Problems

(1) Let A be a C^*-algebra, $a \in A$ and let $p, q \in A$ be orthogonal projections (i.e. selfadjoint idempotents with $pq = 0$). Suppose that a is positive and $pap = 0$. Show that $paq = 0$.

(2) Let A be a C^*-algebra, $a \in A$. Denote by aAa the set of all elements of the form aba with $b \in A$ and by $\overline{aAa}$ the closure of that set. Recall that a C^*-subalgebra $B \subset A$ is *hereditary* if the conditions $0 \le a \le b$ and $b \in B$ imply $a \in B$.
   (a) Check that $\overline{aAa}$ is a C^*-subalgebra for any $a \in A$.
   (b) Let $p \in A$ be a projection. Check that pAp is closed.
   (c) Show that pAp is hereditary for any projection p.
   (d) Show that $\overline{aAa}$ is hereditary for any positive $a \in A$.

(3) Let $X \subset \mathbb{R}$ be the set of points $1, 1/2, 1/3, \ldots$ and 0. For the C^*-algebra M_2 of two-by-two matrices denote by $C(X, M_2)$ the set of all continuous functions on X with values in M_2. Put $B_1 = \{f \in C(X, M_2) : f(0) \text{ is diagonal}\}$, $B_2 = \{f \in C(X, M_2) : f(0) \text{ has the form } \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.
   (a) Show that $C(X, M_2)$, B_1, B_2 are C^*-algebras.
   (b) Find all (two-sided closed) ideals of the C^*-algebras $C(X)$, $C(X, M_2)$, B_1, B_2.

(4) Let A be a C^*-algebra, $J \subset A$ an ideal, and let $a \in A$ be selfadjoint. Show that there exists $j \in J$ such that $\|[a]\| = \|a - j\|$, where $[a] \in A/J$ is the class $a + J$ of a. Hint: decompose $a - \|[a]\| \cdot 1 = a_+ - a_-$ with positive a_+, a_- and show that $a_+ \in J$.

(5) Let A be a C^*-algebra, $a \in A$ selfadjoint. Suppose that the spectrum $\sigma(a)$ is an infinite set. Show that A is infinitedimensional.

(6) Describe the GNS construction for the C^*-algebra $C[0,1]$ and for the positive linear functional φ given by
   (a) $\varphi(f) = f(0)$,
   (b) $\varphi(f) = \frac{1}{2}(f(0) + f(1))$,
   (c) $\varphi(f) = \int_0^1 f(x)\,dx$,
   where $f \in C[0,1]$.

(7) Describe the GNS construction for the C^*-algebra M_n of $n \times n$ complex matrices and for the positive functional φ given by
   (a) $\varphi(A) = a_{11}$,
   (b) $\varphi(A) = \mathrm{tr}(A)$,
   where $A = (a_{ij})_{i,j=1}^n \in M_n$.

(8) Let π, σ be representations of a C^*-algebra A on Hilbert spaces H_π and H_σ respectively and let a partial isometry $U : H_\pi \to H_\sigma$ satisfy $\sigma(a)U = U\pi(a)$ for any $a \in A$. Show that the range and domain subspaces for U are invariant subspaces for $\sigma(A)$ and $\pi(A)$ respectively. (Recall that U is a partial isometry iff U^*U and UU^* both are projections and that the range subspace for U is the image of U in H_σ and the domain subspace for U is the orthogonal complement to the kernel of U in H_π.)

(9) (a) Let $M_n(A)$ denote the set of all $n \times n$ matrices with entries from a C^*-algebra A. Show that there exists a norm on $M_n(A)$ that makes it a C^*-algebra. Hint: consider first the case $A = \mathbb{B}(H)$ and then use the Gelfand–Naimark theorem.
   (b) Let A be a C^*-algebra with the norm $\|\cdot\|$ and let $\|\cdot\|'$ be another norm on A (equivalent to the first norm). Show that if $\|\cdot\|'$ is a C^*-algebra norm on A then the two norms coincide. Use that to show that there is a unique C^*-algebra norm on $M_n(A)$.

(10) Let φ be a state on a C^*-algebra A. Suppose that for some selfadjoint element $a \in A$ one has $\varphi(a^2) = \varphi(a)^2$. Show that this implies that $\varphi(ab) = \varphi(ba) = \varphi(a)\varphi(b)$ for any $b \in A$.

(11) Let $A = c$ be the C^*-algebra of converging sequences of complex numbers, $c = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C}; \lim_{n\to\infty} a_n \text{ exists}\}$. Consider it as a C^*-subalgebra of the algebra $\mathbb{B}(l_2)$ of bounded operators on the Hilbert space l_2 of square-summable sequences. Find the first and the second commutant, A' and A'', of A and (independently) the weak closure of A in $\mathbb{B}(l_2)$.

(12) (a) Show that the weak topology is strictly weaker than the strong topology.
(b) Let $P \subset \mathbb{B}(H)$ denote the set of all projections. Show that the restrictions of the weak and the strong topology on P give the same topology.
(c) Show that the strong limit of a sequence of projections (if it exists) must be a projection.
Remark (not a part of the exercise): (c) is not true for weak limits: with respect to the weak topology, the set P is dense in the set of positive elements of the unit ball of $\mathbb{B}(H)$

(13) Let $H_n \subset H$ be a subspace of the Hilbert space H spanned by the first n vectors of an orthonormal basis. In the set of all sequences $(m_1, m_2, \ldots)$, where $m_k \in \mathbb{B}(H_n) \subset \mathbb{B}(H)$, consider the subset A of all sequences such that
- $\sup_k \|m_k\|$ is finite;
- sequences $(m_1, m_2, \ldots)$ and $(m_1^*, m_2^*, \ldots)$ are convergent with respect to the strong topology.

Show that A is a C^*-algebra. Show that the map $(m_1, m_2, \ldots) \mapsto s-\lim_{k\to\infty} m_k \in \mathbb{B}(H)$ is a surjective $*$-homomorphism $A \to \mathbb{B}(H)$.

(14) Let A be a commutative C^*-algebra and let π be an irreducible representation of A on a Hilbert space H. Show that $\dim H = 1$.

(15) Let $A \subset \mathbb{B}(H)$ be a von Neumann algebra and let $p_1 \leq p_2 \leq p_3 \leq \ldots$, $p_k \in A$, $k \in \mathbb{N}$, be an increasing sequence of projections. Let $H_k \subset H$ be the range of the projection p_k. Show that the projection p onto the closure of the union $\cup_{k\in\mathbb{N}} H_k$ lies in A. Hint: consider the von Neumann subalgebra generated by the given projections.

(16) (Difficult) Let $A \subset \mathbb{B}(H)$ be a von Neumann algebra of infinite dimension. Show that there exists an infinite sequence of mutually orthogonal non-zero projections with sum 1.

(17) (a) Prove that the matrix algebra $M_n(\mathbb{C})$ is simple.
(b) Show that if n doesn't divide m (i.e. $m = kn + r$, $0 < r < n$) then there is no non-degenerate representation of $M_n(\mathbb{C})$ on the Hilbert space of dimension m.

(18) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of a Hilbert space H and let operators a, b be defined by $ae_i = e_{2i}$; $be_i = e_{2i-1}$. Let $E = C^*(a, b) + \mathbb{K}(H) \subset \mathbb{B}(H)$ be the C^*-algebra generated by a, b and compact operators. Let also p_n be the projection onto the linear span of $e_1, \ldots, e_n$.
(a) Check that a, b are bounded operators and that they satisfy $a^*a = b^*b = 1$, $aa^* + bb^* = 1$.
(b) Starting with $\{p_n\}_{n=1}^{\infty}$, construct a quasi-central approximate unit for the pair $\mathbb{K}(H) \subset E$.

(19) Find the multiplier C*-algebra $M(A)$ for the following A:
(a) A is the algebra c_0 of $\mathbb{C}$-valued sequences converging to 0;
(b) A is the algebra of continuous $\mathbb{C}$-valued functions on $\mathbb{R}$ vanishing at $\pm\infty$.

(20) (a) Let Y be a compact Hausdorff space and let X be a dense locally compact subspace, $X \subset Y$. Show that $C_0(X) \subset C(Y)$is an essential ideal.

(b) For a C^*-algebra A let $l^\infty(A) = \{(a_n)_{n=1}^\infty : a_n \in A, \sup_n \|a_n\| < \infty\}$ and $c_0(A) = \{(a_n)_{n=1}^\infty : a_n \in A, \lim_{n\to\infty} \|a_n\| = 0\}$. Show that $c_0(\mathbb{K}(H))$ is an essential ideal both in $l^\infty(\mathbb{K}(H))$ and in $l^\infty(\mathbb{B}(H))$.

(21) Let $\varphi : A \to B$ be a surjective $*$-homomorphism of separable C^*-algebras. Check that if $\lambda : A \to A$ is a left centralizer then the formula $\bar{\varphi}(\lambda)(\varphi(a)) = \varphi(\lambda(a))$ defines a left centralizer for B. Show that this gives rise to a $*$-homomorphism $\bar{\varphi} : M(A) \to M(B)$, which is surjective.

(22) Consider $C[0,1]$ as a C^*-subalgebra in $\mathbb{B}(H)$, where $H = L^2([0,1],\mu)$ with the Lebesgue measure μ (continuous functions act on H by multiplication).
  (a) Check that $C[0,1] \cap \mathbb{K}(H) = 0$;
  (b) For a linear functional φ on $C[0,1]$ defined by $\varphi(f) = f(0)$ find a sequence of unit vectors $\{e_n\}_{n\in\mathbb{N}}$ in H such that it is weakly convergent to 0 in H and $\varphi(f) = \lim_{n\to\infty}\langle fe_n, e_n\rangle$ for any $f \in C[0,1]$.

(23) Let X be a compact Hausdorff space and B a C^*-algebra. A linear map (not necessarily a $*$-homomorphism!) $\varphi : C(X) \to B$ gives rise to a linear map for matrices: $\varphi_n : M_n(C(X)) \to M_n(B)$ (by applying φ to each matrix entry). Show that if φ is a positive map (i.e. takes positive elements of $C(X)$ to positive elements of B) then φ_n is positive map too. Hint: identify $M_n(C(X))$ with $M_n(\mathbb{C})$-valued functions on X; then check positivity for elements of the form $g(x)T \in C(X; M_n(\mathbb{C}))$, where $g \in C(X)$, $T \in M_n(\mathbb{C})$; finally show that linear combinations of such elements are dense in $C(X; M_n(\mathbb{C}))$.

(24) Operators a, b on a Hilbert space H are called compalent if there is a unitary $u \in \mathbb{B}(H)$ such that $u^*au - b \in \mathbb{K}(H)$. Show that selfadjoint operators a, b are compalent if and only if their essential spectra coincide.

(25) Let S be the right (unilateral) shift on l^2 and let A be the unital C^*-algebra generated by S (i.e. A is the closure of linear combinations of polynomials on two non-commuting variables, S and S^*). Let $\alpha \in [0, 2\pi)$ and let $\varphi_\alpha : A \to \mathbb{C}$ be a linear multiplicative map given by $\varphi(S^n) = e^{in\alpha}$; $\varphi((S^*)^n) = e^{-in\alpha}$.
  (a) Check that this map is a $*$-homomorphism and that $\varphi_\alpha(a) = 0$ if $a \in \mathbb{K}(l^2)$;
  (b) Find a sequence of unit vectors $x_n \in l^2$ weakly convergent to 0 such that $\varphi_\alpha(S) = \lim_{n\to\infty}\langle Sx_n, x_n\rangle$.
  (c) Let $\alpha_1, \ldots, \alpha_k \in [0, 2\pi)$. For the homomorphism $\varphi : A \to M_k(\mathbb{C})$ given by $\varphi(a) = diag(\varphi_{\alpha_1}(a), \ldots, \varphi_{\alpha_k}(a))$ and for any $\epsilon > 0$, find an isometry $V : \mathbb{C}^k \to l^2$ such that $\|\varphi(S) - V^*SV\| < \epsilon$.
  (d) Let $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of numbers in $[0, 2\pi)$ and let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space H. For $a \in A$, put $\varphi(a)(e_i) = \varphi_{\alpha_i}(a)e_i$. Then $\varphi : A \to \mathbb{B}(H)$ is a $*$-homomorphism. For any $\epsilon > 0$ find an isometry $V : H \to l^2$ such that $\varphi(S) - V^*SV$ is compact and $\|\varphi(S) - V^*SV\| < \epsilon$.

(26) Show that separability in Glimm's Lemma is necessary:
Let $D = l^\infty$ be represented as diagonal operators on $H = l^2$ and let τ be a non-trivial state on D such that τ equals 0 for any sequence in D that converges to 0. Show that there is no sequence $x_n \in H$ of unit vectors such that $\tau(d) = \lim_{n\to\infty}\langle dx_n, x_n\rangle$.

(27) Show that any AF C^*-algebra contains (if it is not unital) an approximate unit consisting of an increasing sequence of projections.

(28) (a) Show that $C[0,1]$ is not AF. Hint: a finitedimensional C*-subalgebra of $C[0,1]$ consists only of constant functions, hence is 1-dimensional.
  (b) Show that $C[0,1]$ is a C*-subalgebra of the AF algebra $C(K)$ of continuous functions on a Kantor set K. Hint: one can construct a function f on K such that it takes on K all rational values from $[0,1]$. Its spectrum is the closure

of this set, i.e. the whole interval $[0,1]$. Thus, the unital C^*-subalgebra of $C(K)$ generated by f is isometrically $*$-isomorphic to $C(\mathrm{Sp}(f)) = C[0,1]$.

(29) Let $A_n = M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$ and let the embedding $\alpha_n : A_n \to A_{n+1}$ be given by $\alpha_n : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & | & 0 & 0 \\ 0 & a_1 & | & 0 & 0 \\ - & - & - & - & - \\ 0 & 0 & | & a_1 & 0 \\ 0 & 0 & | & 0 & a_2 \end{pmatrix}$, where $a_1, a_2 \in M_{2^n}(\mathbb{C})$.

(a) Determine the Bratteli diagram for the AF algebra $A = \overline{\cup_{n=1}^{\infty} A_n}$;
(b) Using the Bratteli diagram, find all ideals in A;
(c) Determine if A is unital or not.

(30) Let A be an AF algebra.

(a) Show that linear combinations of projections are dense in A;
(b) Show that A has an approximate unit consisting of projections.

(31) A *derivation* of a C^*-algebra A is a bounded linear map $\delta : A \to A$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in A$. It is called *inner* if there exists $x \in A$ such that $\delta = \delta_x$, where $\delta_x(a) = xa - ax$. A derivation δ is *approximately inner* if there exists a net $\{x_\iota\} \subset A$ such that $\delta(a) = \lim_\iota \delta_{x_\iota}(a)$ for any $a \in A$.

(a) Show that every derivation of a finitedimensional C^*-algebra is inner. Hint: take a finite group G of unitaries that span A and use $x = \frac{1}{|G|}\sum_{u \in G} \delta(u)u$.
(b) Show that every derivation of an AF algebra is approximately inner.

(32) Let A, B be C^*-algebras. Two $*$-homomorphisms $\alpha, \beta : A \to B$ are called *homotopic* if there is a family $(\varphi_t)_{t \in [0,1]} : A \to B$ of $*$-homomorphisms such that $\varphi_0 = \alpha$, $\varphi_1 = \beta$ and the map $t \mapsto \varphi_t(a)$ is continuous for any $a \in A$. A C^*-algebra is *contractible* if the identity $*$-homomorphism is homotopic to the zero $*$-homomorphism. Show that the *cone* $CA = \{f \in C([0,1]; A) : f(0) = 0\}$ is contractible for any C^*-algebra A.

(33) Let T be the Toeplitz algebra and $\varphi : T \to T$ its automorphism. Show that $\varphi(\mathbb{K}) \subset \mathbb{K}$, where $\mathbb{K}$ is the ideal of compact operators. Thus, φ induces an automorphism of the quotient C^*-algebra $C(\mathbb{T})$. Show that this induced automorphism, in its turn, induces a homeomorphism of the circle $\mathbb{T}$.

(34) Let $\mathcal{O}_2$ denote the Cuntz algebra on two isometries, s_1 and s_2. Show that $M_3(\mathcal{O}_2)$ is isomorphic to $\mathcal{O}_2$. Hint: check that the matrices $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & s_1 & s_2 \end{pmatrix}$ and $\begin{pmatrix} s_1 & s_2 s_1 & s_2^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ generate $M_3(\mathcal{O}_2)$.

(35) Show that the group C^*-algebra $A = C^*(\mathbb{F}_2)$ of the free group on two generators has no proper projections. Hint: if u and v are the unitaries that correspond to the two generators in a faithful representation of A (on a Hilbert space H) then there are bounded operators a, b such that $u = \exp(ia)$, $v = \exp(ib)$; put $u(t) = \exp(ita)$, $v(t) = \exp(itb)$, $t \in [0,1]$ and consider the C^*-algebra $B = \{\Phi \in C([0,1]; \mathbb{B}(H)) : \Phi(0) \in \mathbb{C} \cdot 1\}$. Then use $u(t)$ and $v(t)$ to include A into B and check that B has no proper projections.

(36) Show that a group homomorphism (of discrete groups) $G_1 \to G_2$ induces a $*$-homomorphism $C^*(G_1) \to C^*(G_2)$ of their group C^*-algebras. Determine the kernel and the range of the $*$-homomorphism induced by the surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.