Lecture 1

1. Multipliers

We will call a C^*-subalgebra $A \subset \mathbb{B}(H)$ *non-degenerate*, if its natural representation in the Hilbert space H is non-degenerate.

Everywhere in this section $A, B \subset \mathbb{B}(H)$.

Definition 1.1. An operator $x \in \mathbb{B}(H)$ is called *left multiplier* A if $xA \subset A$. It is called *right multiplier*, if $Ax \subset A$ and *double (or double-sided) multiplier* or simply *multiplier*, if both conditions are met.

If A is unital, then every left (right) multiplier lies in A.

Since A is weakly dense in A'', we can proceed to the closure $xA \subset A$ and get $xA'' \subset A''$. If A'' contains the unit (which happens when A is non-degenerate) then $x \in A''$.

Definition 1.2. The linear mapping $\sigma : A \to A$ is called *left centralizer*, if $\sigma(ab) = \sigma(a)b$ for any $a, b \in A$. Linear mapping $\sigma : A \to A$ called *right centralizer*, if $\sigma(ab) = a\sigma(b)$ for any $a, b \in A$. Pair (σ_1, σ_2) called *double centralizer*, if σ_1 is a right centralizer, σ_2 is a left centralizer and $\sigma_1(a)b = a\sigma_2(b)$ for any $a, b \in A$.

Lemma 1.3. *Any left centralizer is bounded.*

Proof. Let us assume the opposite. Then for any $n \in \mathbb{N}$ there is an element $x_n \in A$ such that $\|x_n\| < 1/n$ and $\|\sigma(x_n)\| > n$. This means that the series $a = \sum_{n=1}^{\infty} x_n x_n^*$ converges, so $a \in A$ and $x_n x_n^* \leqslant a$. According to Lemma **??**, x_n can be written as $x_n = a^{1/4}u_n$, where $\|u_n\| \leqslant \|a\|^{1/4}$. Therefore, for any n we have $\|\sigma(x_n)\| = \|\sigma(a^{1/4})u_n\| \leqslant \|\sigma(a^{1/4})\| \cdot \|a^{1/4}\|$. A contradiction. □

Theorem 1.4. *If A is non-degenerate, then there is a bijective correspondence between left (right, double) multipliers and left (right, double) centralizers.*

Proof. If x is a left multiplier, then the mapping $A \ni a \mapsto xa \in A$ is a left centralizer. If $xa = ya$ for any $a \in A$, then $(x - y)a = 0$ for any $a \in A''$, so $x = y$ in A''.

Let σ be a left centralizer, and u_λ be an approximate unit for A. Since the directed net $\{\sigma(u_\lambda)\}$ is bounded, then it has a point of accumulation in A'' (bounded sets are weakly compact in $\mathbb{B}(H)$ and accumulation points must lie in the closure of A). Let us denote one of the accumulation points by x. For any $a \in A$, the directed net $\{u_\lambda a\}$ converges in norm to a, so that $\sigma(u_\lambda a) = \sigma(u_\lambda)a$ converges to $\sigma(a)$. Then $xa = \sigma(a) \in A$, so x is a left multiplier. If $xA = 0$, then $\sigma = 0$. Note that if $y \in A''$ is another accumulation point, then $xa = \sigma(a) = ya$ for any $a \in A$, and $(x - y)a = 0$ for any $a \in A''$ (due to the strong density of A in A''), so $x = y$ in A''. Therefore, in this case there is only one point of accumulation.

A similar proof works also for right multipliers and right centralizers.

Let now x be a double multiplier. Then the mappings $\sigma_2 : a \mapsto xa$ and $\sigma_1 : a \mapsto ax$ are left and right multipliers, with $\sigma_1(a)b = (ax)b = a(xb) = a\sigma_2(b)$ for any $a, b \in A$, so x defines a double centralizer. Conversely, if (σ_1, σ_2) is a double centralizer, then, by what has been proven, σ_1 determines a right multiplier x_1, and σ_2 a left multiplier x_2. Since $ax_1b = \sigma_1(a)b = a\sigma_2(b) = ax_2b$ for any $a, b \in A$, we have $x_1 = x_2$, and $x_1 = x_2$ is a double multiplier. □

Problem 1. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Remark 1.5. Accordingly, until the end of this section we will consider non-degenerate $A \subseteq \mathbb{B}(H)$, so that (double) multipliers coincide with double centralizers.

The set of all left (right) multipliers of A is denoted by $LM(A)$ ($RM(A)$), and the set of all multipliers of A by $M(A)$.

Problem 2. Check that $RM(A) = (LM(A))^*$ and that $M(A) = LM(A) \cap RM(A)$, so that $M(A)$ is symmetric with respect to the involution.

It follows directly from the definition that all three sets are norm closed. Thus, $M(A)$ is a C^*-algebra (and the other two are, in the general case, only Banach algebras).

Problem 3. Let X be a locally compact space, and let $C_0(X)$, as before, be the C^*-algebra of continuous functions tending to 0 at infinity. Prove that the algebra $M(C_0(X)) \subset L^\infty(X)$ can be identified with the C^*-algebra $C_b(X)$ of all bounded continuous functions on X.

Example 1.6. If $A = \mathbb{K}(H)$, then $M(\mathbb{K}(H)) \subseteq \mathbb{B}(H)$, but any bounded operator is a multiplier (since $\mathbb{K}(H)$ is an ideal in $\mathbb{B}(H)$), so $M(\mathbb{K}(H)) = \mathbb{B}(H)$.

Definition 1.7. An ideal $A \subset B$ is said to be *essential* if any nonzero ideal B has a nontrivial intersection with A.

Let $A^\perp \subset B$ denote the set $A^\perp = \{x \in B \colon Ax = 0\}$.

Lemma 1.8. *An ideal $A \subset B$ is essential if and only if $A^\perp = 0$.*

Proof. Suppose that $A^\perp = 0$, but A is not essential. Then there is a nonzero ideal $J \subset B$ such that $A \cap J = \{0\}$. Let us take $j \in J$, $j \neq 0$. For any $a \in A$ we have $ja \in J \cap A$, so $ja = 0$ and $0 \neq j \in A^\perp$. A contradiction.

Conversely, let A be essential, but $A^\perp \neq 0$. Then there is an element $x \in A^\perp$ such that $x \neq 0$. Consider the ideal BxB (the closure of the set of all linear combinations of elements of the form $\sum_i b_i x b_i'$, $b_i, b_i' \in B$) and take an arbitrary $y \in BxB \cap A$. As it is known (for example, from Lemma **??**), any element of the C^*-algebra admits a decomposition into the product of two of its elements, so we can write $y = z \cdot a$, where $z, a \in BxB \cap A$. We write $z = \sum_i b_i x b_i'$, so $y = za = \sum_i b_i x (b_i' a) = 0$, since $b_i' a \in A$, hence $x b_i' a = 0$, because $x \in A^\perp$. Therefore, $BxB \cap A = 0$ and we arrive to a contradiction. □

Lemma 1.9. *Let $A \subset B$ be an essential ideal. Then there is an embedding $B \subset M(A)$ that is identical on A.*

Proof. Consider $b \in B$. Then b defines the left and right centralizer A (since A is an ideal) $\sigma_2 : a \mapsto ba$ and $\sigma_1 : a \mapsto ab$, and $\sigma_1(a)a' = (ab)a' = a(ba') = a\sigma_2(a')$ for any $a, a' \in A$, so b defines a double centralizer, and hence a multiplier. So the mapping $\pi : B \to M(A)$ is defined, identical on A. This mapping is obviously a $*$-homomorphism. It remains to check whether π is injective. If $b \in \operatorname{Ker} \pi$, then $\sigma_1 = 0$ and $\sigma_2 = 0$ (see proof of theorem 1.4), so $bA = 0$, $Ab = 0$ and $b \in A^\perp$, which means $b = 0$. □

Note that the correspondence $A \mapsto M(A)$ is not a functor. For example, for $A = \mathbb{K}(H)$ and $B = A^+$, consider the embedding $A \subset B$. Wherein $M(A) = \mathbb{B}(H)$, and $M(B) = B$. Obviously, the embedding does not continue to these multiplier algebras. However, in some cases the transition to multipliers has some functorial properties.

Lemma 1.10. *Let $\varphi : A \to B$ be a surjective $*$-homomorphism of two C^*-algebras. Then it continues to a $*$-homomorphism $\bar{\varphi} : M(A) \to M(B)$.*

Proof. Let $\sigma \in LM(A)$ be a left centralizer. For any $b \in B$ we set $\bar{\varphi}(\sigma)(b) := \varphi(\sigma(\varphi^{-1}(b)))$. It is necessary check that the map is well defined, that is, its independence of the choice of a representative in $\varphi^{-1}(b) \subset A$. Due to linearity, it suffices to check that σ maps $\operatorname{Ker}\varphi$ to itself. Let $a \in \operatorname{Ker}\varphi$. Let us represent it in the form $a = a_1 \cdot a_2$, $a_1, a_2 \in \operatorname{Ker}\varphi$. Then $\varphi(\sigma(a)) = \varphi(\sigma(a_1)a_2) = 0$, since $\operatorname{Ker}\varphi$ is an ideal.

Thus, a left centralizer σ defines the mapping $\bar{\varphi}(\sigma)$, which is the left centralizer of B. The same is done for right and double centralizers (Problem 4). □

Problem 4. Prove that $\bar{\varphi}(\sigma)$ is a left centralizer. Construct an extension of a right centralizer in a similar way. Check that for a double centralizer we get a double centralizer.

Problem 5. Check that in the situation of the previous lemma the extension $\bar{\varphi}$ is also surjective.

Problem 6. Prove that a representation $\pi : A \to \mathbb{B}(H)$ is non-degenerate if and only if for some approximate unit u_λ of the algebra A the following condition is satisfied: for any vector $\xi \in H$ there is a λ such that that $u_\lambda(\xi) \neq 0$.

Lemma 1.11. *Let $B \subset A$ be an algebra and its C^*-subalgebra with a common approximate unit u_λ. Then $M(B) \subset M(A)$.*

Proof. If A is non-degenerate, then B is also non-degenerate by Problem 6. So $M(B) \subset B'' \subset A''$. For any $x \in A$, $y \in M(B)$, $yx = \lim y u_\lambda x \in A$, and similarly, $xy \in A$, so y is a multiplier of A. □

Lecture 2

2. Toeplitz Algebra

Let $\mathbb{T}$ be the unit circle, $e_n = e^{2\pi int} = z^n$, $n \in \mathbb{Z}$, — an orthonormal basis in $H = L^2(\mathbb{T})$, and let $H^2 \subset L^2(\mathbb{T})$ — closed subspace generated by all e_n with non-negative numbers, $n \geqslant 0$. Let us denote by $M_g : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the operator of multiplication by the function $g \in L^\infty(\mathbb{T})$, $M_g(f) = gf$, $f \in L^2(\mathbb{T})$. Let $P_{H^2} \in \mathbb{B}(H)$ denotes the projection onto H^2.

Let us define the operator T_g, where $g \in L^\infty(\mathbb{T})$, on H^2 by the formula $T_g(f) = P_{H^2}M_g(f) = P_{H^2}gf$, $f \in H^2$.

If $g(z) = z$, then the operator T_g can be identified with the right shift operator in l^2 (recall that l^2 and $L^2(\mathbb{T})$ are unitary equivalent in the standard way).

Problem 7. Check this.

Let us denote by $\|T\|_{ess}$ the *essential* norm of the operator T as the norm of its equivalence class in the Calkin algebra. It is easy to see that $\|T\|_{ess} \geq \alpha$ if for any $\varepsilon > 0$ there is an infinite-dimensional closed subspace V such that $\|T\xi\| \geq (\alpha - \varepsilon)\|\xi\|$ for any $\xi \in V$.

Problem 8. Check this.

Lemma 2.1. *Let $g \in L^\infty(\mathbb{T})$. Then $T_{\overline{g}} = T_g^*$ and $\|T_g\|_{ess} = \|T_g\| = \|g\|_\infty$.*

Proof. The first statement is a simple exercise, so let's pass to the second one. It's obvious that $\|T_g\|_{ess} \leq \|T_g\| \leq \|g\|_\infty$. Let's take $\varepsilon > 0$, then, due to the density of polynomials in $L^2(\mathbb{T})$, there is a polynomial $p(z) = \sum_{k=-N}^{N} a_k z^k$ such that $\|p\|_2 = 1$ and $\|gp\|_2 > \|g\|_\infty - \varepsilon$. For any $n > N$, the polynomial $z^n p(z)$ lies in H^2, and $\|z^n p\|_2 = 1$. In addition, for $n = 3N, 6N, 9N, \dots$ the polynomials $z^n p$ are mutually orthogonal.

Let $gp = \sum_{-\infty}^{\infty} b_k e_k$. Then

$$T_g(z^n p) = P_{H^2}(z^n gp) = \sum_{-n}^{\infty} b_k e_{k+n},$$

That's why

$$\lim_{n\to\infty} \|T_g(z^n p)\|_2 = \|gp\|_2 > \|g\|_\infty - \varepsilon,$$

i.e. $\|T_g \xi_n\| \geq \|g\|_\infty - \varepsilon$ for an infinite set of mutually orthogonal vectors $\xi_n = z^n p \in H^2$, hence $\|T_g\|_{ess} \geq \|g\|_\infty$. Comparing this with the previously obtained inequality $\|T_g\|_{ess} \leq \|g\|_\infty$, we obtain the statement of the lemma. □

Set $H^\infty = H^2 \cap L^\infty(\mathbb{T})$.

Lemma 2.2. *If $h \in H^\infty$, then H^2 is an invariant subspace for the operator M_h. For any functions $h \in H^\infty$ and $g \in L^\infty(\mathbb{T})$ the following equalities hold: $T_g T_h = T_{gh}$ and $T_{\overline{h}} T_g = T_{\overline{h}g}$.*

Proof. Let $h = \sum_{n\geq 0} h_n z^n$. Then $M_h z^k = \sum_{n\geq 0} h_n z^{n+k} \in H^2$ for any $k \geq 0$, so $M_h(H^2) \subset H^2$. Therefore $T_h f = hf$ for any function $f \in H^2$, and, therefore, $T_g T_h f = T_g hf = P_{H^2} ghf = T_{gh} f$. □

Lemma 2.3. *The commutator $T_zT_g - T_gT_z$ is a compact operator of rank at most 1 for any function $g \in L^\infty(\mathbb{T})$.*

Proof. Since $z \in H^\infty$, has place equality $T_gT_z = T_{gz}$. Let's look at the operator $T_zT_g - T_{gz}$: its limit on H^2 is

$$P_{H^2}M_zP_{H^2}M_gP_{H^2} - P_{H^2}M_zM_gP_{H_2} = -P_{H^2}M_zP_{H^2}^\perp M_gP_{H^2}.$$

But the operator $P_{H^2}M_zP_{H^2}^\perp$ is of rank 1 (it is equal to $e_0(e_{-1}, \cdot)$). □

Corollary 2.4. *For any functions $g \in L^\infty$ and $f \in C(\mathbb{T})$, the operators $T_fT_g - T_{fg}$ and $T_gT_f - T_{gf}$ are compact.*

Proof. Take $\varepsilon > 0$ and a polynomial $p = \sum_{k=-N}^N a_kz^k$ such that $\|fp\|_\infty < \varepsilon$. It is clear that $T_gT_f - T_{gf} = P_{H^2}M_gP_{H^2}^\perp M_fP_{H^2}$, therefore

$$\|T_fT_g - T_{fg} - T_pT_g + T_{pg}\| = \|P_{H^2}M_gP_{H^2}^\perp(M_f - M_p)P_{H^2}\| \leq \|M_f - M_p\| = \|fp\|_\infty < \varepsilon.$$

It is enough to establish compactness of $P_{H^2}^\perp M_pP_{H^2}$, which follows from the fact that the range of $P_{H^2}^\perp M_pP_{H^2}$ has dimension $\leq N$.

The statement about the second operator is proved in a similar way (it is easier to prove the compactness of its conjugate operator). □

Definition 2.5. The unital C*-algebra $C^*(1, T_z)$ generated by the operator T_z is called *Toeplitz algebra*. Let $\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathbb{K}(H^2)\}$.

Theorem 2.6. *(1) $\mathcal{T} = C^*(1, T_z)$;*
(2) This algebra is irreducible and contains $\mathbb{K}(H^2)$ as the only minimal ideal;
(3) $\mathcal{T}/\mathbb{K}(H^2) \cong C(\mathbb{T})$, and the map $s : C(\mathbb{T}) \to \mathcal{T}$, $s : f \mapsto T_f$, is a continuous lifting (the right inverse) for the quotient map $\mathcal{T} \to \mathcal{T}/\mathbb{K}(H^2)$.

Proof. Let $p \in \mathcal{T}'$ be a projection from commutant of the Toeplitz algebra. It commutes with T_z and with T_z^*, and therefore with $E_0 = 1 - T_zT_z^*$. Then pE_0 is a projection of rank ≤ 1, so it is equal to either E_0 or 0. Since $pE_n = pT_z^nE_0$, we get that $pE_n = E_n$ if $pE_0 = E_0$, and $pE_n = 0$ if $pE_0 = 0$. That is, p is equal to either 1 or 0, which implies the irreducibility of $\mathcal{T}$.

Since $C^*(1, T_z)$ contains a nonzero compact operator, it contains the entire algebra of compact operators $\mathbb{K}(H^2)$ (follows from irreducibility by Corollary **??**). From the density of polynomials in $C(\mathbb{T})$ it follows that $C^*(1, T_z)$ contains all T_f, $f \in C(\mathbb{T})$. Thus $\mathcal{T} \subset C^*(1, T_z)$ is a dense 8-subalgebra (as $T_f^* = T_{\bar{f}}$, $f \in C(\mathbb{T})$).

Let $J \subset \mathcal{T}$ be a nonzero ideal. By the second statement of Lemma **??**, J is irreducible. If $a \in J$, $a \neq 0$, then there exists a compact operator $k \in \mathbb{K}(H^2)$ such that $ak \neq 0$. In this case $ak \in J$. By Corollary **??**, $J \supset \mathbb{K}(H^2)$, i.e. the ideal of compact operators is the smallest.

Let's check that $\mathcal{T}$ is closed. Take a Cauchy sequence $\{T_{f_n} + K_n\}$, $f_n \in C(\mathbb{T})$, $K_n \in \mathbb{K}(H^2)$, $n \in \mathbb{N}$. Then

$$\|f_n - f_m\|_\infty = \|T_{f_n} - T_{f_m}\|_{ess} \leq \|T_{f_n} + K_n - (T_{f_m} + K_m)\|,$$

therefore the sequence $\{f_n\}$ is Cauchy. But then the sequence $\{K_n\}$ is also Cauchy. Since both $C(\mathbb{T})$ and $\mathbb{K}(H^2)$ are norm closed, $\mathcal{T}$ is also closed. And since $\mathcal{T}$ contains T_z and is closed, then it coincides with $C^*(1, T_z)$.

Denote by $\pi : \mathcal{T} \to \mathcal{T}/\mathbb{K}(H^2)$ the quotient $*$-homomorphism. The quotient algebra is obviously commutative. It follows from Lemma 2.1 that the composition $\pi \circ s : C(\mathbb{T}) \to \mathcal{T}/\mathbb{K}(H^2)$ is an isometry. It is also a $*$-homomorphism, so it is a $*$-isomorphism. $\square$

Definition 2.7. An operator $T \in \mathbb{B}(H)$ is called *Fredholm* if its equivalence class $\dot{T}$ in the Calkin algebra is invertible.

Corollary 2.8. *The operator T_f is Fredholm if and only if the function f nowhere equals 0.*

Lecture 3

Lemma 2.9. *Let $T \in \mathbb{B}(H)$ be Fredholm. Then there exists $\varepsilon > 0$ such that if $\|T - S\| < \varepsilon$, then S is also Fredholm, and, for any $K \in \mathbb{K}(H)$, the operator $T + K$ is also Fredholm.*

Proof. The second statement is obvious. To prove the first one, note that $\|\dot{T} - \dot{S}\| = \|T - S\|_{ess} \leq \|T - S\| < \varepsilon$, and ε must be chosen so that so that $\dot{S}$ would be invertible (the set of invertible elements is open). □

Lemma 2.10. *The operator $T \in \mathbb{B}(H)$ is Fredholm if and only if there exists an operator $S \in \mathbb{B}(H)$ such that $TS - 1$ and $ST - 1$ are compact.*

Proof. It is almost obvious: being Fredholm means that in the Calkin algebra there exists an inverse element $\dot{S}$, i.e. the equalities $\dot{T}\dot{S} = 1$ and $\dot{S}\dot{T} = 1$ are satisfied. □

Theorem 2.11 (Atkinson)**.** *The following conditions are equivalent for an operator T:*

(1) T is Fredholm;
(2) The range $\operatorname{Im} T$ of T is closed, and the subspaces $\operatorname{Ker} T$ and $\operatorname{Coker} T = (\operatorname{Im} T)^{\perp}$ are finitedimensional.

Proof. If T is Fredholm, then there is an operator S and a compact operator K such that the equality $ST = 1 + K$ holds. If $\xi \in \operatorname{Ker} T$, then $ST\xi = \xi + K\xi = 0$, i.e. ξ lies in the eigenspace of the compact operator, which must be finitedimensional. Thus $\operatorname{Ker} T$ is finitedimensional. Similarly, $\operatorname{Ker} T^*$ is finitedimensional, as T^* is Fredholm as well.

To prove that $\operatorname{Im} T$ is closed, we approximate K by an operator F of finite rank: let $\|F - K\| < r$. If $\xi \in \operatorname{Ker} F$, then

$$\begin{aligned} \|S\| \cdot \|T\xi\| &\geq \|ST\xi\| = \|\xi + K\xi\| = \|(\xi + F\xi) + (K - F)\xi\| = \|\xi + (K - F)\xi\| \\ &\geq \|\xi\| - \|K - F\|\|\xi\| \geq (1 - r)\|\xi\|. \end{aligned}$$

If $r < 1$, then the operator T is bounded below on $\operatorname{Ker} F$, which implies that $T(\operatorname{Ker} F)$ is closed. Since $(\operatorname{Ker} F)^{\perp} = \operatorname{Im} F^*$ is finitedimensional, then $T((\operatorname{Ker} F)^{\perp})$ is also finitedimensional and therefore closed. Therefore $\operatorname{Im} T = T(H) = T(\operatorname{Ker} F \oplus (\operatorname{Ker} F)^{\perp}) = T(\operatorname{Ker} F) + T((\operatorname{Ker} F)^{\perp})$ is closed.

Problem 9. Let E be a Banach space, $N \subset E$ a closed subspace, $L \subset E$ a finitedimensional subspace. Prove that $N + L$ is closed. Check that if L is only closed insead of finitedimensional then $N + L$ need not be closed.

Conversely, decompose H as the direct sum $H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^{\perp}$. Consider the restriction of T to $(\operatorname{Ker} T)^{\perp}$. This restriction is a bijection onto its range, i.e. on $\operatorname{Im} T$, therefore, by the open mapping theorem, it is invertible. Define an operator S on $\operatorname{Im} T$ as the inverse to $T|_{(\operatorname{Ker} T)^{\perp}}$ and as zero on $(\operatorname{Im} T)^{\perp}$. Then $1 - TS$ is the projection onto $(\operatorname{Im} T)^{\perp}$, and $1 - ST$ is the projection onto $\operatorname{Ker} T$, i.e. they are operators of finite rank; therefore, $\dot{T}$ is invertible in the Calkin algebra. □

Definition 2.12. The difference $\dim \operatorname{Ker} T - \dim \operatorname{Coker} T = \operatorname{ind} T$ is called the *index* of the Fredholm operator T.

Problem 10. Check that the index of the identity operator iz zero. Check that $\operatorname{ind}(T \oplus S) = \operatorname{ind} T + \operatorname{ind} S$ when both T, S are Fredholm.

The index does not change under small deformations, and therefore under homotopies:

Theorem 2.13. *Let T be a Fredholm operator. Then there exists $\varepsilon > 0$ for which the condition $\|TS\| < \varepsilon$ implies that* $\operatorname{ind} T = \operatorname{ind} S$.

Proof. Consider two direct sum decompositions of H: $H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^{\perp}$ and $H = (\operatorname{Im} T)^{\perp} \oplus \operatorname{Im} T$. The first terms in both are finite-dimensional. Write the operator T as a two-dimensional matrix with respect to these two decompositions of H: $T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}$ (here the zeros saty for the maps from $\operatorname{Ker} T$ to $(\operatorname{Im} T)^{\perp}$, from $\operatorname{Ker} T$ to $\operatorname{Im} T$ and from $(\operatorname{Ker} T)^{\perp}$ to $(\operatorname{Im} T)^{\perp}$, respectively, and the operator $T_1 : (\operatorname{Ker} T)^{\perp} \to \operatorname{Im} T$ is invertible).

Write the operator S in the same form (i.e. with respect to the same two decompositions of H into direct sums): $S = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & T_1 + a_{22} \end{pmatrix}$, where all a_{ij} are small. Let us take ε such that the operator $T_1 + a_{22}$ is invertible (recall that T_1 is invertible). Then we set $X = -(T_1 + a_{22})^{-1} a_{21}$ and note that $S \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & T_1 + a_{22} \end{pmatrix}$ (it doesn't matter to us what exactly b_{ij} is equal to, it is important that there is zero at the bottom left, and that $b_{22} = a_{22}$). Similarly, taking $Y = -b_{12}(T_1 + a_{22})^{-1}$, we get

$$\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 \\ 0 & T_1 + a_{22} \end{pmatrix}. \tag{1}$$

The left side of (1) contains the product of three operators, two of which are invertible, so the kernel and cokernel of this product have the same dimensions as the kernel and cokernel of the operator S. This means that $\operatorname{ind} S$ is equal to the index of the right side, which is equal to the sum of the indices of the diagonal elements, i.e. $\operatorname{ind} S = \operatorname{ind} c_{11} + \operatorname{ind}(T_1 + a_{22})$. It follows from invertibility of $T_1 + a_{22}$ that $\operatorname{ind}(T_1 + a_{22}) = 0$, i.e. $\operatorname{ind} S = \operatorname{ind} c_{11}$, where $c_{11} : \operatorname{Ker} T \to (\operatorname{Im} T)^{\perp}$ is an operator mapping one finitedimensional space into another (of different dimensions). So, $\dim \operatorname{Ker} c_{11} = \dim \operatorname{Ker} T - \operatorname{rank} c_{11}$, $\dim \operatorname{Im} c_{11} = \dim \operatorname{Im} T - \operatorname{rank} c_{11}$, hence $\operatorname{ind} c_{11} = \operatorname{ind} T$. □

Problem 11. Prove that if $T : [0, 1] \to \mathbb{B}(H)$ is a continuous map of an interval into the set of Fredholm operators, then $\operatorname{ind} T(0) = \operatorname{ind} T(1)$.

Theorem 2.14. *Let S, T be Fredholm operators. Then* $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$.

Proof. Let's consider continuous family of F_t, $t \in [0, \pi/2]$, operators in $H \oplus H$, $F_t = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \cdot 1 & -\sin t \cdot 1 \\ \sin t \cdot 1 & \cos t \cdot 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} \cos t \cdot 1 & \sin t \cdot 1 \\ -\sin t \cdot 1 & \cos t \cdot 1 \end{pmatrix}$. For each t the operator F_t is Fredholm (since its equivalence class in the Calkin algebra is invertible). Since $F_0 = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, and $F_{\pi/2} = \begin{pmatrix} ST & 0 \\ 0 & 1 \end{pmatrix}$,

$$\operatorname{ind} S + \operatorname{ind} T = \operatorname{ind} F_0 = \operatorname{ind} F_{\pi/2} = \operatorname{ind}(ST) + \operatorname{ind} 1 = \operatorname{ind}(ST).$$

□

Let us consider in more detail the functions $f \in C(\mathbb{T})$ that do not take the value 0, i.e. those for which the operator T_f is Fredholm. They define loops on the complex plane with the punctured point 0, and are characterized, up to homotopy, by the so-called rotation number, or degree. If $f : \mathbb{T} \to \mathbb{C} \setminus \{0\}$, then the function g given by the formula $g(z) = \frac{f(z)}{|f(z)|}$ defines the mapping of the circle into itself, i.e. element of the fundamental group of the circle. This element is called the rotation number for f and is denoted by $\operatorname{wind} f$.

Theorem 2.15. *Let $f \in C(\mathbb{T})$ not vanish anywhere. Then* $\operatorname{ind}(T_f) = -\operatorname{wind} f$.

Proof. From the first item of Theorem 2.13 it follows that the index does not change under small perturbations, therefore, it is homotopy invariant (provided that the perturbation does not go beyond the set of Fredholm operators). Therefore, it is sufficient to check the statement of the theorem for the case $f(z) = z^n$, $n \in \mathbb{N}$. But $\operatorname{ind} T_{z^n} = -n$. $\square$