C^*-algebras and K-theory. Introduction

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April 15, 2025

Contents

Chapter 1

C^*-algebras

Lecture 1

1.1 Definition and first examples

Definition 1.1. An *algebra* A (over a field $\mathbb{K}$) is a ring that is a linear space over $\mathbb{K}$, and the addition in the definition of a ring and in the definition of a linear spaces are the same, and multiplications are connected by the relation $\lambda(ab) = (\lambda a)b$ for all $\lambda \in \mathbb{K}$, $a, b \in A$.

We will consider algebras only over the field of complex numbers $\mathbb{C}$.

Definition 1.2. An algebra A over $\mathbb{C}$ is called a *Banach algebra*, if the underlying linear space is a Banach space and $\|ab\| \leq \|a\|\|b\|$ for any $a, b \in A$.

Problem 1 (easy)**.** Show that in this case the multiplication is continuous (as a mapping $A \times A \to A$).

Definition 1.3. A mapping $* : A \to A$, $a \mapsto a^*$ is called an *involution* if

1. $(a^*)^* = a$;
2. $(a + b)^* = a^* + b^*$;
3. $(\lambda a)^* = \bar{\lambda} a^*$;
4. $(ab)^* = b^* a^*$;
5. $\|a^*\| = \|a\|$

for any $a, b \in A$, $\lambda \in \mathbb{C}$. A Banach algebra with involution is called *involutive Banach algebra.*

Definition 1.4. An involutive Banach algebra A is called a *C^*-algebra,* if the involution satisfies the equality $\|a^* a\| = \|a\|^2$ for all $a \in A$ (this equality is called the *C^*-property*).

Problem 2. Give an example of an involutive Banach algebra that is not a C^*-algebra.

Problem 3 (easy)**.** Show that property (5) of the definition 1.3 follows from properties (1 – 4) and the C^*-property.

Definition 1.5. An element $1 \in A$ is called a (left) *unit*, if $1a = a$ for any $a \in A$. A C^*-algebra A is called *unital* or *algebra with unit*, if it has a (left) unit (identity element).

Problem 4. Show that a left unit element is also a right one, which has $1^* = 1$; that the identity element is unique and that $\|1\| = 1$. It is called the *unit of algebra*.

Problem 5. Verify that the algebra $C(X)$ formed by all continuous complex-valued functions on a compact space X and the algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact space X tending to 0 at infinity (that is, $f : X \to \mathbb{C}$ such that for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $\sup\{|f(x)| \mid x \in K\} < \varepsilon$) are commutative C^*-algebras if the *supremum-norm*: $\|f\| = \sup_{x\in X} |f(x)|$, is taken as the norm and the pointwise multiplication is taken as the multiplication. Moreover, the algebra $C(X)$ is unital.

Problem 6. Verify that the algebra $\mathbb{B}(H)$ of all bounded operators acting on a Hilbert space H is a C^*-algebra with identity. Here as a norm we take the *operator norm* $\|a\| = \sup_{h\in H,\, \|h\|\le 1} \|a(h)\|$, and the multiplication is the composition of operators.

These examples of C^*-algebras are the most important, as we will see later.

1.2 Unitalization, or attaching of a unit

If an involutive Banach algebra A does not have a unit, then it can be embedded into an involutive unital Banach algebra A as follows. Let $A^+ = A \oplus \mathbb{C}$ be a linear space (the direct sum of linear spaces). Let's define a structure of an involutive Banach algebra on A^+ by formulas

$$\begin{aligned} (a,\lambda)(b,\mu) &= (ab+\lambda b+\mu a, \lambda\mu), \\ (a,\lambda)^* &= (a^*, \bar{\lambda}), \\ \|(a,\lambda)\| &= \|a\| + |\lambda| \end{aligned} \tag{1.1}$$

for any $(a,\lambda), (b,\mu) \in A \oplus \mathbb{C}$.

We further assume that the initial algebra A is a C^*-algebra.

Problem 7. Show that this norm turns A^+ into an involutive Banach algebra, but not into a C^*-algebra. (To check completeness, use the completeness of the subspace $A \subset A^+$ of finite codimension (one), this follows from Problem 337(b) in [**?**].)

Definition 1.6. A subset $B \subseteq A$ is called a *subalgebra* of a (Banach) algebra A if B is an algebra with respect to the operations (and norm) of A.

A subset of $B \subseteq A$ is called (C^*-)*subalgebra* C^*-algebras A, if B is a C^*-algebra with respect to the operations and the norm of A. In particular, B is closed in A.

Definition 1.7. A subalgebra $I \subseteq A$ is called a (two-sided) *ideal*, if $aI \subseteq I$ and $Ia \subseteq I$ for any $a \in A$.

Problem 8. Prove that A is an ideal in A^+.

Lemma 1.8. *There is a norm on A^+ such that*

1) *it is equivalent to the one defined above;*

2) *on A it coincides with the original norm:*

3) *it is a C^*-norm.*

Proof. For $b = (b', \lambda) \in A^+$, consider the linear mapping $L_b : A \to A$ according to the formula $L_b(a) = ba$, $a \in A$. For L_b, we can thus define the operator norm: $\|L_b\| := \sup_{a \in A, \|a\| \le 1} \|L_b(a)\|$. If $b \in A$, then $\|L_b\| \le \|b\|$. Since $\|L_b(b^*)\| = \|bb^*\| = \|b\|^2$, we have $\|L_b\| = \|b\|$. For every $b = (b', \lambda) \in A^+$ we set $\|b\|_{new} = \|L_b\|$. Note that the above reasoning shows that $\|.\|_{new}$ satisfies condition 2) from the formulation of the lemma.

First of all, let's check that $\| \cdot \|_{new}$ is a norm. Obviously, the axioms of linearity and triangle are satisfied (that is, this is a *semi-norm*, or *pre-norm*), so it remains to check the nondegeneracy. Let $\|b\|_{new} = 0$ for some $b = (b', \lambda) \in A^+$. It means that $(b', \lambda) \cdot (a, 0) = (b'a + \lambda a, 0) = (0, 0)$ for any $a \in A$, so $-\frac{1}{\lambda} b' \cdot a = a$ and $-\frac{1}{\lambda} b'$ is the unit of A. But A does not have a unit. This means that $\| \cdot \|_{new}$ is the norm. Like any operator norm, the new norm is a norm of a Banach algebra, that is,

$$\|bc\|_{new} \le \|b\|_{new} \|c\|_{new} \tag{1.2}$$

for any $b, c \in A^+$ (Easy problem: check this). We do not claim yet that the involution is an isometry.

This new norm is equivalent to the previously defined norm on A^+, since $A \subset A^+$ is a subspace of codimension 1. Indeed, by the triangle inequality, we have $\|(a, \lambda)\|_{new} \le \|a\| + |\lambda| = \|(a + \lambda)\|$, so it is sufficient to show that there is a constant $c > 0$ such that $\|(a, \lambda)\|_{new} \ge c\|(a, \lambda)\|$. Let's assume the opposite: there are such pairs (a_n, λ_n) and numbers $c_n > 0$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\|(a_n, \lambda_n)\|_{new} \le c_n \|(a_n, \lambda_n)\|.$$

Since none of λ_n is zero, we can assume (by dividing both sides by $\lambda_n \neq 0$ if necessary) that $\lambda_n = 1$. Applying the triangle inequality again, we get

$$\|a_n\| - 1 \le \|(a_n, 1)\|_{new} \le c_n(\|a_n\| + 1),$$

whence $\|a_n\| \le \frac{1+c_n}{1-c_n}$, and for sufficiently large n we have $\|a_n\| < 2$. But then for these n we have $\|(a_n, 1)\|_{new} \le 3c_n$, so

$$\lim_{n\to\infty} (a_n, 1) = 0. \tag{1.3}$$

Now recall that $A^+/A \cong \mathbb{C}$, and all norms on $\mathbb{C}$ are equivalent (moreover, they differ only by a constant multiple). Thus, the quotient norm on A^+/A given by $\|(a, \lambda) + A\|_{new} :=$

$\inf_{a\in A}\|(a,\lambda)\|_{new}$ is equivalent to the usual norm on $\mathbb{C}$. So $\inf_{a\in A}\|(a,1)\|_{new} = \alpha > 0$, what contradicts (1.3).

Now we need to check the C^* property. By definition, for any $b \in A^+$ and any $\varepsilon > 0$ there is an element $a \in A$ such that $\|a\| = 1$ and

$$\|L_b(a)\| \geq (1-\varepsilon)\|L_b\|, \qquad \text{that is} \qquad \|ba\|_{new} = \|ba\| \geq (1-\varepsilon)\|b\|_{new}.$$

We also have

$$\|b^*b\|_{new} \geq \|a^*(b^*b)a\|_{new} = \|a^*(b^*b)a\| = \|(ba)^*(ba)\| = \|ba\|^2 \geq (1-\varepsilon)^2\|b\|^2_{new}$$

(where the first inequality is satisfied by virtue of (1.2), the next equality is satisfied by virtue of that A is an ideal in A^+, and then we apply the C^*-property in A). Passing to the limit as ε tends to zero, we obtain $\|b^*b\|_{new} \geqslant \|b\|^2_{new}$. In particular, $\|b^*\|_{new} \cdot \|b\|_{new} \geqslant \|b^*b\|_{new} \geqslant \|b\|^2_{new}$, so $\|b^*\|_{new} \geqslant \|b\|_{new}$. Therefore $\|b\|_{new} = \|(b^*)^*\|_{new} \geqslant \|b^*\|_{new}$. Thus, the involution is an isometry and, by (1.2), $\|b^*b\|_{new} \leqslant \|b^*\|_{new}\|b\|_{new} = \|b\|^2_{new}$. This means that this is a C^*-norm. □

Definition 1.9. An element $a \in A$ is called *self-adjoint*, if $a^* = a$, *unitary*, if $a^*a = aa^* = 1$ (here the algebra A is assumed to be unital), *normal*, if $a^*a = aa^*$.

Definition 1.10. In a unital C^*-algebra, an element a is called *invertible*, if there is an element $a' \in A$ such that $aa' = a'a = 1$. There is only one a' with this property (check!), called the *inverse to a* and is denoted by a^{-1}. The set of invertible elements $G(A)$ is a group.

Problem 9. Check this.

Lemma 1.11. *If $\|1-a\| < 1$, then a^{-1} exists and is equal to $a' = \sum_{n=0}^{\infty}(1-a)^n$, and the series converges in norm.*

Proof. Convergence immediately follows from the domination by a geometric progression. Next we calculate: $a'a = aa' = -(1-a)a' + a' = -\sum_{n=1}^{\infty}(1-a)^n + \sum_{n=0}^{\infty}(1-a)^n = 1$. □

Problem 10. Prove in a similar way that if $a_0 \in A$ is invertible and $\|a-a_0\| < \frac{1}{\|a_0^{-1}\|}$, then a is also invertible, and $a^{-1} = \sum_{n=0}^{\infty}[a_0^{-1}(a_0-a)]^n a_0^{-1}$.

Corollary 1.12. *The subset $G(A) \subset A$ is an open set. Taking the inverse element $a \mapsto a^{-1}$ is a continuous map $G(A)$ into itself.*

Problem 11. Prove the corollary using the formulas established above.

Lecture 2

1.3 Spectrum and functional calculus

Definition 1.13. Let A be a Banach algebra with identity. For any $a \in A$ we call the set $\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : (a - \lambda 1) \text{ is not invertible}\}$ the *spectrum* of element a. The function $R_a(\lambda) = (a - \lambda 1)^{-1}$ is called the *resolvent* of a. If A is a non-unital C^*-algebra, then *quasi-spectrum* $\mathrm{Sp}'(a)$ of element $a \in A$ is assumed to be equal to the spectrum of a as an element of the algebra A^+.

Problem 12. Show that the quasi-spectrum always contains zero.

Theorem 1.14. *The spectrum of any element is a compact non-empty set. The resolvent is an analytic function outside the spectrum (that is, for any point of the completed complex plane from the complement of the spectrum, it can be represented by a power series with coefficients from algebra, uniformly convergent on some closed disk).*

Proof. If $|\lambda| > \|a\|$, then the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n$ converges in norm, and converges to $-(a - \lambda 1)^{-1}$, since $-(a - \lambda 1) \sum_{n=0}^{k} \lambda^{-(n+1)} a^n = 1 - \lambda^{-(k+2)} a^{k+1}$ converges to 1 . Thus, $R_a(\lambda)$ is analytic in the neighborhood of infinity, defined by $|\lambda| > \|a\|$, and

$$\lim_{\lambda \to \infty} \|R_a(\lambda)\| \leq \lim_{\lambda \to \infty} \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|a\|}{|\lambda|}} = 0.$$

In particular, $\mathrm{Sp}(a)$ is contained in a closed disk of radius $\|a\|$ and thus, is a bounded set.

If $a - \lambda_0 1$ is invertible (that is, $\lambda_0 \notin \mathrm{Sp}(a)$) and $|\lambda - \lambda_0| < \frac{1}{\|(a - \lambda_0 1)^{-1}\|}$ then, the same way,

$$(a - \lambda 1)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (a - \lambda_0 1)^{-n-1}$$

is the Taylor series R_a in a neighborhood of λ_0, so $R_a(\lambda)$ is analytic outside $\mathrm{Sp}(a)$. The same reasoning shows, in particular, that the complement to $\mathrm{Sp}(a)$ is open, that is, $\mathrm{Sp}(a)$ is closed. Since the spectrum is bounded, it is compact.

Since $R_a(\lambda)$ is analytic, then the complex-valued function $\lambda \mapsto \varphi(R_a(\lambda))$ is also analytic for φ being an arbitrary bounded linear functional on A. If $\mathrm{Sp}(a)$ is empty, then $\varphi(R_a(\lambda))$ is an analytic function on the entire $\mathbb{C}$. However, it is bounded because $|\varphi(R_a(\lambda))| \leqslant \|\varphi\| \cdot \|R_a(\lambda)\|$ and $\|R_a(\lambda)\|$ is bounded: we estimate separately on a compact disk of radius $\lambda_0 > \|a\|$ and outside it using the estimation obtained above

$$\|R_a(\lambda)\| \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{|\lambda_0| - \|a\|}.$$

This means $\varphi(R_a(\lambda)) = 0$ for any φ, so $R_a(\lambda) = 0$ (see problem 13 below) for all $\lambda \in \mathbb{C}$. This is impossible (it turns out that $\lambda_1 1 - \lambda_2 1 = 0$ for $\lambda_1 \neq \lambda_2$). $\square$

Problem 13. For an element $x \neq 0$ of a normed space X, there is a continuous linear functional φ that does not vanish on x. (This is a theorem from the standard course, a corollary of the Hahn-Banach theorem, see, for example, Corollary 2 on page 189 in [**?**]).

Problem 14. Let a and b be commuting elements of a Banach algebra. Then the product ab is invertible if and only if each of the elements a and b are invertible.

Theorem 1.15 (Spectral mapping theorem)**.** *If $p(z)$ is a polynomial, then* $\mathrm{Sp}(p(a)) = p(\mathrm{Sp}(a))$.

Proof. For $\alpha \in \mathbb{C}$ we decompose $p(z)-\alpha$ into linear factors $p(z)-\alpha = c\prod_i(z-\beta_i)$. Then $p(a)-\alpha 1 = c\prod_i(a-\beta_i 1)$ and $p(a)-\alpha 1$ are invertible if and only if all $a-\beta_i 1$ are (applying inductively Problem 14). Then $\alpha \in \mathrm{Sp}(p(a))$ if and only if at least one of β_i belongs to $\mathrm{Sp}(a)$. On the other hand, substituting this β_{i_0} into the above representation of p, we obtain $p(\beta_{i_0}) - \alpha = c\prod_i(\beta_{i_0} - \beta_i) = 0$, that is, $\alpha = p(\beta_{i_0})$. So $\mathrm{Sp}(p(a)) \subseteq p(\mathrm{Sp}(a))$. The inverse statement is similar. □

Lemma 1.16. *Let A be a C^*-algebra (unital). Then* $\mathrm{Sp}(a^*) = \overline{\mathrm{Sp}(a)}$. *If a is unitary, then* $\mathrm{Sp}(a) \subset \mathbb{S}^1$, *where* $\mathbb{S}^1 \subset \mathbb{C}$ *is the unit sphere.*

Proof. Since $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$, and similarly, in a different order, then the element is invertible if and only if we invert its conjugate. This gives the first statement.

Now let a be unitary, in particular, invertible. Then $\|a\|^2 = \|a^*a\| = 1$, so for $|\lambda| > 1$ we have $\lambda \notin \mathrm{Sp}(a)$. If $|\lambda| < 1$, then $1 - \lambda a^*$ is invertible, which means that $a(1-\lambda a^*) = a - \lambda 1$ is also invertible. Thus, $\mathrm{Sp}(a) \subset \mathbb{S}^1$. □

Definition 1.17. The number $r(a) = \sup\{|\lambda| \ : \ \lambda \in \mathrm{Sp}(a)\}$ is called the *spectral radius* of a.

Problem 15. Show that $r(a) \leq \|a\|$ for any $a \in A$.

Lemma 1.18. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Proof. Let us expand the resolvent R_a in a neighborhood of infinity: $R_a(\lambda) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$. This function is analytic for $|\lambda| > r(a)$, so for any $\rho > r(a)$, terms tends to zero: $\lim_{n\to\infty} \|\frac{a^n}{\rho^{n+1}}\| = 0$. That's why $\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq \rho$ for any $\rho > r(a)$, so

$$\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq r(a). \tag{1.4}$$

On the other hand, due to compactness of the spectrum, there is $\alpha \in \mathrm{Sp}(a)$ such that $|\alpha| = r(a)$. According to the spectral mapping theorem, $\alpha^n \in \mathrm{Sp}(a^n)$, therefore $|\alpha^n| \leq r(a^n)$. By Problem 15, $r(a^n) \leq \|a^n\|$. Combining, we get

$$r(a) = |\alpha| = (|\alpha^n|)^{1/n} \leq (r(a^n))^{1/n} \leq \|a^n\|^{1/n}$$

for all n, so

$$r(a) \leq \inf_n \|a^n\|^{1/n}. \tag{1.5}$$

Comparing (1.4) and (1.5), we conclude that $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and is equal to $r(a)$. □

1.4 Multiplicative functionals and maximal ideals of commutative Banach algebras

Recall that an algebra A is called *simple*, if it doesn't have proper (i.e., distinct from $\{0\}$ and A) ideals.

Lemma 1.19. *The algebra of complex numbers $\mathbb{C}$ is the only simple commutative algebra.*

Proof. If $A \neq \mathbb{C}$, then there is a non-scalar element $a \in A$ (that is, $a \neq \lambda 1$ for any λ). Let's take some $\alpha \in \mathrm{Sp}(a)$ and set $I = \overline{(a - \alpha 1)A}$, so that $I \neq \{0\}$ is closed (two-sided) ideal in A. For any $b \in A$, the element $(a - \alpha 1)b$ is not invertible (see Problem 14), so by Lemma 1.11 $\|(a - \alpha 1)b - 1\| \geq 1$. Therefore $1 \notin I$ and we arrive to a contradiction with the simplicity of A. □

Definition 1.20. A *multiplicative functional* on A is a nontrivial homomorphism $\varphi : A \to \mathbb{C}$. The set of all multiplicative functionals is denoted by M_A.

Problem 16. Prove that $\varphi(1) = 1$. You can use a simple general observation: image *idempotent* ($p^2 = p$) under homomorphism it is always idempotent.

Lemma 1.21. *A multiplicative functional on a commutative unital Banach algebra A has norm 1. The mapping that associates to each multiplicative functional its kernel is a bijection onto the set of* maximal ideals *of A, that is, such proper ideals that are not contained in any other ideal except the entire algebra A*

Proof. Since, according to Problem 16, $\varphi(1) = 1$, we have $\|\varphi\| \geq 1$. Let $\|\varphi\| > 1$, so there is an element $a \in A$ such that $\|a\| < 1$ and $\varphi(a) = 1$. Then the series $b = \sum_{n=1}^{\infty} a^n$ converges and $a + ab = b$. This means that $\varphi(b) = \varphi(a)(1 + \varphi(b)) = 1 + \varphi(b)$. A contradiction. We obtained $\|\varphi\| = 1$.

Since $\operatorname{Ker} \varphi$ has codimension 1, it is a maximal ideal.

Any functional φ is completely determined by its kernel and the condition $\varphi(1) = 1$ (see Problem 17 below), so the indicated correspondence is a bijection onto the image.

Let us show that it is an epimorphism. If $M \subset A$ is a maximal ideal, then $\mathrm{dist}(M, 1) = 1$, since the unit open ball with center 1 consists of invertible elements (Lemma 1.11), and M cannot contain invertible elements (if $a \in M$ is invertible, then $1 \in M$, and hence $M = A$). Then the closed ideal $\overline{M}$ (the closure of M) also does not contain 1, so by maximality $\overline{M} = M$. Consider the factor algebra A/M being simple (since otherwise M would not be maximal) commutative unital Banach algebra. Therefore, according to Lemma 1.19, $A/M \cong \mathbb{C}$. The corresponding factorization map is a non-zero homomorphism, that is, a multiplicative functional (with kernel M). □

Problem 17. Prove that any functional φ is completely determined by its kernel and condition $\varphi(1) = 1$ (factor by kernel and consider the induced functional on $\mathbb{C}$).

Since multiplicative functionals are bounded by 1, M_A is a subset of the unit ball in the dual Banach space A' (the space of all bounded linear functionals on A). The space A' can be equipped with the $*$-*weak topology*, determined by the prebase of neighborhoods of the form $U_{\varphi_0,\varepsilon,a} = \{\varphi \in A' : |(\varphi - \varphi_0)(a)| < \varepsilon\}$, $\varphi_0 \in A'$, $a \in A$, $\varepsilon > 0$. In terms of directed nets: a directed net φ_α converges to φ if $\varphi_\alpha(a)$ converges to $\varphi(a)$ for each $a \in A$. The unit ball of space A' is compact and Hausdorff with respect to the $*$-weak topology (Banach-Alaoglu theorem, see [**?**, Theorem 5, p. 325]).

Problem 18. Verify that M_A is $*$-weakly closed.

Therefore M_A is also $*$-weakly compact.

Problem 19. The following example describes a typical situation, as it will become clear a little further. Let $A = C[0,1]$, so the multiplicative functionals correspond to points of $[0,1]$. Namely, $\varphi(g) = g(t)$. Show that for the "ordinary" norm of the dual space for any two points $t \neq s$, the distance between φ_t and φ_s is 1. That is the set of multiplicative functionals is discrete, non-compact. There is no limit points, unlike the situation with the $*$-weak topology.

If A does not have unit, consider the Banach algebra $A^+ = A \oplus \mathbb{C}$ (with the original norm (1.2), not the C^*-norm from Lemma 1.8). In it, A is itself a maximal ideal corresponding to the multiplicative functional $\varphi_0((a,\lambda)) = \lambda$. Any other maximal ideal I of the algebra A^+ must have a proper intersection with A. Then $I \cap A$ is an ideal of codimension 1, since I had codimension 1 in A^+. Thus, $I \cap A$ is a maximal ideal in A. Since its codimension is 1, then the quotient mapping is defined being a nonzero homomorphism to $\mathbb{C}$. Conversely, if $\varphi : A \to \mathbb{C}$ is a (nonzero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+ (Problem 21). We obtain a bijective correspondence between M_A and $M_{A^+} \setminus \{\varphi_0\}$.

Problem 20. Prove that M_A is a locally compact Hausdorff space, and M_{A^+} is its one-point compactification.

Problem 21. Check that if $\varphi : A \to \mathbb{C}$ is a (non-zero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+.

Lecture 3

1.5 Gelfand transform

Definition 1.22. For a commutative Banach algebra A we define *Gelfand transform* $\Gamma : A \to C_0(M_A)$ by the relation $\Gamma(a) = \hat{a}$, where $\hat{a}(\varphi) := \varphi(a)$ (the necessary properties will be verified in the next lemma).

Lemma 1.23. *The Gelfand transform is a (non-strict) contraction homomorphism of algebras and the image A separates the points M_A, that is, for any two points M_A there is a function from the image with distinct values at these points.*

Proof. Functions $\hat{a}$ are continuous by the definition of the $*$-weak topology. The mapping is contractive because

$$\|\Gamma(a)\| = \sup_{\varphi \in M_A} |\varphi(a)| \leq \sup_{\varphi \in M_A} \|\varphi\| \cdot \|a\| = \sup_{\varphi \in M_A} \|a\| = \|a\|.$$

The separation of points is obvious since two (multiplicative) functionals are distinct if and only if their values on some element a are distinct. If A is not unital, then we note that $\hat{a}(\tilde{0}) = 0(a) = 0$, where $M_{A^+} = M_A \cup \tilde{0}$. Therefore $\hat{a} \in C_0(M_A)$. $\square$

Problem 22. If an algebra is unital, then $\Gamma(1) = 1$.

Corollary 1.24. *Let A be a commutative unital Banach algebra. Then $a \in A$ is invertible if and only if $\hat{a}$ is invertible, and if and only if $\hat{a}(\varphi) \neq 0$ for any $\varphi \in M_A$. Therefore* $\mathrm{Sp}(a) = \mathrm{Sp}(\hat{a}) = \{\varphi(a) : \varphi \in M_A\}$ *and* $\|\hat{a}\| = r(a)$.

Proof. If a is invertible, then $\hat{a}$ is invertible by Problem 22. If a is not invertible, then consider the ideal $I = \overline{aA}$. As discussed above (see the proof of Lemma 1.19), this ideal cannot contain 1, so it is proper. Let I_M be the maximal ideal containing I (see problem 23 below), and φ be the corresponding I_M multiplicative functional. Then $\varphi(a) = 0$ and $\hat{a}$ is not invertible in $C(M_A)$.

The remaining statements are immediately obtained from what has been proven. $\square$

Problem 23. Any ideal I of a commutative unital Banach algebra is contained in some maximum ideal. Hint: consider the union of J of all proper ideals I_α containing I partially ordered by inclusion. For each chain (a completely ordered subsystem) I_{α_τ} using, as above, by the fact that 1 does not belong to every I_{α_τ}, make sure that it does not belong to $\cup_\tau I_{\alpha_\tau}$, so this is its own ideal. Then apply Zorn's lemma.

Theorem 1.25. *Let A be a commutative C*-algebra. Then the Gelfand transformation is an isometric $*$-isomorphism of A onto $C_0(M_A)$.*

Proof. Let us prove the theorem for a unital algebra. The necessary adaptation for the case without a unit is left to the reader as Problem 24.

Let $\varphi \in M_A$. Let us first consider the self-adjoint element $a^* = a \in A$. Let us set $u_t = \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}$, $t \in \mathbb{R}$. It is easy to check by considering the partial sum and passing to the limit that $u_t^* = u_t^{-1}$, so $u_t \in A$ is unitary. Then

$$1 \geq |\varphi(u_t)| = |\sum_{n=0}^{\infty} \frac{(it\varphi(a))^n}{n!}| = |e^{it\varphi(a)}| = e^{-t \operatorname{Im} \varphi(a)}.$$

Due to the arbitrariness of $t \in \mathbb{R}$ in this estimate, we conclude that $\operatorname{Im} \varphi(a) = 0$, that is, $\varphi(a) \in \mathbb{R}$.

We write an arbitrary element $c \in A$ in the form $c = a + ib$, where $a = (c + c^*)/2$ and $b = (c - c^*)/2i$ are self-adjoint. According to what has been proven, $\varphi(a), \varphi(b) \in \mathbb{R}$, that means $\varphi(c^*) = \varphi(a) - i\varphi(b) = \overline{\varphi(c)}$, so the Gelfand transformation preserves involution and is thus a $*$-homomorphism.

For a self-adjoint element ($a = a^*$) we have $\|a\|^2 = \|a^*a\| = \|a^2\|$, therefore

$$\|\hat{a}\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty} (\|a\|^{2^n})^{1/2^n} = \|a\|.$$

For an element $b \in A$ of general form we have $\|b\|^2 = \|b^*b\| = \|\widehat{b^*b}\| = \|\hat{b}^*\hat{b}\| = \|\hat{b}\|^2$, so the Gelfand transformation is an isometry (onto its image).

Therefore $\Gamma(A)$ is norm closed and involutive subalgebra with identity in $C(M_A)$ separating the points. By theorem Stone-Weierstrass[1], $\Gamma(A) = C(M_A)$. $\square$

Problem 24. Prove the theorem for a non-unital algebra.

So in particular there is an inverse mapping for the transformation Gelfand, which is also an isometric $*$-isomorphism.

Let $a \in A$ be a normal element. Let us denote by $C^*(1, a)$ (respectively, $C^*(a)$) C^*-algebra generated by 1 and a (resp., only a). Due to normality, these algebras are commutative, with the first definition supposing that A is unital.

Let us clarify that by a *C^*-algebra generated by a set* we call the minimal C^*-subalgebra A containing the set, that is, the intersection all C^*-subalgebras of A containing this set.

Problem 25. Verify that the C^*-algebra generated by a set is indeed a C^*-algebra.

Problem 26. If a is an invertible element, then the algebra $C^*(a)$ is unital. In this case $C^*(a) = C^*(1, a)$.

Corollary 1.26. *If $a^* = a$, then* $\operatorname{Sp}(a) \subset \mathbb{R}$.

Proof. As we know, $\operatorname{Sp}(a) = \operatorname{Sp}(\widehat{a})$ and $\widehat{a}^* = \widehat{a}$. But a self-adjoint function is exactly a function with real values, while the spectrum of a function is the set of all its values. $\square$

Corollary 1.27. *The algebra $C^*(1, a)$ is isometrically $*$-isomorphic to the algebra $C(\operatorname{Sp}(a))$ under a mapping that takes a to the function $z(t) = t$, $z : \operatorname{Sp}(a) \subset \mathbb{C} \to \mathbb{C}$. The algebra $C^*(a)$ is mapped onto $C_0(\operatorname{Sp}(a) \setminus \{0\})$.*

[1] The theorem is not always included in the standard course on functional analysis, so we present its proof in Section 1.6.

Proof. For a commutative C^*-algebra $C^*(1,a)$ we find $X = M_{C^*(1,a)}$. Any multiplicative functional $\varphi \in X$ is determined by its value $\varphi(a) = \lambda$ on a. Moreover, due to multiplicativity $\varphi(p(a,a^*)) = p(\lambda,\bar{\lambda})$ for any polynomial p. Thus, X is identified with the set of all possible values λ that $\varphi(a) = \hat{a}(\varphi)$ takes for $\varphi \in X$. According to Corollary 1.24, we have $\hat{a}(X) = \mathrm{Sp}(a)$. We obtain the identification $\mathrm{Sp}(a) \cong X$ using the correspondence $\mathrm{Sp}(a) \ni \lambda \mapsto \varphi_\lambda \in X$, where φ_λ is determined by the condition $\varphi_\lambda(a) = \lambda$.

This identification carries over to functions: every continuous function on X is identified with a continuous function on $\mathrm{Sp}(a)$, namely, the function $\hat{b} = \hat{b}(\varphi)$ is associated with the function argument $\lambda \in \mathrm{Sp}(a)$, specified as $\lambda \mapsto \varphi_\lambda(b)$. For example, if we take the polynomial $p(a,a^*) = b$, then the corresponding function will be $\lambda \mapsto \varphi_\lambda(p(a,a^*)) = p(\lambda,\bar{\lambda})$. In particular, the function $\hat{a}$ is mapped to $\lambda \mapsto \varphi_\lambda(a) = \lambda$, so the Gelfand transform identifies $\hat{a}$ with the identity mapping of $X \subset \mathbb{C}$. By Theorem 1.25, this mapping is an isometric $*$-isomorphism.

If a is invertible, then by Problem 26, $C^*(a)$ isometrically $*$-isomorphic to $C(\mathrm{Sp}(a))$. If a is not is invertible, then $C^*(a)$ does not have unity (see Problem 27). It corresponds under the constructed mapping for $C^*(1,a) \cong C^*(a)^+$ to the ideal $C(\mathrm{Sp}'(a))$ consisting of functions, tending to 0. □

Problem 27. Prove that if a is not invertible, then $C^*(a)$ does not have a unit. Hint: if there is a unit, then it has to be approximated by a polynomial in a and a^*, which cannot be an invertible element.

Corollary 1.28 (continuous functional calculus)**.** *Let a be a normal element of a unital C^*-algebra A, and f is a continuous function on* $\mathrm{Sp}(a)$*. Then the element $f(a) \in A$ is defined as the inverse image of f under the Gelfand transformation: $\Gamma = \Gamma_a : C^*(1,a) \to C(\mathrm{Sp}(a))$, $f(a) := (\Gamma_a)^{-1}(f)$. If $0 \in \mathrm{Sp}(a)$ and $f(0) = 0$, then $f(a) \in C^*(a)$. Moreover, $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ and if g is a continuous function on $f(\mathrm{Sp}(a))$, then $g(f(a)) = (g \circ f)(a)$.*

Proof. Everything has already been proven except the last statement. Let's first consider the polynomial $p(\lambda,\bar{\lambda})$ as $f = f(\lambda)$. Then $\Gamma(p(a,a^*))$ is a function $\lambda \mapsto p(\lambda,\bar{\lambda})$, so $\mathrm{Sp}(p(a,a^*))$ coincides with the set of values of this function, $\{\mu : \mu = p(\lambda,\bar{\lambda}), \lambda \in \mathrm{Sp}(a)\}$. Approximating f by polynomials, we get $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ (Problem 28).

Similarly, consider the polynomial $q(\lambda,\bar{\lambda})$ as g. It's easy to see that

$$q(f(a)) = \lim_{\alpha}(q(p_\alpha(a)) = \lim_{\alpha}(q \circ p_\alpha)(a) = (q \circ f)(a).$$

Now we approximate g by polynomials and use the isometricity of the inverse Gelfand transform. □

Problem 28. Prove that $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ in the proof above, approximating f by polynomials, and correctly stating what it means that the image is continuous under a uniform approximation, and using the isometricity of the Gelfand transform.

Corollary 1.29. *If a is a normal element, then $\|a\| = r(a)$.*

Proof. $\|a\| = \|\hat{a}\| = \sup_{\varphi\in M_A} |\hat{a}(\varphi)| = \sup_{\lambda\in\mathrm{Sp}(a)} |\lambda| = r(a)$. □

1.6 Addition: Stone-Weierstrass theorem

Let us first consider the algebra $C_{\mathbb{R}}(X)$ over $\mathbb{R}$ formed by all continuous real-valued functions on a compact Hausdorff space X.

Theorem 1.30. *Let $A \subseteq C_{\mathbb{R}}(X)$, where X is a compact Hausdorff space, is a closed subalgebra[2] such that A separates the points X and contains $1 \in C_{\mathbb{R}}(X)$ (and hence all constant functions). Then $A = C_{\mathbb{R}}(X)$.*

Proof. First of all, we note that the condition for separating points can be strengthened, namely: for any x and y from X and any u and v from $\mathbb{R}$ there is a function $g \in A$ such that $g(x) = u$ and $g(y) = v$. Indeed, since there is $f \in A$ with the property $u' = f(x) \neq f(y) = v'$, then g can be taken equal

$$g = \frac{u - v}{u' - v'} \cdot f + \frac{u'v - v'u}{u' - v'} \cdot 1.$$

For $f, g \in A$ we define continuous functions $f \vee g$, $f \wedge g$, $\gamma(g)$ as

$$(f \vee g)(s) = \max\{f(s), g(s)\}, \qquad (f \wedge g)(s) = \min\{f(s), g(s)\}, \qquad \gamma(g)(s) = |g(s)|.$$

According to Weierstrass's theorem on the approximation of continuous functions by polynomials, there is a sequence of polynomials p_n such that

$$|\,|\lambda| - p_n(\lambda)| \leqslant \frac{1}{n} \qquad \text{with } -n \leqslant \lambda \leqslant n.$$

Then

$$|\,|g(s)| - p_n(g)(s)| = |\,|g(s)| - p_n(g(s))| \leqslant \frac{1}{n} \qquad \text{for } -n \leqslant g(s) \leqslant n.$$

So $\gamma(g) \in A$. Therefore, $f \vee g \in A$, $f \wedge g \in A$, since

$$f \vee g = \frac{f+g}{2} + \frac{\gamma(f-g)}{2}, \qquad f \wedge g = \frac{f+g}{2} - \frac{\gamma(f-g)}{2}.$$

Let us now consider an arbitrary $F \in C_{\mathbb{R}}(X)$ and, by the remark from the beginning of the proof, find for arbitrary $x, y \in X$ a function $f_{x,y} \in A$ such that $f_{x,y}(x) = F(x)$ and $f_{x,y}(y) = F(y)$. Having temporarily fixed y, we find for each $x \in X$ a neighborhood U_x such that $f_{x,y}(u) > F(u) - \varepsilon$ for $u \in U_x$. Let us choose a finite subcover $U_{x_1}, \dots U_{x_p}$ and define $f_y = f_{x_1,y} \vee \dots \vee f_{x_p,y}$. Then $f_y(u) > F(u) - \varepsilon$ for any $u \in X$. Since $f_{x_i,y}(y) = F(y)$ for any $i = 1, \dots, p$, then $f_y(y) = F(y)$. This means that there is a neighborhood V_y of a point y such that $f_y(u) < F(u) + \varepsilon$ for $u \in V_y$. Let's choose a finite subcover $V_{y_1}, \dots, V_{y_q}$ and define $f := f_{y_1} \wedge \dots \wedge f_{y_q}$. Since every $f_{y_i}(u) > F(u) - \varepsilon$ for any $u \in X$, then $f(u) > F(u) - \varepsilon$ for any $u \in X$. On the other hand, for any $u \in X$ there is $V_{y_i} \ni u$, so $f(u) < f_{y_i}(u) < F(u) + \varepsilon$. Combining the inequalities, we obtain that $|f(u) - F(u)| < \varepsilon$ for any $u \in X$. Due to arbitrariness ε we obtain the required result. □

[2] this condition can be weakened

Theorem 1.31. *Let $A \subseteq C(X)$, where X is a compact Hausdorff space, is a closed involutive subalgebra such that A separates the points X and contains $1 \in C(X)$ (and hence all constant functions). Then $A = C(X)$.*

Proof. The involution has the form $f^*(x) = \overline{f(x)}$. Let A_R consist of real-valued functions belonging to A. Note that this is a unital subalgebra of the algebra $C_{\mathbb{R}}(X)$. Since A_R coincides with the kernel of a continuous $\mathbb{R}$-linear mapping $f \mapsto f - f^*$, then it is closed in A, and hence in $C(X)$. Therefore $A_R = A \cap C_{\mathbb{R}}(X)$ is closed in $C_{\mathbb{R}}(X)$. Finally, A_R separates the points X. Indeed, if $f(x) \neq f(y)$, where $f \in A$, then $f = f_1 + if_2$ for $f_1 = (f + f^*)/2 \in A_R$, $f_1 = (f - f^*)/2i \in A_R$, so at least one of f_1, f_2 separates x and y.

Therefore, by the previous theorem, $C_{\mathbb{R}}(X) = A_R \subset A$. Using the representation $f = f_1 + if_2$ again, but for the entire $C(X)$, we see that $\mathbb{C}$-linear combinations of elements of $C_{\mathbb{R}}(X)$ give $C(X)$ and, at the same time, give A by virtue of what has been proven. So $A = C(X)$. $\square$

Lecture 4

1.7 Positive elements

Definition 1.32. A self-adjoint element a in a unital C^*-algebra A is called *positive,* if $\mathrm{Sp}(a) \subset [0, \infty)$. If A is not unital, then a is called *positive,* if it is positive in A^+.

Positivity is written as $a \geqslant 0$. For two self-adjoint elements $a, b \in A$ we say that $a \geqslant b$ if $a - b \geqslant 0$.

Problem 29. Show that if $a \geqslant 0$ and $0 \geqslant a$, then $a = 0$; and also that $-\|a\|1 \leqslant a \leqslant \|a\|1$ for every self-adjoint a.

Now let's look at applications of continuous functional calculus to positivity.

Corollary 1.33. *Let $a \in A$ be a positive element. Then there exists a unique positive square root b of a, that is, $b \geqslant 0$ such that $b^2 = a$.*

Proof. The function $f(z) = \sqrt{z}$ is defined and continuous on $[0, \infty)$, so $b = f(a)$ is defined. It is self-adjoint and even positive (since f maps $[0, \infty)$ to itself) and $b^2 = f(a)^2 = a$ (by corollary 1.28). If c is another positive square root of a, then $c = f(c^2) = f(a) = b$. $\square$

Corollary 1.34. *Let $a \in A$ be a self-adjoint element. Then there are positive elements $a_+, a_- \in A$, such that $a = a_+ - a_-$ and $a_+ a_- = 0$.*

Proof. Let us define a continuous function $f : \mathbb{R} \to [0, +\infty)$, putting $f(x) = x$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Let's denote $g(x) = f(-x)$. These functions satisfy $f(x) - g(x) = x$ and $f(x)g(x) = 0$. It remains to put $a_+ = f(a)$, $a_- = g(a)$. $\square$

Corollary 1.35. *For a self-adjoint element $a \in A$ the following conditions are equivalent:*

(i) $a \geqslant 0$*;*

(ii) $a = b^2$ *for some self-adjoint b;*

(iii) $\|\mu 1 - a\| \leqslant \mu$ *for every* $\mu \geqslant \|a\|$*;*

(iv) $\|\mu 1 - a\| \leqslant \mu$ *for some* $\mu \geqslant \|a\|$.

Proof. By Corollary 1.33, from (i) it follows (ii). Moreover, (iii) implies (iv) by evident reasons.

Let us show that (ii) implies (iii). By assumption, $a = f(b)$, where $f(x) = x^2$. Moreover, the norm of f on $\mathrm{Sp}(b)$ is equal to $\|a\|$, so $0 \leqslant \mu - x^2 \leq \mu$ for any $\mu \geqslant \|a\|$ and $x \in \mathrm{Sp}(b)$. Since $\mu 1 - a = (\mu - f)(b)$, then $\|(\mu - f)(b)\|$ equals the norm of $\mu - f$ on $\mathrm{Sp}(b)$, which does not exceed μ.

Let us finally show that (iv) implies (i). Since, for some $\mu \geqslant \|a\|$, we have $\|\mu - x\| = \sup_{x \in \mathrm{Sp}(a)} |\mu - x| \leqslant \mu$, then no $x \in \mathrm{Sp}(a)$ can be negative. $\square$

Corollary 1.36. *If a and b are positive, then $a+b$ is positive.*

Proof. Let us choose some $\mu \geqslant \|a\|$, $\nu \geqslant \|b\|$. Then $\|a+b\| \leqslant \mu+\nu$ and the item (iii) of Corollary 1.35 implies

$$\|(\mu+\nu)1-(a+b)\| = \|(\mu 1-a)+(\nu 1-b)\| \leqslant \|\mu 1-a\| + \|\nu 1-b\| \leqslant \mu+\nu.$$

Therefore, by item (iv) of Corollary 1.35, $a+b$ is positive. □

Problem 30. If $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$. *Hint:* As we know, $b \leqslant \|b\| \cdot 1_A$. Thus $a \leqslant \|b\| \cdot 1_A$, i.e. $\mathrm{Sp}(\|b\| \cdot 1_A - a) \subset [0,+\infty)]$. From $\mathrm{Sp}(\mu 1_A - a) = \mu - \mathrm{Sp}(a)$ deduce that $\|a\| = \max\{\lambda \in \mathrm{Sp}(a)\}$ equals $\min\{\mu\colon \mathrm{Sp}(\mu \cdot 1_A - a) \subset [0,+\infty)\}$. This implies the statement.

The following Proposition 1.38 is almost obvious for operators in a Hilbert space, but is very nontrivial for elements of a C^*-algebra. We will need the following result about spectra of products.

Lemma 1.37. *If $a, b \in A$, then* $\mathrm{Sp}(ab) \cup \{0\} = \mathrm{Sp}(ba) \cup \{0\}$.

Proof. Let $0 \neq \lambda \notin \mathrm{Sp}(ab)$. This means that $(ab - \lambda 1) = -\lambda(1-\lambda^{-1}ab)$ is invertible, so there exists an element $u \in A$ such that $(1-\lambda^{-1}ab)u = 1$. Let us set $v = 1 + \lambda^{-1}bua$. Then

$$(1-\lambda^{-1}ba)v = (1-\lambda^{-1}ba)(1+\lambda^{-1}bua) = 1 - \lambda^{-1}ba + \lambda^{-1}bua - \lambda^{-2}babua =$$

$$= 1 - \lambda^{-1}ba + \lambda^{-1}b(1-\lambda^{-1}ab)ua = 1 - \lambda^{-1}ba - \lambda^{-1}ba = 1,$$

so $\lambda \notin \mathrm{Sp}(ba)$. □

Proposition 1.38. *The element a^*a is positive for every $a \in A$.*

Proof. Since a^*a is self-adjoint, we can write $a^*a = b_+ - b_-$ by Corollary 1.34. Let $c := \sqrt{b_-}$, $t := ac$. notice, that $f(0) = 0$ for $f(x) = \sqrt{x}$, so c is approximated by polynomials in b_- without a free term, which means $cb_+ = 0$. We have:

$$-t^*t = -c(b_+ - b_-)c = b_-^2. \tag{1.6}$$

Therefore, $-t^*t$ is positive.

Let us write t in the form $t = x + iy$, where x and y are self-adjoint elements of A (that is $x = (t+t^*)/2$, $y = (t-t^*)/2i$). Then $t^*t + tt^* = 2(x^2+y^2)$ is positive by Corollary 1.36. By the same consequence, we see that the element

$$tt^* = (t^*t + tt^*) - t^*t = (t^*t + tt^*) + b_-^2$$

is also positive, that is, $\mathrm{Sp}(tt^*) \subset [0,\infty)$. By Lemma 1.37, $\mathrm{Sp}(t^*t) \subset [0,\infty)$, so t^*t is positive. But it is also negative by (1.6), so $t^*t = 0$ for problem 29. This means $b_- = 0$, since positive square root is unique. □

Corollary 1.39. *If $b \leqslant c$, then $a^*ba \leqslant a^*ca$ for any $a \in A$.*

Proof. Since $c-b$ is positive, then $c-b=d^2$ for some self-adjoint d. That is why $a^*(c-b)a = a^*d^2a = (da)^*(da) \geqslant 0$. $\square$

Corollary 1.40. *If a and b are invertible and $0 \leqslant a \leqslant b$, then $b^{-1} \leqslant a^{-1}$.*

Proof. Let us first consider the special case $b=1$. Then $\mathrm{Sp}(a) \subseteq [0,1]$. According to the spectral mapping theorem, we see that $\mathrm{Sp}(a^{-1}) \subseteq [1,\infty)$, so $a^{-1} \geqslant 1$. Let's pass to the general case. Since, by Corollary 1.39, $b^{-1/2}ab^{-1/2} \leqslant 1$, then $b^{1/2}a^{-1}b^{1/2} \geqslant 1$ by the first part of the proof. Multiplying this inequality by $b^{-1/2}$ on both sides and again applying corollary of 1.39, we obtain $a^{-1} \geqslant b^{-1}$. $\square$

1.8 Approximate identity

Let Λ be a directed set. A family $(u_\lambda)_{\lambda\in\Lambda}$ elements of the C^*-algebra A is called an *approximate identity (unit),* if $\lim_\Lambda \|xu_\lambda - x\| = 0$ for $x \in A$ (and therefore $\lim_\Lambda \|u_\lambda x - x\| = 0$). If A is unital, then we can take $u_\lambda = 1$ for any λ, so this concept is of interest only for non-unital algebras. In the definition of an approximate unit also include the conditions $0 \leqslant u_\lambda \leqslant 1$ and $u_\lambda \leqslant u_\mu$ for all $\lambda \leqslant \mu$ from Λ.

Theorem 1.41. *Every C^*-algebra has an approximate unit.*

Proof. Let $\Lambda = \{a \in A \mid a \geqslant 0; \|a\| < 1\}$. An order on Λ is given by $\leqslant$. Let us show that the set of elements Λ, indexed tautologically (a has index a), is an approximate unit.

First of all, we need to check that Λ is a directed set, that is, for any two elements $a, b \in \Lambda$ there is a $c \in \Lambda$ such that $a \leqslant c$ and $b \leqslant c$. Let $f(t) := \frac{t}{1-t}$ and $g(t) := \frac{t}{1+t}$. Moreover, the function f is defined on $[0,1)$, the function g is defined on $[0,\infty)$ and $g(f(t)) = t$. Let us put $x := f(a)$, $y := f(a)+f(b)$, $c := g(y)$. Since $0 \leqslant g(t) < 1$, then $c \in \Lambda$. The inequality $x \leqslant y$ implies $1+x \leqslant 1+y$, so $(1+x)^{-1} \geqslant (1+y)^{-1}$ and

$$a = 1-(1+x)^{-1} \leqslant 1-(1+y)^{-1} = c.$$

The inequality $b \leqslant c$ can be obtained similarly, so we have verified that the set Λ is directed.

Now let's check that $\lim_\Lambda \|x - ax\| = 0$ for every $x \in A$. Since by Corollary 1.34 each element can be decomposed into a linear combination of four positive elements:

$$x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i} = \left(\frac{x+x^*}{2}\right)_+ - \left(\frac{x+x^*}{2}\right)_- + i\left(\frac{x-x^*}{2i}\right)_+ - i\left(\frac{x-x^*}{2i}\right)_-, \quad (1.7)$$

then it is sufficient to verify the statement for $x \geqslant 0$. Since for $a \in \Lambda$ we have $0 \leqslant 1-a \leqslant 1$, then by Corollary 1.39, $(1-a)^{1/2}(1-a)(1-a)^{1/2} \leqslant (1-a)^{1/2}(1-a)^{1/2} = 1-a$. That's why (see Problem 30)

$$\|(1-a)x\|^2 = \|x^*(1-a)^2x\| \leqslant \|x^*(1-a)x\|,$$

and it is sufficient to verify that $\lim_\Lambda \|x(1-a)x\| = 0$ for any $x \geqslant 0$ from A, and without loss of generality we can assume that $\|x\| = 1$.

Similar to the previous reasoning, if $a, b \in \Lambda$ and $a \leqslant b$ then $\|x^*(1-b)x\| \leq \|x^*(1-a)x\|$, so, for $x \geqslant 0$,

$$\sup_{b\in\Lambda,\, b\geqslant a} \|x(1-b)x\| = \|x(1-a)x\|.$$

Therefore we need to show that, for any positive $x \in A$ of norm one and for any $\varepsilon > 0$, there is an element $a \in \Lambda$ such that $\|x^*(1-a)x\| < \varepsilon$. Let us put $a_n := g(nx)$, $n \in \mathbb{N}$ (see the beginning of the proof). Then $\|x(1-a_n)x\| = \|h(x)\|$, where $h(t) := t^2(1-g(nt)) = \frac{t^2}{1+nt}$. For any $t \in [0,1]$, we have $0 \leqslant h(t) \leqslant \frac{1}{n}$, so $\|h(x)\| \leqslant \frac{1}{n}$. Hence,

$$\sup_{b\in\Lambda,\, b\geqslant a_n} \|x(1-b)x\| = \|x(1-a_n)x\| \leqslant \frac{1}{n}$$

for any n, so $\lim_{b\in\Lambda} \|x(1-b)x\| = 0$. □

Definition 1.42. An approximate unit is called *countable,* if the set Λ is countable.

Corollary 1.43. *A separable C^*-algebra has a countable approximate unit.*

Proof. Let us choose a dense sequence $(x_n)_{n\in\mathbb{N}}$ in A. Then there is an element $a_1 \in A$, $0 \leqslant a_1$, $\|a_1\| < 1$ such that $\|x_1a_1 - x_1\| \leqslant 1$ (see the proof of the previous theorem). Suppose by induction that we have already found such $a_2, \ldots, a_n$, $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$, such that, for all $k = 1, 2, \ldots, n$, the inequalities $\|a_k\| < 1$ and $\|x_ia_k - x_i\| \leqslant \frac{1}{k}$ are satisfied for $i = 1, 2, \ldots, k$. Now let's find an element $a_{n+1} \geqslant a_n$ with norm $\|a_{n+1}\| < 1$, such that $\|x_ia_{n+1} - x_i\| \leqslant \frac{1}{n+1}$ for $i = 1, 2, \ldots, n+1$. Because the sequence (x_n) is dense in A, then by induction we obtain a non-decreasing sequence $(a_n)_{n\in\mathbb{N}}$ positive elements of the unit ball with $\lim_{n\to\infty} \|xa_n - x\| = 0$ for each $x \in A$. Indeed, for any $\varepsilon > 0$ we can find an element x_i such that $\|x - x_i\| < \varepsilon/3$, and for x_i we can find a number j such that $\|x_ia_k - x_i\| < \varepsilon/3$ for all $k > j$. Then for these k

$$\|xa_k - x\| = \|x_ia_k - x_i + (x - x_i)a_k + x - x_i\| < \varepsilon/3 + \varepsilon/3\|a_k\| + \varepsilon/3 \leqslant \varepsilon.$$

□

Problem 31. Prove that the converse is not true: an algebra with countable approximate unit does not have to be separable.

Lecture 5

1.9 Ideals, factors and homomorphisms

Under an *ideal of C^*-algebra* we will always mean norm-closed two-sided ideal (for maximal ideals in in the commutative case this happens automatically).

Lemma 1.44. *Every ideal I in a C^*-algebra is self-adjoint: $I = I^*$.*

Proof. If $I \subset A$ is an ideal, then $B := I \cap I^* \subseteq A$ is a C^*-subalgebra. In this case, $B \supset I \cdot I^*$. Let (u_λ) is an approximate unit in B, and $j \in I$. Then

$$\lim_{\lambda\in\Lambda} \|j^* u_\lambda - j^*\|^2 = \lim_{\lambda\in\Lambda} \|u_\lambda(jj^*u_\lambda - jj^*) - (jj^*u_\lambda - jj^*)\| \leqslant 2 \lim_{\lambda\in\Lambda} \|jj^*u_\lambda - jj^*\| = 0.$$

Since $u_\lambda \in I$, then $j^*u_\lambda \in I$, so $j^* \in I$, since I is closed. □

The following technical lemma is often used.

Lemma 1.45. *If $x^*x \leqslant a$ is in A, then there is an element $b \in A$ such that $\|b\| \leqslant \|a\|^{1/4}$ and $x = ba^{1/4}$.*

Proof. Let us put $b_n := x(a + \frac{1}{n}1)^{-1/2}a^{1/4}$ (this element lies in A, even if A does not have a unit, but in this case it is convenient for us to carry out calculations in A^+). Let also

$$d_{nm} := \left(a + \frac{1}{n}1\right)^{-1/2} - \left(a + \frac{1}{m}1\right)^{-1/2}, \qquad f_n(t) := t^{3/4}\left(t + \frac{1}{n}\right)^{-1/2}.$$

Then the sequence of functions $\{f_n(t)\}$ converges to $f(t) := t^{1/4}$ uniformly on $[0, \|a\|]$, since due to $u^2 + v^2 \geqslant 2uv$ we have

$$\left(t^{1/4}\left(1 - \frac{t^{1/2}}{(t+1/n)^{1/2}}\right)\right)^2 = t^{1/2}\frac{t + 1/n + t - 2t^{1/2}(t+1/n)^{1/2}}{t+1/n} <$$

$$< t^{1/2}\frac{t + 1/n + t - 2t}{t + 1/n} = t^{1/2}\frac{1/n}{t+1/n} = \frac{2\sqrt{t/n}}{t+1/n} \cdot \frac{1}{2\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}}.$$

We have

$$\begin{aligned} \|b_n - b_m\|^2 &= \|xd_{nm}a^{1/4}\|^2 = \|a^{1/4}d_{nm}x^*xd_{nm}a^{1/4}\| \leqslant \|a^{1/4}d_{nm}ad_{nm}a^{1/4}\| = \\ &= \|d_{nm}a^{3/4}\|^2 = \|f_n(a) - f_m(a)\|^2 = \sup_{t\in[0,\|a\|]} |f_n(t) - f_m(t)|. \end{aligned}$$

Thus, since f_n is a Cauchy sequence, so is b_n. Let us put $b := \lim_{n\to\infty} b_n$. Then $ba^{1/4} = \lim_{n\to\infty} b_n a^{1/4} = \lim_{n\to\infty} x(a + \frac{1}{n}1)^{-1/2}a^{1/2} = x$. □

Problem 32. Verify that the last limit is indeed x. This can be done in a similar way to the calculation for f_n in the proof, using $x^*x \leqslant a$.

Definition 1.46. A subalgebra $B \subset A$ is called *hereditary* if for any positive $b \in B$ and $a \in A$, from the condition $0 \leqslant a \leqslant b$ it follows that $a \in B$.

Problem 33. Prove that a positive element of an arbitrary C^*-subalgebra is a positive element of the entire algebra.

Lemma 1.47. *Let $I \subset A$ be an ideal and $j \in I$ a positive element. If $a^*a \leqslant j$, then $a \in I$. In particular, any ideal is a hereditary subalgebra.*

Proof. Let us represent $a = bj^{1/4}$ in accordance with Lemma 1.45. Moreover, $j^{1/4} \in C^*(j) \subset I$, and therefore $a \in I$. □

If $I \subset A$ is an ideal, then we can define Banach factor algebra A/I with norm $\|a+I\| := \inf_{j\in I} \|a + j\|$. This is an involutive algebra: since I is self-adjoint, then $\|(a + I)^*\| = \|a^* + I\| = \|a + I\|$. To be short we will denote $a + I$ by $\dot{a} \in A/I$.

Theorem 1.48. *The involutive algebra A/I is a C^*-algebra.*

Proof. Only the C^* property needs to be verified. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of I (note that ideals typically do not have a unit, and in any case, a proper ideal does not contain the unit of A, even if the latter exists). Let us first show that

$$\|\dot{a}\| = \lim_{\lambda\in\Lambda} \|a - au_\lambda\|. \tag{1.8}$$

Indeed, since $u_\lambda \in I$, then $\|\dot{a}\| \leq \|a - au_\lambda\|$. To prove the reverse inequality, we choose arbitrarily $\varepsilon > 0$. Then there is an element $j \in I$ such that $\|\dot{a}\| \geqslant \|a - j\| - \varepsilon$. We have

$$\lim_{\lambda\in\Lambda} \|a - au_\lambda\| \leqslant \lim_{\lambda\in\Lambda} (\|a - au_\lambda - (j - ju_\lambda)\| + \|j - ju_\lambda\|) = \lim_{\lambda\in\Lambda} \|a - au_\lambda - (j - ju_\lambda)\|.$$

Writing in A^+, where the equality $a - au_\lambda - (j - ju_\lambda) = (a - j)(1 - u_\lambda)$ holds, we obtain the estimation $\|(a - j)(1 - u_\lambda)\| \leqslant \|a - j\| < \|\dot{a}\| + \varepsilon$. Due to arbitrariness of $\varepsilon > 0$ we obtain (1.8).

Now, calculating in A^+, we find the estimation

$$\begin{aligned} \|\dot{a}^*\dot{a}\| &= \lim_{\lambda\in\Lambda} \|a^*a(1 - u_\lambda)\| \geqslant \lim_{\lambda\in\Lambda} \|(1 - u_\lambda)a^*a(1 - u_\lambda)\| = \\ &= \lim_{\lambda\in\Lambda} \|(a(1 - u_\lambda)\|^2 = \|\dot{a}\|^2. \end{aligned}$$

The inverse inequality $\|\dot{a}^*\dot{a}\| \leqslant \|\dot{a}\|^2$ is true in any involutive Banach algebra. □

Definition 1.49. Let A and B be C^*-algebras. A $*$-*homomorphism* from A to B is any homomorphism φ preserving the involution: $\varphi(a^*) = \varphi(a)^*$. If both algebras are unital, φ is called *unital*, if $\varphi(1_A) = 1_B$.

Problem 34. Let $\varphi : A \to B$ be a $*$-homomorphism of non-unital algebras. Prove that there is a unique unital $*$-homomorphism $\varphi^+ : A^+ \to B^+$, extending φ. Note: The only way to determine φ^+ is the requirement to be unital: $\varphi^+(1) = 1$.

Problem 35. Let $\varphi : A \to B$ be a $*$-homomorphism of algebras, with A non-unital, and B unital. Prove that there is a unique unital $*$-homomorphism $\varphi^{(+)} : A^+ \to B$, extending φ. *Hint:* the same as above.

Theorem 1.50. *Let $\varphi : A \to B$ be a nonzero $*$-homomorphism. Then $\|\varphi\| = 1$ (in particular, it is continuous) and $\varphi(A)$ is a C^*-subalgebra of B. If φ is injective, then it is isometric (on the image).*

Proof. If the algebra A is non-unital, then we will consider φ^+ from Problem 34 or $\varphi^{(+)}$ from problem 35. If the algebra A is unital, then we can assume that B is unital too (if not — then we attach a unity without requiring the homomorphism to be unital). Then $\varphi(1_A) = p$ is a self-adjoint idempotent ($p^2 = p$), the space $B_p := pBp$ is a subalgebra of B (see problem 36) with identity $p = p \cdot 1_B \cdot p$, and φ, considered as a homomorphism in B_p, is unital.

Thus, in the proof we can restrict ourselves to the case of a unital homomorphism $\varphi : A \to B$ of unital algebras.

To distinguish the spectrum of an element in A and B, we will write Sp_A (resp., Sp_B) for the spectrum of elements in A (resp., in B).

Let $a = a^* \in A$. Then $\mathrm{Sp}_B(\varphi(a)) \subset \mathrm{Sp}_A(a)$, since φ is a unital $*$-homomorphism of algebras and $\|\varphi(a)\| = r(\varphi(a)) \leqslant r(a) = \|a\|$. For an element $a \in A$ of general form, we have $\|\varphi(a)\|^2 = \|\varphi(a^*a)\| \leqslant \|a^*a\| = \|a\|^2$, so $\|\varphi\| \leqslant 1$, that is, φ is continuous and does not increase the norm.

Suppose now that φ is injective but not isometric. Then there is an element $a \in A$ such that $\|\varphi(a)\| < \|a\|$. This means $\|\varphi(b)\| < \|b\|$ for $b := a^*a$. Let us denote $\|\varphi(b)\| =: r$ and $\|b\| =: s$. Let h be a continuous real function that satisfies the conditions $q(t) = 0$ for $t \in [0, r]$ and $h(s) = 1$. Then $\|\varphi(h(b))\| = \|h(\varphi(b))\| = \sup_{\lambda \in \mathrm{Sp}_B(\varphi(b))} |h(\lambda)| = 0$, while $\|h(b)\| = \sup_{\lambda \in \mathrm{Sp}_A(b)} |h(\lambda)| \geqslant 1$. A contradiction with injectivity. (The commutation condition is obvious for polynomials, h_n, uniformly approximating h, and in the limit we obtain it for h.)

In the case of a general (not necessarily injective) $*$-homomorphism, note that that $I = \operatorname{Ker} \varphi$ is closed since φ is continuous, so I is an ideal in A. Therefore φ induces an injective $*$-homomorphism $\dot{\varphi} : A/I \to B$ by the rule $\dot{\varphi}(\dot{a}) = \varphi(a)$. Then, by what has been proven, $\dot{\varphi}$ is isometric, and $\varphi(A) = \dot{\varphi}(A/I)$ is closed in B, so it is a C^*-subalgebra. Since φ is non-zero, then there is $a \in A$ with $\varphi(a) \neq 0$. Since $\dot{\varphi}$ is isometric, we have the equality $\|\dot{a}\| = \|\dot{\varphi}(\dot{a})\| = \|\varphi(a)\|$. Moreover, for any $\varepsilon > 0$ there is an element $c \in A$ such that $\dot{c} = \dot{a}$ and $\|c\| < \|\dot{a}\| + \varepsilon$. Thus, $\|\varphi(c)\| > \|c\| - \varepsilon$. Since ε is arbitrary, we obtain $\|\varphi\| \geqslant 1$, so $\|\varphi\| = 1$. □

Problem 36. Prove that the algebra B_p is closed, first obtaining the equality $pBp = \operatorname{Ker}(L_{1-p}) \cap \operatorname{Ker}(R_{1-p})$, where L_{1-p} and R_{1-p} are the linear operators of left and right multiplication by $1 - p$ in B, given by $L_{1-p} : b \mapsto (1-p)b$ and $R_{1-p} : b \mapsto b(1-p)$.

Problem 37. Develop the result of the previous problem by verifying the decomposition into a direct sum of closed subspaces $B = pBp \oplus pB(1-p) \oplus (1-p)Bp \oplus (1-p)B(1-p)$. Moreover, if we write down the quadruple (a, b, c, d), representing an element of a given direct sum in the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplication in B passes under this isomorphism to the matrix multiplication according to the standard rule.

Problem 38. Derive from Theorem 1.50 the statement $\varphi(f(a)) = f(\varphi(a))$ for any normal a and f, which is continuous on the appropriate set (not only for a polynomial).

Problem 39. Obtain a proof of Theorem 1.50 via a reduction to a map of commutative subalgebras.

Corollary 1.51. *Let $I \subset A$ be an ideal, and $B \subset A$ be a C^*-subalgebra. Then $I + B$ coincides with the C^*-subalgebra $C^*(I, B)$ generated by I and B.*

Proof. It is obvious that $I + B \subset C^*(I, B)$ is an involutive subalgebra. Let $q : A \to A/I$ be the $*$-homomorphism of factorization. We know from the previous theorem that $q(B)$ is closed in A/I, so $I + B = q^{-1}(q(B))$ is closed in A. This means that $I + B$ is a C^*-algebra, contained in $C^*(I, B)$. □

So far we were very careful when considering spectrum of an element in a C^*-algebra and its C^*-subalgebra. The next lemma shows that this is not so important.

Lemma 1.52. *Let $B \subset A$ be a unital C^*-subalgebra of a unital C^*-algebra, $1_A = 1_B$, and $a \in B$. Then $\mathrm{Sp}_B(a) = \mathrm{Sp}_A(a)$.*

Proof. Obviously, if an element has an inverse in B, then so does A, whence $\mathrm{Sp}_A(a) \subset \mathrm{Sp}_B(a)$. The reverse inclusion follows from the statement: if a is invertible into A, then its inverse belongs to B. To prove this, consider first the case $a = a^*$. Then the C^*-algebra $C = C^*(a, a^{-1})$ generated by a and a^{-1}, is a commutative unital C^*-subalgebra of A, and therefore it is isomorphic to some algebra of functions $C(X)$. Let $\hat{a}$ denote the image of a under this isomorphism. Then $0 \notin \mathrm{Sp}_{C(X)}(\hat{a}) \subset \mathbb{R}$. Let us choose polynomials p_n such that $p_n(t)$ converges uniformly to t^{-1} on $\mathrm{Sp}_{C(X)}(\hat{a})$. Then $\widehat{a^{-1}} = \lim_{n\to\infty} p_n(\hat{a})$, so $a^{-1} = \lim_{n\to\infty} p_n(a) \in C^*(a) \subset B$.

For a general element a, if a^{-1} exists in A, then $a^{-1}(a^*)^{-1} = (a^*a)^{-1} \in B$ as proven. That is why $a^{-1} = (a^*a)^{-1}a^* \in B$. □

Problem 40. Show with an example that, without the condition $1_A = 1_B$, the previous proposition does not hold.

Lecture 6

1.10 Topologies on $\mathbb{B}(H)$ and von Neumann algebras

Besides the norm topology, there are other useful topologies on the C^*-algebra $\mathbb{B}(H)$.

Definition 1.53. *Strong topology* is defined by a system of seminorms $a \to \|a\xi\|$, $\xi \in H$.
Weak topology is defined by a seminorm system $a \to (a\xi, \eta)$, $\xi, \eta \in H$.

Theorem 1.54. *For a linear functional $\varphi : \mathbb{B}(H) \to \mathbb{C}$ the following conditions are equivalent:*

(i) There exist $\xi_k, \eta_k \in H$, $k = 1, \dots, n$, such that $\varphi(a) = \sum_{k=1}^{n} (a\xi_k, \eta_k)$ for any $a \in \mathbb{B}(H)$;

(ii) φ is weakly continuous;

(iii) φ is strongly continuous.

Proof. It is obvious that (i)$\Longrightarrow$(ii) $\Longrightarrow$(iii). Let us show that (iii) implies (i).

The strong continuity of φ means that the preimage $\{a : |\varphi(a)| < 1\}$ of the open unit disk is an open set in the strong topology, that is, there are positive constants $\varepsilon_1, \dots, \varepsilon_n$ and vectors $\xi_1, \dots, \xi_n$, such that for any $a \in \mathbb{B}(H)$ the condition $\|a\xi_k\| < \varepsilon_k$ (for all $k = 1, \dots, n$) implies $|\varphi(a)| < 1$. Changing the length of these vectors if necessary, we see that in an equivalent way we can say, that there exist vectors $\xi_1, \dots, \xi_n$ such that, for any $a \in \mathbb{B}(H)$, from $\max_k \|a\xi_k\| \leqslant 1$ it follows that $|\varphi(a)| \leqslant 1$. Then

$$|\varphi(a)| \leqslant \left(\sum_{k=1}^{n} \|a\xi_k\|^2 \right)^{1/2}. \tag{1.9}$$

Indeed, if $|\varphi(a)|^2 > \sum_{k=1}^{n} \|a\xi_k\|^2$ for some a, then $|\varphi(a)| > \|a\xi_k\|$ for all k, so since the number of them is finite, one can find $\alpha \in \mathbb{R}$ such that $|\varphi(\alpha a)| > 1$, and $\max_k \|\alpha a\xi_k\| < 1$ (for example, $\alpha^{-1} := (|\varphi(a)| + \max \|a\xi_k\|)/2$). A contradiction.

Let $K := \oplus_{k=1}^{n} H$. The algebra $\mathbb{B}(K)$ can be identified with the algebra of $n \times n$-matrices with elements from $\mathbb{B}(H)$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ map $a \in \mathbb{B}(H)$ to a diagonal matrix with all diagonal elements equal to a.

Let us denote $\xi := \xi_1 \oplus \dots \oplus \xi_n \in K$ and note that, putting $\psi(\rho(a)\xi) = \varphi(a)$, we obtain a linear functional on the closed subspace $L \subset K$, where L is the closure of the space $L_0 := \{\rho(a)\xi \mid a \in \mathbb{B}(H)\}$. Indeed, first we need to verify that ψ is well defined on L_0: if $\rho(a)(\xi) = \rho(b)(\xi)$, then $(a - b)\xi_k = 0$ for $k = 1, \dots, n$. In particular for arbitrarily large $R > 0$ we have $\|R(a - b)\xi_k\| \leqslant 1$, and therefore $|\varphi(R(a - b))| \leqslant 1$. Therefore $|\varphi(a - b)| \leqslant 1/R$. Due to the arbitrariness of R, we obtain that $\varphi(a - b) = 0$ and thus ψ is well defined on L_0. From (1.9) we see that $|\psi(\rho(a))| \leqslant \|\rho(a)\xi\|$, so $|\psi(\zeta)| \leq \|\zeta\|$ for any $\zeta \in L_0$, and hence L. So, ψ is a bounded functional on L. By the Riesz theorem on representation of functionals, there is a vector $\eta \in L \subset K$ such that $\psi(\zeta) = (\zeta, \eta)$

for all $\zeta \in L$ (we assume the Hermitian product to be linear in the first argument), so $\varphi(a) = (\rho(a)\xi, \eta)$. Decomposing it into components $\eta = \eta_1 \oplus \ldots \oplus \eta_n$, we obtain $(\rho(a)\xi, \eta) = \sum_{k=1}^n (a\xi_k, \eta_k)$. □

Corollary 1.55. *In $\mathbb{B}(H)$ a convex set is closed for the weak topology if and only if it is closed for the strong one.*

Proof. This immediately follows from the previous theorem, since, according to the Hahn-Banach theorem, closed convex sets are obtained as the intersection of closed half-spaces, corresponding to linear functionals. □

Definition 1.56. A *von Neumann algebra* is a C^*-subalgebra $\mathbb{B}(H)$ containing unity (identity operator) and closed in the weak topology.

The simplest examples are $\mathbb{C}$ and $\mathbb{B}(H)$ (in fact, the first algebra is a special case of the second).

Definition 1.57. For the set $S \subseteq \mathbb{B}(H)$ we denote by S' its *commutant*, that is, the set of all operators $a \in \mathbb{B}(H)$ such that $as = sa$ for every $s \in S$.

Problem 41. Verify that

- If S is self-adjoint, then so is S'.
- The commutant of any set is a unital algebra.
- The commutant of any set is weakly closed.
- Thus, S' is the von Neumann algebra for any self-adjoint set S.
- If $S_1 \subset S_2$, then $S_1' \supset S_2'$.
- Always $S \subset S''$.
- Therefore $S' = S'''$, $S'' = S''''$, etc.

Theorem 1.58 (von Neumann bicommutant theorem)**.** *Let A be a C^*-subalgebra of $\mathbb{B}(H)$ containing the identity operator. Then the following conditions are equivalent.*

(i) $A = A''$*;*

(ii) A is weakly closed;

(iii) A is strongly closed.

Proof. Since A is a convex subset, then (ii) and (iii) are equivalent as a consequence 1.55. Since A'' is weakly closed, then (ii) follows from (i). It remains to show that (iii) implies (i).

For a vector $\xi \in H$ we denote by p the projection onto the closure V of the linear subspace formed by vectors $a\xi$, $a \in A$.

Thus, $p\eta = \eta$ for $\eta \in V$. Since $1 \in A$, then $\xi \in V$, so $p\xi = \xi$. Therefore $pap\zeta = pa\eta = a\eta = ap\zeta$ for any $\zeta \in H$, where we denote $\eta = p\zeta \in V$. So $pap = ap$ for any $a \in A$. From here $pa = (a^*p)^* = (pa^*p)^* = pap$ and we get $ap = pa$, that is $p \in A'$. Let $b \in A''$. Then $pb = bp$, so $pb\xi = bp\xi = b\xi$ and $b\xi \in V$. Thus, for every $\varepsilon > 0$ there is an element $a \in A$, for which $\|(b-a)\xi\| < \varepsilon$.

Now consider some $\xi_1, \ldots, \xi_n \in H$ and define $\xi := \xi_1 \oplus \ldots \oplus \xi_n \in K := H \oplus \ldots \oplus H$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ be the diagonal embedding. It is easy to see that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$ (Problem 42). Applying the first part of the proof to this situation, we obtain that for any $b \in A''$ and every $\varepsilon > 0$ there is an element $a \in A$ such that $\|(\rho(b) - \rho(a))\xi\| < \varepsilon$. Then $\sum_{k=1}^n \|(b-a)\xi_k\|^2 = \|(\rho(b) - \rho(a))\xi\|^2 < \varepsilon^2$, so we can strongly approximate $b \in A''$ by some $a \in A$. □

Problem 42. Check that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$.

Corollary 1.59. *If A is a von Neumann algebra, then A' is a von Neumann algebra.*

Definition 1.60. The *center* of an algebra is the set of its elements that commute with all its elements.

Corollary 1.61. *If A is a von Neumann algebra, then its center Z is also a von Neumann algebra.*

Proof. For a subalgebra $A \subseteq \mathbb{B}(H)$ we have $Z = A \cap A'$. □

Let $A \subset \mathbb{B}(H)$ be a C^*-algebra containing the identity operator. Then the bicommutant theorem states that A is weakly (strongly) dense in A''. This result has the disadvantage that the approximation is done by elements with, generally speaking, an uncontrollable norm. This is overcome by the following theorem, which we present without the proof, which can be found in [**?**, § 4.3].

Theorem 1.62 (Kaplansky density theorem)**.** *The unit ball A is weakly (strongly) dense in the unit ball A''. The same is true for the sets of positive elements in these unit balls and for sets of unitary elements.*

Definition 1.63. If the center Z of the von Neumann algebra A consists only of scalar operators (that is, $Z = \mathbb{C}1$), then A is called a *factor*.

Remark 1.64. It should be noted that besides the continuous functional calculus for self-adjoint operators, there is a Borel functional calculus: instead of norm approximation of continuous functions by polynomials here Borel functions are approximated by polynomials, and the corresponding operators will converge in the weak topology. More precisely, let the polynomials p_i converge monotonically and pointwise to a Borel function f on the spectrum of a self-adjoint operator $a \in \mathbb{B}(H)$. Then $\{p_i(a)\}$ is a strongly convergent sequence of operators (this is a statement from the standard course, see for example [**?**, §§ 7 and 11]). Since all polynomials commute with the commutator of the self-adjoint operator, then for any Borel function f on the spectrum of a self-adjoint operator a, the operator $f(a)$ lies in $\{a\}''$.

Chapter 2

Representations of C^*-algebras

2.1 Definition and basic properties

Definition 2.1. A *representation* of a C^*-algebra A on a Hilbert space H is a $*$-homomorphism from A to $\mathbb{B}(H)$.

Definition 2.2. A representation of a C^*-algebra A is called *algebraically irreducible*, if there is no proper invariant linear subspace in H (when operated by operators from the image of the representation). A representation is *topologically irreducible*, if there is no proper closed invariant subspaces.

We will see soon that for C^*-algebras these two concepts coincide.

Lemma 2.3. *A representation π is topologically irreducible if and only if $\pi(A)' = \mathbb{C}1$.*

Proof. If $\pi(A)'$ contains something other than scalars, then it also contains a self-adjoint non-scalar operator (this immediately follows from the expansion of a non-scalar operator into a linear combination of two self-adjoint ones $a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$). Using Borel functional calculus (see note 1.64) for this self-adjoint operator b, we can obtain a proper projection p in $\pi(A)'$. Namely, if an operator is nonscalar, then it has at least two distinct points in the spectrum, say, t_0 and t_1, and we need to consider a Borel function f, taking values 0 and 1, and $f(t_0) = 0$, $f(t_1) = 1$ (task 43). (You can also not use calculus, but simply take suitable spectral projections from the standard spectral theorem, that by construction have the necessary commutation properties). Then pH is a closed invariant subspace, since $p \in \pi(A)'$.

Conversely, let $L \subset H$ be a closed $\pi(A)$-invariant subspace, and $p \in \mathbb{B}(H)$ is a projection onto this subspace. Then $\pi(a)p = p\pi(a)p$ for any $a \in A$. Therefore $p\pi(a) = (\pi(a^*)p)^* = (p\pi(a^*)p)^* = p\pi(a)p = \pi(a)p$ and $p \in \pi(A)'$. Moreover, p is not a scalar. $\square$

Problem 43. Verify in the proof above that $f(b)$ is a proper projection, since $f^2 = f$ and $\mathrm{Sp}(f(b)) = \{0, 1\}$.

Problem 44. Prove a more general fact: if a self-adjoint element a in a unital C^*-algebra has $\mathrm{Sp}(a) = \{0, 1\}$, then a is a nonscalar idempotent.

Lemma 2.4. *Let π be a topologically irreducible representation of a C^*-algebra A in a Hilbert space H. Then for any $t \in \mathbb{B}(H)$, a finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\|a\| \leq \|t|_L\|$ and $\|(\pi(a) - t)|_L\| < \varepsilon$.*

Proof. Since π is topologically irreducible, then by Lemma 2.3 $\pi(A)'$ coincides with scalars, hence $\pi(A)'' = \mathbb{B}(H)$. That is why $\pi(A)$ is dense in $\mathbb{B}(H)$ in the weak (strong) topology. Without loss of generality, we can assume that $\|t|_L\| = 1$. Let us put $s = tp_L$, where p_L is the projection onto L. Since L is finite-dimensional, then, by Kaplansky's density theorem, there is $b \in A$ such that $\|\pi(b)\| \leqslant 1$ and $\|(\pi(b) - s)|_L\| < \varepsilon/2$. Then there is an element $c \in A$ such that $\pi(c) = \pi(b)$ and $\|c\| < \|\pi(b)\|(1 + \varepsilon/2)$ (see Theorem 1.50). Let us put $a := \frac{c}{1+\varepsilon/2}$. Then $\|a\| \leqslant 1$ and

$$\|(\pi(a) - t)|_L\| \leqslant \|(\pi(c) - t)|_L\| + \|\pi(a) - \pi(c)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$

Lecture 7

Lemma 2.5. *Let π be a topologically irreducible representation of the C^*-algebra A in the Hilbert space H. Then for any $t \in \mathbb{B}(H)$, finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\pi(a)|L = t|_L$ and $\|a\| \leqslant \|t\| + \varepsilon$.*

Proof. By the previous lemma, there is an element $a_0 \in A$ such that $\|a_0\| \leqslant \|t\|$ and $\|(\pi(a_0) - t)|_L\| < \varepsilon/2$. By induction we can find for each n is an element of $a_n \in A$ such that $\|a_n\| \leqslant 2^{-n}\varepsilon$ and $\|(\sum_{k=0}^{n} \pi(a_n) - t)p_L|\| < 2^{-n-1}\varepsilon$. Indeed, suppose that the elements are found for some n and all the smaller ones. Applying the previous lemma to $s = -\sum_{k=0}^{n} \pi(a_k) + t$, the same subspace L and $2^{-n-2}\varepsilon$, we find an element a_{n+1} such that $\|a_{n+1}\| \leqslant 2^{-n-1}\varepsilon$ and $\|(\sum_{k=1}^{n+1} \pi(a_k) - t)p_L\| < 2^{-n-2}\varepsilon$. Now let's put $a = \sum_{k=0}^{\infty} a_k$. Then $a \in A$ and it is evident that $\|a\| \leqslant \|t\| + \varepsilon$ and $a|_L = t|_L$. $\square$

Theorem 2.6. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

Proof. Let's assume the opposite Let $V \subset H$ be a non-closed invariant space, and $\overline{V}$ is its closure. It is also an invariant subspace (since the action is continuous), so $\overline{V} = H$. Let us take $\eta \in H \setminus V$, of norm 1 for example. Let $\xi \in V$ is a nonzero vector, and t is an operator in H such that $t\xi = \eta$. Then, by the previous lemma, there is an $a \in A$ such that $\pi(a)\xi = \eta$. Contradiction with the invariance of V. $\square$

2.2 Positive linear functionals

Definition 2.7. Linear functional (we do not require continuity, see Lemma 2.10 below) φ on the C^*-algebra A is called *positive*, if $\varphi(a) \geqslant 0$ for any $a \geqslant 0$. If a positive linear functional is continuous and has norm 1, then it is called a *state*.

Example 2.8. If π is a representation of A in the Hilbert space H and $\xi \in H$, then the functional $\varphi(a) := (\xi, \pi(a)\xi)$ is positive. If A is unital and $\|\xi\| = 1$, then such φ is a state.

With every positive linear functional φ we can associate a sesquilinear form on A given by the formula $\langle a, b\rangle := \varphi(a^*b)$, that is, the form $\langle\cdot,\cdot\rangle$ is linear in the second argument is conjugate linear in the first argument. By definition of positivity of the functional $\langle a, a\rangle = \varphi(a^*a) \geqslant 0$ for any $a \in A$. Therefore, by the following lemma it is Hermitian symmetric: $\langle b, a\rangle = \overline{\langle a, b\rangle}$.

Lemma 2.9 (from linear algebra course)**.** *If a sesquilinear form has $\langle a, a\rangle \in \mathbb{R}$ for any a, then it is Hermitian symmetric.*

Proof. Let us write down the polarization identities

$$\langle a+b, a+b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle, \tag{2.1}$$

$$\langle a+ib, a+ib\rangle = \langle a, a\rangle + \langle a, ib\rangle + \langle ib, a\rangle + \langle ib, ib\rangle = \langle a, a\rangle + i(\langle a, b\rangle - \langle b, a\rangle) + \langle b, b\rangle. \tag{2.2}$$

From the first we obtain that $\langle a, b\rangle + \langle b, a\rangle$ is real, and from the second — that $\langle a, b\rangle - \langle b, a\rangle$ is imaginary. So $\overline{\langle a, b\rangle} = \langle b, a\rangle$. $\square$

Thus, $\langle a, b\rangle$ is a positive Hermitian form and, therefore, the Cauchy-(Schwartz-Bunyakovsky) inequality holds for it: $|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle$, that is, $|\varphi(a^*b)|^2 \leqslant \varphi(a^*a)\varphi(b^*b)$.

Lemma 2.10. *Positive linear functionals are continuous. If u_λ is an approximate unit for A, then $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$. In particular, if A is unital, then $\|\varphi\| = \varphi(1)$.*

Proof. Let us first consider the unital case. If $0 \leqslant a \leqslant 1$, then since φ is positive, we obtain that $0 \leqslant \varphi(a) \leqslant \varphi(1)$. For $x \in A$ with $\|x\| \leqslant 1$ we have $0 \leqslant x^*x \leqslant 1$, so $|\varphi(x)|^2 = |\varphi(1 \cdot x)|^2 \leqslant \varphi(1) \cdot \varphi(x^*x) \leqslant \varphi(1)^2$ by the Cauchy-Schwartz-Bunyakovsky inequality. Thus, $\|\varphi\| \leqslant \varphi(1) \leqslant \|\varphi\|$.

Now consider the non-unital case. Suppose that φ is not bounded on the unit ball A. Then it is not restricted on the subset of the unit ball consisting of positive elements (since any element a is decomposable into a linear combination of four positive elements with norms not exceeding $\|a\|$, see (1.7)). Thus, for every $k \in \mathbb{N}$ there is a positive element $a_k \in A$ such that $\|a_k\| \leq 1$ and $\varphi(a_k) > 2^k$. Let us put $a := \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in A$. Then for any $n \in \mathbb{N}$ we have $a \geqslant \sum_{k=1}^{n} \frac{a_k}{2^k}$ and

$$\varphi(a) \geqslant \varphi\left(\sum_{k=1}^{n} \frac{a_k}{2^k}\right) = \sum_{k=1}^{n} \frac{\varphi(a_k)}{2^k} > n,$$

that is impossible. Thus, φ is bounded in the non-unital case as well.

Let $m := \lim_{\lambda\in\Lambda} \varphi(u_\lambda^2)$, and the limit exists since the direction net is increasing and bounded by $\|\varphi\|$ from above. Then, for any $x \in A$ with $\|x\| \leqslant 1$ we have $|\varphi(x)| = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|$ due to the continuity of φ. Therefore, by the Cauchy-Schwartz-Bunyakovsky inequality we have $|\varphi(x)|^2 \leqslant \varphi(u_\lambda^2)\varphi(x^*x) \leqslant m\|\varphi\|$. For any $\varepsilon > 0$ we choose an element $x \in A$ such that $\|\varphi\|^2 < |\varphi(x)|^2 + \varepsilon$. Then $\|\varphi\|^2 < m\|\varphi\| + \varepsilon$. Hence, $\|\varphi\|^2 \leqslant m\|\varphi\|$ and $\|\varphi\| \leqslant m$. Since for any $\varepsilon > 0$ there is u_{λ_0} for which $\varphi(u_{\lambda_0}^2) > m + \varepsilon$, we come to the equality $\|\varphi\| = m$. Since $u_\lambda^2 \leqslant u_\lambda$ and $m \leqslant \lim_{\lambda\in\Lambda} \varphi(u_\lambda) \leqslant \|\varphi\| = m$, we have $\lim_{\lambda\in\Lambda} \varphi(u_\lambda) = \|\varphi\|$. □

Corollary 2.11. *If φ is a state on a unital C^*-algebra, then $\varphi(1) = 1$.*

Proof. By the previous lemma, $1 = \|\varphi\| = \varphi(1)$. □

2.3 GNS-construction (Gelfand-Naimark-Segal)

Definition 2.12. A vector $\xi \in H$ is called *cyclic* for $\pi : A \to \mathbb{B}(H)$, if $\pi(A)\xi$ is dense in H.

Theorem 2.13. *Let φ be a positive linear functional on the C^*-algebra A. Then there exists a representation π_φ of the algebra A on the Hilbert space H and a cyclic vector $\xi_\varphi \in H$ such that $\|\xi_\varphi\|^2 = \|\varphi\|$ and $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ for all $a \in A$.*

Proof. Let $N := \{a \in A : \varphi(a^*a) = 0\}$. Then $N = \{a \in A : \varphi(b^*a) = 0 \text{ for all } b \in A\}$ by the Cauchy-Schwartz-Bunyakovsky inequality. Therefore N is closed as an intersection kernels of continuous functionals $a \mapsto \varphi(b^*a)$). Besides, N is a left ideal, since $\varphi(b^*an) = \varphi((a^*b)^*n) = 0$ for any $a, b \in A$ for $n \in N$, so $an \in N$.

Let us define an Hermitian inner product on the Banach quotient space A/N by the formula $(\dot{a}, \dot{b}) = \varphi(a^*b)$, where $\dot{a}$ denotes the coset class $a+N$. This product is well defined because if $n_1, n_2 \in N$, then $\varphi((a+n_1)^*(b+n_2)) = \varphi(a^*b) + \varphi((a+n_1)^*n_2) + \overline{\varphi(b^*n_1)} = \varphi(a^*b)$. Also $(\dot{a}, \dot{a}) > 0$ holds for $\dot{a} \neq 0$. Let H be the Hilbert space obtained from A/N by the completion w.r.t. the norm given by this inner product. Let us denote by π_0 the representation of A on A/N (here we slightly expand the concept of representation to a pre-Hilbert space) by the formula $\pi_0(a)\dot{x} = (ax)^{\cdot}$, where $\dot{x} \in A/N$. If $n \in N$ then $(a(x+n))^{\cdot} = (ax)^{\cdot}$, so π_0 is well defined. It is involutive, because $(\pi_0(a)\dot{x}, \dot{y}) = \varphi((ax)^*y) = \varphi(x^*(a^*y)) = (\dot{x}, \pi_0(a^*)\dot{y}) = (\pi_0(a^*)^*\dot{x}, \dot{y})$ and $\pi_0(a^*)^* = \pi_0(a)$. In this case, $\|\pi_0\| \leqslant 1$. Really,

$$\begin{aligned} \|\pi_0(a)\|^2 &= \sup_{\|\dot{x}\| \leqslant 1} \|\pi_0(a) \cdot x\|^2 = \sup_{\|\dot{x}\| \leqslant 1} \varphi(x^*a^*ax) \leqslant \\ &\leqslant \sup_{\|\dot{x}\| \leqslant 1} \|a^*a\| \varphi(x^*x) \leqslant \|a\|^2. \end{aligned}$$

Therefore π_0 extends by continuity to a representation π_φ of the algebra A on H.

If the algebra A is unital, then we set $\xi_\varphi := \dot{1}$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ and ξ_φ is cyclic since $\pi_\varphi(A)\xi_\varphi = A/N$ is dense in H. Finally, $\|\varphi\| = \varphi(1) = \|\xi_\varphi\|^2$.

For a general algebra A, consider its approximate unit u_λ. Let us show that $\dot{u}_\lambda$ is a Cauchy directed net. Let us choose an $\varepsilon > 0$. Then there is an index $\alpha \in \Lambda$ such that $\varphi(u_\alpha) > \|\varphi\| - \varepsilon$ (since $\|\varphi\| = \lim_{\lambda \in \Lambda} \varphi(u_\lambda)$ by Lemma 2.10). Now let us find an index $\beta \in \Lambda$ such that $\beta \geqslant \alpha$ and $\|u_\lambda u_\alpha - u_\alpha\| < \varepsilon$ for any $\lambda \geqslant \beta$. Then

$$\mathrm{Re}(\varphi(u_\lambda u_\alpha)) = \varphi(u_\alpha) + \mathrm{Re}(\varphi(u_\lambda u_\alpha - u_\alpha)) > \|\varphi\| - 2\varepsilon.$$

That is why

$$\begin{aligned} \|\dot{u}_\lambda - \dot{u}_\alpha\|^2 &= \varphi((u_\lambda - u_\alpha)^2) = \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2\mathrm{Re}(\varphi(u_\lambda u_\alpha)) \leqslant \\ &\leqslant \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2(\|\varphi\| - 2\varepsilon) \leqslant 4\varepsilon. \end{aligned}$$

This means that for $\lambda, \mu \geqslant \beta$, we have

$$\|\dot{u}_\lambda - \dot{u}_\mu\| \leqslant \|\dot{u}_\lambda - \dot{u}_\alpha\| + \|\dot{u}_\alpha - \dot{u}_\mu\| \leqslant 4\varepsilon^{1/2}.$$

Thus, $\dot{u}_\lambda$ is a Cauchy net. Let $\xi_\varphi := \lim_{\lambda \in \Lambda} \dot{u}_\lambda \in H$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \lim_{\lambda \in \Lambda} \varphi(u_\lambda a u_\lambda) = \varphi(a)$. Since $\pi_\varphi(A)\xi_\varphi = A/N$, then ξ_φ is cyclic. From $\dot{a} = \pi_\varphi(a)\xi_\varphi$ it follows that

$$\lim_{\lambda \in \Lambda} \pi_\varphi(u_\lambda)\dot{a} = \lim_{\lambda \in \Lambda} \pi_\varphi(u_\lambda)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(a)\xi_\varphi = \dot{a}$$

for any $\dot{a} \in A/N$, so the directed net $\pi_\varphi(u_\lambda)$ strongly converges to 1. Therefore $\|\varphi\| = \lim_{\lambda \in \Lambda} \varphi(u_\lambda) = \lim_{\lambda \in \Lambda}(\xi_\varphi, \pi_\varphi(u_\lambda)\xi_\varphi) = \|\xi_\varphi\|^2$. $\square$

2.4 Realization of C^*-algebras as operator algebras on Hilbert space

Corollary 2.14. *Any state φ on a non-unital C^*-algebra A admits a unique extension to a state on A^+.*

Proof. Let π_φ be the representation of A given by the GNS construction. Let us set $\pi_\varphi(1) = 1$. Then π_φ can be extended to a representation of A^+ and $\tilde{\varphi}(a) := (\xi_\varphi, \pi(a)\xi_\varphi)$ is a state. It is unique, since $\tilde{\varphi}(1) = 1$ must hold (Corollary 2.11). □

Problem 45. Let u_λ, $\lambda \in \Lambda$, be some approximate unit in a unital algebra. Prove that $1 = \lim_{\lambda\in\Lambda} u_\lambda$.

Lemma 2.15. *Let $\varphi : A \to \mathbb{C}$ be a continuous linear functional such that $\|\varphi\| = 1 = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$ for some approximate unit u_λ. Then φ is a state.*

Proof. Let us first reduce the proof to the unital case. Let $\tilde{\varphi}$ be some extension (by the Hahn-Banach theorem) functional φ to a continuous functional on A^+. Let $\tilde{\varphi}(1) =: \alpha$. Because the $\|\tilde{\varphi}\| = 1$, then $|\alpha| \leqslant 1$. From inequality $\|2u_\lambda - 1\| \leqslant 1$ it follows that $|2 - \alpha| = \lim_{\lambda\in\Lambda} |\varphi(2u_\lambda - 1)| \leqslant 1$. Thus, $\alpha = 1$. This means that we can assume that A is unital and $\varphi(1) = 1$ (if A was unital from the very beginning, then we use the problem 45).

Let us now show that $\varphi(a) \in \mathbb{R}$ if $a = a^*$ (and therefore contained in $[-\|a\|, \|a\|]$). Let a — self-adjoint element of norm 1. Then $\|a \pm in1\|^2 = \|a^2 + n^2 1\| = n^2 + 1$, so $|\varphi(a) \pm in| \leqslant \sqrt{n^2+1}$ for any $n \in \mathbb{N}$. This means that $\varphi(a)$ is contained in the intersection of all disks with centers at $\pm in$ and radii $\sqrt{n^2+1}$. This intersection is equal to the real interval $[-1, 1]$.

If $0 \leq a \leq 1$, then $\|2a - 1\| \leq 1$. Applying the previous reasoning to the self-adjoint element $2a - 1$, we obtain that $-1 \leqslant 2\varphi(a) - 1 \leqslant 1$, so $\varphi(a) \geqslant 0$ and φ is positive. □

Lecture 8

Lemma 2.16. *Let $a \in A$ be a self-adjoint element. Then there is a state φ on A such that $|\varphi(a)| = \|a\|$.*

Proof. If A is non-unital, then we will work in A^+. Consider the commutative C^*-algebra $C^*(a)$. Then there is a multiplicative linear functional φ_0 on $C^*(a)$ such that $|\varphi_0(a)| = \|a\|$ (we must take as φ_0 the mapping, which is the taking of the value of functions at that point of $\mathrm{Sp}(a)$, where the function $\hat{a}$ reaches its maximum). Then $\varphi_0(1) = 1 = \|\varphi_0\|$. Consider the extension of φ_0 by the Hahn-Banach theorem to a functional φ on A^+. Then, since $\|\varphi\| = 1 = \varphi(1)$, then φ is a state by Lemma 2.15. □

Corollary 2.17. *For any $a \in A$ there exists a representation π and a unit vector ξ in the space of representation such that $\|\pi(a)\xi\| = \|a\|$.*

Proof. By the previous lemma, we find a state φ such that $\varphi(a^*a) = \|a\|^2$. Let $\pi = \pi_\varphi$ and $\xi = \xi_\varphi$ were obtained for φ using the GNS construction. Then $\|\pi(a)\xi\|^2 = (\xi, \pi(a^*a)\xi) = \varphi(a^*a) = \|a\|^2$. □

Theorem 2.18 (Gelfand-Naimark)**.** *Any C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H. If A is separable, then H can be chosen to be separable.*

Proof. Let us set $\pi = \oplus_\varphi \pi_\varphi$, where the direct sum is taken over all states on A. More precisely, we consider the Hilbert direct sum $H := \oplus_\varphi H_\varphi$ (completion with respect to the ℓ_2 norm of the space of compactly supported mappings $\varphi \mapsto h_\varphi \in H_\varphi$, that is, the sets $h = \{h_\varphi\}$, $h_\varphi \in H_\varphi$, and only a finite number h_φ is nonzero, and the norm is defined as $\|h\|^2 = \sum_\varphi \|h_\varphi\|^2$) with diagonal action $\pi(a)(\{h_\varphi\}) = \{\pi_\varphi(a)(h_\varphi)\}$. Then, as can be seen from the proof of the previous consequences, $\|\pi(a)\| = \sup_\varphi \|\pi_\varphi(a)\| = \|a\|$. If A is separable, then it is sufficient to take the sum over a countable set $\{\varphi_n\}$, where $\|\pi_{\varphi_n}(a_n)\| = \|a_n\|$, for elements a_n forming a dense subset in A. Then $\pi = \oplus_{n\in\mathbb{N}} \pi_{\varphi_n}$, and the corresponding Hilbert space is separable, since each H_{φ_n} is separable (as a completion of a factor-space of a separable space). □

Definition 2.19. The representation constructed in the theorem (in its first part) is called the *universal representation* of A. The von Neumann algebra $\pi(A)''$, where π is the universal representation, contains $\pi(A) \cong A$ as a dense subset and is called the *enveloping von Neumann algebra* for A.

2.5 Jordan decomposition

Lemma 2.20. *Let φ be a linear functional on A. Then $\varphi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are self-adjoint.*

Proof. Let us take, in the same way as we did for elements of algebra, $\psi_1(a) = (\varphi(a) + \overline{\varphi(a^*)})/2$ and $\psi_2(a) = (\varphi(a) - \overline{\varphi(a^*)})/2i$. □

Let A_{sa} denote the set of all self-adjoint elements of A. Then it is evident that A_{sa} is a real Banach space.

Problem 46. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa}.

To prove the Jordan decomposition theorem, we need the following statement, which is of independent interest.

Theorem 2.21 (on extension of positive functionals)**.** *Let $B \subset A$ be a C^*-subalgebra, and $\varphi : B \to \mathbb{C}$ be a positive functional. Then there exists a positive functional $\varphi' : A \to \mathbb{C}$ such that that $\varphi'|_B = \varphi$ and $\|\varphi'\| = \|\varphi\|$.*

Proof. The following cases are possible:

a) both algebras have a common unit,

b) A has one, but B does not,

c) both algebras do not have a unit,

d) B has one, but A does not.

e) both algebras with 1, but $1_A \neq 1_B$.

By Corollary 2.14, (c) and (b) can be reduced by adjoining 1 to (a) (for (b) it should be noted that $B^+ \cong B \oplus \mathbb{C}\, 1_A$). In turn, (d) obviously reduces to (e).

In case (a) we extend φ (using the Hahn-Banach theorem) to some $\varphi' : A \to \mathbb{C}$ of the same norm. Then by Lemma 2.10, $\|\varphi'\| = \|\varphi\| = \varphi(1) = \varphi'(1)$ and φ' is positive by Lemma 2.15.

In case (e), consider the C^*-subalgebra $B_1 := B \oplus \mathbb{C}\, 1_A = B \oplus \mathbb{C}\,(1_A - 1_B)$ and extend φ to $\varphi_1 : B_1 \to \mathbb{C}$, setting $\varphi_1(1_A - 1_B) = 0$. Then $\varphi_1(a) = \varphi(1_B \cdot a)$, where $a \in B_1$. Indeed, if $a \in B$, then $\varphi_1(a) = \varphi(1_B \cdot a) = \varphi(a)$, and if $a = 1_A - 1_B$, then $\varphi_1(a) = \varphi(1_B(1_A - 1_B)) = \varphi(0) = 0$. In this case, the unit of B_1 is 1_A. Moreover, $\|\varphi_1\| \leqslant \|\varphi\| \cdot \|1_B\| = \|\varphi\|$, and $\varphi_1(1_A) = \varphi(1_B) = \|\varphi\|$. This means that $\|\varphi_1\| = \|\varphi\| = \varphi_1(1_A) = \varphi_1(1_{B_1})$ and, by Lemma 2.15, φ_1 is positive. Thus, case (e) is also reduced to the proven case (a). □

The Jordan theorem about decomposition of a measure in the sum of positive and negative ones [**?**, Ch. VI, §5, Theorem 1] in the functional language (in the sense of the Riesz-Markov-Kakutani theorem [**?**, Ch. I, §6, Theorem 4]) can be written as: for any bounded real linear functional $\tau : C(\Omega, \mathbb{R}) \to \mathbb{R}$ there are positive linear functionals τ_+ and τ_- such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$, where Ω is a compact Haudorff space and $C(\Omega, \mathbb{R})$ is the real algebra of all real continuous functions on Ω.

Theorem 2.22 (Jordan decomposition)**.** *Let ψ be a self-adjoint linear functional on A. Then $\psi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive linear functionals on A and $\|\psi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Denote by K the set of all self-adjoint linear functionals of norm $\leqslant 1$, i.e., $K \subset (A^*)_{sa}$. Then K is a $*$-weak closed subset of the unit ball and hence it is $*$-weak compact. Define an $\mathbb{R}$-linear map

$$\theta : A_{sa} \to C(K, \mathbb{R}), \qquad \theta(a)(\tau) = \tau(a),$$

so, if $a \in A$, $a \geqslant 0$, then $\theta(a) \geqslant 0$ in K. By Lemma 2.16 the mapping θ is an isometry onto its image.

There is a natural isometry $\tau \mapsto \tau'$ of real spaces $(A^*)_{sa}$ and $(A_{sa})^*_{\mathbb{R}}$ (real functionals) (see Problem46). By the Hahn-Banach theorem there is a functional $\rho \in (C(K, \mathbb{R}))^*_{\mathbb{R}}$ such that $\rho \circ \theta = \psi'$ and $\|\rho\| = \|\psi'\|$ (an extension of a functional from the closed subspace $\theta(A_{sa})$). Then by the Jordan theorem for measures (as it is explained above before the formulation) there are positive functionals ρ_+ and ρ_- such that $\rho = \rho_+ - \rho_-$ and $\|\rho\| = \|\rho_+\| + \|\rho_-\|$. Consider $\varphi'_+ := \rho_+ \circ \theta$ and $\varphi'_- := \rho_- \circ \theta$. These are functionals from $(A_{sa})^*_{\mathbb{R}}$. Let φ_+ and φ_- correspond to them under the identification with $(A^*)_{sa}$. Evidently they satisfy all the conditions, except maybe the norm property. Let us verify it:

$$\|\psi\| = \|\psi'\| = \|\rho\| = \|\rho_+\| + \|\rho_-\| \geqslant \|\varphi'_+\| + \|\varphi'_-\| = \|\varphi_+\| + \|\varphi_-\| \geqslant \|\psi\|.$$

□

2.6 Linear topological spaces

Definition 2.23. A subset M of a linear space is called *balanced*, if for any $v \in M$ the vector λv belongs to M for any $|\lambda| \leqslant 1$. In particular, M is a star set relative to the zero of space.

Definition 2.24. A subset M of a linear space is called *absorbing*, if for any vector v of the space there is a number $\alpha > 0$ such that $v \in \beta M$ for $|\beta| \geqslant \alpha$.

Definition 2.25. A linear space equipped with a topology is called *linear topological space* (LTS), if the operations of linear space are continuous.

In the basic course of functional analysis, the following simple statements are proved: (see [**?**, Chapter III, §5]):

Proposition 2.26. *1). A base of LTS consists of shifts of neighborhoods of zero.*

2). Any vector of LTS and a closed set not containing it have disjoint neighborhoods.

Definition 2.27. An LTS L satisfies the *homothety condition*, if for any neighborhood of zero W its homothety λW is also a neighborhood of zero for any $\lambda \neq 0$ from the main field.

Remark 2.28. Obviously, the topology of a normed space satisfies the homothety condition.

Proposition 2.29. *For any neighborhood of zero U of an LTS L with the homothety condition, there is a balanced neighborhood contained in it.*

Proof. Consider the continuous mapping $\mathbb{C} \times L \to L$ (multiplication) mapping $(0, 0_L) \mapsto 0_L$. Then, by virtue of continuity, there are $\delta > 0$ and a neighborhood of zero W such that $\lambda W \subseteq U$ for $|\lambda| \leqslant \delta$ (a non-strict inequality can be achieved by reducing δ from the standard definition). Let $W' := \cup_{0<|\lambda|\leqslant 1} \lambda W$. By virtue of 2.27, this W' is what we are looking for. □

Remark 2.30. In fact, it can be proven that the base of neighborhoods of zero of an LTS L can be chosen from absorbing balanced sets, and also that the homothety condition is in fact not a condition, but we will not need this (see [**?**, Chapter II, §4]).

We will need the following important result.

Theorem 2.31. *Let L be a finite-dimensional space, $\dim L = n$. Then any Hausdorff topology τ making L a linear topological space L_τ with the homothety condition coincides with the topology of the Euclidean norm $\|v\|^2 = \sum_{i=1}^n |v^i|^2$, where $e_1, \dots, e_n$ is some base of L, and $v = v^1 e_1 + \cdots v^n e_n$.*

Proof. The space L with Euclidean (or unitary) topology will be denoted by L_u, and neighborhoods of zero of two topologies (τ and Euclidean) will be denoted by T and U, respectively.

Consider an arbitrary T. Then there is a neighborhood T_0 such that $T_0 + \cdots + T_0 \subset T$ (n terms) due to the continuity of the addition operation. For every k there is $\varepsilon_k > 0$ such that $v^k e_k \in T_0$ for $|v_k| < \varepsilon_k$ ($k = 1, \dots, n$). Let $\varepsilon := \min_k \varepsilon_k$, and $U := \{v \in L \mid \|v\| < \varepsilon\}$. Then $v^k e_k \in T_0$ for any $v \in U$ and any $k = 1, \dots, n$. Thus, $U \subset T$. From what has been proved, in particular, it follows that the identity mapping $\iota : L_u \to L_\tau$ is continuous.

Conversely, let U be an arbitrary neighborhood, we can assume that $U = B(0, \varepsilon)$ is an open ball of radius ε with boundary (sphere) S, which is a compact set. Then $S = \iota(S)$ is compact in L_τ. This means that it is closed, since the topology is Hausdorff. Then there is a stellar neighborhood of zero T (for example, balanced) that does not intersect S by virtue of propositions 2.26 and 2.29. Moreover, $T \subseteq U$, since otherwise there exists a vector $v \in T$ such that $\|v\| \geqslant \varepsilon$, and if we put $\alpha := \varepsilon/\|v\|$, $w := \alpha v$, then $\alpha \leqslant 1$, so $w \in T$ by the star property. But $\|w\| = \varepsilon$, so $w \in T \cap S = \varnothing$. A contradiction. □

Lecture 9

2.7 Finite-dimensional C^*-algebras

Consider the $*$-weak topology on A defined by the seminorm system $a \mapsto |\varphi(a)|$ for all linear functionals φ. From Lemma 2.20 and Theorem 2.22 it follows that the same topology can be obtained by using only seminorms, defined by states.

Note also that the corresponding LTS has the homothety property 2.27.

Lemma 2.32. *A finite-dimensional C^*-algebra is always unital.*

Proof. If A is finite-dimensional, then the topology of the norm coincides with the $*$-weak topology according to Theorem 2.31. Let u_n be an approximate unit of the algebra A. Then for any state φ the sequence $\varphi(u_n)$ is non-decreasing and bounded. Therefore u_n converges in $*$-weak topology, and therefore in norm. Thus, there is a limit $\lim_n u_n = a$. Then $ax = xa = x$ for any $x \in A$, so $a = 1$. $\square$

Lemma 2.33. *Let $I \subset A$ be an ideal in a finite-dimensional C^*-algebra A. Then $I = Ap$ for some central projection (=idempotent from the center) p.*

Proof. Since I is finite-dimensional, it is unital by Lemma 2.32. Let $p \in I$ be the unit of I. Then for every $x \in A$, one has $xp \in I$, so $p(xp) = xp$. Hence $px^*p = x^*p$ for any $x \in A$, whence $xp = pxp = px$ and p belongs to the center of A. Obviously, $p^2 = p$. $\square$

Lemma 2.34. *A simple finite-dimensional C^*-algebra A is isometrically $*$-isomorphic to the matrix algebra M_n for some n.*

Proof. First of all, note that $aAb \neq 0$ for any non-zero $a, b \in A$. Indeed, AaA is a non-zero ideal (since A is unital and $0 \neq a = 1 \cdot a \cdot 1 \in A$), so by simplicity, $AaA = A$. Therefore $1 = \sum_i x_i a y_i$ and $b = \sum_i x_i a y_i b$. Hence, if $ayb = 0$ for any $y \in A$, then $b = \sum_i x_i (a y_i b) = 0$. This contradicts the assumption.

Let B be some maximal commutative subalgebra of A. Then it can be identified with $C(X) = \mathbb{C}^n = \mathbb{C} \cdot e_1 \oplus \ldots \oplus \mathbb{C} \cdot e_n$ for some n, where X consists of n points, and $e_i \in B$ denotes the element corresponding to the characteristic functions at point i. Here e_i are projections with the relations $e_i e_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1$. Since $e_i A e_i \cdot e_j = e_j \cdot e_i A e_i = 0$ and B is maximal, then $e_i A e_i \subset B$. Therefore $e_i A e_i = \mathbb{C} \cdot e_i$ (since, obviously, $0 \neq e_i A e_i \ni e_i$, or you can use the statement from the beginning of the proof).

For any i, j there is $x = x_{ij} \in A$ such that $x = e_i x e_j \neq 0$, $\|x\| = 1$. Indeed, by virtue of the statement from the beginning of the proof, $e_i A e_j \neq 0$, so we have $x = e_i y e_j$ with $\|x\| = 1$. In this case $e_i x e_j = e_i e_i y e_j e_j = e_i y e_j = x$. Then $x^* x = e_j x^* e_i e_i x e_j \in e_j A e_j$, and therefore, according to what has been proven, this element has the form αe_j, $\alpha \in \mathbb{C}$. Since $x^* x$ is a positive element with norm equal to one, then $\alpha = 1$, so $x^* x = e_j$. Likewise, $x x^* = e_i$. Let us denote such $x = x_{ij}$ for $j = 1$ by u_i, so that $u_i = e_i x e_1 = e_i u_i e_1$. Then $u_i^* u_i = e_1$, $u_i u_i^* = e_i$, $i = 1, \ldots, n$. Let us set $u_{ij} := u_i u_j^*$. In this case, $u_i e_1 u_i^* = u_i u_i^* u_i u_i^* = e_i e_i = e_i$, So $u_{ij}u_{ji} = u_i u_j^* u_j u_i^* = u_i e_1 u_i^* = e_i$. Also $e_j u_{ji} = u_j u_j^* u_j u_i^* = u_j e_1 u_i^* = u_j u_i^* u_i u_i^* = u_{ji} e_i$, and $e_i u_{ij} = u_i u_i^* u_i u_j^* = u_i e_1 u_j^* = u_i u_j^* u_j u_j^*$.

If $x \in e_i A e_j$, that is, $x = e_i a e_j$, then $x u_{ji} = e_i a e_j u_{ji} = e_i a u_{ji} e_i \in e_i A e_i$, so $x u_{ji} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Then $x = x e_j = x u_{ji} u_{ij} = \lambda e_i u_{ij} = \lambda u_{ij}$, so for any $x \in A$ there is a number $\lambda_{ij}(x) \in \mathbb{C}$ such that $e_i x e_j = \lambda_{ij}(x) u_{ij}$. Thus, $x = \sum_{i,j} e_i x e_j = \sum_{ij} \lambda_{ij}(x) u_{ij}$. The correspondence $x \mapsto (\lambda_{ij}(x))$ defines an isomorphism $\kappa : A \to M_n$ (Problem 47). $\square$

Problem 47. Check the bijectivity and necessary algebraic properties of κ.

Theorem 2.35. *If A is finite-dimensional, then $A = \oplus_k A p_k$, where p_k are central projections, and each $A p_k$ is a matrix algebra $M_{n(k)}$.*

Proof. For a simple algebra, the result follows from Lemma 2.34. If A is not simple, then $I = Ap$ by Lemma 2.33, where p is a central projection. Then $A = I \oplus J$, where $J := A(1-p)$. Then J is also an ideal, since $(1-p)$ is also a central projection, so $A(1-p)A = AA(1-p) \subseteq A(1-p)$. In this case, the center of A, being a finite-dimensional commutative algebra, is isomorphic to $\mathbb{C}^m$ (functions on finite set), and characteristic functions correspond to the projections. Next, we argue by induction, reducing the dimension, until we arrive to the sum of simple algebras. $\square$

2.8 Non-degenerate representations

Definition 2.36. Let π be a representation of a C^*-algebra A on a Hilbert space H. We denote by $\pi(A)H$ the (possibly non-closed) linear space of finite linear combinations of the form $\sum_i \pi(a_i)\xi_i$, where $a_1, \dots, a_n \in A$, $\xi_1, \dots, \xi_n \in H$. A representation π is called *non-degenerate*, if $\pi(A)H$ is dense in H.

Problem 48. If A is unital, then π is non-degenerate if and only if $\pi(1) = 1$.

Lemma 2.37. *Let $I \subset A$ be an ideal and π a non-degenerate representation of I on a Hilbert space H. Then there is a unique extension π to a representation $\tilde{\pi}$ of the entire algebra A on H.*

Proof. Let us first define $\tilde{\pi}$ on vectors from the dense subspace $\pi(I)H \subset H$ by the formula

$$\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right) := \sum_i \pi(a j_i)\xi_i. \tag{2.3}$$

This is well-defined because if $\sum_i \pi(j_i)\xi_i = \sum_i \pi(j_i')\xi_i'$, then

$$\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \tilde{\pi}(a)\left(\sum_i \pi(u_\lambda j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \pi(a u_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)$$

and, similarly, $\tilde{\pi}(a)(\sum_i \pi(j_i')\xi_i') = \lim_{\lambda\in\Lambda} \pi(a u_\lambda)(\sum_i \pi(j_i')\xi_i')$, where $u_\lambda \in I$ is an approximate unit of I. Note that the existence of the last limit in the chain follows from the existence of the penultimate limit. Hence, for each of the two cases it should be proved separately. Since

$$\begin{aligned}\left\|\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right)\right\| &= \lim_{\lambda\in\Lambda}\left\|\pi(au_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)\right\| \leqslant \sup_{\lambda\in\Lambda}\|\pi(au_\lambda)\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| \leqslant \\ &\leqslant \|a\|\cdot\sup_{\lambda\in\Lambda}\|u_\lambda\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| = \|a\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\|,\end{aligned}$$

$\tilde{\pi}$ is bounded, i.e., $\tilde{\pi}(a)$ extends to a bounded operator in H.

At the same time, it is easy to check $\tilde{\pi}(ab) = \tilde{\pi}(a)\tilde{\pi}(b)$ and $\tilde{\pi}(a^*) = \tilde{\pi}(a)^*$ for any $a, b \in A$, so $\tilde{\pi}$ is a representation of A. Uniqueness follows from the fact that any extension of π has to satisfy (2.3). □

Lemma 2.38. *Under the conditions of Lemma 2.37 the representation π is irreducible if and only if $\tilde{\pi}$ is irreducible.*

Proof. Let π be reduced by a proper invariant subspace $L \subset H$. Then, due to non-degeneracy, $H = \overline{\pi(I)(L + L^\perp)} \subseteq \overline{\pi(I)L} + \overline{\pi(I)L^\perp}$. Since $L^\perp$ is also invariant, then $\overline{\pi(I)L^\perp} \subset L^\perp$, so $\overline{\pi(I)L} = L$. Then $\tilde{\pi}(A)L = \tilde{\pi}(A)\overline{\pi(I)L} = \overline{\pi(I)L} = L$ and L reduces $\tilde{\pi}$. The opposite statement is trivial. □

Lemma 2.39. *Let π be a representation of A on a Hilbert space H, and $I \subset A$ is an ideal. Then the orthogonal projection p onto $\overline{\pi(I)H}$ lies in the center of $\pi(A)''$. If π is irreducible and $\pi(I) \neq 0$, then $\pi|_I$ is also irreducible.*

Proof. Since $\pi(A)\pi(I)H = \pi(I)H$, then $\overline{\pi(I)H}$ is an invariant space for $\pi(A)$, hence $p \in \pi(A)'$ (see the end of proof of Lemma 2.3). If $x \in \pi(I)'$, then $x\pi(j)\xi = \pi(j)x\xi \in \pi(I)H$ for any $j \in I$, $\xi \in H$, so pH is an invariant subspace of $\pi(I)'$ and, therefore, $p \in \pi(I)''$. So,

$$p \in \pi(I)'' \cap \pi(A)' \subset \pi(A)'' \cap \pi(A)',$$

that is the center of $\pi(A)''$.

If π is irreducible, then p is a scalar operator (that is, 0 or 1) (cf. Lemma 2.3), and since $\pi(I) \neq 0$, then $p = 1$. Thus, $\pi|_I$ is non-degenerate. So by Lemma 2.38 it is irreducible. □

Chapter 3

Special classes of C^*-algebras

3.1 C^*-algebra of compact operators

In this section we will consider C^*-subalgebras of C^*-algebra $\mathbb{K}(H)$ of compact operators on the Hilbert space H. We will say that C^*-subalgebra of the algebra $\mathbb{B}(H)$ *irreducible*, if its identical representation is irreducible.

Definition 3.1. The projection p is called *minimal*, if there is no projection $q \neq 0$, $q \neq p$ such that $qp = q$. In other words, p does not *dominate* any non-trivial projection.

Lemma 3.2. *Any nonzero C^*-algebra A consisting of compact operators contains a minimal projection e and $eAe = \mathbb{C} \cdot e$. If A is irreducible, then e is a rank 1 projection (as a projection in Hilbert space).*

Proof. Since A is nonzero, it contains a nonzero positive operator (see (1.7)), which (as is known from the basic course of functional analysis, see [**?**, Theorem 1, p. 360]), has a discrete spectrum (except of 0) with eigenvalues of finite multiplicities. Let us consider the spectral projection for a non-zero point of the spectrum. Since the characteristic function of this isolated point is continuous on the spectrum, then this projection belongs to A. Then among the nonzero projections dominated by it there is some projection $e \in A$ of minimal rank among the dominated (since they have finite ranks). Then e is minimal (the uniqueness of the minimal and even the equality of ranks of different minimal projections is not supposed). If eAe consists not only of $\mathbb{C} \cdot e$, then in the same way we can construct a projection dominated by e and arrive to a contradiction.

Now suppose that A is irreducible, but the rank of e is greater than 1. Let us choose a pair of nonzero orthogonal vectors ξ, η in the image e. Since for any a there is a number $\lambda \in \mathbb{C}$ such that $eae = \lambda e$, we have $(\xi, a\eta) = (e\xi, ae\eta) = (\xi, eae\eta) = \lambda(\xi, \eta)$, that is $a\eta \perp \xi$ for any $a \in A$. Considering all ξ from the image e being orthogonal to η, we see that the subspace $\overline{A\eta}$ is a proper invariant subspace. A contradiction. □

Lemma 3.3. *The only irreducible C^*-subalgebra of $\mathbb{K}(H)$ is itself.*

Proof. Let A be an irreducible C^*-subalgebra of $\mathbb{K}(H)$, and $e \in A$ a minimal projection of rank 1. Then there is a unit vector $\xi \in H$ such that $e\eta = \xi(\xi, \eta)$ for any η (we take ξ from the image of e). Due to irreducibility, for any $\eta, \zeta \in H$ there are elements $a, b \in A$ such that $a\xi = \eta$, $b\xi = \zeta$ (see Lemma 2.5). Moreover, $A \ni aeb^*$ and $aeb^*(\kappa) = a\xi(\xi, b^*\kappa) = \eta(\zeta, \kappa)$, $\kappa \in H$. Thus, A contains all operators of rank 1. Such operators generate $\mathbb{K}(H)$ (any compact operator is approximated by finite-dimensional), so $A = \mathbb{K}(H)$. □

Corollary 3.4. *The algebra* $\mathbb{K}(H)$ *is simple.*

Proof. Since $\mathbb{K}(H)$ is irreducible, then any non-zero ideal is also irreducible (by Lemma 2.39), so it coincides with $\mathbb{K}(H)$ (by Lemma 3.3). □

Corollary 3.5. *Let A be an irreducible C^*-subalgebra of $\mathbb{B}(H)$ containing a nonzero compact operator. Then $\mathbb{K}(H) \subseteq A$.*

Proof. Since $A \cap \mathbb{K}(H)$ is a nonzero ideal of A, it is irreducible by Lemma 2.39. By Lemma 3.3 this subalgebra of $\mathbb{K}(H)$ should coincide with the entire $\mathbb{K}(H)$. □

Lecture 10

3.2 AF-algebras

Definition 3.6. Let us call a C^*-algebra an *AF-algebra* (approximately finite-dimensional), if it is the closure of the union of an increasing sequence of its finite-dimensional C^*-subalgebras.

Problem 49. Prove that the matrix algebra M_n is simple for any n (this does not follow from Lemma 2.34, from which one can deduce that M_n is simple for some n). *Hint:* for any ideal $I \neq \{0\}$ consider a matrix from it with $a_{ij} \neq 0$. By multiplying on the left and right by matrices with 1's in one place and zeros in the rest, you obtain a matrix of I with a single nonzero element a_{ij}. Multiplying by permutation matrices, get similar matrices with all possible i, j. Their linear combinations give the entire M_n algebra.

Problem 50. Deduce from Problem 49 and Lemma 2.34 that the image of the matrix algebra M_n under a $*$-homomorphism is either a zero algebra or an algebra isomorphic to M_n.

Problem 51. Prove the following almost obvious fact: if p and q are projections of the same rank in M_n, then there exists a unitary matrix u such that $q = u^*pu$.

Lemma 3.7. *Let $\varphi : M_n \to M_k$ be a non-zero $*$-homomorphism, so that $p := \varphi(1_n)$ is a self-adjoint projection, where 1_n is the unit of M_n. Then $\mathrm{rk}(p) = \mathrm{Trace}(p)$ is divided by $n = \mathrm{rk}(1_n) = \mathrm{Trace}(1_n)$.*

Proof. Consider some one-dimensional orthogonal (self-adjoint) projection $e \in M_n$. Then $\varphi(e)$ is a self-adjoint projection in M_k. Its rank does not depend on the choice of e, since any other e' is equal to u^*eu (by problem 51), where u is unitary, so

$$\mathrm{Trace}(\varphi(e')) = \mathrm{Trace}(\varphi(u^*eu)) = \mathrm{Trace}(\varphi(u^*)\varphi(e)\varphi(u)) =$$

$$= \mathrm{Trace}(\varphi(u)\varphi(u^*)\varphi(e)) = \mathrm{Trace}(\varphi(uu^*e)) = \mathrm{Trace}(\varphi(e)).$$

If this (one for all) rank is zero, then φ would be zero. This means it is equal to $c \geqslant 1$. Let us now consider an orthonormal basis $e_1, \dots, e_n$ (for example, canonical) in $\mathbb{C}^n$ and denote the corresponding one-dimensional orthoprojections by $[e_i]$, so $[e_j]\,[e_i] = 0$ for $i \neq j$. Then, since $\varphi([e_i])\varphi([e_j]) = \varphi([e_ie_j]) = 0$ for $i \neq j$, we get

$$\mathrm{Trace}(p) = \mathrm{rk}(\varphi(1_n)) = \mathrm{rk}(\varphi([e_1] \oplus \cdots \oplus [e_n])) = \mathrm{rk}(\varphi([e_1])) + \cdots + \mathrm{rk}(\varphi([e_n])) = cn.$$

□

Definition 3.8. The ratio $c := \frac{\mathrm{rk}(p)}{n}$ will be called *multiplicity* of φ.

Along with the standard left action of M_n on $\mathbb{C}^n$, we consider the left action of M_n on itself by multiplication, so the canonical expansion

$$M_n \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{n \text{ times}} = M_n[e_1] \oplus \cdots \oplus M_n[e_n]$$

is a decomposition into simple modules (=irreducible representations), where $[e_i] \in M_n$ is an orthogonal projection onto the basis vector e_i of the standard basis. Another way to write it is $[e_i] = e_i \otimes e_i^*$ (considering matrices as endomorphisms), where e^* is the Hermitian conjugate functional for e, so $[e_i]v = (e_i \otimes e_i^*)v = e_i(e_i, v)$. For different vectors we get the matrix unit $e_{ij} = e_i \otimes (e_j)^*$, so $[e_i] = e_{ii}$.

Lemma 3.9. *Any irreducible left module M in M_n has the form $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$, where g, f are some unit (can be taken to be unit) vectors.*

Proof. For the left action, the module $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$ is isomorphic to $\mathbb{C}^n$ with the standard action, and therefore is irreducible. Therefore, if $g \otimes f^* \in M$, then $M = M_n(g \otimes f^*)$. It remains to show that M contains an element of the form $g \otimes f^*$. But this form describes any operator of rank 1. Indeed, if a is an operator of rank 1, then we must take as f the unit vector perpendicular to its kernel, and $g = a(f)$. Finally, if M is nonzero and $0 \neq b \in M$, then choose $f \neq 0$ from its image. Then $(f \otimes f^*)b$ is an operator of rank 1 from M. □

Theorem 3.10. *Let φ be a (unital) $*$-automorphism of the C^*-algebra M_n. Then it is inner: $\varphi(a) = vav^*$ for any a, where $v \in M_n$ is unitary.*

Proof. Note that φ is an isomorphism between M_n, considered as a left module over M_n with the standard action $a \cdot x$, and M_n, considered as a module with the action $a * x = \varphi(a) \cdot x$, since $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) = a * \varphi(x)$. Since $\varphi(M_n) = M_n$, then the invariant and irreducible modules for both actions are the same (the latter are described by Lemma 3.9), and $\varphi(\mathbb{C} \otimes e_i) = \mathbb{C} \otimes h_i$, moreover, since the automorphism takes a direct sum to a direct sum, then h_i form a basis in $\mathbb{C}^n$ and, thus, an isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $u : e_i \to h_i$ (even if we assume $\|h_i\| = 1$, then u is uniquely defined only up to multiplication by a diagonal matrix of complex numbers modulo one). Thus, $\varphi(e_{ij}) = r_{ij} \otimes h_j$, where $r_{ij} \in \mathbb{C}^n$ are some elements. Similar reasoning with right modules shows that $\varphi(e_{ij}) = g_i \otimes (s_{ij})^*$, where $v : \mathbb{C}^n \to \mathbb{C}^n$, $v : e_i \to g_i$, is an isomorphism, and $s_{ij} \in \mathbb{C}^n$ are some elements. From these two relations we obtain that $\varphi(e_{ij}) = \lambda_{ij} g_i \otimes (h_j)^*$, where λ_{ij} are some numbers. Moreover (see the proof of Lemma 3.7) $\varphi([e_i]) = \lambda_{ii} g_i \otimes (h_i)^*$ is a self-adjoint projection, so $h_i = \mu g_i$ and $g_i = (\lambda_{ij}\overline{\mu})(g_i \otimes (g_i)^*)(g_i) = (\lambda_{ij}\overline{\mu}) g_i$ and $\lambda_{ij}\overline{\mu} = 1$. Thus, we can assume that $h_i = g_i$, $u = v$ and (new) λ_{ij} satisfy $\lambda_{ii} = 1$ for any i. In this case, g_i form an orthonormal basis (see the proof of Lemma 3.7), so $u : e_i \mapsto g_i$ is unitary. Since the homomorphism φ preserves the equalities $e_{ij} e_{ji} = e_{ii} = [e_i]$ and $e_{ij}^* = e_{ji}$, we obtain that $\lambda_{ij}\lambda_{ji} = 1$, $\overline{\lambda_{ij}} = \lambda_{ji}$, so, in particular, λ_{ij} are complex numbers of norm 1.

Consider the matrix $\Lambda = \|\lambda_{ij}\|$. Passing, if necessary, from vectors g_i to $(\lambda_{1i})^{-1} g_i = \overline{\lambda_{1i}} g_i$ for $i = 2, \ldots, n$, we can consider $\lambda_{1i} = \lambda_{i1} = 1$ for $i = 2, \ldots, n$. At the same time, the image $\varphi(a)$ of a matrix $a = \|a_{ij}\|$ can be written down with respect to the orthonormal basis $\{g_i\}$ as $\|\lambda_{ij}a_{ij}\|$, and if the matrix a was unitary, then $\varphi(a)$ must also be unitary. Let some $\lambda_{ij} \neq 1$ (that is possible only for $i \neq j$, $i \neq 1$). Taking for these i, j the unitary matrix a with $a_{1i} = 1/\sqrt{2}$, $a_{1j} = 1/\sqrt{2}$, $a_{ii} = 1/\sqrt{2}$, $a_{ij} = -1/\sqrt{2}$ (and the rest in these rows and columns, of course, is formed with zeros), we obtain the orthogonality condition for these rows in the form $0 = (1/\sqrt{2}\cdot 1)(\overline{1/\sqrt{2}\cdot 1}) + (-1/\sqrt{2}\cdot\lambda_{ij})(\overline{1/\sqrt{2}\cdot 1}) = (1-\lambda_{ij})/2$, so $\lambda_{ij} = 1$. Contradiction.

So, for any i, j we get that $\varphi(e_i \otimes (e_j)^*) = v(e_i) \otimes (v(e_j))^* = v\cdot(e_i \otimes (e_j)^*)\cdot v^*$. Since any matrix is a linear combination of such matrix units, by linearity we obtain the required equality $\varphi(a) = vav^*$. □

Lemma 3.11. *Let the homomorphism φ be unital under the conditions of Lemma 3.7. Then φ is determined by multiplicity up to a unitary equivalence (conjugation) in M_k.*

Proof. Let $f_i^1, \dots, f_i^c$ be an orthonormal basis in the image of the projection $\varphi(e_i)$, $s = 1, \dots, n$. Denoting by $[f_i^j]$ the corresponding one-dimensional pairwise orthogonal projections, we have $\varphi(e_i) = [f_i^1] + \dots + [f_i^c]$. Then $\{f_i^j\}$, $i = 1, \dots, n$, $j = 1, \dots, c$, is an orthobasis $\mathbb{C}^k$, $1_k = \sum_{i,j}[f_i^j]$. If $u \in M_k$ is a unitary matrix taking $\{f_i^j\}$ to the canonical basis of $\mathbb{C}^k$, where $uf_i^j = e_{(i-1)n+j}$, then

$$[e_{(i-1)n+j}]x = e_{(i-1)n+j}(e_{(i-1)n+j}, x) = uf_i^j(uf_i^j, x) = uf_i^j(f_i^j, u^*x) = u[f_i^j]u^*x \qquad (3.1)$$

for any $x \in \mathbb{C}^k$. That is why

$$\varphi([e_i]) = [f_i^1] + \dots + [f_i^c] = \sum_{j=1}^{c} u^*[e_{(i-1)n+j}]u = u^*\left(\sum_{j=1}^{c}[e_{(i-1)n+j}]\right)u. \qquad (3.2)$$

Thus,

$$u\varphi(a)u^* = \begin{pmatrix} \varphi_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_c(a) \end{pmatrix} \text{ (block diagonal matrix)}, \qquad (3.3)$$

where $\varphi_i : M_n \to M_n$ is a non-zero homomorphism, and therefore an isomorphism ($i = 1, \dots, c$). Therefore, we can apply Theorem 3.10 to it and find a unitary $v_i \in M_n$, such that $\varphi_i(a) = v_i^* a v^i$. Denoting $v = v_1 \oplus \dots \oplus v_c$ (a unitary element from M_k), we obtain that $vu\varphi(a)u^*v^* = S_c(a)$ for any $a \in M_n$, where $S_c : M_n \to M_k$, $k = cn$, is the standard homomorphism of multiplicity c:

$$S_c(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix} \text{ (block diagonal matrix with } c \text{ blocks equal to } a\text{)},$$

as desired. □

Lemma 3.12. *Let φ be a unital $*$-homomorphism of a finite-dimensional C^*-algebra $A = M_{n_1} \oplus \ldots \oplus M_{n_k}$ into a finite-dimensional C^*-algebra $B = M_{m_1} \oplus \ldots \oplus M_{m_l}$. Then φ is given (up to unitary equivalence in B) by some $l \times k$-matrix $C = (c_{ij})$ with nonnegative elements, and $\sum_{j=1}^{k} c_{ij} n_j = m_i$.*

Proof. Let $\epsilon_i : B \to M_{m_i}$ be the canonical epimorphism, and $\sigma_j : M_{n_j} \to A$ the canonical embedding, $i = 1, \ldots, l$, $j = 1, \ldots, k$. Then $\epsilon_i \circ \varphi$ is a unital homomorphism of A into M_{m_i}. Let c_{ij} be the multiplicity of $\varphi_{ij} = \epsilon_i \circ \varphi \circ \sigma_j : M_{n_j} \to M_{m_i}$ in the sense of Definition 3.8.

Note that $\sigma_j(1_{n_j})$ are pairwise orthogonal self-adjoint projections in A with their sum equal to one, so $p_{ij} := \varphi_{ij}(1_{n_j})$ are pairwise orthogonal self-adjoint projections in M_{m_i}, also with their sum equal to one. Therefore, as before, their direct sum is unitarily equivalent with the help of u_i in M_{m_i} to the sum of the corresponding diagonal projections $u_i p_{ij} u_i^*$. Applying Lemma 3.11 to each of $a \mapsto u_i \varphi_{ij}(a) u_i^*$, we find that $\epsilon_i \circ \varphi$ is unitarily equivalent (in M_{m_i}) to the homomorphism $\mathrm{id}_{n_1}^{c_{i1}} \oplus \ldots \oplus \mathrm{id}_{n_k}^{c_{ik}} = S_{c_{i1}} \oplus \cdots \oplus S_{c_{ik}}$. Comparison of dimensions gives the equality $\sum_{j=1}^{k} c_{ij} n_j = m_i$. Since φ is determined by the direct sum $\epsilon_i \circ \varphi$, $i = 1, \ldots, l$, the statement is proven. $\square$

Problem 52. Suppose that in the previous lemma we exclude the requirement of unitality. Prove an analogue of the lemma in this case. Namely, take instead of B the subalgebra $\varphi(A) = p\varphi(A)p$ in B, where $p = \varphi(1_A)$, apply the previous lemma to $\varphi : A \to \varphi(A)$ and obtain the statement of the lemma with inequalities $\sum_{j=1}^{k} c_{ij} n_j \leqslant m_i$ instead of equalities.

Lecture 11

Definition 3.13. Any A $*$-homomorphism between finite-dimensional C^*-algebras can be represented in the following graphical way. Let's represent A in the form k-tuple $\{(1,1)=n_1,\dots,(1,k)=n_k\}$ corresponding $A\cong M_{n_1}\oplus\dots\oplus M_{n_k}$, and B — in the form l-tuple $\{(2,1)=m_1,\dots,(2,l)=m_l\}$ corresponding to $B\cong M_{m_1}\oplus\dots\oplus M_{m_l}$. Let us represent φ using arrows between the elements of sets, and from $(1,j)$ to $(2,i)$ we draw c_{ij} arrows, the number is equal to the partial multiplicity. A sequence of such pictures for a sequence of homomorphisms $A_1\subset A_2\subset\dots\subset A_p\subset\cdots$ is called the *Bratteli diagram of this sequence*. It is sometimes called the Bratteli diagram of an algebra, but the same algebra can be obtained from different sequences.

Problem 53. Draw Bratteli diagrams (for some defining sequences) of the following AF-algebras:

1) of the algebra of compact operators $\mathbb{K}(H)$;
2) of its unitization $\mathbb{K}(H)^+$;
3) the closure of the union of $A_p=M_{2^p}$, with embeddings $A_p\subset A_{p+1}$ of multiplicity 2 according to the formula $a\mapsto\begin{pmatrix}a&0\\0&a\end{pmatrix}$ (CAR algebra);
4) $C(K)$, where K is the Cantor set obtained from $[0,1]$ by successive removing the middle third of the corresponding intervals. If K_p is a set, obtained at the pth step of this process, then A_p is an algebra of continuous functions constant on intervals of K_p;
5) $C(X)$, where $X:=\{0\}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, and A_k consists of all functions constant on $[0,1/2^k]$.

Lemma 3.14. *If two Bratteli diagrams coincide, then the corresponding AF-algebras are isometrically $*$-isomorphic.*

Proof. Let A_n and B_n be two sequences of finite-dimensional C^*-algebras with inclusions $\alpha_n:A_n\to A_{n+1}$, $\beta_n:B_n\to B_{n+1}$. Since the Bratteli diagrams are the same, then for each n there is an isomorphism $\varphi_n:A_n\to B_n$. Consider $\varphi_{n+1}\circ\alpha_n$ and $\beta_n\circ\varphi_n:A_n\to B_{n+1}$. They can differ only by unitary $u_{n+1}\in B_{n+1}$, that is $\beta_n\circ\varphi_n=\mathrm{Ad}_{u_{n+1}}\varphi_{n+1}\circ\alpha_n$, where $\mathrm{Ad}_{u_{n+1}}(a)=u_{n+1}a(u_{n+1})^*$.

Let's put $\psi_1=\varphi_1$, $v_1=1$. Let us define inductively $v_{n+1}=\beta_n(v_n)u_{n+1}\in B_{n+1}$, $\psi_{n+1}=\mathrm{Ad}_{v_{n+1}}\varphi_{n+1}$. Then

$$\begin{aligned}\beta_n\psi_n &= \beta_n\,\mathrm{Ad}_{v_n}\varphi_n=\mathrm{Ad}_{\beta_n(v_n)}\beta_n\varphi_n=\mathrm{Ad}_{\beta(v_n)}\mathrm{Ad}_{u_{n+1}}\varphi_{n+1}\alpha_n\\ &= \mathrm{Ad}_{\beta_n(v_n)u_{n+1}}\varphi_{n+1}\alpha_n=\psi_{n+1}\alpha_n.\end{aligned}$$

In this case $\cup_{n=1}^\infty\psi_n:\cup_{n=1}^\infty A_n\to\cup_{n=1}^\infty B_n$ is an isometric $*$-isomorphism, so the closures are also isomorphic. $\square$

One should not think that AF-algebras are "small" and that C^*-subalgebras of AF-algebras are AF-algebras again. For example, $C[0,1]$ is not an AF-algebra (since its only finite-dimensional C^*-subalgebra consists of constant functions), but it is a C^*-subalgebra of the AF-algebra $C(K)$ functions on the Cantor set. Indeed, let f be a function on K that has a dense set of rational numbers in the interval $[0,1]$ as its values. For example, the restriction of the Cantor function $f : [0,1] \to [0,1]$ [**?**, Ch. VI, §4] on K has all rationals of the form $p/2^s$ as its values. Its spectrum is a closure of this set, that is, equal to the entire interval $[0,1]$. Thus, $C^*(1,f) \subset C(K)$ is isometrically $*$-isomorphic to $C(\mathrm{Sp}(f)) = C[0,1]$.

Lecture February 17

3.3 Multipliers

We will call a C^*-subalgebra $\mathbb{B}(H)$ *non-degenerate*, if its natural representation in the Hilbert space H is non-degenerate.

Everywhere in this section $A, B \subset \mathbb{B}(H)$.

Definition 3.15. An operator $x \in \mathbb{B}(H)$ is called *left multiplier* A if $xA \subset A$. It is called *right multiplier*, if $Ax \subset A$ and *double (or double-sided) multiplier* or simply *multiplier*, if both conditions are met.

If A is unital, then every left (right) multiplier lies in A.

Since A is weakly dense in A'', we can proceed to the closure $xA \subset A$ and get $xA'' \subset A''$. If A'' is equal to one, then $x \in A''$.

Definition 3.16. The linear mapping $\sigma : A \to A$ is called *left centralizer*, if $\sigma(ab) = \sigma(a)b$ for any $a, b \in A$. Linear mapping $\sigma : A \to A$ called *right centralizer*, if $\sigma(ab) = a\sigma(b)$ for any $a, b \in A$. Pair (σ_1, σ_2) called *double centralizer*, if σ_1 is a right centralizer, σ_2 is a left centralizer and $\sigma_1(a)b = a\sigma_2(b)$ for any $a, b \in A$.

Lemma 3.17. *Any left centralizer is bounded.*

Proof. Let us assume the opposite. Then for any $n \in \mathbb{N}$ there is an element $x_n \in A$ such that $\|x_n\| < 1/n$ and $\|\sigma(x_n)\| > n$. This means that the series $a = \sum_{n=1}^{\infty} x_n x_n^*$ converges, so $a \in A$ and $x_n x_n^* \leqslant a$. According to Lemma 1.45, x_n can be written as $x_n = a^{1/4}u_n$, where $\|u_n\| \leqslant \|a\|^{1/4}$. Therefore, for any n we have $\|\sigma(x_n)\| = \|\sigma(a^{1/4})u_n\| \leqslant \|\sigma(a^{1/4})\| \cdot \|a^{1/4}\|$. A contradiction. $\square$

Theorem 3.18. *If A is non-degenerate, then there is a bijective correspondence between left (right, double) multipliers and left (right, double) centralizers.*

Proof. If x is a left multiplier, then the mapping $A \ni a \mapsto xa \in A$ is a left centralizer. If $xa = ya$ for any $a \in A$, then $(x - y)a = 0$ for any $a \in A''$, so $x = y$ in A''.

Let σ be a left centralizer, and u_λ be an approximate unit for A. Since the directed net $\{\sigma(u_\lambda)\}$ is bounded, then it has a point of accumulation in A'' (bounded sets are weakly compact in $\mathbb{B}(H)$ and accumulation points must lie in the closure of A). Let us denote one of the accumulation points by x. For any $a \in A$, the directed net $\{u_\lambda a\}$ converges in norm to a, so that $\sigma(u_\lambda a) = \sigma(u_\lambda)a$ converges to $\sigma(a)$. Then $xa = \sigma(a) \in A$, so x is a left multiplier. If $xA = 0$, then $\sigma = 0$. Note that if $y \in A''$ is another accumulation point, then $xa = \sigma(a) = ya$ for any $a \in A$, and $(x - y)a = 0$ for any $a \in A''$ (due to the strong density of A in A''), so $x = y$ in A''. Therefore, in this case there is only one point of accumulation.

A similar proof works also for right multipliers and right centralizers.

Let now x be a double multiplier. Then the mappings $\sigma_2 : a \mapsto xa$ and $\sigma_1 : a \mapsto ax$ are left and right multipliers, with $\sigma_1(a)b = (ax)b = a(xb) = a\sigma_2(b)$ for any $a, b \in A$, so x

defines a double centralizer. Conversely, if (σ_1, σ_2) is a double centralizer, then, by what has been proven, σ_1 determines a right multiplier x_1, and σ_2 a left multiplier x_2. Since $ax_1b = \sigma_1(a)b = a\sigma_2(b) = ax_2b$ for any $a, b \in A$, we have $x_1 = x_2$, and $x_1 = x_2$ is a double multiplier. □

Problem 54. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Remark 3.19. Accordingly, until the end of this section we will consider non-degenerate $A \subseteq \mathbb{B}(H)$, so that (double) multipliers coincide with double centralizers.

The set of all left (right) multipliers of A is denoted by $LM(A)$ $(RM(A))$, and the set of all multipliers of A by $M(A)$.

Problem 55. Check that $RM(A) = (LM(A))^*$ and that $M(A) = LM(A) \cap RM(A)$, so that $M(A)$ is symmetric with respect to the involution.

It follows directly from the definition that all three sets are norm closed. Thus, $M(A)$ is a C^*-algebra (and the other two are, in the general case, only Banach algebras).

Problem 56. Let X be a locally compact space, and let $C_0(X)$, as before, be the C^*-algebra of continuous functions tending to 0 at infinity. Prove that the algebra $M(C_0(X)) \subset L^\infty(X)$ can be identified with the C^*-algebra $C_b(X)$ of all bounded continuous functions on X.

Example 3.20. If $A = \mathbb{K}(H)$, then $M(\mathbb{K}(H)) \subseteq \mathbb{B}(H)$, but any bounded operator is a multiplier (since $\mathbb{K}(H)$ is an ideal in $\mathbb{B}(H)$), so $M(\mathbb{K}(H)) = \mathbb{B}(H)$.

Definition 3.21. An ideal $A \subset B$ is said to be *essential* if any nonzero ideal B has a nontrivial intersection with A.

Let $A^\perp \subset B$ denote the set $A^\perp = \{x \in B \colon Ax = 0\}$.

Lemma 3.22. *An ideal $A \subset B$ is essential if and only if $A^\perp = 0$.*

Proof. Suppose that $A^\perp = 0$, but A is not essential. Then there is a nonzero ideal $J \subset B$ such that $A \cap J = \{0\}$. Let us take $j \in J$, $j \neq 0$. For any $a \in A$ we have $ja \in J \cap A$, so $ja = 0$ and $0 \neq j \in A^\perp$. A contradiction.

Conversely, let A be essential, but $A^\perp \neq 0$. Then there is an element $x \in A^\perp$ such that $x \neq 0$. Consider the ideal BxB (the closure of the set of all linear combinations of elements of the form $\sum_i b_i x b_i'$, $b_i, b_i' \in B$) and take an arbitrary $y \in BxB \cap A$. As it is known (for example, from Lemma 1.45), any element of the C^*-algebra admits a decomposition into the product of two of its elements, so we can write $y = z \cdot a$, where $z, a \in BxB \cap A$. We write $z = \sum_i b_i x b_i'$, so $y = za = \sum_i b_i x(b_i' a) = 0$, since $b_i' a \in A$, hence $xb_i' a = 0$, because $x \in A^\perp$. Therefore, $BxB \cap A = 0$ and we arrive to a contradiction. □

Lemma 3.23. *Let $A \subset B$ be an essential ideal. Then there is an embedding $B \subset M(A)$ that is identical on A.*

Proof. Consider $b \in B$. Then b defines the left and right centralizer A (since A is an ideal) $\sigma_2 : a \mapsto ba$ and $\sigma_1 : a \mapsto ab$, and $\sigma_1(a)a' = (ab)a' = a(ba') = a\sigma_2(a')$ for any $a, a' \in A$, so b defines a double centralizer, and hence a multiplier. So the mapping $\pi : B \to M(A)$ is defined, identical on A. This mapping is obviously a $*$-homomorphism. It remains to check whether π is injective. If $b \in \operatorname{Ker} \pi$, then $\sigma_1 = 0$ and $\sigma_2 = 0$ (see proof of theorem 3.18), so $bA = 0$, $Ab = 0$ and $b \in A^\perp$, which means $b = 0$. $\square$

Note that the correspondence $A \mapsto M(A)$ is not a functor. For example, for $A = \mathbb{K}(H)$ and $B = A^+$, consider the embedding $A \subset B$. Wherein $M(A) = \mathbb{B}(H)$, and $M(B) = B$. Obviously, the embedding does not continue to these multiplier algebras. However, in some cases the transition to multipliers has some functorial properties.

Lemma 3.24. *Let $\varphi : A \to B$ be a surjective $*$-homomorphism of two C^*-algebras. Then it continues to a $*$-homomorphism $\bar{\varphi} : M(A) \to M(B)$.*

Proof. Let $\sigma \in LM(A)$ be a left centralizer. For any $b \in B$ we set $\bar{\varphi}(\sigma)(b) := \varphi(\sigma(\varphi^{-1}(b)))$. It is necessary check that the map is well defined, that is, its independence of the choice of a representative in $\varphi^{-1}(b) \subset A$. Due to linearity, it suffices to check that σ maps $\operatorname{Ker} \varphi$ to itself. Let $a \in \operatorname{Ker} \varphi$. Let us represent it in the form $a = a_1 \cdot a_2$, $a_1, a_2 \in \operatorname{Ker} \varphi$. Then $\varphi(\sigma(a)) = \varphi(\sigma(a_1)a_2) = 0$, since $\operatorname{Ker} \varphi$ is an ideal.

Thus, a left centralizer σ defines the mapping $\bar{\varphi}(\sigma)$, which is the left centralizer of B. The same is done for right and double centralizers (Problem 57). $\square$

Problem 57. Prove that $\bar{\varphi}(\sigma)$ is a left centralizer. Construct an extension of a right centralizer in a similar way. Check that for a double centralizer we get a double centralizer.

Problem 58. Check that in the situation of the previous lemma the extension $\bar{\varphi}$ is also surjective.

Problem 59. Prove that a representation $\pi : A \to \mathbb{B}(H)$ is non-degenerate if and only if for some approximate unit u_λ of the algebra A the following condition is satisfied: for any vector $\xi \in H$ there is a λ such that that $u_\lambda(\xi) \neq 0$.

Lemma 3.25. *Let $B \subset A$ be an algebra and its C^*-subalgebra with a common approximate unit u_λ. Then $M(B) \subset M(A)$.*

Proof. If A is non-degenerate, then B is also non-degenerate by Problem 59. So $M(B) \subset B'' \subset A''$. For any $x \in A$, $y \in M(B)$, $yx = \lim yu_\lambda x \in A$, and similarly, $xy \in A$, so y is a multiplier of A. $\square$

Lecture February 24

3.4 Calkin algebra

If a Hilbert space H is not separable, then $\mathbb{B}(H)$ can be quite complex, in particular, have more than just one proper ideal. For example, the set of operators with separable image is an ideal. This does not happen in the case of separable H, which we will usually restrict ourselves to.

Definition 3.26. The quotient C^*-algebra $\mathbb{Q}(H) = \mathbb{B}(H)/\mathbb{K}(H)$ is called *Calkin algebra.*

Definition 3.27. The quotient C^*-algebra $M(A)/A$ is called *algebra of outer multipliers,* or *generalized Calkin algebra.*

For an operator $T \in \mathbb{B}(H)$, denote by $\|T\|_{ess}$ the *essential* norm of the operator T, which is defined as the norm of its equivalence class in the Calkin algebra. It is easy to see that $\|T\|_{ess} \geq \alpha$ if for any $\varepsilon > 0$ there is an infinite-dimensional closed subspace V such that $\|T\xi\| \geq (\alpha - \varepsilon)\|\xi\|$ for any $\xi \in V$.

Problem 60. Check this.

3.5 Toeplitz Algebra

Let $\mathbb{T}$ be the unit circle, $e_n = e^{2\pi i n t} = z^n$, $n \in \mathbb{Z}$, — an orthonormal basis in $H = L^2(\mathbb{T})$, and let $H^2 \subset L^2(\mathbb{T})$ — closed subspace generated by all e_n with non-negative numbers, $n \geqslant 0$. Let us denote by $M_g : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the operator of multiplication by the function $g \in L^\infty(\mathbb{T})$, $M_g(f) = gf$, $f \in L^2(\mathbb{T})$. Let $P_{H^2} \in \mathbb{B}(H)$ denotes the projection onto H^2.

Let us define the operator T_g, where $g \in L^\infty(\mathbb{T})$, on H^2 by the formula $T_g(f) = P_{H^2}M_g(f) = P_{H^2}gf$, $f \in H^2$.

If $g(z) = z$, then the operator T_g can be identified with the right shift operator in l^2 (recall that l^2 and $L^2(\mathbb{T})$ are unitary equivalent in the standard way).

Problem 61. Check this.

Lemma 3.28. *Let $g \in L^\infty(\mathbb{T})$. Then $T_{\overline{g}} = T_g^*$ and $\|T_g\|_{ess} = \|T_g\| = \|g\|_\infty$.*

Proof. The first statement is a simple exercise, so let's pass to the second one. It's obvious that $\|T_g\|_{ess} \leq \|T_g\| \leq \|g\|_\infty$. Let's take $\varepsilon > 0$, then, due to the density of polynomials in $L^2(\mathbb{T})$, there is a polynomial $p(z) = \sum_{k=-N}^{N} a_k z^k$ such that $\|p\|_2 = 1$ and $\|gp\|_2 > \|g\|_\infty - \varepsilon$. For any $n > N$, the polynomial $z^n p(z)$ lies in H^2, and $\|z^n p\|_2 = 1$. In addition, for $n = 3N, 6N, 9N, \ldots$ the polynomials $z^n p$ are mutually orthogonal.

Let $gp = \sum_{-\infty}^{\infty} b_k e_k$. Then

$$T_g(z^n p) = P_{H^2}(z^n gp) = \sum_{-n}^{\infty} b_k e_{k+n},$$

That's why

$$\lim_{n\to\infty} \|T_g(z^n p)\|_2 = \|gp\|_2 > \|g\|_\infty - \varepsilon,$$

i.e. $\|T_g \xi_n\| \geq \|g\|_\infty - \varepsilon$ for an infinite set of mutually orthogonal vectors $\xi_n = z^n p \in H^2$, hence $\|T_g\|_{ess} \geq \|g\|_\infty$. Comparing this with the previously obtained inequality $\|T_g\|_{ess} \leq \|g\|_\infty$, we obtain the statement of the lemma. □

Set $H^\infty = H^2 \cap L^\infty(\mathbb{T})$.

Lemma 3.29. *If $h \in H^\infty$, then H^2 is an invariant subspace for the operator M_h. For any functions $h \in H^\infty$ and $g \in L^\infty(\mathbb{T})$ the following equalities hold: $T_g T_h = T_{gh}$ and $T_{\overline{h}} T_g = T_{\overline{h}g}$.*

Proof. Let $h = \sum_{n\geq 0} h_n z^n$. Then $M_h z^k = \sum_{n\geq 0} h_n z^{n+k} \in H^2$ for any $k \geq 0$, so $M_h(H^2) \subset H^2$. Therefore $T_h f = hf$ for any function $f \in H^2$, and, therefore, $T_g T_h f = T_g hf = P_{H^2} ghf = T_{gh} f$. □

Lemma 3.30. *The commutator $T_z T_g - T_g T_z$ is a compact operator of rank at most 1 for any function $g \in L^\infty(\mathbb{T})$.*

Proof. Since $z \in H^\infty$, has place equality $T_g T_z = T_{gz}$. Let's look at the operator $T_z T_g - T_{gz}$: its limit on H^2 is

$$P_{H^2} M_z P_{H^2} M_g P_{H^2} - P_{H^2} M_z M_g P_{H_2} = -P_{H^2} M_z P_{H^2}^\perp M_g P_{H^2}.$$

But the operator $P_{H^2} M_z P_{H^2}^\perp$ is of rank 1 (it is equal to $e_0(e_{-1}, \cdot)$). □

Corollary 3.31. *For any functions $g \in L^\infty$ and $f \in C(\mathbb{T})$, the operators $T_f T_g - T_{fg}$ and $T_g T_f - T_{gf}$ are compact.*

Proof. Take $\varepsilon > 0$ and a polynomial $p = \sum_{k=-N}^{N} a_k z^k$ such that $\|fp\|_\infty < \varepsilon$. It is clear that $T_g T_f - T_{gf} = P_{H^2} M_g P_{H^2}^\perp M_f P_{H^2}$, therefore

$$\|T_f T_g - T_{fg} - T_p T_g + T_{pg}\| = \|P_{H^2} M_g P_{H^2}^\perp (M_f - M_p) P_{H^2}\| \leq \|M_f - M_p\| = \|fp\|_\infty < \varepsilon.$$

It is enough to establish compactness of $P_{H^2}^\perp M_p P_{H^2}$, which follows from the fact that the range of $P_{H^2}^\perp M_p P_{H^2}$ has dimension $\leq N$.

The statement about the second operator is proved in a similar way (it is easier to prove the compactness of its conjugate operator). □

Definition 3.32. The unital C*-algebra $C^*(1, T_z)$ generated by the operator T_z is called *Toeplitz algebra.* Let $\mathcal{T} = \{T_f + K : f \in C(\mathbb{T}), K \in \mathbb{K}(H^2)\}$.

Theorem 3.33. *1. $\mathcal{T} = C^*(1, T_z)$;*

2. This algebra is irreducible and contains $\mathbb{K}(H^2)$ as the only minimal ideal;

3. $\mathcal{T}/\mathbb{K}(H^2) \cong C(\mathbb{T})$, and the map $s : C(\mathbb{T}) \to \mathcal{T}$, $s : f \mapsto T_f$, is a continuous lifting (the right inverse) for the quotient map $\mathcal{T} \to \mathcal{T}/\mathbb{K}(H^2)$.

Proof. Let $p \in \mathcal{T}'$ be a projection from commutant of the Toeplitz algebra. It commutes with T_z and with T_z^*, and therefore with $E_0 = 1 - T_z T_z^*$. Then pE_0 is a projection of rank ≤ 1, so it is equal to either E_0 or 0. Since $pE_n = pT_z^n E_0$, we get that $pE_n = E_n$ if $pE_0 = E_0$, and $pE_n = 0$ if $pE_0 = 0$. That is, p is equal to either 1 or 0, which implies the irreducibility of $\mathcal{T}$.

Since $C^*(1, T_z)$ contains a nonzero compact operator, it contains the entire algebra of compact operators $\mathbb{K}(H^2)$ (follows from irreducibility by Corollary 3.5). From the density of polynomials in $C(\mathbb{T})$ it follows that $C^*(1, T_z)$ contains all T_f, $f \in C(\mathbb{T})$. Thus $\mathcal{T} \subset C^*(1, T_z)$ is a dense 8-subalgebra (as $T_f^* = T_{\overline{f}}$, $f \in C(\mathbb{T})$).

Let $J \subset \mathcal{T}$ be a nonzero ideal. By the second statement of Lemma 2.39, J is irreducible. If $a \in J$, $a \neq 0$, then there exists a compact operator $k \in \mathbb{K}(H^2)$ such that $ak \neq 0$. In this case $ak \in J$. By Corollary 3.5, $J \supset \mathbb{K}(H^2)$, i.e. the ideal of compact operators is the smallest.

Let's check that $\mathcal{T}$ is closed. Take a Cauchy sequence $\{T_{f_n} + K_n\}$, $f_n \in C(\mathbb{T})$, $K_n \in \mathbb{K}(H^2)$, $n \in \mathbb{N}$. Then

$$\|f_n - f_m\|_\infty = \|T_{f_n} - T_{f_m}\|_{ess} \leq \|T_{f_n} + K_n - (T_{f_m} + K_m)\|,$$

therefore the sequence $\{f_n\}$ is Cauchy. But then the sequence $\{K_n\}$ is also Cauchy. Since both $C(\mathbb{T})$ and $\mathbb{K}(H^2)$ are norm closed, $\mathcal{T}$ is also closed. And since $\mathcal{T}$ contains T_z and is closed, then it coincides with $C^*(1, T_z)$.

Denote by $\pi : \mathcal{T} \to \mathcal{T}/\mathbb{K}(H^2)$ the quotient $*$-homomorphism. The quotient algebra is obviously commutative. It follows from Lemma 3.28 that the composition $\pi \circ s : C(\mathbb{T}) \to \mathcal{T}/\mathbb{K}(H^2)$ is an isometry. It is also a $*$-homomorphism, so it is a $*$-isomorphism. □

Definition 3.34. An operator $T \in \mathbb{B}(H)$ is called *Fredholm* if its equivalence class $\dot{T}$ in the Calkin algebra is invertible.

Corollary 3.35. *The operator T_f is Fredholm if and only if the function f nowhere equals 0.*

Lemma 3.36. *Let $T \in \mathbb{B}(H)$ be Fredholm. Then there exists $\varepsilon > 0$ such that if $\|T - S\| < \varepsilon$, then S is also Fredholm, and, for any $K \in \mathbb{K}(H)$, the operator $T + K$ is also Fredholm.*

Proof. The second statement is obvious. To prove the first one, note that $\|\dot{T} - \dot{S}\| = \|T - S\|_{ess} \leq \|T - S\| < \varepsilon$, and ε must be chosen so that so that $\dot{S}$ would be invertible (the set of invertible elements is open). □

Lemma 3.37. *The operator $T \in \mathbb{B}(H)$ is Fredholm if and only if there exists an operator $S \in \mathbb{B}(H)$ such that $TS - 1$ and $ST - 1$ are compact.*

Proof. It is almost obvious: being Fredholm means that in the Calkin algebra there exists an inverse element $\dot{S}$, i.e. the equalities $\dot{T}\dot{S} = 1$ and $\dot{S}\dot{T} = 1$ are satisfied. □

Theorem 3.38 (Atkinson)**.** *The following conditions are equivalent for an operator T:*

1. *T is Fredholm;*

2. *The range* $\operatorname{Im} T$ *of* T *is closed, and the subspaces* $\operatorname{Ker} T$ *and* $\operatorname{Coker} T = (\operatorname{Im} T)^\perp$ *are finitedimensional.*

Proof. If T is Fredholm, then there is an operator S and a compact operator K such that the equality $ST = 1 + K$ holds. If $\xi \in \operatorname{Ker} T$, then $ST\xi = \xi + K\xi = 0$, i.e. ξ lies in the eigenspace of the compact operator, which must be finitedimensional. Thus $\operatorname{Ker} T$ is finitedimensional. Similarly, $\operatorname{Ker} T^*$ is finitedimensional.

To prove that $\operatorname{Im} T$ is closed, we approximate K by an operator F of finite rank: let $\|FK\| < r$. If $\xi \in \operatorname{Ker} F$, then

$$\begin{aligned} \|S\| \cdot \|T\xi\| &\geq \|ST\xi\| = \|\xi + K\xi\| = \|(\xi + F\xi) + (K - F)\xi\| = \|\xi + (K - F)\xi\| \\ &\geq \|\xi\| - \|K - F\|\|\xi\| \geq (1 - r)\|\xi\|. \end{aligned}$$

If $r < 1$, then the operator T is bounded below on $\operatorname{Ker} F$, which implies that $T(\operatorname{Ker} F)$ is closed. Since $(\operatorname{Ker} F)^\perp = \operatorname{Im} F^*$ is finitedimensional, then $T((\operatorname{Ker} F)^\perp)$ is also finitedimensional and therefore closed. Therefore $\operatorname{Im} T = T(H) = T(\operatorname{Ker} F \oplus (\operatorname{Ker} F)^\perp)$ is closed. Moreover, $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp = \operatorname{Coker} T$ is finitedimensional.

Conversely, decompose H as the direct sum $H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^\perp$. Consider the restriction of T to $(\operatorname{Ker} T)^\perp$. This restriction is a bijection onto its range, i.e. on $\operatorname{Im} T$, therefore, by the open mapping theorem, it is invertible. Define an operator S on $\operatorname{Im} T$ as the inverse to $T|_{(\operatorname{Ker} T)^\perp}$ and as zero on $(\operatorname{Im} T)^\perp$. Then $1 - TS$ is the projection onto $(\operatorname{Im} T)^\perp$, and $1 - ST$ is the projection onto $\operatorname{Ker} T$, i.e. they are operators of finite rank; therefore, $\dot{T}$ is invertible in the Calkin algebra. $\square$

Definition 3.39. The difference $\dim \operatorname{Ker} T - \dim \operatorname{Coker} T = \operatorname{ind} T$ is called the *index* of the Fredholm operator T.

The index does not change under small deformations, and therefore under homotopies:

Theorem 3.40. *Let* T *be a Fredholm operator. Then there exists* $\varepsilon > 0$ *for which the condition* $\|T - S\| < \varepsilon$ *implies that* $\operatorname{ind} T = \operatorname{ind} S$.

Proof. Consider two direct sum decompositions of H: $H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^\perp$ and $H = (\operatorname{Im} T)^\perp \oplus \operatorname{Im} T$. The first terms in both are finite-dimensional. Write the operator T as a two-dimensional matrix with respect to these two decompositions of H: $T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}$ (here the zeros stay for the maps from $\operatorname{Ker} T$ to $(\operatorname{Im} T)^\perp$, from $\operatorname{Ker} T$ to $\operatorname{Im} T$ and from $(\operatorname{Ker} T)^\perp$ to $(\operatorname{Im} T)^\perp$, respectively, and the operator $T_1 : (\operatorname{Ker} T)^\perp \to \operatorname{Im} T$ is invertible).

Write the operator S in the same form (i.e. with respect to the same two decompositions of H into direct sums): $S = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & T_1 + a_{22} \end{pmatrix}$, where all a_{ij} are small. Let us take ε such that the operator $T_1 + a_{22}$ is invertible (recall that T_1 is invertible). Then we set $X = -(T_1 + a_{22})^{-1} a_{21}$ and note that $S \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & T_1 + a_{22} \end{pmatrix}$ (it doesn't matter to us what exactly b_{ij} is equal to, it is important that there is zero at the bottom left, and that $b_{22} = a_{22}$). Similarly, taking $Y = -b_{12}(T_1 + a_{22})^{-1}$, we get

$$\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 \\ 0 & T_1 + a_{22} \end{pmatrix}. \tag{3.4}$$

The left side of (3.4) contains the product of three operators, two of which are invertible, so the kernel and cokernel of this product have the same dimensions as the kernel and cokernel of the operator S. This means that $\operatorname{ind} S$ is equal to the index of the right side, which is equal to the sum of the indices of the diagonal elements, i.e. $\operatorname{ind} S = \operatorname{ind} c_{11} + \operatorname{ind}(T_1 + a_{22})$. It follows from invertibility of $T_1 + a_{22}$ that $\operatorname{ind}(T_1 + a_{22}) = 0$, i.e. $\operatorname{ind} S = \operatorname{ind} c_{11}$, where $c_{11} : \operatorname{Ker} T \to (\operatorname{Im} T)^{\perp}$ is an operator mapping one finitedimensional space into another (of different dimensions). So, $\dim \operatorname{Ker} c_{11} = \dim \operatorname{Ker} T - \operatorname{rank} c_{11}$, $\dim \operatorname{Im} c_{11} = \dim \operatorname{Im} T - \operatorname{rank} c_{11}$, hence $\operatorname{ind} c_{11} = \operatorname{ind} T$. □

Problem 62. Prove that if $T : [0, 1] \to \mathbb{B}(H)$ is a continuous map of an interval into the set of Fredholm operators, then $\operatorname{ind} T(0) = \operatorname{ind} T(1)$.

Theorem 3.41. *Let S, T be Fredholm operators. Then* $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$.

Proof. Let's consider continuous family of F_t, $t \in [0, \pi/2]$, operators in $H \oplus H$, $F_t = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \cdot 1 & -\sin t \cdot 1 \\ \sin t \cdot 1 & \cos t \cdot 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} \cos t \cdot 1 & \sin t \cdot 1 \\ -\sin t \cdot 1 & \cos t \cdot 1 \end{pmatrix}$. For each t the operator F_t is Fredholm (since its equivalence class in the Calkin algebra is invertible). Since $F_0 = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, and $F_{\pi/2} = \begin{pmatrix} ST & 0 \\ 0 & 1 \end{pmatrix}$,

$$\operatorname{ind} S + \operatorname{ind} T = \operatorname{ind} F_0 = \operatorname{ind} F_{\pi/2} = \operatorname{ind}(ST) + \operatorname{ind} 1 = \operatorname{ind}(ST).$$

□

Let us consider in more detail the functions $f \in C(\mathbb{T})$ that do not take the value 0, i.e. those for which the operator T_f is Fredholm. They define loops on the complex plane with the punctured point 0, and are characterized, up to homotopy, by the so-called rotation number, or degree. If $f : \mathbb{T} \to \mathbb{C} \setminus \{0\}$, then the function g given by the formula $g(z) = \frac{f(z)}{|f(z)|}$ defines the mapping of the circle into itself, i.e. element of the fundamental group of the circle. This element is called the rotation number for f and is denoted by $\operatorname{wind} f$.

Theorem 3.42. *Let $f \in C(\mathbb{T})$ not vanish anywhere. Then* $\operatorname{ind}(T_f) = -\operatorname{wind} f$.

Proof. From the first item of Theorem 3.40 it follows that the index does not change under small perturbations, therefore, it is homotopy invariant (provided that the perturbation does not go beyond the set of Fredholm operators). Therefore, it is sufficient to check the statement of the theorem for the case $f(z) = z^n$, $n \in \mathbb{N}$. But $\operatorname{ind} T_{z^n} = -n$. □

Lecture March 3

3.6 C*-extensions and their Busby invariant

If I is an ideal in a C*-algebra E with the corresponding quotient C*-algebra $A = E/I$, then this can be written as a short exact sequence of C*-algebras $0 \to I \to E \to A \to 0$. Exactness means that the kernel of each $*$-homomorphism (except the first one) is equal to the range of the previous $*$-homomorphism. Short exact sequences are called extensions (for precision, extensions of the C*-algebra A by the C*-algebra I). The Toeplitz algebra can be included in the extension $0 \to \mathbb{K}(H^2) \to \mathcal{T} \to C(\mathbb{T}) \to 0$.

Definition 3.43. Two extensions, $0 \to I \to E \to A \to 0$ and $0 \to I \to E' \to A \to 0$ are *strongly equivalent* if there is a $*$-isomorphism $\alpha : E \to E'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\alpha"'] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commutes.

They are *equivalent* if there is a $*$-isomorphism $\alpha : E \to E'$ such that $\alpha(I) \subset I$, $\alpha|_I$ is an automorphism of I, and the induced factor map $\dot{\alpha} : A \to A$ is identical.

Let $0 \to I \to E \to A \to 0$ be an extension, $e \in E$, $x \in I$. Put $i_l(e)(x) = ex \in I$, $i_r(e) = xe \in I$, then $i_l(e)$ is a left centralizer and $i_r(e)$ is a right centralizer, and the pair $(i_l(e), i_r(e))$ is a double centralizer for the ideal I. The formula $e \mapsto (i_l(e), i_r(e))$ defines a map $i : E \to M(I)$ into the multiplier algebra of I. Obviously, this map is a $*$-homomorphism. Since $i(I) = I$, the map i defines a $*$-homomorphism of quotient algebras, $\varphi : A \to Q(I)$. This homomorphism is called the *Busby invariant* of the given extension.

For a $*$-homomorphism $\varphi : A \to Q(I)$ we set

$$E' = \{(a, m) : a \in A, m \in M(I), \varphi(a) = \pi(m)\},$$

where $\pi : M(I) \to M(I)$ denotes the quotient map. E' is a C*-algebra, because it is a closed $*$-subalgebra of C*-algebra $A \oplus M(I)$. The set of all pairs of the form $(0, x)$, $x \in I$, forms an ideal in E' isomorphic to I, and there is a $*$-homomorphism $E' \to A$, given by the formula $(a, m) \mapsto a$, whose kernel coincides with this ideal, so we obtain the extension $0 \to I \to E' \to A \to 0$.

Lemma 3.44. *If $\varphi : A \to Q(I)$ is the Busby invariant of the extension $0 \to I \to E \to A \to 0$, then the extension $0 \to I \to E' \to A \to 0$ constructed above is strongly equivalent to the original extension.*

Proof. Let us denote the quotient map $E \to A$ by q. Define the map $\alpha : E \to E'$ by the formula $\alpha(e) = (q(e), i(e))$. The commutativity of the corresponding diagram is obvious. It is also easy to see that $\operatorname{Ker} \alpha = 0$ ($\operatorname{Ker} \alpha \subset I$, and if $x \in I$, then $i(x) = 0$ only if $x = 0$). It follows from the Five Homomorphisms Lemma (5-Lemma) that α is an isomorphism. $\square$

Corollary 3.45. *Two extensions (with the same ideal I and quotient A) are strongly equivalent if and only if their Busby invariants coincide.*

Definition 3.46. The extension $0 \to I \to E \to A \to 0$ *is essential* if I is an essential ideal in E.

Problem 63. Prove that an extension is essential if and only if its Busby invariant is injective.

For a number of reasons, some of which we will discuss later, extensions in which $I = \mathbb{K}(H)$, are especially important. For convenience, we will write $\mathbb{K}$ instead of $\mathbb{K}(H)$, since C*-algebras $\mathbb{K}(H_1)$ and $\mathbb{K}(H_2)$ are isomorphic for any separable Hilbert spaces H_1 and H_2.

Lemma 3.47. *Let $\varphi : \mathbb{K} \to \mathbb{K}$ be an automorphism. Then $\varphi = \mathrm{Ad}_U$ (i.e. $\varphi(K) = UKU^*$, $K \in \mathbb{K}$) for some unitary operator $U \in \mathbb{B}(H)$.*

Proof. Let $\xi \in H$ be a unit vector. Then the operator $\theta_{\xi,\xi}$ given by the formula $\theta_{\xi,\xi}(\zeta) = \xi(\xi, \zeta)$ is a minimal projection. Therefore, $\varphi(\theta_{\xi,\xi})$ is also a minimal projection, i.e. it has the form $\varphi(\theta_{\xi,\xi}) = \theta_{\eta,\eta}$ for some unit vector $\eta \in H$. Let us define the operator U by the equality $U\zeta := \varphi(\theta_{\zeta,\xi})\eta$, where $\zeta \in H$, $\theta_{\zeta,\xi}(cdot) = \zeta(\xi, \cdot)$.

As

$$\|U\zeta\|^2 = (\varphi(\theta_{\zeta,\xi}\eta), \varphi(\theta_{\zeta,\xi}\eta)) = (\varphi(\theta^*_{\zeta,\xi}\theta_{\zeta,\xi})\eta, \eta) = (\varphi(\theta_{\xi,\xi})\|\zeta\|^2\eta, \eta) = (\theta_{\eta,\eta}\eta, \eta)\|\zeta\|^2 = \|\zeta\|^2,$$

the operator U is an isometry.

Let $\zeta \in H$, then there exists an operator $K \in \mathbb{K}$ such that $\varphi(K) = \theta_{\zeta,\eta}$. Then

$$U(K\xi) = \varphi(\theta_{K\xi,\xi})\eta = \varphi(K)\varphi(\theta_{\xi,\xi})\eta = \theta_{\zeta,\eta}\theta_{\eta,\eta}\eta = \zeta,$$

therefore U is surjective, hence unitary.

Take arbitrary $K \in \mathbb{K}$ and $\zeta \in H$. Then

$$UKU^*\zeta = \varphi(\theta_{KU^*\zeta,\xi})\eta = \varphi(K)\varphi(\theta_{U^*\zeta,\xi})\eta = \varphi(K)U(U^*\zeta) = \varphi(K)\zeta,$$

i.e. $\varphi = \mathrm{Ad}_U$. □

Lemma 3.48. *Two essential extensions of a C*-algebra A by an ideal $\mathbb{K}$ are equivalent if and only if their Busby invariants are unitarily equivalent (via some unitary operator $U \in \mathbb{B}(H)$).*

Proof. Note that if the extension is essential, then the Busby invariant $\varphi : A \to \mathbb{Q}(H)$ is injective, hence $E = \pi^{-1}(\varphi(A))$, where $\pi : \mathbb{B}(H) \to \mathbb{Q}(H)$ is the quotient map. Let φ_1, φ_2 be the Busby invariants for two essential extensions E_1, E_2. If there exists a unitary operator $U \in \mathbb{B}(H)$ such that $\varphi_2 = \mathrm{Ad}_{\pi(U)}\varphi_1$, then $\mathrm{Ad}_U E_2 = \mathrm{Ad}_U \pi^{-1}(\varphi_2(A)) = \pi^{-1}(\mathrm{Ad}_{\pi(U)}\varphi_2(A)) = \pi^{-1}(\varphi_1(A)) = E_1$, and Ad_U defines an isomorphism $E_2 \to E_1$.

Conversely, suppose that E_1 and E_2 are equivalent via a $*$-isomorphism $\alpha : E_2 \to E_1$. Then $\alpha|_{\mathbb{K}}$ is an automorphism of $\mathbb{K}$, so there exists a unitary operator U such that $\alpha|_{\mathbb{K}} = \mathrm{Ad}_U$. For any $T \in E_2$ and $K \in \mathbb{K}$ we have

$$\alpha(T)(UKU^*) = \alpha(T)\alpha(K) = \alpha(TK) = UTKU^* = UTU^* \cdot UKU^*.$$

Since the union of the images of all UKU^* (when K runs over a set of compact operators) is dense in H, we obtain that $\alpha(T) = UTU^*$, i.e. the Busby invariants for E_1 and E_2 are related by $\mathrm{Ad}_{\pi(U)}$. □

Let us define another equivalence relation for essential extensions of an arbitrary C*-algebra by the algebra of compact operators.

Definition 3.49. Essential extensions are *weakly equivalent* if their Busby invariants φ_1, φ_2 are related by the formula $\varphi_2 = \mathrm{Ad}_u \varphi_1$, where $u \in \mathbb{Q}(H)$ is a unitary in the Calkin algebra.

Note that if $U \in \mathbb{B}(H)$ is unitary, then $\pi(U) \in \mathbb{Q}(H)$ is also unitary, but not for every unitary element $u \in \mathbb{Q}(H)$ there is unitary operator $U \in \mathbb{B}(H)$ for which $\pi(U) = u$. For example, if T_z is a one-sided shift operator, then $\pi(T_z)$ is unitary; if there were a unitary operator U with $\pi(U) = \pi(T_z)$, then $U = T_z + K$, where K is a compact operator. Then $T(t) = T_z + tK$, $t \in [0, 1]$, defines a continuous family of Fredholm operators, therefore $\mathrm{ind}\, T_z = \mathrm{ind}\, T(0) = \mathrm{ind}\, T(1) = \mathrm{ind}\, U$, but $\mathrm{ind}\, T_z = -1$, and $\mathrm{ind}\, U = 0$.

Due to the fact that $H \oplus H \cong H$, we can define the sum of extensions as the direct sum of their Busby invariants. Let $i : H \to H \oplus H$ be an isomorphism of Hilbert spaces. We denote by $\imath : \mathbb{B}(H) \to \mathbb{B}(H \oplus H) = M_2(\mathbb{B}(H))$ the isomorphism of algebras induced by the isomorphism i (by $M_n(A)$ we denote the matrix algebra with coefficients from A), which in turn induces isomorphisms $\imath_{\mathbb{K}} : \mathbb{K} \to M_2(\mathbb{K})$ and $\bar{\imath} : \mathbb{Q}(H) \to M_2(\mathbb{Q}(H))$.

If $\varphi_1, \varphi_2 : A \to \mathbb{Q}(H)$ are two Busby invariants of essential extensions (i.e., two $*$-monomorphisms), then their sum $\varphi_1 + \varphi_2$ is defined by the formula $(\varphi_1 + \varphi_2)(a) = \bar{\imath}^{-1}(\varphi_1(a) \oplus \varphi_2(a))$, $a \in A$. Note that this is also a monomorphism.

Problem 64. Let $j : H \to H \oplus H$ be another isomorphism of Hilbert spaces, $\bar{\jmath} : \mathbb{Q}(H) \to M_2(\mathbb{Q}(H))$ be an induced $*$-isomorphism of algebras . Show that there is a unitary operator $U \in \mathbb{B}(H)$ such that $\bar{\jmath}^{-1}(\varphi_1(a) \oplus \varphi_2(a)) = \mathrm{Ad}_U \bar{\imath}^{-1}(\varphi_1(a) \oplus \varphi_2(a))$, i.e. the sum does not depend on the choice of isomorphism i up to equivalence.

Problem 65. Show that the sum for essential extensions is commutative and associative up to equivalence.

Definition 3.50. An essential extension $0 \to \mathbb{K} \to E \to A \to 0$ *is trivial* if there is a $*$-homomorphism $A \to E$ which is a right inverse for the quotient map $E \to A$.

Trivial extensions exist. Let $\pi : A \to \mathbb{B}(H)$ be a faithful representation of the C*-algebra A (it exists by the Gelfand–Naimark theorem). Let $\overline{H} = H \oplus H \oplus \cdots$ be the direct sum of a countable number of Hilbert spaces H, $\overline{\pi} : A \to \mathbb{B}(\overline{H})$ the direct sum of representations, $\overline{\pi}(a) = \pi(a) \oplus \pi(a) \oplus \cdots$. Let $\varphi(a) = q(\overline{\pi}(a))$, where $q : \mathbb{B}(\overline{H}) \to Q(\overline{H})$ is the quotient homomorphism. Then $\mathrm{Ker}\, \varphi = \{0\}$.

Definition 3.51. Essential extensions of $\varphi, \psi : A \to \mathbb{Q}(H)$ *are stably equivalent* (resp. *weakly stably equivalent*), if there exist trivial essential extensions $\lambda, \mu : A \to \mathbb{Q}(H)$ such that $\varphi + \lambda$ and $\psi + \mu$ are equivalent (respectively weakly equivalent).

Note that in this definition we can require the existence, instead of two trivial extensions λ and μ, of a single trivial extension $(\lambda + \mu) + (\lambda + \mu) + \cdots$.

Definition 3.52. We denote by $\mathrm{Ext}(A)$ (resp. $\mathrm{Ext}_w(A)$) the set of classes of stable equivalence (resp. stable weak equivalence) of essential extensions of the form $0 \to \mathbb{K} \to E \to A \to 0$.

These sets have the structure of an Abelian semigroup with the $+$ operation.

Problem 66. The zero element is given by any trivial extension.

Typically, if A is unital, one considers only unital extensions (that is, those for which the Busby invariant $\varphi : A \to \mathbb{Q}(H)$ is unital), although one can also work with non-unital extensions.

Lemma 3.53. Ext *and* Ext_w *are contravariant functors.*

Proof. Both statements are similar; here we will prove the first, i.e. functoriality of Ext.

Let $\alpha : B \to A$ be a $*$-homomorphism of C*-algebras, $\varphi : A \to \mathbb{Q}(H)$ a Busby invariant. It would be natural to take the composition $\varphi \circ \alpha : B \to \mathbb{Q}(H)$, but the Busby invariant defined this way may not be injective (and the corresponding extension may not be essential). But this is easy to fix: let $\lambda : B \to \mathbb{Q}(H)$ be the Busby invariant of the trivial extension. Let $\alpha^*([\varphi]) = [\varphi \circ \alpha + \lambda]$, where $[\cdot]$ denotes the equivalence class.

Let $\varphi, \psi : A \to \mathbb{Q}(H)$ be stably equivalent, i.e. there exist trivial extensions defined by $*$-homomorphisms $\mu, \nu : A \to \mathbb{Q}(H)$ and a unitary operator $U \in \mathbb{B}(H)$ such that $\varphi + \mu = \mathrm{Ad}_U(\psi + \nu)$. Then

$$
\begin{aligned}
(\varphi \circ \alpha + \lambda) + (\mu \circ \alpha + \lambda) &= (\varphi + \mu) \circ \alpha + \lambda + \lambda = \mathrm{Ad}_U(\psi + \nu) \circ \ alpha + \lambda + \lambda \\
&= \mathrm{Ad}_U(\psi \circ \alpha + \lambda') + \mathrm{Ad}_U(\nu \circ \alpha + \lambda'),
\end{aligned}
$$

where $\lambda' = \mathrm{Ad}_{U^*} \lambda$. It is clear that $\mu \circ \alpha + \lambda$ and $\mathrm{Ad}_U(\nu \circ \alpha + \lambda')$ are trivial, so $\varphi \circ \alpha + \lambda$ and $\psi \circ \alpha + \lambda'$ are equivalent. Adding λ' to the first of them, and λ to the second, we find that they are stably equivalent. Thus, the map $\alpha^* : \mathrm{Ext}(A) \to \mathrm{Ext}(B)$ is well defined. Note that the difficulties with adding trivial extensions are due to the fact that all Busby invariants must be monomorphisms.

The homomorphity of α^* is trivially verified. It remains to check that if $\beta : C \to B$ is a $*$-homomorphism of C*-algebras, then $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. We leave this as an exercise. $\square$

Problem 67. Compute $\mathrm{Ext}(\mathbb{C})$.

For more meaningful calculations we will need a classification of isometries.

Lemma 3.54. *Let $V \in \mathbb{B}(H)$ be an isometry. Then there exists a decomposition of $H = H_1 \oplus H_2$ into the direct sum of two invariant subspaces (one of which may be zero) such that $V|_{H_1}$ is the direct sum of the one-sided shift operators S, and $V|_{H_2}$ is a unitary operator. If V is Fredholm, then $V|_{H_1}$ is unitarily equivalent to S^m, where $m = \dim \operatorname{Coker} V$.*

Proof. If V is unitary, then the statement is obvious (with H_1 zero). If V is a non-unitary isometry, then we set $N = (\operatorname{Im} V)^\perp$ and prove the following equality:

$$\cap_{n=0}^{\infty} \operatorname{Im} V^n = \cap_{n=0}^{\infty} (V^n(N))^\perp. \tag{3.5}$$

Let $L \subset H$ be an arbitrary closed subspace, $\xi \in L^\perp$, $\eta \in L$. Then $V\xi \in V(L^\perp)$, $V\eta \in V(L)$, and $V\xi \perp V\eta$ as V is an isometry. Therefore $V(L^\perp) \subset V(L)^\perp$. Take here $L = N$, and replace V with V^n: $V^n(N^\perp) \subset V^n(N)^\perp$. Then $\operatorname{Im} V^{n+1} = V^n(V(H)) = V^n(N^\perp) \subset V^n(N)^\perp$.

To prove the reverse inclusion in (3.5), take $\xi \in \cap_{n=0}^{\infty}(V^n(N))^\perp$ and let's prove, by induction, that $\xi \in \operatorname{Im} V^n$ for all $n \in \mathbb{N}$. For $n = 0$ this is trivial. For the induction step, assume that $\xi \in \operatorname{Im} V^n$, i.e. that there exists a vector η such that $\xi = V^n\eta$. By assumption, $\xi \in (V^n N)^\perp$ for any $n \in \mathbb{N}$, therefore $\xi = V^n\eta \perp V^n N$, and from the isometricity of V^n we obtain, that $\eta \perp N$, i.e. $\eta \in \operatorname{Im} V$, therefore, $\xi = V^n\eta \in \operatorname{Im} V^{n+1}$. So, the equality (3.5) is proven.

Note that the subspace $H_2 = \cap_{n=0}^{\infty} \operatorname{Im} V^n$ is obviously invariant for V, and from the equality (3.5) it follows that $H_1 = H_2^\perp$ is the closure of the linear hull of all $V^n(N)$, which is also invariant for V, and $V = V|_{H_1} \oplus V|_{H_2}$. The second summand here is unitary, because is an isometry whose image coincides with the domain, and the first summand is the direct sum of one-sided shifts S in an amount equal to $\dim N$.

If V is a Fredholm isometry of index $-m$ then $\dim N = m$ is finite, and it is easy to see that $S \oplus \cdots \oplus S$ (m terms) is unitary equivalent to S^m.

□

Lemma 3.55. *Let $u \in \mathbb{Q}(H)$ be a unitary element of the Calkin algebra. Then there is an isometry or co-isometry $U \in \mathbb{B}(H)$ such that $q(U) = u$, where $q : \mathbb{B}(H) \to \mathbb{Q}(H)$ is the quotient $*$-homomorphism. If $V \in \mathbb{B}(H)$ and $q(V) = u$, then* $\operatorname{ind} V = \operatorname{ind} U$.

Proof. Let $T \in \mathbb{B}(H)$, $T \in q^{-1}(u)$. Then T is Fredholm. Consider the case $\operatorname{ind} T \leq 0$ (the case of positive index is similar, only instead of isometry we get co-isometry). Then $\dim \operatorname{Ker} T \leq \dim(\operatorname{Im} T)^\perp$. Define an operator of finite rank K to be zero on $(\operatorname{Ker} T)^\perp$ and an injective map to $\operatorname{Ker} T$, mapping it to $(\operatorname{Im} T)^\perp$. Then $R = T + K$ is Fredholm (since $q(R) = q(T) = u$) and has zero kernel. Therefore R^*R is invertible. Set $U = R(R^*R)^{-1/2}$ (the polar decoposition) and note that $q(U) = u(u^*u)^{-1/2} = u$, and $U^*U = (R^*R)^{-1/2}R^*R(R^*R)^{-1/2} = 1$, i.e. U is an isometry. If $V \in q^{-1}(u)$, then $q(U - V) = 0$, and $K = U - V$ is compact. Continuity of the index for the path $[0, 1] \ni t \mapsto U + tK$ gives $\operatorname{ind} U = \operatorname{ind} V$.

□

Note that the last statement of the Lemma means that the index is correctly defined for unitary elements of the Calkin algebra.

Theorem 3.56. $\mathrm{Ext}(C(\mathbb{T})) = \mathbb{Z}$.

Proof. $C(\mathbb{T})$, as a C*-algebra, is singly generated by the function z, therefore any Busby invariant φ is determined by its value on this generator. Let $\varphi(z) = u \in \mathbb{Q}(H)$. Since $|z| = 1$, this function is a unitary element of the algebra $C(\mathbb{T})$, so u is also unitary (we consider only unital Busby invariants). Let $U \in \mathbb{B}(H)$ be an isometry or co-isometry, $q(U) = u$. Let us show that the map $u \mapsto \operatorname{ind} U$ defines an isomorphism $\mathrm{Ext}(C(\mathbb{T})) \to \mathbb{Z}$. If $\operatorname{ind} U \leq 0$, then by Lemma 3.54, U is unitarily equivalent to S^n or $S^n \oplus V$, where S is a one-sided shift operator, and V is unitary. If $\operatorname{ind} U \geq 0$, then U is a co-isometry unitarily equivalent to $(S^*)^n$ or $(S^*)^n \oplus V$. Let $\varphi_1, \varphi_2 : C(\mathbb{T}) \to \mathbb{Q}(H)$ be two unital Busby invariants, $\varphi_i(z) = q(U_i) \in \mathbb{Q}(H)$, $i = 1.2$. If $\operatorname{ind} U_1 = \operatorname{ind} U_2$, then φ_1 and φ_2 are stably equivalent to the Busby invariant given by the formula $\varphi(z) = q(S^n \oplus V)$ (or $\varphi(z) = q((S^*)^n \oplus V))$.

Note that the Busby invariant of a trivial extension has the form $\lambda(z) = q(V)$, where V is a unitary operator.

If the Busby invariants φ_1 and φ_2 are stably equivalent, then $\varphi_1(z) \oplus \lambda(z)$ and $\varphi_2(z) \oplus \mu(z)$ are unitary equivalent. Let T, R be Fredholm operators, $q(T) = \varphi_1(z)$, $q(R) = \varphi_2(z)$. Let also U, V be unitary operators, $\lambda(z) = q(U)$, $\mu(z) = q(V)$. Then the Fredholm operators $T \oplus U$ and $R \oplus V$ are unitarily equivalent. And unitarily equivalent Fredholm operators have equal indices, since the dimensions of the kernel and cokernel are the same. $\square$

Lecture March 10

3.7 Group C*-algebras

Let G be a group. For simplicity, we will consider only discrete groups (i.e. without topology) and we will assume that they are countable.

Let $\mathbb{C}[G]$ be the set of all complex-valued functions on G of finite support. Let δ_g denote the characteristic function of the point $\{g\} \subset G$, then any function from $\mathbb{C}[G]$ can be written as a finite sum $\sum_g \alpha_g \delta_g$, $\alpha_g \in \mathbb{C}$. For $f, h \in \mathbb{C}[G]$ we define their product (*convolution*) by the formula $(f \star h)(g) = \sum_{\gamma \in G} f(\gamma)h(\gamma^{-1}g)$. Involution is defined by $\mathbb{C}[G]$ formula $f^*(g) = \overline{f(g^{-1})}$. It is easy to see that $\mathbb{C}[G]$ with these operations is a $*$-algebra, and the function δ_e, where $e \in G$ is a neutral (unit) element, is the unit in $\mathbb{C}[G]$.

Let π be a unitary representation of G on a Hilbert space H. It can be extended to $\tilde{\pi} : \mathbb{C}[G] \to \mathbb{B}(H)$ by the formula $\tilde{\pi}(\sum_g \alpha_g \delta_g) = \sum_g \alpha_g \pi(g)$. As $\tilde{\pi}(f \star h) = \tilde{\pi}(f)\tilde{\pi}(h)$ and $\tilde{\pi}(f^*) = \tilde{\pi}(f)^*$, we see that $\tilde{\pi}$ is a representation of a $*$-algebra. Conversely, any unitary $*$-representation $\tilde{\pi}$ of the algebra $\mathbb{C}[G]$ defines a unitary representation of the group G by the formula $\pi(g) = \tilde{\pi}(\delta_g)$ (unitarity follows from the equality $\delta_g \star \delta_{g^{-1}} = \delta_e$).

For any unitary representation π of G in the Hilbert space H we define the C*-algebra $C^*_\pi(G)$ as the closure (in norm) of the set all operators of the form $\sum_g \alpha_g \pi(g)$. In other words, we can define the seminorm $\|\cdot\|_\pi$ on $\mathbb{C}[G]$ by the formula $\|\sum_g \alpha_g \delta_g\|_\pi := \|\sum_g \alpha_g \pi(g)\|$. Then $N_\pi = \{f \in \mathbb{C}[G] : \|f\|_\pi = 0\}$ is an ideal, with the quotient $\mathbb{C}[G]/N_\pi$. It satisfies all the axioms of C*-algebra except completeness, and we can complete it using the norm $\|\cdot\|_\pi$. This completion is a C*-algebra. We will denote it by $C^*_\pi(G)$.

We will consider only unitary representations of groups.

Any group has a trivial representation τ, mapping all elements of the group to 1, i.e. $\tau(g) = 1$ for any $g \in G$. It is easy to see that $C^*_\tau(G) \cong \mathbb{C}$.

Another special representation of the group G is called the *regular* representation. Let $H = l_2(G)$ be the set of all square integrable complex-valued functions on G. Then the formula $\lambda(g)f(\gamma) = f(g^{-1}\gamma)$ defines the unitary representation λ. The C*-algebra $C^*_r(G) = C^*_\lambda(G)$ is called the *reduced* C*-algebra of the group G.

Another representation of u can be obtained by taking the direct sum *all* irreducible unitary representations. This representation is called the *universal* representation, and the corresponding C*-algebra $C^*(G) = C^*_u(G)$ is called the universal C*-algebra (or the full C*-algebra) of the group G.

Definition 3.57. Let π and σ be representations of the group G. It is said that σ *weakly contains* π if $\|f\|_\pi \leq \|f\|_\sigma$ for any $f \in \mathbb{C}[G]$. This gives partial order on the set of representations of the group G.

Problem 68. Show that if σ weakly contains π, then the identity mapping $\mathbb{C}[G]$ onto itself extends to a surjective $*$-homomorphism $C^*_\sigma(G) \to C^*_\pi(G)$. In particular, the identity map extends to the surjective $*$-homomorphism $C^*(G) \to C^*_r(G)$.

Group C*-algebras of commutative groups

If G is commutative, then its Pontryagin dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ is defined as a group of characters of group G (i.e. one-dimensional unitary representations).

Lemma 3.58. *If G is commutative then the canonical surjection $C^*(G) \to C^*_r(G)$ is an isomorphism. In addition, $C^*(G) \cong C(\widehat{G})$.*

Proof. $C^*(G)$ is commutative and unital, so the Gelfand transformation gives an isomorphism $\Gamma : C^*(G) \to C(X)$, and all that remains is to identify the compact Hausdorff space X, which is the space of multiplicative linear functionals on $C^*(G)$. But the latter are in one-to-one correspondence with one-dimensional unitary representations (characters) of the group G, i.e. $X = \widehat{G}$. It remains to check that the topology on X (given by Gelfand duality) coincides with the topology on $\widehat{G}$ given by Pontryagin duality, which is left to the reader.

The isomorphism $\Gamma : C^*(G) \to C(\widehat{G})$ can be described in terms of the Fourier transform $\mathcal{F}$. $\widehat{f}(\chi) = (\mathcal{F}f)(\chi) = \sum_{g\in G} f(g)\chi(g) = \chi(f)$, where $f \in \mathbb{C}[G]$, $\chi \in \widehat{G}$. From here it follows that $\|f\|_{C^*(G)} = \|\widehat{f}\|_{C(\widehat{G})}$.

Recall that the Fourier transform $\mathcal{F}$ extends to the unitary operator $U : l_2(G) \to L^2(\widehat{G})$. The regular representation λ extended to the representation $\tilde{\lambda}$ of the algebra $\mathbb{C}[G]$ is unitarily equivalent to $U\tilde{\lambda}(f)U^*\widehat{h} = U\tilde{\lambda}(f)h = \mathcal{F}(f \star h) = \widehat{f}\,\widehat{h}$, i.e. the operator $\tilde{\lambda}(f)$ is unitarily equivalent to the operator $M_{\widehat{f}}$ of multiplication by the function $\widehat{f}$. Therefore $\|f\|_\lambda = \|\widehat{f}\|_{C(\widehat{G})}$. But $\|\widehat{f}\|_{C(\widehat{G})} = \|f\|_{C^*(G)}$, hence the norms on $C^*(G)$ and on $C^*_r(G)$ coincide. $\square$

Here is another argument directly showing that the $*$-homomorphism $i : C^*(G) \to C^*_r(G)$ induced by the identity map is an isomorphism. As both C^*-algebras are commutative, by Gelfand duality, there exist topological spaces X and Y, dual to $C^*(G)$ and $C^*_r(G)$, respectively, and the dual map $Y \to X$ is injective, hence we can view Y as a subset of X. We already have shown that $X = \widehat{G}$. Suppose that $Y \neq \widehat{G}$. Then there exists a non-zero function $\widehat{a} \in C(\widehat{G})$, whose restriction onto Y vanishes. Equivalently, there exists $a \in C^*(G)$ (dual to $\widehat{a}$) such that $a \neq 0$, but $i(a) = 0$. Then there exists $b = \sum_{g\in G} b_g g \in \mathbb{C}[G] \subset C^*(G)$ close to a such that $b \neq 0$, and we still have $i(b) = 0$. But, as $i|_{\mathbb{C}[G]}$ is the identity map, we get a contradiction.

Group C*-algebras of free groups

The free group $\mathbb{F}_2$ is generated by two generators without relations. Any unitary representation of the group $\mathbb{F}_2$ is determined by two unitary operators, U and V. Let (U_α, V_α) be the set of irreducible unitary pairs of operators on a separable Hilbert space. Then the group C*-algebra is isomorphic to $C^*(\oplus_\alpha U_\alpha, \oplus_\alpha V_\alpha)$.

Theorem 3.59. *There is a sequence π_n, $n \in \mathbb{N}$, of finite-dimensional representations of the C^*-algebra $C^*(\mathbb{F}_2)$ such that the representation $\oplus_{n\in\mathbb{N}}\pi_n$ is faithful.*

Proof. Since $\mathbb{C}[\mathbb{F}_2]$ is dense in $C^*(\mathbb{F}_2)$, the group C*-algebra is separable, so there is a faithful representation σ of it on a separable Hilbert space H. Let $U = \sigma(u)$, $V = \sigma(v)$, where u, v are generators of $\mathbb{F}_2$.

Let $L_n \subset H$ be an increasing sequence of finite-dimensional subspaces such that $\cup_{n\in\mathbb{N}} L_n$ is dense in H, and let P_n be the projection onto L_n. Set $A_n = P_n U|_{L_n}$, $B_n = P_n V|_{L_n}$ and define operators on $L_n \oplus L_n$ by

$$U_n = \begin{pmatrix} A_n & (1_n - A_n A_n^*)^{1/2} \\ (1_n - A_n^* A_n)^{1/2} & -A_n^* \end{pmatrix}, \quad V_n = \begin{pmatrix} B_n & (1_n - B_n B_n^*)^{1/2} \\ (1_n - B_n^* B_n)^{1/2} & -B_n^* \end{pmatrix},$$

where 1_n denotes the identity operator on L_n. It is easy to check that U_n and V_n are unitaries.

Let us put $\pi_n(u) = U_n$, $\pi_n(v) = V_n$ and $\pi = \oplus_{n\in\mathbb{N}} \pi_n$. It is easy to see that the sequences U_n and V_n converge in the strong topology, and $s\text{-}\lim_{n\to\infty} U_n = \begin{pmatrix} U & 0 \\ 0 & -U^* \end{pmatrix}$, $s\text{-}\lim_{n\to\infty} V_n = \begin{pmatrix} V & 0 \\ 0 & -V^* \end{pmatrix}$. So, if $f = \sum_{g\in\mathbb{F}_2} \alpha_g \delta_g$ is a finite sum then $s\text{-}\lim_{n\to\infty} \pi_n(f) = \begin{pmatrix} \sigma(f) & 0 \\ 0 & * \end{pmatrix}$ (no matter what is in the right lower corner — it may only increase the norm of the matrix, but not decrease it). Since the strong limit does not increase the norm, we obtain that

$$\|f\| \geq \|\pi(f)\| = \sup_{n\in\mathbb{N}} \|\pi_n(f)\| \geq \|s\text{-}\lim_{n\to\infty} \pi_n(f)\| \geq \|\sigma(f)\| = \|f\|.$$

From the density of finite sums in $C^*(\mathbb{F}_2)$ it follows that $\|\pi(f)\| = \|f\|$ for any $f \in C^*(\mathbb{F}_2)$. □

Theorem 3.60. *$C^*(\mathbb{F}_2)$ does not contain any projections other than 0 and 1.*

Proof. Let $A = \{F \in C([0,1], \mathbb{B}(H)) : F(0) \in \mathbb{C}1\}$. Let us show that this C*-algebra does not contain any projections other than 0 and 1. Let F be a projection, then $F(t)$ is a projection for each $t \in [0,1]$. But, for $t = 0$, $F(0)$ can be equal to either 0 or 1. Since any non-trivial projection is separated from 0 and 1 by a distance of 1, $F(t)$ must be equal to $F(0)$ for any t.

Let σ be a faithful representation of the algebra $C^*(\mathbb{F}_2)$, $U = \sigma(u)$, $V = \sigma(v)$. By the spectral theorem, there exist self-adjoint operators A, B such that $U = e^{iA}$, $V = e^{iB}$. Let us put $F(t) = e^{itA}$, $G(t) = e^{itB}$. Then $F, G \in A$ are unitaries, and the pair (F, G) defines a $*$-homomorphism from $C^*(\mathbb{F}_2)$ to A. This $*$-homomorphism is injective, since its composition with the evaluation homomorphism at the point $t = 1$ is injective. This means that we have constructed an embedding of $C^*(\mathbb{F}_2)$ into A. Since there are no nontrivial projections in A, there are none in $C^*(\mathbb{F}_2)$. □

Let us now turn to the reduced group C*-algebra of a free group. Recall that we use the notation δ_s, $s \in \mathbb{F}_2$, for the characteristic function of the point s.

For any $a \in C_r^*(\mathbb{F}_2)$ we put $\tau(a) = (\delta_e, a\delta_e)$. This defines the linear functional on $C_r^*(\mathbb{F}_2)$ satisfying the condition $\|\tau\| = 1$.

Lemma 3.61. *The functional τ is a faithful trace on $C_r^*(\mathbb{F}_2)$.*

Proof. Let $f = \sum_{s\in\mathbb{F}_2} \alpha_s\delta_s$, $h = \sum_{s\in\mathbb{F}_2} \beta_s\delta_s$ be finite sums. Then $\tau(\lambda(f)) = \alpha_e$, $\tau(\lambda(h)) = \beta_e$. That's why

$$\tau(f \star h) = \tau(\lambda(\sum_{s,t\in\mathbb{F}_2} \alpha_s\beta_t\delta_{st})) = \sum_{s\in\mathbb{F}_2} \alpha_s\beta_{s^{-1}} = \tau(\lambda(h \star f)).$$

By continuity, we conclude that $\tau(ab) = \tau(ba)$ for any $a, b \in C_r^*(\mathbb{F}_2)$.

If $a \in C_r^*(\mathbb{F}_2)$, $a \geq 0$ and $\tau(a) = 0$, then

$$(\delta_s, a\delta_s) = (\lambda(s)\delta_e, a\lambda(s)\delta_e) = (\delta_e, \lambda(s^{-1})a\lambda(s)\delta_e) = \tau(\lambda(s)^{-1}a\lambda(s)) = \tau(a) = 0$$

and $|(\delta_t, a\delta_s)|^2 \leq (\delta_t, a\delta_t) \cdot (\delta_s, a\delta_s) = 0$ (this is the Cauchy–Bunyakovsky inequality for the possibly degenerate scalar product $\langle x, y\rangle = (x, ay)$), so $a = 0$. $\square$

Corollary 3.62. *The map $\Phi : a \mapsto \tau(a) \cdot 1$ is a faithful conditional expectation from $C_r^*(\mathbb{F}_2)$ to the subalgebra $\mathbb{C}1$.*

Note that Lemma 3.61 and Corollary 3.62 are true for any group G. However, in the case of a free group there is an explicit formula for Φ (see Theorem 3.65 below).

Lemma 3.63. *Let $H = L \oplus L^\perp$, let B have the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ with respect to this decomposition, and let U_i, $i = 1, \ldots, n$, are unitary operators such that $U_iU_j^*$ have the form $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ for $i \neq j$. Then $\|\frac{1}{n}\sum_{i=1}^n U_i^*BU_i\| \leq \frac{2}{\sqrt{n}}\|B\|$.*

Proof. First suppose that B and C are of the form $B = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$, $C = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$. Then $\|B + C\|^2 = \|(B + C)^*(B + C)\| = \|B^*B + C^*C\| \leq \|B\|^2 + \|C\|^2$.

If $i \neq j$, then $(U_iU_j^*)^*B(U_iU_j^*)$ has the form $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}\begin{pmatrix} * & * \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$.
Hence

$$\begin{aligned} \|\sum_{i=1}^n U_i^*BU_i\|^2 &= \|U_1^*(B + \sum_{i=2}^n U_1U_i^*BU_iU_1^*)U_1\|^2 \\ &\leq \|B\|^2 + \|\sum_{i=2}^n U_1U_i^*BU_iU_1^*\|^2 = \|B\|^2 + \|\sum_{i=2}^n U_i^*BU_i\|^2. \end{aligned}$$

Inductively separating terms one by one, we get that $\|\sum_{i=1}^n U_i^*BU_i\| \leq n\|B\|^2$, so $\|\frac{1}{n}\sum_{i=1}^n U_i^*BU_i\| \leq \frac{1}{\sqrt{n}}\|B\|$.

The same result is obtained if B has the form $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$. For the general case, B can be written as a sum of two terms of the form $\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} + \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. $\square$

Lemma 3.64. *Let $s \in \mathbb{F}_2$, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \lambda(u^{-i})\lambda(s)\lambda(u^i) = \begin{cases} \lambda(s) & \text{if } s = u^k \text{ for some } k \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that any element of a free group is a word of generators and their inverses, and if one cannot shorten the word by cancelling subwords of the form xx^{-1} then this word is called reduced. There is a bijection between the reduced words and the elements of the free group. Let s does not equal any power of the generator u. Then it can be written in the form $s = u^m s_0 u^l$, where $m, l \in \mathbb{Z}$ and s_0 is equal to $v^{\pm 1}$ or $v^{\pm 1} t v^{\pm 1}$ (in the reduced form). Put

$$L = \mathrm{Span}\{\delta_s : s = e \ \text{ or } \ s = u^{\pm 1} r \text{ (in the reduced form)}\},$$

Then

$$L^{\perp} = \mathrm{Span}\{\delta_s : s = v^{\pm 1} r \text{ (in the reduced form)}\}.$$

Note that $\lambda(s_0)L \subset L^{\perp}$ and $\lambda(u^k)L^{\perp} \subset L$ for any $k \neq 0$. This means that $\lambda(s_0) = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ and $\lambda(u^i)\lambda(u^j)^* = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ for $i \neq j$ with respect to the expansion $l^2(\mathbb{F}_2) = L \oplus L^{\perp}$. By the previous Lemma, if $s = u^m s_0 u^l$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum\nolimits_{i=1}^{n} \lambda(u^{-i})\lambda(s)\lambda(u^i) = \lambda(u^m)\Big(\lim_{n\to\infty} \frac{1}{n} \sum\nolimits_{i=1}^{n} \lambda(u^{-i})\lambda(s_0)\lambda(u^i)\Big)\lambda(u^l) = 0.$$

On the other hand, if $s = u^k$, then the equality $\frac{1}{n}\sum_{i=1}^{n} \lambda(u^{-i})\lambda(s)\lambda(u^i) = \lambda(s)$ is satisfied in the obvious way for any $n \geq 1$. □

Theorem 3.65. *For any $a \in C_r^*(\mathbb{F}_2)$ one has*

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{n} \lambda(u^{-i}v^{-j}) a \lambda(v^j u^i) = \tau(a)\cdot \tag{3.6}$$

Proof. Subject $f = \sum_{g\in\mathbb{F}_2} \alpha_g \delta_g \in \mathbb{C}[\mathbb{F}_2]$. By the previous Lemma,

$$\lim_{n\to\infty} \frac{1}{n} \sum\nolimits_{i=1}^{n} \lambda(v^{-j})\lambda(f)\lambda(v^j) = \sum\nolimits_k \alpha_{v^k}\lambda(v^k),$$

i.e. from f, only the terms with $s = v^k$, $k \in \mathbb{Z}$, remain. Let us denote $f_0 = \sum_k \alpha_{v^k}\lambda(v^k)$. Then

$$\lim_{m\to\infty} \frac{1}{m} \sum\nolimits_{i=1}^{m} \lambda(u^{-i})\lambda(f_0)\lambda(u^i) = \alpha_e \cdot 1 = \tau(a)\cdot 1 = \Phi(a),$$

and the formula (3.6) is proven for the case when $a \in \mathbb{C}[\mathbb{F}_2]$. The general case is obtained by continuity. □

Corollary 3.66. *The algebra $C_r^*(\mathbb{F}_2)$ is simple.*

Proof. If $a \neq 0$, then $\tau(a^*a) \neq 0$, so any ideal containing a also contains 1. □

Corollary 3.67. *The full and the reduced group C*-algebras of the group $\mathbb{F}_2$ are not isomorphic.*

Proof. The first one has many ideals (for example, all kernels of finite-dimensional representations), while the second one is simple. □

Chapter 4

K-theory of C*-algebras

Lecture 17.03.2025

4.1 Homotopies in some classes of operators

Let $\pi : A^+ \to \mathbb{C}$, $(a, \lambda) \mapsto \lambda$, be the map that "forgets" elements of the algebra A. The same can be done for the matrix algebras over A. An element $x \in A^+$ is *normalized* if $\pi(x) = 1_{\mathbb{C}}$ (or the identity matrix in the matrix case). Let us denote by $\mathrm{GL}^+(A)$ and $U^+(A)$ the sets of normalized elements of the group $\mathrm{GL}(A^+)$ of invertibles and $U(A^+)$ of unitaries, respectively. Replacing A by the matrix algebra $M_n(A)$ over A (and π by $\pi_n : M_n(A)^+ \to \mathbb{C}$), we write $\mathrm{GL}_n^+(A)$ and $U_n^+(A)$ for the groups of normalized elements of $\mathrm{GL}_n(A^+)$ and $U_n(A^+)$, respectively. We will write $\tilde{A}$ to denote the C^*-algebra A if it already contains the unit, and the C^*-algebra A^+ if A is not unital.

Recall that in $\mathbb{B}(H)$ the polar decomposition exists for arbitrary operators, but in the general unital C*-algebra A the polar decomposition exists, generally speaking, only for invertible elements. It represents the invertible element $z \in A$ as the product of a unitary and a positive element: $z = z(z^*z)^{-1/2} \cdot (z^*z)^{1/2}$.

Lemma 4.1. *Let A be a unital C^*-algebra. Then the polar decomposition determines the deformation retraction of the group $GL(A)$ of invertible elements onto the group of unitary elements $U(A)$. A similar statement is true for matrix groups and for normalized elements (in the case of a non-unital algebra).*

Proof. If $z \in \mathrm{GL}(A)$ then $z^*z \in \mathrm{GL}(A)$, $z^*z \geqslant 0$, and $|z|^{-1} := (z^*z)^{-1/2}$ is defined. Also $u := z(z^*z)^{-1/2} \in U(A)$. Indeed,

$$u^*u = (z^*z)^{-1/2}z^*z(z^*z)^{-1/2} = 1_A,$$

$$uu^* = z(z^*z)^{-1/2}(z^*z)^{-1/2}z^* = z(z^*z)^{-1} = z \cdot z^{-1} \cdot (z^*)^{-1} \cdot z^* = 1_A.$$

Define the homotopy

$$z_t := z(z^*z)^{-1/2} \cdot (t \cdot 1_A + (1-t) \cdot (z^*z)^{1/2}), \qquad t \in [0,1].$$

Since the first factor is unitary, and the second is positive, and $\geqslant \min(1, \|(z^*z)^{-1/2}\|)\, 1_A$, the homotopy lies in the set of invertible elements. This (linear) homotopy is continuous and connects $z_0 = z$ with $z_1 = u$. For continuity of the retraction, it should also be noted that, for $z \in \mathrm{GL}(A)$, the maps $z \mapsto (z^*z)^{1/2}$ and $z \mapsto (z^*z)^{-1/2}$ are norm continuous.

If $z \in A^+$ is normalized then $(z^*z)^{1/2}$ and $(z^*z)^{-1/2}$ are also normalized. □

Definition 4.2. Let us call *symmetry* a self-adjoint unitary element: $a = a^* = a^{-1}$.

Proposition 4.3. *Any symmetry is homotopic (in $U(A)$) to the unit element.*

Proof. Let u be a symmetry, then $1 - u \in A_{sa}$, $(1_A - u)/2 \leqslant 1_A$, and we can define a norm continuous path in $U(A)$:

$$u_t := \exp\left(i\pi \cdot t \cdot \frac{1_A - u}{2}\right), \qquad t \in [0,1], \qquad u_0 = 1_A.$$

Further,

$$(1_A - u)^2 = 1_A - 2u + u^2 = 1_A - 2u + 1_A = 2(1_A - u).$$

By induction we get $(1_A - u)^n = 2^{n-1}(1_A - u)$. Therefore,

$$\begin{aligned}
u_1 &= \exp\left(i\pi \cdot \frac{1_A - u}{2}\right) = 1_A + \sum_{n=1}^{\infty} \frac{\left[i\pi \cdot \frac{1_A - u}{2}\right]^n}{n!} = \\
&= 1 + \sum_{n=1}^{\infty} \frac{\left[\frac{i\pi}{2}\right]^n \cdot 2^{n-1} \cdot \frac{1_A - u}{2}}{n!} = 1_A + \left[\sum_{n=1}^{\infty} \frac{(i\pi)^n}{n!}\right] \cdot \frac{1_A - u}{2} = \\
&= 1_A + \left[-1 + \sum_{n=0}^{\infty} \frac{(i\pi)^n}{n!}\right] \cdot \frac{1_A - u}{2} = 1 + [-1 + \exp(i\pi)] \cdot \frac{1_A - u}{2} = \\
&= 1_A + [-1 - 1] \cdot \frac{1_A - u}{2} = 1_A - 1_A + u = u.
\end{aligned}$$

□

Theorem 4.4. *The following elements in $M_2(\tilde{A})$*

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \qquad \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

are homotopic in $GL_2(\tilde{A})$, (if $u, v \in \tilde{A}$ are invertible), and in $U_2(\tilde{A})$, (if $u, v \in \tilde{A}$ are unitaries). The homotopy can be chosen to lie in the set of normalized elements.

Proof. Consider the rotation matrix

$$R_t := \begin{pmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix}, \qquad t \in [0,1].$$

Define, for $t \in [0,1]$, the homotopy

$$w_t := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} R_t^*, \qquad z_t := R_t \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} R_t^*. \tag{4.1}$$

Then w_t and z_t are continuous paths of invertible (in the second case, unitary) elements, and

$$w_0 = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned} w_1 &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & -u \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & v \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \end{aligned}$$

$$z_0 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

$$z_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -v \\ u & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

This gives two of the three necessary homotopies. The third one permutes u and v in w_t. Obviously, the homotopy consists of normalized elements. □

Denote by $\mathrm{Exp}(A)$ the subgroup $\mathrm{GL}(\tilde{A})$, consisting of all finite products of elements of the form $\exp(a)$, $a \in A$. Let $\mathrm{GL}(\tilde{A})_0$ be the connected component of the unit element (it is a clopen set).

Lemma 4.5. *Let G be a topological group and H its subgroup. If H is open in G, then H is closed in G.*

Proof. Since $H = G \setminus \cup_{g \notin H} gH$, then H is the complement of an open set. □

Proposition 4.6. *The element $z \in \tilde{A}$ is invertible and homotopic to the unit element if and only if $z = \prod_{k=1}^n \exp(a_k)$, $a_1, \dots a_n \in A$. Thus,* $\mathrm{GL}(\tilde{A})_0 = \mathrm{Exp}(A)$

Proof. Obviously, a product of this type is invertible, since the exponents are invertible (the exponent of an element close to zero is close to the unit due to the explicit representation in the form of a series, and for distant elements is determined through normalization and close elements). Since $z_t := \prod_{k=1}^n \exp(ta_k)$, $t \in [0,1]$ is a continuous path of invertible elements connecting z with 1, we have $\mathrm{Exp}(A) \subset \mathrm{GL}(\tilde{A})_0$. By Lemma 4.5, it remains to prove openness of $\mathrm{Exp}(A)$. (Here we use that, for open subsets of a Banach space, connectness is equivalent to path connectedness.)

Take an arbitrary element $z_0 = \prod_{k=1}^n \exp(a_k)$ in $\mathrm{Exp}(A)$. We must show that any invertible element z sufficiently close to z_0 is represented as a product of exponents. Assume that $\|z - z_0\| < \|z_0^{-1}\|^{-1}$, and set $z' := zz_0^{-1}$. Then

$$\|z' - 1\| \leqslant \|z - z_0\| \cdot \|z_0^{-1}\| < 1,$$

so the spectrum of $z' - 1$ is contained in the open unit disk, and the spectrum z' lies to the right of the imaginary axis. Therefore the logarithm is holomorphic in a neighborhood of the spectrum of z' and we can write $z' = \exp(\log(z'))$ (note that z' need not be normal, so we use holomorphic rather than continuous functional calculus). Hence $z = z'z_0 = \exp(\log(z')) \prod_{k=1}^n \exp(a_k) \in \mathrm{Exp}(A)$. □

Corollary 4.7. *Let A and B be C^*-algebras, and $\alpha : A \to B$ be a surjective homomorphism. Then α extends to a unital surjective homomorphism $\widetilde{\alpha} : \widetilde{A} \to \widetilde{B}$ such that the invertible elements in the connected component of 1 in $\widetilde{B}$ can be lifted to invertible elements in the connected component of 1 in $\widetilde{A}$. The same is true if we replace 'invertible' by 'unitary'.*

In particular, $\alpha^+(\mathrm{GL}_n^+(A)_0) = \mathrm{GL}_n^+(B)_0$ and $\alpha^+(U_n^+(A)_0) = U_n^+(B)_0$, where $\alpha^+ : A^+ \to B^+$ is the standard extension of α.

Proof. The homomorphism $\widetilde{\alpha}$ is defined as follows. If both A and B are unital then $\widetilde{\alpha} := \alpha$. If they are both non-unital then $\widetilde{\alpha}(a, \lambda) := (\alpha(a), \lambda)$. If A is non-unital, and B has the unit 1_B, then $\widetilde{\alpha}(a + \lambda) := \alpha(a) + \lambda\, 1_B$. The remaining case is excluded, since α is surjective.

Let $y \in B^\sim$ be invertible and $y \sim_h 1$, then according to Proposition 4.6 $y = \prod_{k=1}^n \exp(z_k)$ for some $z_1, \ldots, z_n \in B^\sim$. Since $\alpha^\sim$ is surjective, we find the lifts $x_k \in A^\sim$ for z_k, $\alpha^\sim(x_k) = z_k$, $k = 1, \ldots, n$. The element $a := \prod_{k=1}^n \exp(x_k)$ lies in the connected component of the unit of $A^\sim$, is invertible, and satisfies

$$\alpha^\sim(a) = \alpha^\sim\left(\prod\nolimits_{k=1}^n \exp(x_k)\right) = \prod\nolimits_{k=1}^n \exp\left(\alpha^\sim(x_k)\right) = \prod\nolimits_{k=1}^n \exp(z_k) = y.$$

If y is unitary, then consider $u := a(a^*a)^{-1/2}$. Then u is a unitary element homotopic to a (and therefore to 1) by Lemma 4.1. We also have $\alpha^\sim(u) = y(y^*y)^{-1/2} = y$.

The remaining statements are more or less obvious. ☐

Corollary 4.8. *If J is an ideal in A and u is a unitary element in $(A/J)^\sim$, then $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ can be lifted to a unitary element $w \in M_2(A^\sim)$, homotopic to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. Apply Corollary 4.7 and Theorem 4.4. ☐

4.2 Projections and partial isometries

Definition 4.9. A *projection* is a self-adjoint idempotent: $p = p^* = p^2$. Two projections, p and q, are *orthogonal* if $pq = 0$ (and therefore $qp = q^*p^* = (pq)^* = 0$). The sum of two orthogonal projections $p \oplus q = p + q$ is also a projection:

$$(p+q)(p+q) = p^2 + qp + pq + q^2 = p + 0 + 0 + q = p + q.$$

We write $p \leqslant q$ if $qp = pq = p$.

Proposition 4.10. *1) Let $p \in A$ be a projection, then $1 - p \in A^\sim$ is also a projection. It is orthogonal to p and is called the* orthogonal complement *of p.*

2) More generally: if $p \leqslant q$, then qp is a projection, orthogonal to p.

Proof. 1) $(1-p)(1-p) = 1 - 2p + p^2 = 1 - 2p + p = 1 - p$, so $1 - p$ is an idempotent, and $(1-p)^* = 1 - p^* = 1 - p$. Also

$$p\,(1-p) = pp^2 = pp = 0.$$

2) $(q-p)^2 = q^2 - qp - pq + p^2 = q - p - p + p = q - p$,
$p(q-p) = pq - p^2 = p - p = 0.$ ☐

Problem 69. For any projection p one has $0 \leqslant p \leqslant 1$. *Note:* $p = p^*p \geqslant 0$, $\|p\| = \|p^*p\| = \|p\|^2$, $\|p\| = 0$ or 1.

Problem 70. For any projection p the element $2p - 1 \in A^{\sim}$ is a symmetry. *Note:* $(2p-1)(2p-1) = 4p^2 - 2p - 2p + 1 = 1$.

Problem 71. If v is a symmetry, then $\frac{v+1}{2}$ is a projection. (This projection, unlike the symmetry from the previous problem, does not have to be homotopic, in the set of projections, to the identity.)

Problem 72. Let $p \neq 0$ be a projection. Then $\|p\| = 1$ (see problem 69).

Problem 73. Let $p \neq 0, 1$ be a projection. Then $\operatorname{Sp} p = \{0, 1\}$. Conversely, let v be a normal element and $\operatorname{Sp} v = \{0, 1\}$. Then v is a projection. *Note:* use spectrum mapping theorem for $f(t) = t^2$. Back: the function of a normal operator is determined by the values on the spectrum of this operator.

Lemma 4.11. *Let p, q be projections. Then the following conditions are equivalent:*

(1) *$p + q$ is a projection;*

(2) *p and q are orthogonal;*

(3) *$p + q \leqslant 1$.*

Proof. (2)⇒(1): was proven above. (1)⇒(3): see problem 69. (3)⇒(2): we have $p(p + q)p \leqslant p^2$, so $p + pqp \leqslant p$, $pqp \leqslant 0$, whence $pqp = 0$, since $pqp \geqslant 0$. This implies $0 = pqp = pqqp = p^*q^*qp = (qp)^*qp$ and $qp = 0$. □

Definition 4.12. The element $v \in A$ is called a *partial isometry*, if v^*v is a projection. If A is unital and $v^*v = 1_A$, then v is called an *isometry.* A projection $p := v^*v$ is called the *domain* of v, and $q = vv^*$ is called the *range* of v.

Lemma 4.13. *The element $q = vv^*$ is a projection.*

Proof. Obviously, q is a self-adjoint element, $q \in A_{sa}$. Further,

$$q^4 = vv^*vv^*vv^*vv^* = vp^3v^* = vpv^* = vv^*vv^* = q^2.$$

Thus, q^2 is a projection, and its spectrum lies in $\{0, 1\}$. Then, by the spectral mapping theorem, the spectrum q is also of this form. □

Proposition 4.14. *The following relations hold:*

1. *$v = vv^*v = vp = qv$,*
2. *$v^* = v^*vv^* = pv^* = v^*q$.*

Proof. The second relation is obtained from the first one by conjugation. The second and the third equality in the first item are obvious. Let us now prove for example, that $v = vp$, i.e. $v(1-p) = 0$. Indeed, $v(1-p)(1-p)^*v^* = v(1-p)v^* = q - q^2 = 0$. □

Lemma 4.15. *Let v_1 and v_2 be partial isometries such that projections $p_1 := v_1^* v_1$ and $p_2 := v_2^* v_2$ are orthogonal: $p_1 p_2 = 0$. Then $v_1 v_2^* = v_2 v_1^* = 0$.*

Proof. Indeed, $v_1^* v_1 v_2^* v_2 = 0$, whence, multiplying, we get

$$0 = v_1 v_1^* v_1 v_2^* v_2 v_2^* = v_1 v_2^*$$

by 4.14. □

Proposition 4.16. *Let $a \in A$ and $0 \leqslant a \leqslant 1_A$. If $\|a^2 - a\| < \varepsilon \leqslant 1/4$, then there is a projection $p \in A$ such that $\|p - a\| < 2\varepsilon \leqslant 1/2$.*

Proof. Let $t \in spec(a)$. Then t is real and $0 \leqslant t - t^2 < \varepsilon \leqslant \frac{1}{4}$, hence, $t \in \left[0, \frac{1}{2} - \delta\right] \cup \left[\frac{1}{2} + \delta, 1\right]$, where $\delta := \frac{1}{2}\sqrt{1 - 4\varepsilon}$. Set $p := f(a)$, where

$$f(t) = \begin{cases} 0, & \text{if } t < \frac{1}{2}; \\ 1, & \text{if } t > \frac{1}{2} \end{cases}$$

is a continuous function on the spectrum of a. Since $f = \overline{f} = f^2$, then p is a projection. Further,

$$\sup\{\, |f(t) - t| \,,\, t \in spec(a)\} \leqslant \frac{1}{2} - \delta,$$

So

$$\|p - a\| = \|f(a) - \mathrm{Id}(a)\| \leqslant \frac{1}{2} - \delta = \frac{1}{2}(1 - \sqrt{1 - 4\varepsilon}) < 2\varepsilon,$$

since $\varepsilon < \frac{1}{4}$, and $1 - 4\varepsilon < \sqrt{1 - 4\varepsilon}$. □

Lecture 24.03.2025

Definition 4.17. The projections p and q in the C^*-algebra A are called

- *equivalent*, or *Murray-von Neumann equivalent*, if $p = v^*v$, $q = vv^*$ for some partial isometry $v \in A$;
- *unitarily equivalent*, if $p = u^*qu$ for some unitary element $u \in \tilde{A}$;
- *homotopic*, if p and q can be are connected by a norm-continuous homotopy of projections.

These relations will be denoted by $\sim$, $\sim_u$, and $\sim_h$, and the classes are $[.]$, $[.]_u$, and $[.]_h$, respectively.

Problem 74. In commutative algebra, equivalent projections are equal to each other.

Problem 75. The domain and the range projections of a partial isometry are equivalent.

Problem 76. The zero projection is always equivalent only to itself. The unit projection is unitarily equivalent only to itself, but it can be Murray-von Neumann equivalent to smaller projections.

Proposition 4.18. *Unitary equivalence implies Murray-von Neumann equivalence.*

Proof. We have $p = u^*qu$ for some unitary element $u \in \tilde{A}$. Put $v = qu$. Then

$$v^*v = u^*qqu = u^*qu = p, \quad vv^* = quu^*q = qq = q.$$

□

Proposition 4.19. *If p_1, p_2, q_1, q_2 are projections in A, and*

$$p_1 \sim q_1, \quad p_2 \sim q_2, \quad p_1 \perp p_2, \quad q_1 \perp q_2,$$

then $p_1 \oplus p_2 \sim q_1 \oplus q_2$.

Proof. Let $p_1 = v_1^*v_1$, $q_1 = v_1v_1^*$, $p_2 = v_2^*v_2$, $q_2 = v_2v_2^*$. By lemma 4.15

$$v_1v_2^* = v_2v_1^* = v_1^*v_2 = v_2^*v_1 = 0.$$

That's why

$$(v_1 + v_2)^*(v_1 + v_2) = v_1^*v_1 + v_2^*v_2 = p_1 \oplus p_2,$$
$$(v_1 + v_2)(v_1 + v_2)^* = v_1v_1^* + v_2v_2^* = q_1 \oplus q_2.$$

□

Proposition 4.20. *Let $q = zpz^{-1}$, where $p, q \in A$ are projections, and $z \in \tilde{A}$ is an invertible element. Then $p \sim_u q$.*

Proof. As $qz = zp$, we have $z^*q = pz^*$ and $pz^*z = z^*qz = z^*zp$. Thus, p commutes with z^*z, and hence with functions of this positive element, in particular, with $|z|^{-1} = (z^*z)^{-1/2}$. For unitary $u := z|z|^{-1}$ we have

$$upu^* = z|z|^{-1}p|z|^{-1}z = zp|z|^{-2}z = qz|z|^{-2}z = q.$$

□

Proposition 4.21. *Projections p and q in A are unitarily equivalent to ($p \sim_u q$) if and only if $p \sim q$ and $1 - p \sim 1 - q$.*

Proof. Let v and w be partial isometries in $A^{\sim}$ such that

$$v^*v = p, \quad vv^* = q, \quad w^*w = 1 - p, \quad ww^* = 1 - q.$$

Then, by Lemma 4.15, applied to p and $1 - p$, we have $vw^* = wv^* = 0$, and by this Lemma applied to q and $1 - q$, we have $v^*w = w^*v = 0$. So, putting $u := v + w$, we have

$$u^*u = (v^* + w^*)(v + w) = v^*v + w^*w = p + (1 - p) = 1,$$

$$uu^* = (v + w)(v^* + w^*) = vv^* + ww^* = q + (1 - q) = 1.$$

Thus u is unitary. Next, by Proposition 4.14 $w(1-p) = w$, so $wp = 0$, as well as $pw^* = 0$. That's why

$$upu^* = (v + w)p(v^* + w^*) = vpv^* = q.$$

The converse implication follows from Proposition 4.18. □

Proposition 4.22. *Let p and q be projections such that $\|p - q\| < 1$, then they are homotopic.*

More precisely, on the set of projections q satisfying $\|p - q\| < 1$, a continuous map $q \mapsto u_q$ to the set of unitary elements of $A^{\sim}$ is defined, such that the following conditions hold:

(1) $q = u_q p u_q^*$,

(2) *u_q is homotopic to 1 inside the set of unitary elements.*

If A is non-unital then u_q can be chosen to be normalized: $\pi(u_q) = 1$.

Proof. Define the following two symmetries in $A^{\sim}$:

$$v_p := 2p - 1, \qquad v_q := 2q - 1, \qquad \text{and put} \qquad z_q := v_q v_p + 1.$$

Then

$$qz_q = q(2q-1)(2p-1)+q = 4qp+2q-2qp-q+q = 2qp = (2q-1)(2p-1)p+p = z_q p. \quad (4.2)$$

The element z_q is invertible in $A^{\sim}$, since when $\|p - q\| < 1$ we have

$$\|z_q - 2\| = \|v_q v_p - 1\| = \|v_q(v_p - v_q)\| \leqslant \|v_p - v_q\| = 2\|p - q\| < 2. \quad (4.3)$$

Therefore, rewriting (4.2) in the form $q = z_q p z_q^{-1}$ and denoting $u_q := z_q|z_q|^{-1}$ (the unitary from the polar decomposition of z_q), we get $q = u_q p u_q^*$ (this follows from the proof of Proposition 4.20).

As

$$\|z_{q_1} - z_{q_2}\| = \|v_{q_1}v_p - v_{q_2}v_p\| \leqslant \|v_{q_1} - v_{q_2}\|\,\|v_p\| = 2\|q_1 - q_2\|,$$

the map $q \mapsto Z_q$ is continuous. As the map $z \mapsto z|z|^{-1}$ is continuous on $\mathrm{GL}(A^\sim)$, by (4.3), the map $q \mapsto u_q$ is well defined and continuous on the set $\{q \in A \,|\, q = q^2 = q^*,\ \|p - q\| < 1\}$.

Moreover, the correspondence $t \mapsto z_{t,q} := tz_q + 2 - 2t$, $t \in [0, 1]$, defines a homotopy in the set of invertible elements for every q close to p, since $\|z_{t,q} - 2\| = t\|z_q - 2\| < 2$. This homotopy connects $z_{0,q} = 2$ with $z_{1,q} = z_q$. Take the unitary part $u_{t,q}$ of the polar decomposition of $z_{t,q}$. We obtain a homotopy within the unitary elements between 1 and u_q, so $u_q \in U(A^\sim)_0$. The family $u_{t,q} p u_{t,q}^*$, $t \in [0, 1]$, defines a homotopy of projections connecting p and q.

Finally, if A is not unital, then as $p, q \in A$, we have $\pi v_p = \pi v_q = -1$, so $\pi z_q = 2$, and $\pi u_q = 1$. □

Corollary 4.23. *Let $(q_\lambda)_{\lambda\in\Lambda}$ be a directed set of projections in A, converging to the projection p, and $\|p - q_\lambda\| < 1$ for all $\lambda \in \Lambda$. Then there is such a directed set $(u_\lambda)_{\lambda\in\Lambda}$ of unitary elements in A such that $\|1 - u_\lambda\| \to 0$ and $q_\lambda = u_\lambda p u_\lambda^*$ for each $\lambda \in \Lambda$.*

Proof. According to the construction from Proposition 4.22, the map $q_\lambda \mapsto u_\lambda$ is continuous, therefore u_λ converges to $u_p = 1$, since

$$z_p = (2p - 1)(2p - 1) + 1 = 4p - 2p - 2p + 1 + 1 = 2.$$

□

Corollary 4.24. *Let p and q be projections in A.*

1. *Then p is homotopic to q if and only if there exists a homotopy u_t of unitary elements $A^\sim$ such that $u_0 = 1$ and $p = u_1 q u_1^*$.*
2. *Moreover, if some homotopy p_t of projections is given between p and q, then it can be given by $p_t = u_t p u_t^*$ for some homotopy u_t of unitary elements $A^\sim$.*
3. *If A is non-unital then each u_t can be chosen to be normalized.*
4. *If A is non-unital, if the projection p_0 or p_1 belongs to A and if they are connected by the homotopy p_t of projections in $A^\sim$, then the homotopy actually lies in A: $p_t \in A$ for any $t \in [0, 1]$.*

Proof. The sufficiency of the first statement is obvious. For the opposite direction, let us divide the segment $[0, 1]$ into small segments $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|p_{t_i} - p_t\| < 1$ for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$. For each of the segments apply Proposition 4.22. We get continuous families of unitary elements $A^\sim$:

$$u_t^i, \quad t \in [t_i, t_{i+1}], \quad u_{t_i}^i = 1, \quad p_t = u_t^i p_{t_i} (u_t^i)^*.$$

For $t \in [t_j, t_{j+1}]$ we set $u_t := u_t^j u_{t_j}^{j-1} \dots u_{t_1}^0$. This is the required homotopy that proves the second statement, from which a weaker one follows: the necessity in the first statement.

Normalization follows from the explicit construction in Proposition 4.22.

Finally, the fourth statement follows (we assume that $p_0 \in A$) from the fact that $abc \in A$ if $b \in A$ and $a, c \in \tilde{A}$ (take $b = p_0$, $a = u_t$, $c = u_t^*$). $\square$

Proposition 4.25. *$p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q$.*

Proof. The first implication is proven in Corollary 4.24, and the second implication follows from Proposition 4.18. $\square$

Problem 77. Prove that the reverse implications do not hold. For the second one, consider $A = B(H)$, the identical projection and a non-identical projection with infinite-dimensional range, and for the first one consider $A = M_2(C(S^3))$.

Proposition 4.26.

$$p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

$$p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \tag{4.5}$$

Proof. Let v be a partial isometry such that $p = v^*v$ and $q = vv^*$. Consider $u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \in M_2(\tilde{A})$. Then (see Proposition 4.14)

$$\begin{aligned} u^*u &= \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix}\begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} = \\ &= \begin{pmatrix} v^*v+1-p & v^*-v^*q+v^*-pv^* \\ v-qv+v-vp & 1-q+vv^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} uu^* &= \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}\begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} = \\ &= \begin{pmatrix} vv^*+1-q & v-vp+v-qv \\ v^*-pv^*+v^*-v^*q & 1-p+v^*v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus, u is unitary, but not normalized. Besides,

$$\begin{aligned} u\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}u^* &= \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} = \\ &= \begin{pmatrix} vp & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} = \begin{pmatrix} vpv^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The first part has been proven.

Let now $q = upu^*$. In accordance with Theorem 4.4 choose a homotopy $w_t \in U_2(\tilde{A})$, connecting $w_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ and $w_1 = \begin{pmatrix} uu^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $p_t := w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*$ is a homotopy of projections, connecting

$$p_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} upu^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

with $p_1 = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. At the same time, homotopy does not lie in the whole $M_2(\tilde{A})$, but in the smaller set $M_2(A)$, since the factor $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ lies there. □

Proposition 4.27. *If p and q are projections, then projections $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ and $\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$ are homotopic.*

Proof. Define a homotopy similarly to z_t in (4.1). □

4.3 Representing elements as matrices

Let $p \in A$ be a projection, then we have the decomposition

$$p \oplus (1 - p) = 1 \in \tilde{A},$$

and accordingly, for any $a \in A$:

$$a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p). \tag{4.6}$$

This can be written as

$$\varphi_p : A \to M_2(A), \qquad a \mapsto \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}.$$

Let's define

$$\psi : M_2(A) \to A, \qquad \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \mapsto a_1 + a_2 + a_3 + a_4.$$

Problem 78. Check that φ_p defines a C^*-isomorphism on its image, and the inverse map is given by ψ. Make sure that for the whole $M_2(A)$ the map ψ is not an isomorphism, or even a homomorphism.

Of course, in the general situation the representation (4.6) (or φ_p) is far from representing A as a matrix algebra: "the terms are too different from each other". Therefore it is reasonable to limit ourselves to the following situation: $p \sim (1-p)$ and A unital. Then we can choose a partial isometry v such that $v^*v = p$, $vv^* = 1-p$, and define

$$\psi_v : M_2(A) \to A, \quad \psi_v \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := pa_{11}p + a_{12}v^* + va_{21} + va_{22}v^*,$$

$$\varphi_v : A \to M_2(pAp), \quad f_v(a) := \begin{pmatrix} pap & pav \\ v^*ap & v^*av \end{pmatrix}$$

(matrix elements belong to pAp, since $v = vp$, $v^* = pv^*$).

Problem 79. Check that

$$\psi_v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \psi_v \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = p, \quad \psi_v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \psi_v \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = p,$$

$$\psi_v \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \psi_v \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = v^*, \quad \psi_v \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \psi_v \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} = v.$$

Prove that $\varphi_v : A \to M_2(pAp)$ is a C^*-isomorphism, and the inverse is given by ψ_v.

Proposition 4.28. *Let $p_1, \dots, p_n$ be pairwise orthogonal equivalent projections in the unital algebra A, and $p_1 \oplus \dots \oplus p_n = 1$. Then A is isomorphic to $M_n(p_1Ap_1)$.*

Proof. Let us choose partial isometries v_k, $k = 1, \dots, n$, such that $v_1 = p_1$, $v_k^*v_k = p_1$, $v_kv_k^* = p_k$. Then $v_k^* = v_k^*p_k = v_k^*v_kv_k^* = p_1v_k^*$ and $v_k = v_kp_1$ (see proposal 4.14) so $n \times n$-matrix

$$\varphi_n(a) := \begin{pmatrix} v_1^*av_1 & v_1^*av_2 & \dots & v_1^*av_n \\ \vdots & \vdots & & \vdots \\ v_n^*av_1 & v_n^*av_2 & \dots & v_n^*av_n \end{pmatrix}$$

belongs to $M_n(p_1Ap_1)$. Let $\psi_n : M_n(p_1Ap_1) \to A$ assigns to a matrix $\|b_{ij}\|$ the sum $\sum_{i,j=1}^n v_ib_{ij}v_j^*$. Then

$$\psi_n \circ \varphi_n(a) = \sum\nolimits_{i,j=1}^n v_iv_i^*av_jv_j^* = \sum\nolimits_{i,j=1}^n p_iap_j = \left(\sum\nolimits_{i=1}^n p_i\right) a \left(\sum\nolimits_{j=1}^n p_j\right) = a.$$

Backwards,

$$[\varphi_n \circ \psi_n(a_{ij})]_{rs} = v_r^* \left(\sum\nolimits_{i,j} v_ia_{ij}v_j^*\right) v_s = p_1a_{rs}p_1 = a_{rs},$$

as

$$v_r^*v_i = \begin{cases} p_1, & r = i; \\ v_r^*p_rp_iv_i = 0, & r \neq i. \end{cases}$$

So, φ_n and ψ_n are mutually inverse, and therefore bijective. Obviously, they preserve the linear space structure and conjugation. Let's check multiplicativity (it suffices to check it for φ_n):

$$[\varphi_n(a)\varphi_n(b)]_{rs} = \sum\nolimits_{k=1}^n v_r^*av_kv_k^*bv_s = v_r^*a\left(\sum\nolimits_{k=1}^n p_k\right) bv_s = v_r^*abv_s = [\varphi_n(ab)]_{rs}.$$

□

Lemma 4.29. *Let A be unital, and let $w \in A$ be an isometry. Let $p := ww^*$. Then the map $a \mapsto waw^*$ defines a $*$-isomorphism from A to pAp.*

Proof. Since $waw^* = ww^*waw^*ww^* = pwaw^*p$ (by Proposition 4.14), then the image of the map $a \mapsto waw^*$ is contained in pAp. It's obvious that it is a $*$-homomorphism. Its multiplicativity follows from $w^*w = 1$. Surjectivity follows from $pap = ww^*aww^* = w(w^*aw)w^*$. To prove injectivity, note that if $waw^* = 0$, then $0 = w^*(waw^*)w = a$. □

So we have to impose another constraint for projections and come to the following definition.

Definition 4.30. A projection p in the unital C^*-algebra A is called a *half*, or *proper* if p and $1 - p$ are equivalent to 1.

Remark 4.31. It is obvious that in this situation $p \sim (1-p)$ and one can write $p = v^*v$, where v is an isometry, not a partial isometry.

4.4 Equivalence of projections in infinite-dimensional matrix algebras

Definition 4.32. Let $M_n(A) \hookrightarrow M_{n+1}(A)$ be an embedding in the upper left corner (bordered by zeros). Put $M_\infty(A) := \cup_n M_n(A)$. This is involutive algebra, but not complete.

Definition 4.33. Two projections $p, q \in M_\infty(A)$ are called *equivalent*, $p \sim q$, if there is $v \in M_\infty(A)$ such that $p = v^*v$ and $q = vv^*$. We denote by $[.]$ the corresponding classes, and by $V(A)$ the set of such classes,

$$V(A) := \{[p] \, : \, p = p^* = p^2 \in M_\infty(A)\}.$$

Define the addition by:

$$[p] + [q] := [\mathrm{diag}(p, q)] = [p' \oplus q'],$$

if $p' \sim p$, $q' \sim q$, $p' \perp q'$.

Remark 4.34. Let $p, q \in M_n(A)$ and $p \sim q$ in $M_\infty(A)$. Then v appearing in the definition satisfies $v = vv^*vv^*v = qvp$, so $v \in M_n(A)$, and in this case we have equivalence of p and q in $M_n(A)$. Further, note that by Propositions 4.25 and 4.26, different types of projection equivalence in the algebra $M_\infty(A)$ give the same equivalence.

Proposition 4.35. *For every C^*-algebra A, $V(A)$ is an Abelian semigroup with the neutral element $0 = [0]$ (we use the additive notation).*

For any morphism of C^*-algebras $\alpha : A \to B$ the induced map α_* is given by

$$\alpha_* : V(A) \to V(B), \qquad \alpha_*[(a_{ij})] := [(\alpha(a_{ij}))],$$

and is a well-defined semigroup homomorphism.

The correspondence $A \mapsto V(A)$, $\alpha \mapsto \alpha_*$, is a covariant functor from the category of C^*-algebras to the category of Abelian semigroups.

Proof. The first statement is obvious.

Since α is a $*$-homomorphism of C^*-algebras, then, passing to matrices, it maps self-adjoint matrices to self-adjoint ones, and idempotents to idempotents, in particular, projections to projections. The same for v shows that α preserves equivalence. Obviously, the direct sums are preserved. Thus, α_* is a well defined homomorphism.

It remains to check functoriality on morphisms: if $\beta : B \to C$ is another morphism, then $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$, which is checked immediately at the level of representatives. Finally, the identity morphism $\mathrm{Id} : A \to A$ induces the identity map of $V(A)$. $\square$

Example 4.36. 1. Let $A = \mathbb{C}$. Any element of $V(\mathbb{C})$ is given by a projection in a sufficiently large matrix algebra $M_n(\mathbb{C})$. Projections in $M_n(\mathbb{C})$ are equivalent precisely when their images have the same dimension, so $V(\mathbb{C}) = \mathbb{N} \cup \{0\}$.

2. Since $M_n(M_k) = M_{nk}$, then from the previous example it follows that $V(M_k) = \mathbb{N} \cup \{0\}$.

3. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the algebra of compact operators, and $B = B(\mathcal{H})$ be the algebra of all bounded operators. Then $M_n(\mathcal{K}(\mathcal{H})) \cong \mathcal{K}(\mathcal{H}^n) \cong \mathcal{K}(\mathcal{H})$ and $M_n(B(\mathcal{H})) \cong B(\mathcal{H})$. In $\mathcal{K}(\mathcal{H})$ every projection is finite-dimensional, and $B(\mathcal{H})$ contains infinite-dimensional projections. Projections are equivalent if and only when they have the same image dimensions. To verify this, just select orthonormal basis in the image of each of the projections. Moreover, partial isometries defining the equivalence in the case of $\mathcal{K}(\mathcal{H})$, are compact operators. Thus,

$$V(\mathcal{K}) = \mathbb{N} \cup \{0\}, \qquad V(B) = \mathbb{N} \cup \{0, \infty\}.$$

4. Let $A = B/\mathcal{K}$ (the Calkin algebra). Modulo $\mathcal{K}$ all finite-dimensional projections in B are equivalent. Thus, $V(B/\mathcal{K}) = \{0, \infty\}$.

Lecture 31.03.2025

4.5 Grothendieck Group

Let us formalize the process of immersion of a semigroup into a group, for example, $\mathbb{N} \hookrightarrow \mathbb{Z}$.

Let H be a commutative semigroup (the set with associative and commutative operation), and K — its subsemigroup. We will use multiplicative notation, although the situation is commutative.

Let us define the following equivalence relation $\equiv$ on $H \times K$:

$$\begin{aligned} (h_1, k_1) \equiv (h_2, k_2) \quad &\Leftrightarrow \quad \exists y_1, y_2 \in K : \quad (h_1 y_1, k_1 y_1) = (h_2 y_2, k_2 y_2) \\ &\Leftrightarrow \quad \exists x \in K : \quad h_1 k_2 x = k_1 h_2 x. \end{aligned}$$

A pair (h, k) can be thought of as $\frac{h}{k}$.

Problem 80. If we simply define $(h_1, k_1) \equiv (h_2, k_2) \Leftrightarrow h_1 k_2 = h_2 k_1$, then there will be no transitivity.

Lemma 4.37. *The conditions are indeed equivalent and define an equivalence relation.*

Proof. Let the second be done. Let's take $y_1 = k_2 x$, $y_2 = k_1 x$. Then

$$h_1 y_1 = h_1 k_2 x = k_1 h_2 x = h_2 y_2, \quad k_1 y_1 = k_1 k_2 x = y_2 k_2,$$

and the first one is done. Conversely, let the first one be done. Let's take $x = y_1 y_2$. Then

$$h_1 k_2 x = h_1 k_2 y_1 y_2 = (h_1 y_1)(k_2 y_2) = h_2 y_2 k_1 y_1 = h_2 k_1 x,$$

and the second is done. Reflexivity and symmetry (from of the second definition) are obvious. Let $(h_1, k_1) \equiv (h_2, k_2)$ and $(h_2, k_2) \equiv (h_3, k_3)$, those. there exist $x_1, x_2 \in K$ such that $h_1 k_2 x_1 = k_1 h_2 x_1$, $h_2 k_3 x_2 = k_2 h_3 x_2$. Then

$$(h_1 k_3)(k_2 x_1 x_2) = k_1 h_2 x_1 k_3 x_2 = k_1 x_1 k_2 h_3 x_2 = (k_1 h_3)(k_2 x_1 x_3),$$

those. $h_1 k_3 x = k_1 h_3 x$ for $x = k_2 x_1 x_2 \in K$ and the second condition is satisfied, so $(h_1, k_1) \equiv (h_3, k_3)$. Transitivity is established. $\square$

Let us denote the set of equivalence classes by $[H][K]^{-1} := (H \times K)/\equiv$, and equivalence classes by $[.]$. Notice, that on classes, multiplication is correctly defined by the formula

$$[(h_1, k_1)]\,[(h_2, k_2)] := [(h_1 h_2, k_1, k_2)]$$

(like multiplying fractions). Indeed, let $(h_1, k_1) \equiv (h_1', k_1')$, $(h_2, k_2) \equiv (h_2', k_2')$:

$$h_1 k_1' x_1 = k_1 h_1' x_1, \qquad h_2 k_2' x_2 = k_2 h_2' x_2, \qquad x_1, x_2 \in K.$$

Then

$$(h_1h_2)(k_1'k_2')(x_1x_2) = (h_1k_1'x_1)(h_2k_2'x_2) = (k_1h_1'x_1)(k_2h_2'x_2) = (h_1'h_2')(k_1k_2)(x_1x_2).$$

We say that H is a *semigroup with cancellation* if $h_1h = h_2h$ implies $h_1 = h_2$.

Note that always $(x,x) \equiv (y,y)$, $x, y \in K$. Let us denote the corresponding class by 1. Then for any $h \in H$, $k \in K$,

$$1[(h,k)] = [(xh,xk)] = [(h,k)], \text{ since } (xh)kk' = h(xk)k' \text{ for any } k' \in K.$$

This means that $[H][K]^{-1}$ is an Abelian semigroup with identity (monoid). Moreover, $[(k_2,k_1)]$ is the inverse to $[(k_1,k_2)]$ if $k_1, k_2 \in K$. Really,

$$[(k_1,k_2)]\,[(k_2,k_1)] = [(k_1k_2,k_2k_1)] = 1.$$

In particular, $[H][H]^{-1}$ is an Abelian group.

Definition 4.38. The Abelian group $G(H) := [H][H]^{-1}$ is called *Grothendieck group* H.

Let's define the map

$$\iota : H \to [H][K]^{-1}, \quad \iota(h) := [(hk,k)], \quad k \in K.$$

Obviously, the mapping ι is well defined (not depends on k) and is consistent with intuition for $\mathbb{N} \to \mathbb{Q}$, $n \mapsto \frac{nk}{k}$.

Proposition 4.39. *1. The mapping ι is a homomorphism that is injective if and only if H is a semigroup with reductions to elements K^3.*

2. For any $k \in K$ the element $\iota(k)$ is invertible in $[H][K]^{-1}$.

3. Every element $[H][K]^{-1}$ can be written in the form $\iota(h)\iota(k)^{-1}$, $h \in H$, $k \in K$.

Proof. Homomorphy is obvious:

$$\iota(h)\iota(g) = [(hk,k)]\,[(gk,k)] = [((hg)k^2,k^2)] = \iota(hg).$$

Let $\iota(h) = \iota(h')$ if and only if $h = h'$. This means that $h = h'$ if and only if there exists such $x \in K$ such that $hkkx = h'kkx$, as required in the first item.

The second item immediately follows from the argument preceeding Definition 4.38.

Finally, the formula

$$[(h,k)] = [(h,k)]\,[(k,k)]\,[(k,k)] = [(hk,k)]\,[(k,k^2)] = [(hk,k)]\,[(kk,k)]^{-1} = \iota(h)\iota(k)^{-1}$$

proves the third item. $\square$

Theorem 4.40. *The semigroup $[H][K]^{-1}$ has the following universal property. Let S be a semigroup with identity and let $\phi : H \to S$ be a semigroup homomorphism such that K maps into the set of invertible elements of S. Then ϕ factorizes through ι, and the factorization is unique, i.e. there exists a unique homomorphism $\psi : [H][K]^{-1} \to S$ such that the diagram*

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[dr, "\iota"'] & S \\ & {[H][K]^{-1}} \arrow[u, "\psi"'] \end{tikzcd}$$

commutes.

Proof. Let's start with uniqueness. uppose that there exists a homomorphism ψ with the required properties. Then its value on any element is uniquely defined:

$$\psi[(h,k)] = \psi(\iota(h)\iota(k)^{-1}) = (\psi \circ \iota)(h)\,((\psi \circ \iota)(k))^{-1} = \phi(h)\phi(k)^{-1}$$

(we use here the fact that K is mapped into the set of invertibles). Thus, uniqueness is proven and the formula gives the only possible definition of ψ: $\psi[(h,k)] = \phi(h)\phi(k)^{-1}$. Then

$$(\psi \circ \iota)(h) = \psi[(hk,k)] = \phi(hk)\phi(k)^{-1} = \phi(h)\phi(k)\phi(k)^{-1} = \phi(h).$$

We need to check correctness and homomorphity. Let $(h',k') \equiv (h,k)$, so $h'kx = k'hx$, $x \in K$. Then

$$\phi(h)\phi(k')\phi(x) = \phi(h')\phi(k)\phi(x), \qquad \phi(h)\phi(k)^{-1} = \phi(h')\phi(k')^{-1}.$$

Here we use that $\phi(K)$ consists of invertibles. Finally,

$$\psi[(hh',kk')] = \phi(hh')\phi(kk')^{-1} = \phi(h)\phi(k)^{-1}\phi(h')\phi(k')^{-1} = \psi[(h,k)]\psi[(h',k')].$$

□

Corollary 4.41. *Let $\phi : H_1 \to H_2$ be a homomorphism of commutative semigroups with identities. Then there exists a unique homomorphism $\psi : G(H_1) \to G(H_2)$ making the following diagram commute:*

$$\begin{tikzcd} H_1 \arrow[r, "\phi"] \arrow[d, "\iota"'] & H_2 \arrow[d, "\iota"] \\ G(H_1) \arrow[r, "\psi"'] & G(H_2). \end{tikzcd}$$

Proof. Take $S = G(H_2)$ in the previous theorem, and $\iota\phi$ as ϕ, and use that $\iota(H_2)$ consists of invertible elements. □

Proposition 4.42. *Let K contain an element ∞ such that $h \cdot \infty = \infty$ for all $h \in H$. Then $[H][K]^{-1} = 0$.*

Proof. For any $h \in H$, $k \in K$ we have

$$(h, k) \equiv (h \cdot \infty, k \cdot \infty) = (\infty, \infty) \equiv 1.$$

□

Example 4.43. (=exercises)

1. Let $\mathbb{N} = (\mathbb{N}, +)$. Then $G(\mathbb{N}) = [\mathbb{N}][\mathbb{N}]^{-1} = \mathbb{Z}$, and $\iota : \mathbb{N} \to \mathbb{Z}$ is injective. The elements of $\mathbb{Z}$ are represented as differences of natural numbers.

2. Let $\mathbb{N}^* = (\mathbb{N}, \times)$. Then $G(\mathbb{N}^*) = \mathbb{Q}_+$ (positive rational numbers by multiplication), and $[\mathbb{Z}][\mathbb{N}]^{-1} = \mathbb{Q}$.

3. If we add zero to $\mathbb{N}^*$, then $G(\mathbb{N}^* \cup \{0\}) = 0$.

4.6 $K_0(A)$

Let $GV(A)$ be the Grothendieck group of $V(A)$, and

$$\iota_A : V(A) \to GV(A), \qquad \iota_A[p] := [p] - [0],$$

the canonical homomorphism. It is injective if and only if $V(A)$ has cancellation (since in this case $K = H = V(A)$ has the neutral element and $K^3 = H^3 = H = V(A)$).

Thus, we obtain a covariant functor from the category of C^*-algebras to the category of groups (Corollary 4.41 and Proposition 4.35).

Definition 4.44. Let $\pi : A^+ \to \mathbb{C}$ be the quotient map. Let us define the K-group

$$K_0(A) := \mathrm{Ker}(\pi_* : GV(A^+) \to \mathbb{Z}) \subset GV(A^+).$$

We still have $\iota_A : V(A) \to K_0(A)$, $[p] \mapsto [p] - [0]$.

Example 4.45. To emphasize the role of unitalization, consider $A = C_0(\mathbb{R}^2)$. Neither A nor $M_\infty(A)$, do not have non-trivial projections, but it will follow from Bott's periodicity theorem that $K_0(A) \cong \mathbb{Z}$.

Proposition 4.46. *For every C^*-algebra A there is a split exact sequence*

$$0 \to K_0(A) \xrightarrow{i_A} GV(A^+) \underset{i_*}{\overset{\pi_*}{\rightleftarrows}} \mathbb{Z} \to 0$$

(i.e. $\pi_* \circ i_* = \mathrm{Id}_\mathbb{Z}$). In particular, $GV(A^+) \cong K_0(A) \oplus \mathbb{Z}$.

Proof. The only thing that doesn't follow from the definition is surjectivity of π_*. But it follows from the fact that $\pi \circ i = \mathrm{Id}_\mathbb{C}$. This also gives the splitting.

Decomposition into a direct sum is a well-known fact for split exact sequences. We will prove that $GV(A^+) = i_A(K_0(A)) \oplus i_*(\mathbb{Z})$. Let's decompose any element $g \in GV(A^+)$ as follows:

$$g = (g - i_* \circ \pi_*(g)) + i_* \circ \pi_*(g).$$

Then $\pi_*(g - i_* \circ \pi_*(g) = \pi_*(g) - \pi_*(g) = 0$, since $\pi_* \circ i_* = \mathrm{Id}_{\mathbb{Z}}$. Therefore for some $s \in K_0(A)$ we have $i_A(s) = g - i_* \circ \pi_*(g)$. So that $g = i_A(s) + i_*(\pi_*(g))$ and $GV(A^+) = i_A(K_0(A)) + i_*(\mathbb{Z})$. It remains to show that $i_A(K_0(A)) \cap i_*(\mathbb{Z}) = 0$. Let $b \in i_A(K_0(A)) \cap i_*(\mathbb{Z})$, i.e. $b = i_A(c) = i_*(z)$, $c \in K_0(A)$, $z \in \mathbb{Z}$. Then $z = \pi_* \circ i_*(z) = \pi_* \circ i_A(c) = 0$, whence $b = i_*(z) = 0$. $\square$

Proposition 4.47. *The correspondence $A \mapsto K_0(A)$ is a covariant functor from the category of C^*-algebras to the category of Abelian groups, wherein to the morphism $\alpha : A \to B$, there corresponds a homomorphism given by the formula*

$$\alpha_*([(x_{ij})] - [(y_{ij})]) := [(\alpha^+ x_{ij})] - [(\alpha^+ y_{ij})],$$

where $[(x_{ij})]$ and $[(y_{ij})]$ are matrices of projections in $M_\infty(A^+)$, and $\alpha^+ : A^+ \to B^+$ is the unital morphism defined by the formula $\alpha^+(a + \lambda) := \alpha(a) + \lambda$.

Proof. By Proposition 4.35 and functoriality of passing to the Grothendieck group, we have functoriality at the level of $GV(A)$. For $K_0(A)$, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(A) & \xrightarrow{i_A} & GV(A^+) & \xrightarrow{\pi_*^A} & \mathbb{Z} \\ & & & & \downarrow{\scriptstyle \alpha_*^+} & & \downarrow{\scriptstyle \mathrm{Id}} \\ 0 & \longrightarrow & K_0(B) & \xrightarrow{i_B} & GV(B^+) & \xrightarrow{\pi_*^B} & \mathbb{Z} \end{array}$$

with exact rows and commuting right square. Then for any element $\gamma \in K_0(A)$ we have

$$\pi_*^B \alpha_*^+ i_A(\gamma) = \pi_*^A i_A(\gamma) = 0.$$

This means that α_*^+ restricts to $K_0(A)$ and defines the required homomorphism. Obviously, α_*^+ itself is given by the formula from the theorem. $\square$

Proposition 4.48. *Let $\pi_k : A_1 \oplus A_2 \to A_k$, $k = 1, 2$, be the projections onto the summands of the direct sum of C^*-algebras A_1 and A_2. Then the induced maps π_{k*} are isomorphisms*

$$\pi_{1*} \oplus \pi_{2*} : V(A_1 \oplus A_2) \cong V(A_1) \oplus V(A_2),$$

$$\pi_{1*} \oplus \pi_{2*} : GV(A_1 \oplus A_2) \cong GV(A_1) \oplus GV(A_2),$$

$$\pi_{1*} \oplus \pi_{2*} : K_0(A_1 \oplus A_2) \cong K_0(A_1) \oplus K_0(A_2),$$

Proof. Let p_1 and p_2 be projections in $M_\infty(A_1)$ and $M_\infty(A_2)$, respectively. Then $p_1 \oplus p_2$ (where the direct sum is taken for each matrix element) is a projection in $M_\infty(A_1 \oplus A_2)$ and

$$\begin{aligned} \pi_{1*} \oplus \pi_{2*}[p_1 \oplus p_2] &= \pi_{1*}[p_1 \oplus p_2] \oplus \pi_{2*}[p_1 \oplus p_2] = \\ &= [\pi_1(p_1 \oplus p_2)] \oplus [\pi_2(p_1 \oplus p_2)] = [p_1] \oplus [p_2]. \end{aligned}$$

This shows that $\pi_{1*} \oplus \pi_{2*}$ in the first line (for $V(A)$) is surjective. Let now check injectivity. Suppose that $\pi_{1*} \oplus \pi_{2*}(p_1 \oplus p_2)$ is equivalent to $\pi_{1*} \oplus \pi_{2*}(q_1 \oplus q_2)$, and let $v_i \in M_\infty(A_i)$,

$i = 1, 2$, implement the equivalence between p_i and q_i. Then $v = v_1 \oplus v_2$ implements the equivalence between $p_1 \oplus p_2$ and $q_1 \oplus q_2$. Thus, we have an isomorphism for $V(A)$. The transition to Grothendieck groups is functorial (with respect to to morphisms of semigroups — in this case the morphism is not induced by a mapping of algebras), so we have an isomorphism for $GV(A)$.

With K_0 the situation is more complicated. Moreover, the map itself is not yet determined. Let's look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(A_1 \oplus A_2) \arrow[r, "i_A"] \arrow[dr, "s"] \arrow[dd] & GV((A_1 \oplus A_2)^+) \arrow[r, "\pi_*^{A_1 \oplus A_2}"] \arrow[d, "\mu_*"] & \mathbb{Z} \arrow[d, "\Delta_*"] \\
 & & GV(A_1^+ \oplus A_2^+) \arrow[r, "\nu_*"] \arrow[d, "r", "\cong"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\mathrm{Id}"] \\
0 \arrow[r] & K_0(A_1) \oplus K(A_2) \arrow[r, "i_{A_1} \oplus i_{A_2}"] & GV(A_1^+) \oplus GV(A_2^+) \arrow[r, "\pi_*^{A_1} \oplus \pi_*^{A_2}"] & \mathbb{Z} \oplus \mathbb{Z}.
\end{tikzcd}$$

To define the map (the left vertical arrow) and to prove its isomorphism, we will take the following steps:

1) check the commutativity of the triangle and the two right squares,

2) prove exactness of the middle row, i.e. that s is injective and $\operatorname{Im} s = \operatorname{Ker} \nu_*$,

3) check that the image of $r \circ s$ is contained in the image of $i_{A_1} \oplus i_{A_2}$ (which will finally define the required map),

4) apply to the middle and the bottom rows of the diagram the "five homomorphisms Lemma" (lemma 4.49) to prove that this map is an isomorphism.

Let's start by carefully defining the maps involved. Recall that $\mathbb{Z}$ is actually $GV(\mathbb{C})$, and $\mathbb{Z} \oplus \mathbb{Z} = GV(\mathbb{C} \oplus \mathbb{C}) = GV(\mathbb{C}) \oplus GV(\mathbb{C})$ (by the proven parts of Theorem). The homomorphisms of the upper right square are defined by the maps of algebras:

$$\pi^{A_1 \oplus A_2} : (A_1 \oplus A_2)^+ \to \mathbb{C}, \qquad (a_1, a_2, \lambda) \mapsto \lambda,$$

$$\Delta : \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}, \qquad \lambda \mapsto (\lambda, \lambda),$$
$$\mu : (A_1 \oplus A_2)^+ \to A_1^+ \oplus A_2^+, \qquad (a_1, a_2, \lambda) \mapsto (a_1, \lambda, a_2, \lambda),$$
$$\nu : A_1^+ \oplus A_2^+ \to \mathbb{C} \oplus \mathbb{C}, \qquad (a_1, \lambda_1, a_2, \lambda_2) \mapsto (\lambda_1, \lambda_2).$$

(The only thing that can raise doubts is the multiplicativity of μ. Let's check:

$$\begin{aligned}
\mu((a_1, a_2, \lambda)(b_1, b_2, \tau) &= \mu(a_1 b_1 + \tau a_1 + \lambda b_1, a_2 b_2 + \tau a_2 + \lambda b_2, \lambda\tau) = \\
&= (a_1 b_1 + \tau a_1 + \lambda b_1, \lambda\tau, a_2 b_2 + \tau a_2 + \lambda b_2, \lambda\tau),
\end{aligned}$$

$$\begin{aligned}
\mu(a_1, a_2, \lambda)\, \mu(b_1, b_2, \tau) &= (a_1, \lambda, a_2, \lambda)\,(b_1, \tau, b_2, \tau) = \\
&= (a_1 b_1 + \tau a_1 + \lambda b_1, \lambda\tau, a_2 b_2 + \tau a_2 + \lambda b_2, \lambda\tau),
\end{aligned}$$

This immediately implies the commutativity of the upper right triangle at the level of algebras, and therefore at the level of their Grothendieck groups. The homomorphism s

is defined as the composition $\mu_* \circ i_A$ and the triangle is patently commutative. Finally, r is an isomorphism from the previous argument.

1) It remains to check commutativity of the lower right square, which can be done at the level of the matrix elements of M_∞, which is obvious. (You can also show that for GV the isomorphism $\pi_{1*} \oplus \pi_{2*}$ is functorial with respect to the maps $A_1 \to B_1$, $A_2 \to B_2$.)

2) First of all, it is obvious that $\nu_* \circ s = 0$ due to commutativity. Next, let's consider

$$\mu' : A_1^+ \oplus A_2^+ \to (A_1 \oplus A_2)^+, \qquad (a_1, \lambda_1, a_2, \lambda_2) \mapsto (a_1, a_2, \lambda_1),$$

so $\mu' \circ \mu = \mathrm{Id}$. Passing to GV, we get injectivity of μ_*, and hence of s. Let us now consider $(a^1_{ij}, \lambda^1_{ij}, a^2_{ij}, \lambda^2_{ij})$ and $(b^1_{ij}, \tau^1_{ij}, b^2_{ij}, \tau^2_{ij})$ from $M_\infty(A_1^+ \oplus A_2^+)$, such that their difference gives the class $x \in GV(A_1^+ \oplus A_2^+)$. If the class x at ν_* is equal to zero, then there are partial isometries v_1 and v_2 in $M_\infty(\mathbb{C})$ such that

$$(\lambda^1_{ij}) = v_1^* v_1, \quad (\tau^1_{ij}) = v_1 v_1^*, \qquad (\lambda^2_{ij}) = v_2^* v_2, \quad (\tau^2_{ij}) = v_2 v_2^*.$$

Therefore, $(a^1_{ij}, \tau^1_{ij}, a^2_{ij}, \tau^2_{ij}) \in M_\infty(A_1^+ \oplus A_2^+)$ is equivalent to $(a^1_{ij}, \lambda^1_{ij}, a^2_{ij}, \lambda^2_{ij})$ (using the partial isometry $(\mathrm{Id}, v_1, \mathrm{Id}, v_2)$). This means that x can be represented by the difference of $(a^1_{ij}, 0, a^2_{ij}, 0)$ and $(b^1_{ij}, 0, b^2_{ij}, 0)$. And this difference already lies in the image μ. So x lies in the image of μ_*, and hence of s.

3) We need to check that if $\pi_*^{A_1 \oplus A_2}(x) = 0$, then $\pi_*^{A_1} \oplus \pi_*^{A_2} \circ r \circ \mu_*(x) = 0$. This immediately follows from the commutativity of the right squares.

4) Now we are ready to apply the 5 homomorphisms Lemma. □

Lemma 4.49 (5 homomorphisms)**.** *Let*

$$\begin{array}{ccccccccc}
G_5 & \xrightarrow{\alpha_5} & G_4 & \xrightarrow{\alpha_4} & G_3 & \xrightarrow{\alpha_3} & G_2 & \xrightarrow{\alpha_2} & G_1 \\
\downarrow{\scriptstyle\gamma_5} & & \downarrow{\scriptstyle\gamma_4} & & \downarrow{\scriptstyle\gamma_3} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_1} \\
H_5 & \xrightarrow{\beta_5} & H_4 & \xrightarrow{\beta_4} & H_3 & \xrightarrow{\beta_3} & H_2 & \xrightarrow{\beta_2} & H_1
\end{array}$$

be a commuting diagram, where both rows are exact and γ_1, γ_2, γ_4 and γ_5 are isomorphisms. Then γ_3 is an isomorphism.

Proof. Let us show that γ_3 is a monomorphism. Let $\gamma_3(g_3) = 0$. Then $\gamma_2\alpha_3(g_3) = \beta_3\gamma_3(g_3) = 0$. Therefore, $\alpha_3(g_3) = 0$. This means that there exists $g_4 \in G_4$ such that $\alpha_4(g_4) = g_3$. Then $\beta_4\gamma_4(g_4) = 0$ and there exists an element $h_5 \in H_5$ such that $\beta_5(h_5) = \gamma_4(g_4)$. Consider an element $g_5 \in G_5$ such that $\gamma_5(g_5) = h_5$. Then $\gamma_4(\alpha_5(g_5)) = \gamma_4(g_4)$ and this means that $g_4 = \alpha_5(g_5)$. Therefore, $g_3 = \alpha_4\alpha_5(g_5) = 0$.

Let us show that γ_3 is an epimorphism. Let $h_3 \in H_3$. There exists an element $g_2 \in G_2$ such that $\gamma_2(g_2) = \beta_3(h_3)$. Then $\gamma_1\alpha_2(g_2) = \beta_2\beta_3(h_3) = 0$. Therefore, $\alpha_2(g_2) = 0$, and there exists $g_3 \in G_3$ such that $\alpha_3(g_3) = g_2$. Then $\beta_3(h_3 - \gamma_3(g_3)) = 0$ and there exists $h_4 \in H_4$ such that $\beta_4(h_4) = h_3 - \gamma_3(g_3)$. Let an element $g_4 \in G_4$ be such that $\gamma_4(g_4) = h_4$. Then $g_3 + \alpha_4(g_4) \in G_3$ and $\gamma_3(g_3 + \alpha_4(g_4)) = \gamma_3(g_3) + \beta_4(h_4) = h_3$. □

Proposition 4.50. *If A be unital, then $K_0(A) = GV(A)$.*

Proof. In this case $A^+ \cong A \oplus \mathbb{C}$, so as suggested by 4.48 $GV(A^+) \cong GV(A) \oplus \mathbb{Z}$. By Proposition 4.46 $GV(A^+) \cong K_0(A) \oplus \mathbb{Z}$. $\square$

Problem 81. Prove that
1) $K_0(\mathbb{C}) = \mathbb{Z}$,
2) $K_0(B(\mathcal{H})) = 0$,
3) $K_0(B(\mathcal{H})/\mathcal{K}(\mathcal{H})) = 0$,

Lemma 4.51. *Let p be a projection in $M_k(A^+)$, and let $\pi_{\mathbb{C}}(p)$ is equivalent to p_n in $M_k = M_k(\mathbb{C})$, where p_n is a projection onto the first n basis vectors, i.e. matrix of the form*

$$p_n = \left(\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & \\ \hline 0 & & & 0 \end{array} \right).$$

Then p is unitarily equivalent in $M_k(A^+)$ to a projection q such that $\pi_{\mathbb{C}}(q) = p_n$, i.e. $q - p_n \in M_k(A)$.

Proof. Note that from the rules of multiplication in A^+ it immediately follows that $\pi_{\mathbb{C}}(p)$ is a projection. Since in M_k the projections are equivalent if and only if their ranks are equal, then $\pi_{\mathbb{C}}(p) \sim p_n$ implies $1-\pi_{\mathbb{C}}(p) \sim 1-p_n$, which means by Proposition 4.21, $\pi_{\mathbb{C}}(p) \sim_u p_n$. Thus, there is a unitary matrix $u \in M_k$ such that $u\pi_{\mathbb{C}}(p)u^* = p_n$. Put $q := upu^*$. Then q is a projection in $M_k(A^+)$, $p \sim_u q$ and $\pi_{\mathbb{C}}(q) = \pi_{\mathbb{C}}(upu^*) = u\pi_{\mathbb{C}}(p)u^* = p_n$. $\square$

Lemma 4.52. *Let $a \in M_k(A)$. Then a commutes with p_n for some $n \leqslant k$ if and only if a has the block-diagonal form* $\mathrm{diag}(a_1, a_2)$*, $a_1 \in M_n(A)$, $a_2 \in M_{kn}(A)$.*

Moreover, a is invertible (respectively, unitary) if and only if a_1 and a_2 are also invertible (respectively, unitary).

Moreover, $ap_n = p_n a = 0$ if and only if a has the block-diagonal form $\mathrm{diag}(0, a_2)$*, $0 \in M_n(A)$, $a_2 \in M_{kn}(A)$.*

Proof. Split a into blocks: $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, so that $a_{11} \in M_n(A)$, i.e. the block size iss $n \times n$. Accordingly, $p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$ap_n = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}, \quad p_n a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}.$$

From here we immediately obtain the first statement. The second statement in the unitary case immediately follows from the first one. Further, $a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$ is invertible if and only if there exists an element $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{22}b_{21} & a_{22}b_{22} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & b_{12}a_{22} \\ b_{21}a_{11} & b_{22}a_{22} \end{pmatrix}.$$

This is equivalent to the invertibility of a_{11} and a_{22}: $b_{11} = a_{11}^{-1}$, and $b_{22} = a_{22}^{-1}$ and also (using this) that $b_{12} = b_{21} = 0$.

The last statement immediately follows from the first one. □

Theorem 4.53. *For any C^*-algebra A, $K_0(A)$ is an Abelian group.*

1) *Elements of $K_0(A)$ can be represented by differences $[p] - [q]$, where p and q are projections from $M_k(A^+)$ for some $k \in \mathbb{N}$ such that $p - q \in M_k(A)$.*

2) *If A is unital then p and q can be chosen in $M_k(A) \subset M_k(A^+)$.*

3) *Moreover, each element of $K_0(A)$ can be represented as $[p]-[p_n]$, where p is a projection in $M_k(A^+)$ for some $k \geqslant n$, and $p - p_n \in M_k(A)$.*

4) *If $[p] - [q] = 0 \in K_0(A)$, where $p, q \in M_k(A^+)$ then for suitable $m \leqslant n$*

$$\operatorname{diag}(p, p_m) \sim_h \operatorname{diag}(q, p_m) \text{ in } M_{k+n}(A^+)$$

and vice versa.

Proof. 1) Every element $K_0(A)$ is represented by a formal difference of $[p'] - [q']$ where $[p']$, $[q']$ are from $V(A^+)$. We want to show that there exist representatives $p \in [p']$, $q \in [q']$ such that their scalar parts match, i.e. $p - q \in M_\infty(A)$. Apriory,

$$\pi_*([p'] - [q']) = [\pi_{\mathbb{C}}(p')] - [\pi_{\mathbb{C}}(q')] = 0 \in K_0(\mathbb{C}) = \mathbb{Z}. \tag{4.7}$$

Since $V(\mathbb{C}) = \mathbb{N} \cup \{0\}$ is a semigroup with cancellation, then from (4.7) it follows that $[\pi_{\mathbb{C}}(p')] = [\pi_{\mathbb{C}}(q')] =: n \in V(\mathbb{C})$. Thus, $\pi_{\mathbb{C}}(p') \sim p_n \sim \pi_{\mathbb{C}}(q')$ in $M_\infty(A^+)$. Applying the lemma 4.51, we find projections $p, q \in M_\infty(A^+)$ such that

$$p \sim p', \qquad q \sim q', \qquad \pi_{\mathbb{C}}(p') = p_n = \pi_{\mathbb{C}}(q').$$

It is clear that p and q satisfy the requirements. All calculations can be implemented in $M_k(A^+)$ for a sufficiently large k.

2) If A is unital then $K_0(A) = GV(A)$ by Proposition 4.50 and the elements are already presented in the required form.

3) Let $x = [q_1] - [q] \in K_0(A)$, where the projections q_1 and q are from $M_\infty(A^+)$. Then $q \leqslant p_n$ for some $n \in \mathbb{N}$, so $p_n - q$ is a projection. We can assume that the same holds for q_1. Then we can consider the projection q_2 in $M_{2n}(A^+) \subset M_\infty(A^+)$, which has the same nontrivial block as q_1, but located in the lower right $n \times n$-angle. Then $q_2 \sim_u q_1$ and $q_2 \perp p_n$. Note that q_2, $p_n - q$ and q are mutually orthogonal, and $q_3 := q_2 \oplus (p_n - q)$ is a projection. Also

$$\begin{aligned} [q_3] - [p_n] &= [q_2 \oplus (p_n - q)] - [p_n] = [q_2] + [p_n - q)] + [q] - [q] - [p_n] = \\ &= [q_1] + [(p_n - q) \oplus q] - [q] - [p_n] = [q_1] + [p_n] - [q] - [p_n] = [q_1] - [q] = x. \end{aligned}$$

Then $0 = \pi_*(x) = [\pi_{\mathbb{C}}(q_3)] - [\pi_{\mathbb{C}}(p_n)] = [\pi_{\mathbb{C}}(q_3)] - [p_n]$, so, by Lemma 4.51 we conclude that $p \sim_u q_3$, and we have $p - p_n \in M_\infty(A)$ and $x = [p] - [p_n]$.

4) By the definition of the Grothendieck group, $[p] - [q] = 0 \in K_0(A)$ means that $[p] + [r] = [q] + [r]$ in $V(A^+)$ for some projection $r \in M_m(A^+)$. Then $r \leqslant p_m$ in $M_\infty(A^+)$ and $r \perp (p_m - r)$, so

$$[\mathrm{diag}(p, p_m)] = [\mathrm{diag}(p, r \oplus (p_m - r))] = [p] + [r] + [p_m - r] = [\mathrm{diag}(q, p_m)].$$

Assuming that $p, q \in M_k(A^+)$ for some $k \in \mathbb{N}$, we find that from the last equality it follows that $\mathrm{diag}(p, p_m) \sim_h \mathrm{diag}(q, p_m)$ in $M_{k+n}(A^+)$ for sufficiently large $n \geqslant m$. The opposite is obvious. $\square$

4.7 K_0 and inductive limits

Let $\{B_i, \varphi_{ij}\}$ be a directed system of C^*-algebras, and φ_{ij} are injective, which implies that they are isometric. Then the C^*-inductive limit $C^*[\varinjlim B_i]$ is the C^*-completion of the algebraic limit (=union) of B_i.

The most important example for us will be $B_i = M_i(A) = A \otimes M_i(\mathbb{C})$, $\cup_i B_i = A \otimes M_\infty(\mathbb{C})$, $C^*[\varinjlim B_i] = A \otimes \mathcal{K}$, as well as matrix algebras over them. Therefore, we restrict ourselves to injective homomorphisms φ_{ij}, although the general situation can also be considered.

Let φ_k denote the canonical embeddings $B_k \to \cup_i B_i \subset C^*[\varinjlim B_i]$.

Lemma 4.54. *Let p be a projection in $C^*[\varinjlim B_i]$. For any $\varepsilon > 0$ there are $k \in \mathbb{N}$ and a projection $q \in B_k$ such that $\|p - \varphi_k q\| < \varepsilon$.*

Proof. By the definition of the C^*-inductive limit we find k and $b \in B_k$ with sufficiently small $\|p - \varphi_k b\|$. This will also be true for the positive element $a = (b^*b)^{1/2}$. Then the spectrum of $\varphi_k a$ is close to the spectrum of p, i.e. concentrated on $[0, +\infty)$ in the neighborhood of 0 and 1. The spectrum of a is also like this (may differ by 0). Let $f : [0, +\infty) \to [0, 1]$ be a continuous function equal to 0 in the vicinity of 0 and to 1 in the vicinity of 1. Then $q := f(a)$ is a projection, and $\varphi_k q = \varphi_k f(a) = f(\varphi_k a)$ is close to p. Our argument is close to the proof of Proposition 4.16, from where you can extract more details. $\square$

Lemma 4.55. *Let all B_i be unital, and let φ_{ij} be unital. Then $C^*[\varinjlim B_i]$ is unital.*

Let u be unitary in $C^*[\varinjlim B_i]$. For any $\varepsilon > 0$ there are $k \in \mathbb{N}$ and a unitary $v \in B_k$ such that $\|u - \varphi_k v\| < \varepsilon$.

Proof. The unit is already present in $\varinjlim B_i$, and even more so, in $C^*[\varinjlim B_i]$ (it is equal to $\varphi_i(1_i)$, where 1_i is the unit of B_i, for any i). In particular, all φ_i are unital.

Let us choose k and $a \in B_k$ such that $\|u - \varphi_k a\|$ is small. In particular, $\varphi_k a$ is invertible. Therefore, a is invertible since φ_k is unital. Let $v := a(a^*a)^{-1/2}$ be a unitary from the polar decomposition of a. Then $\varphi_k v = (\varphi_k a)((\varphi_k a)^* \varphi_k a)^{-1/2}$, since φ_k is unital. Since $((\varphi_k a)^* \varphi_k a)^{-1/2}$ is close to $(u^*u)^{-1/2} = 1$, $\varphi_k v$ is close to u. $\square$

Lemma 4.56. *Let $\{B_i, \varphi_{ij}\}$ be an injective directed system and $B = C^*[\varinjlim B_i]$. Then $\{B_i^+, \varphi_{ij}^+\}$ is an injective directed system with unital φ_{ij}^+ and $B^+ = C^*[\varinjlim B_i^+]$.*

Proof. Morphisms φ_{ij}^+ are unital and obviously injective if φ_{ij} are injective. It is also obvious that $\varinjlim B_i^+ = \varinjlim B_i \oplus \mathbb{C}$. Passing to C^*-completions, we obtain the required result. $\square$

Lemma 4.57. *Close unitary elements are homotopic.*

Proof. Let $u \in U(A) \subset GL(A)$. Because $GL(A)$ is open, then there exists an open ball $B_\varepsilon(u) \subset GL(A)$. Let u' be unitary and $\|u - u'\| < \varepsilon$, i.e. $u' \in B_\varepsilon(u)$. Then the segment (linear homotopy) connects u and u' in $GL(A)$. All that remains is to apply a deformational retraction of $GL(A)$ onto $U(A)$ (Lemma 4.1). $\square$

In fact, it is enough to take $\varepsilon = 1$, because $\|u^{-1}\| = \|u\| = 1$.

Theorem 4.58. *Let $\{A_i, \Phi_{ij}\}$ be a directed system of C^*-algebras with injective homomorphisms. Then $\{K_0(A_i), \Phi_{ij*}\}$ is a directed system of groups, and*

$$K_0(C^*[\varinjlim A_i]) \cong \varinjlim K_0(A_i). \tag{4.8}$$

Proof. Denote $A := C^*[\varinjlim A_i]$. Due to functoriality, $\Phi_{ij*} \circ \Phi_{jk*} = \Phi_{ik*}$ and the direct limit of semigroups $H := \varinjlim \{V(A_i), \Phi_{ij*}\}$ is defined. Let $\Phi_i : A_i \to A$ and $\Psi_i : V(A_i) \to H$ be the canonical homomorphisms. Consider the commutative diagram

$$\begin{tikzcd} V(A_j) \arrow[r, "\Phi_{ij*}"] \arrow[drr, "\Phi_{j*}"'] & V(A_i) \arrow[r, "\Psi_i"] \arrow[dr, "\Phi_{i*}"] & H \arrow[d, "\theta"] \\ & & V(A), \end{tikzcd}$$

where θ is (uniquely) defined by the universal property of the direct limit, namely, $\theta(\Psi_i(x)) := \Phi_{i*}(x)$. Obviously all constructions are natural with respect to unitalization, therefore, to prove the theorem it is enough to check that θ is a semigroup isomorphism (and consider A_i^+ instead A_i, etc.).

Let us check surjectivity of θ. Let $p \in M_n(A)$ be a projection. By Lemma 4.54, we choose the projection $q \in M_n(A_i)$ with small $\|p - \Phi_i q\|$ for sufficiently large i. Here we used the obvious equality $M_n(C^*[\varinjlim B_i]) = C^*[\varinjlim M_n(B_i)]$ for fixed n. Then (for example, as suggested by 4.22)

$$[p] = [\Phi_i q] = \Phi_{i*}[q] = \theta \circ \Psi_i[q],$$

so $[p]$ lies in the image of θ.

Let's check that θ is injective (this doesn't hold automatically, since Φ_{i*} do not have to be injective). Let $x, y \in H$. By choosing sufficiently large n and j_0, we can assume that $x = \Psi_{j_0}[p_{j_0}]$, $y = \Psi_{j_0}[q_{j_0}]$ for some projections $p_{j_0}, q_{j_0} \in M_n(A_{j_0})$. Suppose that $\theta(x) = \theta(y)$. Then (see the commutative diagram above) we can assume (increasing n if necessary) that $p := \Phi_{j_0}(p_{j_0}) \sim_u q := \Phi_{j_0}(q_{j_0})$ in $M_n(A)$ via some $u \in U_n(A^+)$, i.e.

$q = upu^*$. By Lemmas 4.56 and 4.55, let us approximate u by some unitary element from $M_n(A^+_{j_1})$:

$$\|u - \Phi^+_{j_1} u_{j_1}\| < \varepsilon, \qquad u_{j_1} \in U_n(A^+_{j_1}).$$

Choosing $j_2 = \max(j_1, j_0)$, we define

$$p_{j_2} := \Phi_{j_2 j_0}(p_{j_0}), \quad q_{j_2} := \Phi_{j_2 j_0}(q_{j_0}), \quad u_{j_2} := \Phi^+_{j_2 j_1}(u_{j_1}),$$

so $p = \Phi_{j_2}(p_{j_2})$, $q = \Phi_{j_2}(q_{j_2})$ and $\|u - \Phi^+_{j_2} u_{j_2}\| < \varepsilon$. Put

$$q'_{j_2} := u_{j_2} p_{j_2} u^*_{j_2}, \qquad u' := \Phi^+_{j_2}(u_{j_2}). \tag{4.9}$$

Then $\|u - u'\| < \varepsilon$ and

$$\|\Phi_{j_2}(q'_{j_2}) - \Phi_{j_2}(q_{j_2})\| = \|u' p u'^* - upu^*\| \leqslant 2\,\|u' - u\| < 2\varepsilon.$$

Recall that we limited ourselves to injective limits, therefore the inclusions of the matrix algebras is also isometric, for example Φ_{j_2} (unlike $\Phi_{j_2 *}$, about which we can say nothing so far). So

$$\|q'_{j_2} - q_{j_2}\| < 2\varepsilon.$$

Assuming that we have chosen $\varepsilon < 1/2$, we get that $q_{j_2} \sim_u q'_{j_2} \sim_u p_{j_2}$ and

$$x = \Psi_{j_0}[p_{j_0}] = \Psi_{j_2} \circ \Phi_{j_2 j_0 *}[p_{j_0}] = \Psi_{j_2}[p_{j_2}] = \Psi_{j_2}[q_{j_2}] = \Psi_{j_2} \circ \Phi_{j_2 j_0 *}[q_{j_0}] = \Psi_{j_0}[q_{j_0}] = y.$$

□

Lemma 4.59. *Let A be a C^*-algebra. Let*

$$\iota_{n1} : A \hookrightarrow M_n(A), \qquad a \mapsto \operatorname{diag}(a, 0, \ldots, 0)$$

be the canonical embedding. Then the induced map

$$\iota_{n1*} : K_0(A) \hookrightarrow K_0(M_n(A))$$

is an isomorphism.

Proof. It is enough to prove that $\iota_{n1*} : V(A) \to V(M_n(A))$ is an isomorphism. Let (a_{ij}) be a projection in $M_\infty(A)$. Then the projection $\iota_{n1}(a_{ij})$ is equivalent to (a_{ij}). Indeed, it is necessary to apply unitary transformations (homotopic to the identity), which swap distant rows (resp., columns) (which are zeroes) and first rows (resp., columns) (with possibly non-zero entries).

Thus, $\iota_{n1*}[(a_{ij})] = [\iota_{n1}(a_{ij})] = [(a_{ij})]$. This shows that ι_{n1*} is bijective. □

Theorem 4.60 (stability K_0)**.** *The map $A \to A \otimes \mathcal{K}$, $a \mapsto a \otimes e_{11}$, induces an isomorphism $K_0(A) \cong K_0(A \otimes \mathcal{K})$. Here e_{11} denotes a projection of rank 1 in $\mathcal{K}$.*

In particular, if A and B are stably isomorphic, i.e. $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then $K_0(A) \cong K_0(B)$.

Proof. Let $\iota_{nm} : M_m(A) \hookrightarrow M_n(A)$, $a \mapsto \mathrm{diag}(a, 0)$, $n \geqslant m$, be the canonical embedding. Then $A \otimes \mathcal{K}$ is a C^*-inductive limit system $\{M_n(A), \iota_{nm}\}$. Apply K_0 to commutative diagram

$$\begin{tikzcd} M_m(A) \arrow[rr, "\iota_{nm}"] & & M_n(A) \\ & A. \arrow[ul, "\iota_{m1}"] \arrow[ur, "\iota_{n1}"'] & \end{tikzcd}$$

We obtain the following commutative diagram of isomorphisms (by the previous Lemma), where we denoted by ψ_m and ψ_n the maps inverse to ι_{m1*} and ι_{n1*}, respectively:

$$\begin{tikzcd} K_0(M_m(A)) \arrow[rr, "\iota_{nm*}"] \arrow[dr, "\psi_m"'] & & K_0(M_n(A)) \arrow[dl, "\psi_n"] \\ & K_0(A). & \end{tikzcd}$$

Due to universality of the inductive limit, there is a unique limit homomorphism $\theta : \varinjlim K_0(M_n(A)) \to K_0(A)$. On the other hand, by Theorem 4.58, $\varinjlim K_0(M_n(A)) = K_0(A \otimes \mathcal{K}$. A commutative diagram arises:

$$\begin{tikzcd} K_0(M_m(A)) \arrow[r, "\iota_{m*}"] \arrow[d, "\iota_{nm*}"'] \arrow[dr, "\psi_m"] & K_0(A \otimes \mathcal{K}) \arrow[d, "\theta"] \\ K_0(M_n(A)) \arrow[r, "\psi_n"'] \arrow[ur, "\iota_{n*}"] & K_0(A), \end{tikzcd}$$

where $\iota_n : M_n(A) = A \otimes M_n \to A \otimes \mathcal{K}$ are the canonical morphisms. Then θ is isomorphism. (This is an algebraic fact, as we here do not pass to the C^*-limits.) Indeed, since ψ_n is an isomorphism, then θ is an epimorphism. Let $\theta(x) = \theta(y)$, $x, y \in K_0(A \otimes \mathcal{K})$. Then for sufficiently large n_x and n_y

$$x = \iota_{n_x*}(a_x), \qquad y = \iota_{n_y*}(a_y), \qquad a_x \in K_0(M_{n_x}(A)), \quad a_y \in K_0(M_{n_y}(A)).$$

Let $n \geqslant \max(n_x, n_y)$, and $x_n = \iota_{nn_x}(a_x)$, $y_n = \iota_{nn_y}(a_y)$. Then $x = \iota_{n*}(x_n)$, $y = \iota_{n*}(y_n)$, and

$$\psi_n(x_n) = \theta \circ \iota_{n*}(x_n) = \theta(x) = \theta(y) = \theta \circ \iota_{n*}(y_n) = \psi_n(y_n).$$

Since ψ_n is injective (in fact even an isomorphism), then $x_n = y_n$, whence $x = \iota_{n*}(x_n) = \iota_{n*}(y_n) = y$. $\square$

Corollary 4.61. $K_0(\mathcal{K}) = K_0(\mathbb{C} \otimes \mathcal{K}) = K_0(\mathbb{C}) = \mathbb{Z}$.

Lecture 07.04.2025

4.8 Short exact sequence

Lemma 4.62. *Let J be an ideal in A, and hence in A^+, and let $\pi_J : A^+ \to A^+/J$ be the quotient map. If $x \in A^+$ then*

1. *$x \in J$ if and only if $\pi_J(x) = 0$;*
2. *$x \in J^+$ if and only if $\pi_J(x) \in \mathbb{C}$.*

Proof. If $x = a + \lambda$, $x \in A$, $\lambda \in \mathbb{C}$, then $\pi_J(x) = \pi_J(a) + \lambda$. □

Theorem 4.63 (semi-exactness of $K_0(A)$)**.** *The short exact sequence*

$$0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

where J is an ideal in A, $i : J \hookrightarrow A$ is an embedding, π the quotient map, induces the exact (in the middle) sequence of K_0-groups

$$K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

Proof. We must prove that $\operatorname{Ker} \pi_* = \operatorname{Im} i_* \subset K_0(A)$. If $x \in K_0(J)$, then we can write $x = [p] - [p_n]$, where $p \in M_\infty(J^+)$ is a projection, and $p - p_n \in M_\infty(J)$ (see item 3 of Theorem 4.53). Then

$$\pi_* \circ \iota_*(x) = [\pi_J(p)] - [\pi_J(p_n)] = 0,$$

since $p - p_n \in M_\infty(J)$. This shows that $\operatorname{Ker} \pi_* \supset \operatorname{Im} i_*$.

Conversely, if $y \in K_0(A)$, then it can be written as $y = [q] - [p_n]$, where $q \in M_k(A^+)$ and $q - p_n \in M_k(A)$, $k \geqslant n$. Condition $\pi_*(y) = 0$ means

$$\operatorname{diag}(\pi_J(q), p_d) \sim_u \operatorname{diag}(p_n, p_d) \text{ in } M_m((A/J)^+)$$

for some d and m, $m \geqslant k + d$. This follows from item 4 of Theorem 4.53. Accordingly, let $u \in M_m((A/J)^+)$ be a unitary element such that

$$u \operatorname{diag}(\pi_J(q), p_d)\, u^* = \operatorname{diag}(p_n, p_d).$$

By Corollary 4.8, we find a unitary lifting $w \in M_{2m}(A^+)$ of the element $\operatorname{diag}(u, u^*)$. Let us define the projection $r \in M_{2m}(A^+)$ by the formula

$$r := w \operatorname{diag}(q, p_d)\, w^*.$$

Then

$$\pi_J(r) = \operatorname{diag}(u, u^*) \operatorname{diag}(\pi_J(q), p_d) \operatorname{diag}(u^*, u) = \operatorname{diag}(p_n, p_d).$$

By Lemma 4.62, $r \in M_{2m}(J^+)$. But $[r] = [\operatorname{diag}(q, p_d)]$, so

$$y = [q] - [p_n] = [\operatorname{diag}(q, p_d)] - [\operatorname{diag}(p_n, p_d)] = [r] - [p_{n+d}]$$

lies in the image i_*. □

4.9 Homotopy invariance

Definition 4.64. Let A and B be C^*-algebras. Two morphisms $\alpha, \beta : A \to B$ are called *homotopic,* if there is such a path $\gamma_t : A \to B$, $t \in [0,1]$, in the set of $*$-homomorphisms, such that $t \mapsto \gamma_t(a)$ is a norm continuous path in B for each fixed $a \in A$, and $\gamma_0 = \alpha$, $\gamma_1 = \beta$. Notation: $\alpha \sim_h \beta$.

The morphism $\alpha : A \to B$ is called *equivalence,* if there is a morphism $\beta : B \to A$ such that $\beta \circ \alpha \sim_h \mathrm{Id}_A$, $\alpha \circ \beta \sim_h \mathrm{Id}_B$.

If $\beta \circ \alpha \sim_h \mathrm{Id}_A$, $\alpha \circ \beta = \mathrm{Id}_B$, then α is called a *deformation retraction,* and B is called a *deformation retract* of A.

A is *contractible,* if Id_A is homotopic to the zero map.

Theorem 4.65 (homotopy invariance)**.** *Let $\alpha_0, \alpha_1 : A \to B$ be homotopic morphisms. Then the induced homomorphisms coincide:*

$$\alpha_{0*} = \alpha_{1*} : K_0(A) \to K_0(B).$$

Proof. Let α_t, $t \in [0,1]$, be a homotopy. For determining homomorphisms of K-groups we first define unitalization maps of matrix algebras $\alpha_t^+ : M_n(A^+) \to M_n(B^+)$, and then we define

$$\alpha_{t*} : K_0(A) \to K_0(B), \qquad [p] - [q] \mapsto [\alpha_t^+(p)] - [\alpha_t^+(q)].$$

Note that since n is finite, then (for a fixed p) the map $t \mapsto \alpha_t^+(p)$ is a continuous path in matrix projections. This means that all these projections are equivalent, and the same is true for q. □

Definition 4.66. The *cone* of a C*-algebra A is the C*-algebra $CA = \{f \in C([0,1]; A) : f(0) = 0\}$. Its *suspension* is the C*-subalgebra $SA = \{f \in C([0,1]; A) : f(0) = f(1) = 0\}$ of the cone. *The cone of a homomorphism* $\alpha : A \to B$ is the C*-algebra $C_\alpha = \{(a, f) \in A \oplus CB : f(1) = \alpha(a)\}$.

Lemma 4.67. *Sequences $0 \to SA \to CA \to A \to 0$ and $0 \to SB \xrightarrow{i} C_\alpha \xrightarrow{\pi} A \to 0$, where $i(f) = (0, f)$, $\pi(a, f) = a$, are exact.*

Proof. Obvious. □

Proposition 4.68. *The cone CA is contractible for any C^*-algebra A.*

Proof. Let us define $*$-homomorphisms $\gamma_t : CA \to CA$, $t \in [0,1]$, by the formula $\gamma_t(f)(s) = f(st)$. Then $\gamma_1 = \mathrm{id}_{CA}$, $\gamma_0 = 0$. □

4.10 Group K_1

The mapping $x \mapsto \mathrm{diag}(x, 1)$ defines the embedding of the groups $\mathrm{GL}_n(A) \to \mathrm{GL}_{n+1}(A)$ and $U_n(A) \to U_{n+1}(A)$. Passing to the direct limit, we obtain the topological groups $\mathrm{GL}_\infty(A) = \varinjlim \mathrm{GL}_n(A)$ and $U_\infty(A) = \varinjlim U_n(A)$ (as well as the groups $\mathrm{GL}_\infty^+(A)$ and $U_\infty^+(A)$). The metric on them is given by the norm on the algebras $\mathbb{M}_n(A)$. We denote by G_0 the connected component of the unit of the group G. Recall that this is a normal subgroup.

For convenience, we will consider matrices of finite size to be elements of the direct limit, i.e. if $u \in \mathrm{GL}_n(A)$, then we can denote by u the element $\mathrm{diag}(u, 1_\infty)$ of the group $\mathrm{GL}_\infty(A)$.

Definition 4.69. Let $K_1(A) = \mathrm{GL}^+_\infty(A)/\mathrm{GL}^+_\infty(A)_0 = \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0 = U^+_\infty(A)/U^+_\infty(A)_0 = U_\infty(A^+)/U_\infty(A^+)_0$.

We will denote the equivalence classes in the quotient group by $[\cdot]$.

Lemma 4.70. *$K_1(A)$ is a commutative group, and $[u][v] = [uv] = [\mathrm{diag}(u, v)]$ for any $u \in U^+_n(A)$, $v \in U^+_m(A)$.*

Proof. The statement follows from the homotopy connecting $\begin{pmatrix} u & \\ & v \end{pmatrix}$ with $\begin{pmatrix} uv & \\ & 1 \end{pmatrix}$ using standard rotation.

□

Problem 82. Let $\alpha : A \to B$ be a $*$-homomorphism of C*-algebras. Check that the map $\alpha_* : K_1(A) \to K_1(B)$ is well defined by $\alpha_*([u]) = [\alpha_n(u)]$, where $u \in U^+_n(A)$, and $\alpha_n : M_n(A) \to M_n(B)$ is a homomorphism of matrix algebras induced by the homomorphism α.

Problem 83. Let $x, y \in U^+_n(A)$, and $\mathrm{diag}(x, 1_\infty)$ and $\mathrm{diag}(y, 1_\infty)$ be homotopic in $U^+_\infty(A)$. Then there exists $k \in \mathbb{N}$ and a homotopy $u : [0, 1] \to U^+_{n+k}(A)$ such that $u(0) = \mathrm{diag}(x, 1_k)$, $u(1) = \mathrm{diag}(y, 1_k)$. Therefore $U^+_\infty(A)/U^+_\infty(A)_0 = \varinjlim(U^+_n(A)/U^+_n(A)_0)$ (and this is also true for the remaining quotient groups from the definition of K_1).

Problem 84. Check homotopy invariance of K_1; check that $K_1(M_n(A)) \cong K_1(A)$.

Problem 85. Compute $K_1(\mathbb{C})$, $K_1(\mathbb{K})$.

Problem 86. Compute $K_1(C(\mathbb{T}^1))$, where $\mathbb{T}^1$ denotes a circle.

Lecture 14.04.2025

4.11 Relationship between K_0 and K_1

Maps from the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ are usually called loops. We will need the following loop classes:

$$SA = \{f \in C(\mathbb{T}; A) : f(1) = 0\};$$

$$(SA)^+ = \{f \in C(\mathbb{T}; A^+) : f(1) = \pi_{\mathbb{C}} f(z) = \mathrm{Const} \in \mathbb{C} \ \ \forall z \in \mathbb{T}\};$$

Let us define a homomorphism $\theta_A : K_1(A) \to K_0(SA)$.

Let $[u] \in K_1(A)$, $u \in U_n^+(A)$ for some n. As we know, then there is a homotopy $t \mapsto w_t \in U_{2n}^+(A)$ connecting $w_0 = 1_{2n}$ and $w_1 = \mathrm{diag}(u, u^*)$. This homotopy defines a family of projections $q_t = w_t p_n w_t^* \in M_{2n}(A^+))$, where $p_n = \mathrm{diag}(1_n, 0_n)$ is a diagonal matrix with n units and n zeros on the diagonal. Since $q_0 = p_n$, $q_1 = w_1 p_n w_1^* = \mathrm{diag}(u 1_n u^*, 0) = p_n$, this family of projections is a loop. Since $\pi_{\mathbb{C}} w_t = 1_{2n}$ for any $t \in [0, 1]$, we obtain that $\pi_{\mathbb{C}} q_t = p_n$ for any $t \in [0, 1]$, from which we obtain $q_t - p_n \in M_{2n}(A)$, and thus q can be considered as an element of the algebra $M_{2n}((SA)^+)$, where $q - p_n \in M_{2n}(SA)$. Note that, unlike q, w is not a loop, because $w_1 \neq w_0$. Let $\theta_A([u]) = [q] - [p_n] \in K_0(SA)$.

Lemma 4.71. *This map is well defined.*

Proof. Let us first check the independence of the choice of u and w. Let $v \in U_n^+(A)$, and let u and v be connected by homotopy $t \mapsto a_t \in U_n^+(A)$ ($a_0 = u$, $a_1 = v$). Along with the homotopy $t \mapsto w_t \in U_{2n}^+(A)$, we consider the homotopy $t \mapsto z_t \in U_{2n}^+(A)$ connecting $z_0 = 1_{2n}$ and $z_1 = \mathrm{diag}(v, v^*)$. We get two projections, $q_u = w p_n w^*$ and $q_v = z p_n z^*$. Let us show that these projections are unitarily equivalent in $M_{2n}((SA)^+)$. Put $x_t = w_t \cdot \mathrm{diag}(u^* a_t, u a_t^*) \cdot z_t^*$. Then

$$x_0 = w_0 \cdot \mathrm{diag}(u^* a_0, u a_0^*) \cdot z_0^* = 1_{2n},$$

$$x_1 = w_1 \cdot \mathrm{diag}(u^* a_1, u a_1^*) \cdot z_1^* = \mathrm{diag}(u, u^*) \cdot \mathrm{diag}(u^* v, u v^*) \cdot \mathrm{diag}(v, v^*) = 1_{2n}.$$

Also $\pi_{\mathbb{C}} x_t = 1_{2n}$ for any $t \in [0, 1]$, so the loop $x : t \mapsto x_t$ can be considered as an element of $U_{2n}^+(SA)$.

The unitary equivalence of the projections q_u and q_v follows from the equality

$$x q_v x^* = x(z p_n z^*) x^* = w \cdot \mathrm{diag}(u^* a, u a^*) \cdot \mathrm{diag}(1_n, 0_n) \cdot \mathrm{diag}(a^* u, a u^*) \cdot w^* = w p_n w^* = q_u.$$

It remains to check the independence of the dimension n. Instead of $u \in U_n^+(A)$, you can work with $\mathrm{diag}(u, 1_m) \in U_{n+m}^+(A)$. Let us denote the corresponding projection by q'. Let $y \in M_{2n+2m}(\mathbb{C})$ be a matrix (permutation of rows and columns) such that

$$y \cdot \mathrm{diag}(u, u^*, 1_m, 1_n) \cdot y^* = \mathrm{diag}(u, 1_m, u^*, 1_m).$$

Set $z_t = y \cdot \mathrm{diag}(w_t, 1_{2m}) \cdot y^* \in U_{2n+2m}^+(A)$. Then $z_0 = 1_{2n+2m}$, $z_1 = \mathrm{diag}(u, 1_m, u^*, 1_m)$, and to define the projection q' we can choose the homotopy z_t (we have already proven that the unitary equivalence class of the projection q' does not depend on the choice of such a homotopy): $q'_t = z_t p_{n+m} z_t^*$. Then the projection

$$q' = y \cdot \mathrm{diag}(w, 1_{2m}) \cdot y^* \cdot p_{n+m} \cdot y \cdot \mathrm{diag}(w^*, 1_{2m}) \cdot y^*$$

is homotopic to the projection

$$q'' = \mathrm{diag}(w, 1_{2m}) \cdot y^* \cdot p_{n+m} \cdot y \cdot \mathrm{diag}(w^*, 1_{2m})$$

(you can get rid of the right and left factors, since $U_k(\mathbb{C})$ is connected for any dimension), i.e. $[q'] - [p_{n+m}] = [q''] - [p_{n+m}]$. But

$$q'' = \mathrm{diag}(w, 1_{2m}) \cdot \mathrm{diag}(p_n, p_m) \cdot \mathrm{diag}(w^*, 1_{2m}) = \mathrm{diag}(wp_nw^*, 1_m) = \mathrm{diag}(q, p_m),$$

therefore, $[q'] - [p_{n+m}] = [\mathrm{diag}(q, p_m)] - [\mathrm{diag}(p_n, p_m)] = [q] - [p_n]$, which is what needed to be proved . $\square$

Theorem 4.72. *1. The map θ_A constructed above is functorial, i.e. if $\alpha : A \to B$ is a $*$-homomorphism of C*-algebras, then the diagram*

$$\begin{tikzcd} K_1(A) \arrow[r, "\alpha_*"] \arrow[d, "\theta_A"'] & K_1(B) \arrow[d, "\theta_B"] \\ K_0(SA) \arrow[r, "S\alpha_*"'] & K_0(SB) \end{tikzcd}$$

commutes, where α_* and $S\alpha_*$ are group homomorphisms induced by the homomorphism α.

2. The map θ_A is a group isomorphism for any C-algebra A.*

Proof. 1. Let $u \in U_n^+(A)$, w_t be a homotopy in $U_{2n}^+(A)$ connecting 1_{2n} and $\mathrm{diag}(u, u^*)$. Then $\alpha^+(w_t)$ is a homotopy in $U_{2n}^+(B)$ connecting 1_{2n} with $\mathrm{diag}(\alpha(u), \alpha(u)^*)$. That's why

$$\theta_B \circ \alpha_*([u]) = \theta_B([\alpha^+(u)]) = [\alpha^+(w)p_n\alpha^+(w)^*] - [p_n] = \alpha_*([wp_nw^*] - [p_n]) = \alpha_* \circ \theta_A([u]).$$

2. Let's start with injectivity. Let $\theta_A([u]) = \theta_A([v])$. We can assume in advance that $u, v \in U_n^+(A)$ with common n. Let (w_t^0), (z_t^0) be homotopies in $U_{2n}^+(A)$ connecting 1_{2n} with $\mathrm{diag}(u, u^*)$ and with $\mathrm{diag}(v, v^*)$ respectively. Supplementing them with constant direct summands, we can define for each $k \in \mathbb{N}$ the homotopies (w_t^k), (z_t^k) in $U_{2n+2k}^+(A)$, connecting 1_{2n+2k} to $\mathrm{diag}(u, 1_k, u^*, 1_k)$ and to $\mathrm{diag}(v, 1_k, v^*, 1_k)$, respectively. Put

$$q_t^k = w_t^k p_{n+k} w_t^{k*}, \qquad r_t^k = z_t^k p_{n+k} z_t^{k*}, \qquad k = 0, 1, 2, \ldots.$$

It is clear that q^k is homotopic to $\mathrm{diag}(q^0, p_k)$, and r^k is homotopic to $\mathrm{diag}(r^0, 1_k)$ in $M_{2n+2k}((SA)^+)$.

The assumption $\theta_A([u]) = \theta_A([v])$ means that $[q^0] - [r^0] = 0$ in $K_0(SA)$, hence there exist $k, m \in \mathbb{N}$, $k < m$, such that the projections $\mathrm{diag}(q^0, p_k)$ and $\mathrm{diag}(r^0, p_k)$ are unitarily equivalent in $M_{2n+m}((SA)^+)$. By adding units and zeros on the diagonals, we can make it so that $m = 2k$. Let's assume that this has already been done. Unitary equivalence in $M_{2n+2k}((SA)^+)$ means existence of a $x \in U^+_{2n+2k}(SA)$ such that $xq^kx^* = r^k$, i.e. $x_t(w^k_0 p_{n+k} w^{k*}_t)x^*_t = z^k_t p_{n+k} z^{k*}_t$ for any $t \in [0,1]$. This can be written as

$$p_{n+k}(w^{k*}_t x^*_t z^k_t) = (w^{k*}_t x^*_t z^k_t)p_{n+k}. \tag{4.10}$$

A matrix commuting with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ must be diagonal, therefore from the equality (4.10) it follows that $y_t = w^{k*}_t x^*_t z^k_t$ is diagonal for all t, i.e. $y_t = \mathrm{diag}(a_t, b_t)$ for some $a, b \in U^+_{n+k}(SA)$. Knowing that $x_0 = x_1 = 1_{2n+2k}$, and that $w^k_0 = z^k_0 = 1_{2n+2k}$, $w^k_1 = \mathrm{diag}(u, 1,u^*, 1_k)$, $z^k_1 = \mathrm{diag}(v, 1_k, v^*, 1_k)$, we get that

$$y_0 = 1_{2n+2k}, \qquad y_1 = w^{k*}_1 x_1 z^k_1 = \mathrm{diag}(u^*v, 1_k, uv^*, 1_k),$$

therefore $a_0 = 1_{n+k}$, $a_1 = \mathrm{diag}(u^*v, 1_k)$. Then $t \mapsto \mathrm{diag}(u, 1_k)a_t$ defines a homotopy in $U^+_{2n+2k}(A)$ connecting $\mathrm{diag}(u, 1_k)$ and $\mathrm{diag}(v, 1_k)$. Existence of such a homotopy means that $[u] = [v]$ in $K_1(A)$.

Now let's prove surjectivity. Let $x \in K_0(SA)$. Then it can be written in the form $x = [r] - [p_n]$, where $r : t \mapsto r_t$ is a projection in $M_k((SA)^+)$, $k \geq n$, and $r - p_n \in M_k(SA)$. Write r as a matrix, $r_t = \big((r_t)_{ij}\big)$, where the matrix elements have the form $(r_t)_{ij} = (\tilde{r}_t)_{ij} + \lambda_{ij}$, $i, j = 1, \ldots, k$, $(\tilde{r}_t)_{ij} \in SA$, $\lambda_{ij} \in \mathbb{C}$. Since $r \in M_k((SA)^+)$, r_0 and r_1 are scalar matrices, it means that $r_0 = r_1 = (\lambda_{ij})$, and since $r_t = p_n \in M_k(SA)$ for all t, then $r_0 = r_1 = p_n$. Consider a continuous family of projections r_t, $t \in [0,1]$. There is a continuous family $w_t \in U^+_k(A)$ such that $w_0 = 1_k$ and $r_t = w_t p_n w^*_t$ for all $t \in [0,1]$. About w_1 we cannot say that $w_1 = 1_k$, but since $p_n = r_1 = w_1 p_n w^*_1$ (i.e. $p_n w_1 = w_1 p_n$), then w_1 is diagonal, $w_1 = \ diag(u, v)$, where $u \in U^+_n(A)$, $v \in U^+_{k-n}(A)$. We claim that $\theta_A([u]) = x$, which will prove surjectivity.

Let $(Z_t)_{[0,1]}$ be a homotopy in $U^+_{2k}(A)$ connecting 1_{2k} and $\mathrm{diag}(u, 1_{kn}, u^*, 1_{kn})$. Let us set $q_t = Z_t p_n Z^*_t$, $q \in M_{2k}((SA)^+)$. For brevity, let's denote $W_t = \mathrm{diag}(w_t, w^*_t) \in U^+_{2k}(A)$. Let $(a_t)_{[0,1]}$ be a path in $U^+_{2k-n}(A)$ (arbitrary for now, we will choose it later). Then $\mathrm{diag}(1_n, a_t)p_n = p_n \, \mathrm{diag}(1_n, a_t) = p_n$, so

$$\begin{aligned} \mathrm{diag}(r_t, 0) &= \mathrm{diag}(w_t p_n w^*_t, 0) = W_t p_n W^*_t = W_t \cdot \mathrm{diag}(1_n, a_t) \cdot p_n \cdot \mathrm{diag}(1_n, a^*_t) \cdot W^*_t \\ &= W_t \cdot \mathrm{diag}(1_n, a_t) \cdot Z^*_t \cdot (Z_t p_n Z^*_t) \cdot Z_t \cdot \mathrm{diag}(1_n, a^*_t) \cdot W^*_t \\ &= W_t \cdot \mathrm{diag}(1_n, a_t) \cdot Z^*_t q_t Z_t \cdot \mathrm{diag}(1_n, a^*_t) \cdot W^*_t \ = \ Y_t q_t Y^*_t, \end{aligned}$$

where $Y_t = W_t \cdot \mathrm{diag}(1_n, a_t) \cdot Z^*_t \in U^+_{2k}(A)$.

Now we need to choose a_t so that $Y : t \mapsto Y_t$ is an element of $M_{2k}((SA)^+)$ — then the projections r and q would be unitarily equivalent, i.e. $[r] = [q]$ in $K_0(SA)$ (and then $[r] - [p_n] = [q] - [p_n] = x = \theta_A([u])$). To do this, it suffices to ensure that $Y_0 = Y_1 = 1_{2k}$. Since $W_0 = Z_0 = 1_{2k}$, $W_1 = \mathrm{diag}(u, v, u^*, v^*)$, $Z_1 = \mathrm{diag}(u, 1_{kn}, u^*, 1_{kn})$, we obtain that $Y_0 = \mathrm{diag}(1_n, a_0)$, $Y_1 = \mathrm{diag}(1_n, X)$, where $X = \mathrm{diag}(v, U^*, v^*) \cdot a_1 \cdot \mathrm{diag}(1_{kn}, u, 1_{kn})$. This gives conditions on a_t: $a_0 = 1_{2k-n} = X$, i.e. $a_1 = \mathrm{diag}(v^*, 1_n, v)$. But $\mathrm{diag}(v^*, 1_n, v) \in U^+_{2k-n}(A)_0$, so there is a homotopy connecting a_0 and a_1. Let's take it as $(a_t)_{[0,1]}$. □

This theorem allows us to give the following definition (especially useful in algebraic K-theory).

Definition 4.73. $K_n(A) = K_0(S^n A)$.

Let's connect K_0 and K_1 in one more way. Let $0 \to J \to A \to A/J \to 0$ be a short exact sequence of C*-algebras. Let $x \in K_1(A/J)$, $x = [u]$, where $u \in U_n^+(A/J)$. Let $v \in U_k^+(A/J)$ be such that $\mathrm{diag}(u, v) \in U_{n+k}^+(A/J)_0$ (for example, we can take $v = u^*$). Since elements from the connected component of identity admit unitary liftings, there exists $w \in U_{n+k}^+(A)$ such that $\pi_J^{(n+k)}(w) = \mathrm{diag}(u, v)$, where $\pi_J^{(n)} : M_n(A) \to M_n(A/J)$ is the quotient map of matrix algebras.

Definition 4.74. The map $\delta : K_1(A/J) \to K_0(J)$, $\delta(x) = [wp_n w^*] - [p_n]$, is called the boundary homomorphism.

Theorem 4.75. *The boundary homomorphism is well defined.*

Proof. Let's start by checking that $\delta(x) = [wp_n w^*] - [p_n] \in K_0(J)$. Note that $\pi_J^{(n+k)}(wp_n w^*) = \mathrm{diag}(u, v) \cdot p_n \cdot \mathrm{diag}(u^*, v^*) = p_n$, therefore $wp_n w^* - p_n \in M_{n+k}(J)$, i.e. specifies an element in $K_0(J)$.

Let us check the independence of the choice of w. Let w' be another unitary lift for $\mathrm{diag}(u, v)$. Let us set $z = w'w^* \in U_{n+k}^+(A)$. Then $\pi_J^{(n+k)}(z) = \mathrm{diag}(u, v) \cdot \mathrm{diag}(u^*, v^*) = 1_{n_k}$, so $z \in U_{n+k}^+(J)$. Since $z(wp_n w^*)z^* = w'p_n w'^*$, we get that $[wp_n w^*] - [p_n] = [w'p_n w'^*] - [p_n]$ in $K_0(J)$.

Next, let's check independence from the dimensions of u and v. Write w as a matrix, $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$, where $pi_J^{(n)}(w_{11}) = u$, $\pi_J^{(k)}(w_{22}) = v$. Replace u and v with $\mathrm{diag}(u, 1_m)$ and $\mathrm{diag}(v, 1_j)$. Then $W = \begin{pmatrix} w_{11} & 0 & w_{12} & 0 \\ 0 & 1_m & 0 & 0 \\ w_{21} & 0 & w_{22} & 0 \\ 0 & 0 & 0 & 1_j \end{pmatrix} \in U_{n+k+m+j}^+(A)$ is a unitary lifting for $\mathrm{diag}(u, 1_m, v, 1_j)$. Conjugating with scalar unitary matrices and rearranging the blocks, we obtain that

$$\begin{aligned} Wp_{n+m}W^* &\sim \mathrm{diag}(w, 1_{m+j}) \cdot p_{n+m} \cdot \mathrm{diag}(w^*, 1_{m+j}) \\ &\sim \mathrm{diag}(w, 1_{m+j}) \cdot \mathrm{diag}(1_n, 0_k, 1_m, 0_j) \cdot \mathrm{diag}(w^*, 1_{m+j}) \\ &= \mathrm{diag}(wp_n w^*, p_m), \end{aligned}$$

and

$$[Wp_{n+m}W^*] - [p_{n+m}] = [\mathrm{diag}(wp_n w^*, p_m)] - [\mathrm{diag}(p_n, p_m)] = [wp_n w^*] - [p_n].$$

Finally, let us check independence from the choice of u and v. Let $u' \in U_n^+(A/J)$, $u' \sim_h u$, where $\sim_h$ denotes existence of a unitary homotopy between the right and left sides, and let $v' \in U_k^+(A/J)$ be such that $\mathrm{diag}(u', v') \in U_{n+k}^+(A/J)_0$. The condition $u' \sim_h u$ is equivalent to $u^*u' \sim_h 1_n$, and for u^*u' there is a unitary lift $a \in U_n^+(A)_0$, i.e. $\pi_J^{(n)}(a) = u^*u'$. Likewise, since

$$\mathrm{diag}(v^*v', 1_n) \sim_h \mathrm{diag}(1_n, v^*v') \sim_h \mathrm{diag}(u^*u', v^*v') \sim_h 1_{n+k},$$

there is a unitary lift $b \in U_{n+k}^+(A)_0$ for $\mathrm{diag}(v^*v', 1_n)$. Then $z = \mathrm{diag}(w, 1_n) \cdot \mathrm{diag}(a, b)$ is a unitary lift for $\mathrm{diag}(u', v'1_n)$ (not for $\mathrm{diag}(u', v')$, but this does not matter, since we have already checked independence from the matrix size). Using u and v we get $\delta([u]) = [wp_nw^*] - [p_n]$, and using u' and $\mathrm{diag}(v', 1_n)$ we get $\delta([u]) = [zp_nz^*] - [p_n]$. Let's show that they are equal. Since $\mathrm{diag}(a, b)$ commutes with p_n, we get

$$\begin{aligned} zp_nz^* &= \mathrm{diag}(w, 1_n) \cdot \mathrm{diag}(a, b) \cdot p_n \cdot \mathrm{diag}(a^*, b^*) \cdot (w^*, 1_n) = \mathrm{diag}(w, 1_n) \cdot p_n \cdot \mathrm{diag}(w^*, 1_n) \\ &= \mathrm{diag}(wp_nw^*, 0_n). \end{aligned}$$

So, the map δ is well defined. It remains to check that it is a group homomorphism. Let $u_1, u_2 \in U_n^+(A/J)$. Let $w_1, w_2 \in U_{2n}^+(A)$ be lifts for $\mathrm{diag}(u_1, u_1^*)$ and $\mathrm{diag}(u_2, u_2^*)$ respectively. Then a lift of w_3 for $\mathrm{diag}(u_1, u_2, u_1^*, u_2^*)$ can be obtained from $\mathrm{diag}(w_1, w_2)$ by permuting blocks, which is carried out using block-scalar matrices. Homomorphity follows from the calculation

$$\begin{aligned} \delta([u_1] + [u_2]) &= \delta([u_1u_2]) = \delta([\mathrm{diag}(u_1, u_2)]) = [w_3p_{2n}w_3^*] - [p_{2n}] \\ &= [\mathrm{diag}(w_1, w_2) \cdot p_{2n} \cdot \mathrm{diag}(w_1^*, w_2^*)] - [p_{2n}] \\ &= [\mathrm{diag}(w_1, w_2) \cdot \mathrm{diag}(p_n, p_n) \cdot \mathrm{diag}(w_1^*, w_2^*)] - [p_n] - [p_n] \\ &= [w_1p_nw_1^*] - [p_n] + [w_2p_nw_2^*] - [p_n] = \delta([u_1]) + \delta([u_2]). \end{aligned}$$

□

The following theorem is proved by a similar argument to the previous ones.

Theorem 4.76. *Let $J \subset A$ be an ideal, $\pi : A \to A/J$ the quotient homomorphism, then the sequence*

$$K_1(J) \to K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\delta} K_0(J) \to K_0(A) \xrightarrow{\pi_*} K_0(A/J)$$

is exact.

Bott's periodicity theorem is proved by fundamentally different methods. Unfortunately, limited time does not allow us to present its proof here.

Theorem 4.77. *The groups $K_0(A)$ and $K_2(A) = K_0(S^2(A))$ are canonically isomorphic.*

Corollary 4.78. *Let $J \subset A$ be an ideal, $\pi : A \to A/J$ the quotient homomorphism, then the sequence*

$$\begin{tikzcd} K_1(J) \arrow[r] & K_1(A) \arrow[r, "\pi_*"] & K_1(A/J) \arrow[d, "\delta"] \\ K_0(A/J) \arrow[u] & K_0(A) \arrow[l, "\pi_*"'] & K_0(J) \arrow[l] \end{tikzcd}$$

is exact.

Let's give an example of using this sequence for calculations. The *dimension drop* algebra D_n is the C*-algebra of functions on the interval $[0, 1]$ with values in the matrix algebra M_n, $n \geq 2$, such that they are equal to 0 at 0 and are scalar at 1, i.e. $D_n = \{f \in C([0, 1]; M_n) : f(0) = 0, f(1) = \mathbb{C} \cdot 1\}$. Let $J \cong SM_n$ be the ideal of matrix-valued functions equal to 0 at both ends of the interval $[0,1]$. Then $K_0(J) = 0$, $K_1(J) = \mathbb{Z}$, $D_n/J \cong \mathbb{C}$, and $K_0(D_n/J) \cong \mathbb{Z}$, $K_1(D_n/J) = 0$. Consider the ideal $SJ = S^2M_n$ in the algebra SD_n and write down the exact sequence for them

$$\begin{tikzcd} 0 \arrow[r] & K_1(SD_n) \arrow[r, "\pi_*"] & \mathbb{Z} \arrow[d, "\delta"] \\ 0 \arrow[u] & K_0(SD_n) \arrow[l, "\pi_*"'] & \mathbb{Z} \arrow[l] \end{tikzcd}$$

i.e.

$$0 \to K_1(SD_n) \to \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \to K_0(SD_n) \to 0,$$

The groups $K_i(SD_n) = K_{1-i}(D_n)$ depend on the explicit form of the homomorphism $\delta : K_1(S\mathbb{C}) \to K_0(S^2M_n)$, which must be calculated explicitly (useful exercise!). It turns out that δ is multiplication by n, which means that $K_0(D_n) = 0$, $K_1(D_n) = \mathbb{Z}/n\mathbb{Z}$.