ON A QUESTION RAISED BY MICHAEL SKEIDE

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We choose a dense countable subset $Q = \{q(n) \mid n \in \mathbb{N}\} \subseteq [0,1]$ such that $q(n) > 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we consider the indicator function $\chi_{[0,q(n))} : [0,1] \to [0,1]$ for the half-open interval $[0, q(n))$ and we choose a sequence $\{\psi_m(n)\}_{m=1}^\infty$ of continuous functions $\psi_m(n) : [0,1] \to [0,1]$ such that

$$\sum_{m=1}^\infty \psi_m(n)^2(t) = \chi_{[0,q(n))}(t) \tag{1}$$

for all $t \in [0,1]$.

We let $\ell^2(\mathbb{N}, C([0,1]))$ denote the standard Hilbert C^*-module over the unital C^*-algebra $C([0,1])$ (of continuous functions on the interval $[0,1]$). For each $m \in \mathbb{N}$, we let $e_m \in \ell^2(\mathbb{N}, C([0,1]))$ denote the sequence, which is constant equal to one in position $m \in \mathbb{N}$ and zero elsewhere.

For each $n \in \mathbb{N}$, we define the $C([0,1])$-linear bounded operator

$$\Psi(n) : \ell^2(\mathbb{N}, C([0,1])) \to C([0,1]) \quad \Psi(n)(e_m) := \psi_m(n).$$

Lemma 0.1. *Let $n \in \mathbb{N}$. The $C([0,1])$-linear bounded operator $\Psi(n) : \ell^2(\mathbb{N}, C([0,1])) \to C([0,1])$ is a non-adjointable contraction.*

Proof. For each $m \in \mathbb{N}$, we let $P_m : \ell^2(\mathbb{N}, C([0,1])) \to \ell^2(\mathbb{N}, C([0,1]))$ denote the orthogonal projection onto the closed submodule $e_m \cdot C([0,1]) \subseteq \ell^2(\mathbb{N}, C([0,1]))$. For each $M \in \mathbb{N}$, we equip $\mathbb{C}^M$ with the standard Hilbert space structure and the norm of $\Psi(n) \sum_{m=1}^M P_m : \ell^2(\mathbb{N}, C([0,1])) \to C([0,1])$ then agrees with the norm of the continuous map

$$\begin{pmatrix} \psi_1(n) \\ \psi_2(n) \\ \vdots \\ \psi_M(n) \end{pmatrix} : [0,1] \to \mathbb{C}^M$$

which in turn agrees with $\sup_{t \in [0,1]} \left\{ \sqrt{\sum_{m=1}^M \psi_m(n)^2(t)} \right\}$. By (1) this norm is dominated by 1 and it follows that $\Psi(n) : \ell^2(\mathbb{N}, C([0,1])) \to C([0,1])$ is a contraction.

Since the sum $\sum_{m=1}^\infty \psi_m(n)^2$ is only pointwise convergent and not convergent with respect to the C^*-algebra norm on $C([0,1])$ we see that $\Psi(n) : \ell^2(\mathbb{N}, C([0,1])) \to C([0,1])$ is not adjointable. Indeed, the sequence $\{\psi_m(n)\}_{m=1}^\infty$ does not belong to $\ell^2(\mathbb{N}, C([0,1]))$. □

We let X denote the Hilbert C^*-module direct sum

$$X := C([0,1]) \oplus \Big(\oplus_{n=1}^\infty \ell^2(\mathbb{N}, C([0,1])) \Big). \tag{2}$$

For each $n \in \mathbb{N}$ we then have the adjointable isometric inclusion $\iota(n) : \ell^2(\mathbb{N}, C([0,1])) \to X$ of the Hilbert C^*-module $\ell^2(\mathbb{N}, C([0,1]))$ into the n^{th} component of X and we also consider the adjointable isometric inclusion $\iota(0) : C([0,1]) \to X$ of the Hilbert C^*-module $C([0,1])$ into the 0^{th} component of X.

We let $\Psi(0) := 1 : C([0,1]) \to C([0,1])$ denote the identity operator. We equip X with the $C([0,1])$-linear bounded operator

$$\Psi : X \to C([0,1]) \quad \Psi = \sum_{n=0}^{\infty} 2^{-n} \Psi(n)\iota(n)^*,$$

where the infinite sum converges absolutely in operator norm.

Now, for each $m, n \in \mathbb{N}$ we define the vector

$$\zeta_m(n) := \iota(0)\big(2^{-n}\psi_m(n)\big) - \iota(n)(e_m) \in X$$

and we define the subspace $S \subseteq X$ as the span of all these vectors, thus

$$S := \operatorname{span}_{\mathbb{C}}\big\{\zeta_m(n) \mid m, n \in \mathbb{N}\big\}.$$

Lemma 0.2. *It holds that $\Psi(\zeta) = 0$ for all $\zeta \in S$.*

Proof. This is immediate since

$$\Psi(\zeta_m(n)) = 2^{-n}\psi_m(n) - 2^{-n}\Psi(n)(e_m) = 0$$

for all $m, n \in \mathbb{N}$. □

Lemma 0.3. *It holds that $S^\perp = \{0\}$.*

Proof. Let $x = \{x(n)\}_{n=0}^\infty \in S^\perp$ be given. For each $n \in \mathbb{N}_0$ we have that $x(n) = \sum_{m=1}^\infty e_m x_m(n) \in \ell^2(\mathbb{N}, C([0,1]))$ for some continuous functions $x_m(n)$, $m \in \mathbb{N}$, whereas $x(0) \in C([0,1])$. For each $n, m \in \mathbb{N}$ we compute that

$$2^{-n}\psi_m(n)x(0) - x_m(n) = 2^{-n}\psi_m(n)x(0) - \langle e_m, x(n)\rangle = \langle \zeta_m(n), x\rangle = 0$$

and hence that

$$x_m(n) = 2^{-n}\psi_m(n) \cdot x(0) \quad \text{for all } n, m \in \mathbb{N}. \tag{3}$$

Suppose now for contradiction that $x \neq 0$. By (3) we must have that $x(0)(t_0) \neq 0$ for some $t_0 \in [0,1]$ and then by continuity there exists an open set $U \subseteq [0,1]$ such that $x(0)(t) \neq 0$ for all $t \in U$. Since the countable set $Q \subseteq [0,1]$ is dense we may then find an $n_0 \in \mathbb{N}$ such that $q(n_0) \in U$. Since $x \in X$ we know that $\iota(n_0)^*(x) = x(n_0) \in \ell^2(\mathbb{N}, C([0,1]))$. However, from (3) we see that

$$x(n_0) = \sum_{m=1}^{\infty} e_m \cdot 2^{-n_0}\psi_m(n_0) \cdot x(0).$$

This is a contradiction since the sum

$$\sum_{m=1}^{\infty} 2^{-2n_0}\psi_m(n_0)^2 \cdot |x(0)|^2$$

does not converge in supremum norm. Indeed, by (1) the above sum converges pointwise to the function

$$2^{-2n_0}\chi_{[0,q(n_0))} \cdot |x(0)|^2 : [0,1] \to [0,\infty)$$

which has a discontinuity at the point $q(n_0) \in [0,1]$. This proves the lemma. □

We gather what we have obtained so far:

Proposition 0.1. *Let $E = \ell^2(\mathbb{N}, C([0,1]))$ denote the standard Hilbert C^*-module over the unital C^*-algebra $C([0,1])$. Then there exists a non-trivial $C([0,1])$-linear bounded operator $\Phi : E \to C([0,1])$ and a subspace $S \subseteq E$ such that*

$$\Phi(\zeta) = 0 \quad \text{for all } \zeta \in S \quad \text{and} \quad S^\perp = \{0\}.$$

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