (Wegge-Olsen, Thm 15.3.7)**

*The following statements are equivalent for every $T \in \mathcal{L}(H)$:*

(i) $T$ *has the polar decomposition* $T = U|T|$, *where* $U \in \mathcal{L}(H)$ *is a partial isometry for which* $\mathcal{N}(U) = \mathcal{N}(T)$ *and*

$$\mathcal{N}(U^*) = \mathcal{N}(T^*),\ \mathcal{R}(U) = \overline{\mathcal{R}(T)},\ \mathcal{R}(U^*) = \overline{\mathcal{R}(T^*)};$$

(ii) $H = \mathcal{N}(|T|) \dotplus \overline{\mathcal{R}(|T|)}$ *and* $H = \mathcal{N}(T^*) \dotplus \overline{\mathcal{R}(T)}$;

(iii) *Both* $\overline{\mathcal{R}(T)}$ *and* $\overline{\mathcal{R}(|T|)}$ *are orthogonally complemented in* $H$.

**Remark**

(1) For densely defined Hilbert $C^*$-module operators, see M. Frank and K. Sharifi, J. Operator Theory 64 (2) (2010), 377–386. (2) Conditions in item (i) can be simplified.

## A simplified version

**Theorem (Liu-Luo-Xu, 2018)**

*The following statements are equivalent for every* $T \in \mathcal{L}(H,K)$*:*

(i) $\overline{\mathcal{R}(T^*)}$ *and* $\overline{\mathcal{R}(T)}$ *are both orthogonally complemented;*

(ii) *There exists a (unique)* $U \in \mathcal{L}(H,K)$ *such that*

$$T = U|T| \quad \text{with} \quad U^*U = P_{\overline{\mathcal{R}(T^*)}}. \tag{2.2}$$

*In each case,* $T^* = U^*|T^*|$ *with* $UU^* = P_{\overline{\mathcal{R}(T)}}$.

**Remark**

In the Hilbert $C^*$-module case, condition (2.2) can be treated as the definition of the polar decomposition, which can not be simplified as (2.1).

## The orthogonal complemented case

### Proposition

*Let $T \in \mathcal{L}(H,K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented. Let $U \in \mathcal{L}(H,K)$ be a partial isometry satisfying*

$$T = U|T| \quad \text{with} \quad \mathcal{N}(T) \subseteq \mathcal{N}(U). \tag{2.3}$$

*Then $T = U|T|$ is the polar decomposition of $T$.*

### Remark

In the Hilbert space case, $\overline{\mathcal{R}(T^*)}$ is always orthogonally complemented, so if a partial isometry $U$ satisfies (2.3), then $U|T|$ is exactly the polar decomposition of $T$.

## An interesting example

### Proposition

*There exists a matrix $T \in M_3$ with the polar decomposition $T = U|T|$ such that for each $n \in \mathbb{N}$,*

(i) $T^{2n-1} = U^{2n-1}|T^{2n-1}|$ *is the polar decomposition of* $T^{2n-1}$;

(ii) $T^{2n} \neq U^{2n}|T^{2n}|$.

### Example

$$T = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

## $n$-centered operators

**Definition**

Suppose that $T \in \mathcal{L}(H)$ has the polar decomposition $T = U|T|$, and $n$ is a natural number. Then $T$ is said to be $n$-centered if for all $1 \leq i \leq n$,

$$T^k = U^k|T^k|$$

is the polar decomposition of $T^k$. If $T$ is $n$-centered for every $n \in \mathbb{N}$, then $T$ is said to be a centered operator.

**Proposition (Liu-Luo-Xu, 2019)**

*For every $n \geq 2$, there exist a Hilbert space $H$ and $T \in \mathcal{L}(H)$ such that $T$ is $n$-centered, whereas $T$ is not $(n+1)$-centered.*

Now we move to the third topic: the generalized parallel sum.

## The case of positive semi-definite matrices

Let $A, B \in \mathbb{C}^{n\times n}$ be positive semi-definite matrices. The parallel sum of $A$ and $B$ is defined by

$$A : B = A(A+B)^\dagger B,$$

where $(A+B)^\dagger$ is the Moore-Penrose inverse of $A+B$. It is so named because of its origin in and application to the electrical network theory that

$$\left(r_1^{-1}+r_2^{-1}\right)^{-1} = r_1(r_1+r_2)^{-1}r_2$$

is the resistance arising from resistors $r_1$ and $r_2$ in parallel.

- W. N. Anderson, Jr. and R. J. Duffin, J. Math. Anal. Appl. 26 (1969), 576–594.

## The parallel sum in other settings

- Nonsquare matrices under certain conditions of range inclusions [Mitra-Odell, 1986];
- Positive operators $A$ and $B$ on a Hilbert space such that the range of $A+B$ is closed [Anderson-Schreiber, 1972];
- General positive operators on a Hilbert space without the range restrictions [Fillmore-Williams, 1971; Morley, 1989];
- Nonnegative forms [Hassi-Sebestyén-Snoo, 2009];
- Unbounded operators [Kosaki, 2017];
- States of a $C^*$-algebra [Tarcsay, 2015];
- Linear relations [Arlinskiĭ, 2020];
- Adjointable operators $A$ and $B$ on a Hilbert $C^*$-module such that the range of $A+B$ is closed [Luo-Song-Xu, 2019].

## The background

Let $H$ and $K$ be Hilbert $C^*$-modules. Suppose that $T \in \mathcal{L}(H,K)$ is Moore-Penrose invertible, that is, $\mathcal{R}(T)$ is closed. Denote by

$$T\{1\} = \big\{X \in \mathcal{L}(K,H) : TXT = T\big\}.$$

Each element $T^-$ in $T\{1\}$ is called a $\{1\}$-inverse of $T$.

**Definition**

Let $A, B \in \mathcal{L}(H)$ be such that $A+B$ is Moore-Penrose invertible. Then $A$ and $B$ are said to be parallel summable if $A(A+B)^-B$ is invariant under any choice of $(A+B)^-$. In such case, the parallel sum of $A$ and $B$ is denoted by

$$A : B = A(A+B)^{\dagger}B.$$

## Parallel summable condition

### Proposition

*Let $A, B \in \mathcal{L}(H)$ be such that $A + B$ is Moore-Penrose invertible. Then $A$ and $B$ are parallel summable if and only if*

$$\mathcal{R}(A) \subseteq \mathcal{R}(A+B), \quad \mathcal{R}(A^*) \subseteq \mathcal{R}\big[(A+B)^*\big]. \tag{3.1}$$

### Remark

Suppose that $A, B \in \mathcal{L}(H)_+$ are such that $A + B$ is Moore-Penrose invertible. Then $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$, and thus (3.1) is satisfied.

### Proposition

*Let $A, B \in \mathcal{L}(H)$ be such that $|A+B| = |A| + |B|$ and $A + B$ is Moore-Penrose invertible. Then $A$ and $B$ are parallel summable.*

## Fillmore-Williams's approach

Let $H$ be a Hilbert space and $T \in \mathbb{B}(H)$. A direct application of the Douglas theorem (or the polar decomposition of $T$) gives

$$\mathcal{R}(T) = \mathcal{R}\big[(TT^*)^{\frac{1}{2}}\big]. \tag{3.2}$$

Now, given any $A, B \in \mathbb{B}(H)_+$, let

$$T = \begin{pmatrix} 0 & 0 \\ A^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix} \in \mathbb{B}(H \oplus H).$$

Utilizing (3.2) one can obtain the following proposition.

### Proposition

*Let $H$ be a Hilbert space and let $A, B \in \mathbb{B}(H)_+$. Then*

$$\mathcal{R}(A^{\frac{1}{2}}) + \mathcal{R}(B^{\frac{1}{2}}) = \mathcal{R}\big[(A+B)^{\frac{1}{2}}\big].$$

## New definition

Suppose that $H$ is a Hilbert space and $A, B \in \mathbb{B}_+(H)$. Let $C$ and $D$ be the reduced solutions of

$$A^{\frac{1}{2}} = (A+B)^{\frac{1}{2}} X \quad \text{and} \quad B^{\frac{1}{2}} = (A+B)^{\frac{1}{2}} Y.$$

Then the parallel sum of $A$ and $B$ is defined by

$$A : B = A^{\frac{1}{2}} C^* D B^{\frac{1}{2}}.$$

In the special case that $\mathcal{R}(A+B)$ is closed, this parallel sum is exactly equal to $A(A+B)^{\dagger} B$.

- P. A. Fillmore and J. P. Williams, Adv. Math. 7 (1971), 254–281.

## The idea

As is shown above, the main tool employed in [Fillmore-Williams] for the parallel sum is the Douglas theorem for bounded linear operators on Hilbert spaces. This tool is only conditionally applicable in the case of Hilbert $C^*$-module.

To deal with the parallel sum, some new phenomena may happen if

Hilbert spaces
Positive operators
Douglas theorem/the polar decomposition
$\downarrow$
Hilbert $C^*$-modules
Adjointable operators
The polar decomposition & the generalized Douglas theorem

## The tractable pair

### Definition

*Let $A \in \mathcal{L}(H,K)$ and let $B \in \mathcal{L}(K)$. Put*

$$T_{A,B} := \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix} \in \mathcal{L}(H \oplus K).$$

*The pair $(A,B)$ is said to be tractable if $T_{A,B}$ has the polar decomposition.*

### Theorem (Fu-Moslehian-Xu-Zamani, 2020)

*Let $S_{A,B} = (AA^* + BB^*)^{\frac{1}{2}}$ for $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}(K)$. Then the pair $(A,B)$ is tractable if and only if the following conditions are satisfied:*

(i) $\overline{\mathcal{R}(S_{A,B})}$ *is orthogonally complemented in $K$;*

(ii) $\mathcal{R}(A) \subseteq \mathcal{R}(S_{A,B})$ *and* $\mathcal{R}(B) \subseteq \mathcal{R}(S_{A,B})$.

## The generalized parallel sum

Suppose that items (i) and (ii) are satisfied. By the (conditional) Douglas theorem, both of the systems

$$S_{A,B}X = A \ \big(X \in \mathcal{L}(H,K)\big), \quad S_{A,B}Y = B \ \big(Y \in \mathcal{L}(K)\big) \quad (3.3)$$

have the reduced solutions, which are denoted by $C$ and $D$, respectively.

### Definition

*Let $A \in \mathcal{L}(H,K)$, $B \in \mathcal{L}(K)$, and suppose that the pair $(A,B)$ is tractable. The generalized parallel sum of $A$ and $B$ is defined by*

$$A \circledast B = AC^*(BD^*)^*,$$

*where $C$ and $D$ are the reduced solutions to the systems (3.3).*

## Some properties

### Proposition

*Let $A \in \mathcal{L}(H,K)$, $B \in \mathcal{L}(K)$, and suppose that the pair $(A,B)$ is tractable. Then the following statements are valid:*

(i) $0 \le A \circledast B = B \circledast A$.

(ii) $A \circledast B \le AA^*$ *and* $A \circledast B \le BB^*$.

(iii) $A \circledast B \le \frac{1}{4}(AA^* + BB^*)$.

(iv) $\mathcal{R}(AA^*) \cap \mathcal{R}(BB^*) \subseteq \mathcal{R}(A \circledast B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$.

(v) $\overline{\mathcal{R}(A \circledast B)} = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$.

(vi) *If* $A_n := (AA^* + \frac{1}{n}I)^{\frac{1}{2}}\ (n \in \mathbb{N})$, *then* $\lim\limits_{n\to\infty} A_n \circledast B = A \circledast B$ *in the strict topology.*

⋮

## The relationship

**Proposition**

*Let $A \in \mathcal{L}(H,K)$, $B \in \mathcal{L}(K)$, and suppose that $A$ and $B$ have the polar decompositions. Then $(A,B)$ is tractable if and only if $(|A^*|,|B^*|)$ is tractable. In this case,*

$$A \circledast B = |A^*| \circledast |B^*| = AA^* : BB^*.$$

It leads a definition as follows:

**Definition**

*If $A,B \in \mathcal{L}(H)_+$ are such that $(A^{\frac{1}{2}}, B^{\frac{1}{2}})$ is tractable. Then $A^{\frac{1}{2}} \circledast B^{\frac{1}{2}}$ and $2(A^{\frac{1}{2}} \circledast B^{\frac{1}{2}})$ are denoted by $A : B$ and $A!B$, and are called the parallel sum and the harmonic mean of $A$ and $B$.*

## A problem

**Proposition**

*There exist Hilbert $C^*$-modules $H$ and $K$, and some $A \in \mathcal{L}(H, K)$ and $B \in \mathcal{L}(K)$ such that $(|A^*|, |B^*|)$ is tractable, whereas $(A, B)$ is not tractable.*

**Problem**

*Is it possible to find $A \in \mathcal{L}(H, K)$ and $B \in \mathcal{L}(K)$ for some Hilbert $C^*$-modules $H$ and $K$ such that $(A, B)$ is tractable, whereas $(|A^*|, |B^*|)$ is not tractable?*

Up to now, we have no answer.

Now we move to the last topic: the Halmos' two projections theorem.

## The Friedrichs angle

**Objects:** $M$ and $N$ are two closed subspaces of a Hilbert space $H$.

**Notations:**

$$\widetilde{M} = M \cap (M \cap N)^{\perp}, \quad \widetilde{N} = N \cap (M \cap N)^{\perp},$$
$$c(M, N) = \sup \left\{ |\langle x, y \rangle| : x \in \widetilde{M}, y \in \widetilde{N}, \|x\| \leq 1, \|y\| \leq 1 \right\}.$$

**Definition**

The Friedrichs angle [Friedrichs, 1937], denoted by $\alpha(M, N)$, is the unique angle in $[0, \frac{\pi}{2}]$ whose cosine is equal to $c(M, N)$.

## A natural question

For every closed subspaces $M$ and $N$ of a Hilbert space $H$, it is known [Deutsch, 1994] that

$$c(M,N)=\big\|P_M P_N\big(I-P_{M\cap N}\big)\big\|.$$

Hence

$$\begin{aligned} c(M^\perp,N^\perp)=&\big\|P_{M^\perp}P_{N^\perp}\big(I-P_{M^\perp\cap N^\perp}\big)\big\| \\ =&\big\|(I-P_M)(I-P_N)\big(I-P_{M^\perp\cap N^\perp}\big)\big\|. \end{aligned}$$

**Natural question:** Whether is it true that

$$c(M,N)=c(M^\perp,N^\perp)?$$

## Hilbert space case

A positive answer can be given based on the following norm equation:

$$\left\|PQ - P_{\mathcal{R}(P)\cap\mathcal{R}(Q)}\right\| = \left\|(I-P)(I-Q) - P_{\mathcal{N}(P)\cap\mathcal{N}(Q)}\right\|, \quad (4.1)$$

where $P$ and $Q$ are any projections on a Hilbert space $H$.

- F. Deutsch, The angle between subspaces of a Hilbert space, Approximation theory, wavelets and applications (Maratea, 1994), 107–130,

The proof of equation (4.1) given in [Deutsch, 1994] relies on the Pythagorean theorem, that is, for every projection $P$ on $H$, and every element $x$ in $H$, it has

$$\|x\|^2 = \|P(x)\|^2 + \|(I-P)x\|^2.$$

## The motivation

The Pythagorean theorem is no longer true for a general Hilbert $C^*$-module $H$, since in this case only an inequality can be obtained:

$$\|x\|^2 = \|a^2 + b^2\| \leq \|a\|^2 + \|b\|^2 = \|P(x)\|^2 + \|(I-P)x\|^2,$$

in which $a = \langle Px, Px\rangle^{\frac{1}{2}}$ and $b = \langle (I-P)x, (I-P)x\rangle^{\frac{1}{2}}$.

Suppose that $H$ is a Hilbert $C^*$-module, $P, Q \in \mathcal{L}(H)$ are two projections such that $\mathcal{R}(P) \cap \mathcal{R}(Q)$ and $\mathcal{N}(P) \cap \mathcal{N}(Q)$ are both orthogonally complemented. Is it always true that

$$\left\| PQ - P_{\mathcal{R}(P)\cap\mathcal{R}(Q)} \right\| = \left\| (I-P)(I-Q) - P_{\mathcal{N}(P)\cap\mathcal{N}(Q)} \right\| ? \quad (4.2)$$

### Remark

This leads us to study the validity of the above norm equation by constructing unitary operators based on the generalized Halmos' two projections theorem.

## A space decomposition

Let $K$ be a Hilbert space, $P, Q \in \mathbb{B}(K)$ be two projections. Put

$$K_1 = \mathcal{R}(P) \cap \mathcal{R}(Q), \quad K_2 = \mathcal{R}(P) \cap \mathcal{N}(Q),$$
$$K_3 = \mathcal{N}(P) \cap \mathcal{R}(Q), \quad K_4 = \mathcal{N}(P) \cap \mathcal{N}(Q),$$

and

$$K_5 = \mathcal{R}(P) - K_1 \oplus K_2, \quad K_6 = \mathcal{N}(P) - K_3 \oplus K_4.$$

With the notation as above, we have

$$P = \begin{pmatrix} I & & & & & \\ & I & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix}, \ Q = \begin{pmatrix} I & & & & \\ & 0 & & & \\ & & I & & \\ & & & 0 & \\ & & & & T \end{pmatrix}.$$

## Halmos' two projections theorem

### Theorem (Halmos, 1969)

*Suppose that $K$ is a Hilbert space and $P, Q \in \mathbb{B}(K)$ are projections. Then $K_5 = 0$ if and only if $K_6 = 0$. When $K_5 \neq 0$, the operator $T$ can be written as*

$$T = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} U_0 \\ U_0^* Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} & U_0^*(I_{K_5} - Q_0) U_0 \end{pmatrix} \in \mathbb{B}(K_5 \oplus K_6),$$

*where $U_0$ is a unitary operator from $K_6$ to $K_5$, both $Q_0$ and $I_{K_5} - Q_0$ are positive, injective and contractive.*

- P. Halmos, Two subspaces, Trans. Amer. Math. Soc., 1969
- C. Deng and H. Du, Acta Math. Sinica (Chinese Series), 2006
- A. Böttcher and I. M. Spitkovsky, Linear Algebra Appl., 2010

## An application of Halmos' theorem

### Proposition

*Let $K$ be a Hilbert space and $P, Q \in \mathbb{B}(K)$ be projections. If*

$$\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\} \text{ and } \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\},$$

*then there exists a unitary operator $U \in \mathbb{B}(K)$ such that*

$$(I - P)(I - Q) = U \cdot PQ \cdot U^*,$$
$$I - P + I - Q = U \cdot (P + Q) \cdot U^*.$$

**Outline of the proof** By assumption $K_1 = \{0\}$ and $K_4 = \{0\}$.

**Case 1:** $K_5 \neq \{0\}$; equivalently, $K_6 \neq \{0\}$. In this case,

$$P = I_{K_2} \oplus 0_{K_3} \oplus I_{K_5} \oplus 0_{K_6},$$
$$Q = 0_{K_2} \oplus I_{K_3} \oplus T.$$

It follows immediately that

$$PQ = 0 \oplus 0 \oplus T_1, (I-P)(I-Q) = 0 \oplus 0 \oplus T_2,$$
$$P + Q = I_{K_2} \oplus I_{K_3} \oplus T_3, I - P + I - Q = I_{K_2} \oplus I_{K_3} \oplus T_4,$$

in which

$$T_1 = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} U_0 \\ 0 & 0 \end{pmatrix},$$
$$T_2 = \begin{pmatrix} 0 & 0 \\ -U_0^* Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} & U_0^* Q_0 U_0 \end{pmatrix},$$
$$T_3 = \begin{pmatrix} I_{K_5} + Q_0 & Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} U_0 \\ U_0^* Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} & U_0^*(I_{K_5} - Q_0) U_0 \end{pmatrix},$$
$$T_4 = \begin{pmatrix} I_{K_5} - Q_0 & -Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} U_0 \\ -U_0^* Q_0^{\frac{1}{2}}(I_{K_5} - Q_0)^{\frac{1}{2}} & U_0^*(I_{K_5} + Q_0) U_0 \end{pmatrix}.$$

Then the operator $U$ defined by

$$U = I_{K_2} \oplus I_{K_3} \oplus \widetilde{U} \in \mathbb{B}(K_2 \oplus K_3 \oplus K_5 \oplus K_6)$$

will meet the demanding, where

$$\widetilde{U} = \begin{pmatrix} 0 & -U_0 \\ U_0^* & 0 \end{pmatrix} \in \mathbb{B}(K_5 \oplus K_6).$$

**Case 2:** $K_5 = \{0\}$; equivalently, $K_6 = \{0\}$. In this case,

$$PQ = 0, (I-P)(I-Q) = 0,$$
$$P + Q = I_{K_2} \oplus I_{K_3} = I - P + I - Q.$$

## The pair of harmonious projections

### Definition

*Two projections $P$ and $Q$ on a Hilbert $C^*$-module $H$ are said to be harmonious if the four closures*

$$\overline{\mathcal{R}(P+Q)}, \overline{\mathcal{R}(P+I-Q)}, \overline{\mathcal{R}(I-P+Q)}, \overline{\mathcal{R}(I-P+I-Q)}$$

*are all orthogonally complemented in $H$.*

### Proposition

*Two projections $P$ and $Q$ on a Hilbert $C^*$-module $H$ are harmonious if and only if $\overline{\mathcal{R}\big[PQ(I-P)\big]}$ and $\overline{\mathcal{R}\big[(I-P)QP\big]}$ are both orthogonally complemented in $H$.*

## The generalized Halmos' two projections theorem

In the Hilbert $C^*$-module case, an additional condition of

$$\overline{\mathcal{R}\big[Q_0(I_{K_5}-Q_0)\big]}=K_5$$

has to be added, where $Q_0$ appears in the expression of $T$. This condition will be satisfied automatically in the Hilbert space case, since $\mathcal{N}\big[Q_0(I_{K_5}-Q_0)\big]=\{0\}$ and $\overline{\mathcal{R}\big[Q_0(I_{K_5}-Q_0)\big]}$ is always orthogonally complemented in $H_5$.

**Theorem (Luo-Moslehian-Xu, 2019; Xu-Yan, 2020)**

*Let $P,Q$ be projections on a Hilbert $C^*$-module $H$. Then the Halmos' two projections theorem is valid if and only if $P$ and $Q$ are harmonious.*

**Remark**

Based on the Halmos' two projections theorem, a positive answer can be given to the validity of the norm equation (4.2).

## Concluding remark

It is known that a closed submodule of a Hilbert C*-module may fail to be orthogonally complemented. Due to this weakness, some new phenomena may happen compared with that in a Hilbert space.

In the framework of adjointable operators on Hilbert C*-modules, we have reported some of our recent joint works on the generalized Douglas theorem, the polar decomposition, the generalized parallel sum and the generalized Halmos’ two projections theorem.

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Thank you for your attention