Resampling and distributions of functionals of random processes

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Abstract

We introduce the notion of weak approaching and conditionally weak approaching sequences of random processes. This notion generalizes the conventional weak convergence, and has been proposed for real valued random variables in Belyaev (1995). Some of the standard tools for an investigation of the behavior of weak approaching sequences of random elements in metric spaces are developed. The spaces of smoothed and right continuous functions with left-hand limits are considered. This technique allows us to use the resampling approach for an evaluation of distributions of certain continuous functionals on realizations of random processes. Two numerical examples are presented for such functionals as supremum and number of level crossings.[1]

Keywords: resampling, random processes, weak approaching, weak convergence, metric spaces, Gaussian processes, Shot effect processes, distributions of supremum, level crossings

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1 Introduction

Most of research related to distribution of functionals on realizations of random processes (e.g., supremum, number of level(s) crossings, sojourn) are carried out in the frames of probability theory (see e.g., Leadbetter, Lindgren and Rootzén (1983)). In this paper, we consider the statistical problem of nonparametric estimation of distributions of such functionals in one special case, namely, when the functionals are defined on realizations of the sum of many "small" independent random processes. In this case, the statistical data consist of observations of these "small" processes. Each "small" process can be observed only once. One meets this situation in many different applications, e.g., in an analysis of data related to banks activity, and in evaluating risks of floods.

We consider the problem of estimation of distributions of some functionals in the asymptotic scheme, when the number of these "small" observed processes tends to infinity. This problem is solved by obtaining consistent estimators of the distributions. We apply the resampling technique, which supposes very mild assumptions on the observed random processes. Our approach is non-parametric and allows us to deal with non-stationary random processes. The most essential restriction concerns mathematical expectations of the observed "small" processes. We assume that their mathematical expectations are constant. Of course, this assumption is valid when the random processes are stationary.

The following notation will be used: $\mathbf{C}[0,1]$ ($\mathbf{C}^k[0,1]$) is the space of all real-valued continuous functions $f(\cdot)$ (with continuous kth derivative $f^{(k)}(\cdot)$); $\mathbf{D}[0,1]$ the space of all real-valued right-hand continuous functions with left-hand limits; $\#A$ the number of elements in a set A; $o(1)$ ($o_P(1)$) the members of a sequence that converges to zero (in probability); $\mathcal{L}(X)$ the distribution law of a random element X; $\mathcal{L}(X|Z)$ the standard regular conditional distribution law of a random element X given Z; and $(X|Z)$ and $\mathrm{E}[f(X)|Z]$ the random variable X and the conditional mathematical expectation of $f(X)$ given Z corresponding to $\mathcal{L}(X \mid Z)$, respectively. The sign "$\star$" is related to random elements obtained by using the standard random sampling with replacement.

Let $(\mathbf{S}, d)$ be a separable and complete normed space, and $\mathfrak{S}$ a σ-field generated by open sets. Denote by $\mathbf{C}_b(\mathbf{S})$ the space of all bounded and continuous real-valued functions on $\mathbf{S}$. Let X_n and Y_n be $\mathbf{S}$-valued random elements with distribution laws $\mathcal{L}(X_n)$ and $\mathcal{L}(Y_n)$, respectively. Note that X_n and Y_n can be defined on different probability spaces. We say that sequences $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ *approach each other (in distribution)* if, for every function $c(\cdot) \in \mathbf{C}_b(\mathbf{S})$,

$$\mathrm{E}c(X_n) - \mathrm{E}c(Y_n) = \mathrm{o}(1) \text{ as } n \to \infty, \tag{1}$$

and write $X_n \overset{ad}{\leftrightarrow} Y_n$. Correspondingly, we say $\{\mathcal{L}(X)\}_{n\geq 1}$ and $\{\mathcal{L}(X)\}_{n\geq 1}$ *weakly approach each other* if (1) holds. This notion generalizes the classical weak convergence for a sequence of random elements with a unique weak limit element ("attractor"), $Y_n \equiv Y_0$, as treated in Billingsley (1968). The further generalization is the following: let $\{Z_n\}_{n\geq 1}$ be a sequence of $\mathcal{Z}_n$-valued random elements for given spaces $\{\mathcal{Z}_n\}_{n\geq 1}$. Assume that Z_n is defined on the same basic probability space $(\Omega, \mathcal{F}, \mathrm{P})$ as X_n for all n. By definition, the sequence $\{(X_n|Z_n)\}_{n\geq 1}$ with the conditional distributions $\{\mathcal{L}(X_n|Z_n)\}_{n\geq 1}$ *approaches* $\{Y_n\}_{n\geq 1}$ *in probability along* Z_n, if for every $c(\cdot) \in \mathbf{C}_b(\mathbf{S})$,

$$\mathrm{E}[c(X_n)|Z_n] - \mathrm{E}c(Y_n) = o_P(1) \text{ as } n \to \infty, \tag{2}$$

then we write $(X_n|Z_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} Y_n$.

This type of convergence can be useful in many statistical problems arising in intensive computer methods of statistical data analysis (e.g., in the bootstrap and resampling, see Efron and Tibshirani (1993), Hjorth (1994)). It has been introduced and investigated for real-valued and vector-valued random elements in Belyaev (1995, 1996b), and Belyaev and Sjöstedt (1996), respectively. The theory of weak convergence and the weak approaching of sequences have a lot of useful applications in probability theory and statistics.

This research was done during the visit of the second author to the Department of Mathematical Statistics, Umeå University, in January - February 1997. Theorems 3 and 5 were obtained together by both authors. Theorems 1, 2, and 4 by second author.

The basic results of the conventional theory of weak convergence, include Continuous Mapping Theorem and the necessary tools to establish conditions for weak convergence in certain functional spaces (e.g., in the space of continuous functions). This paper is organized as follows: in Section 2, we develop a similar technique for conditionally weak approaching sequences of random elements and consider two cases: the space of smooth functions and the space of right continuous functions with left-hand limits. In Section 3, we consider resampling for an array of realizations of independent random processes. Then, we apply these results for an evaluation of the distribution of certain functionals on realizations of random processes. Two numerical examples (for the analytical Gaussian process with the covariance e^{-t^2} and a Shot effect process) are presented. Section 4 contains the proofs of the statements from the previous sections.

2 Weak approaching sequences of random processes

First, we introduce some additional notation. We recall that a sequence of random elements, $\{X_n\}_{n\geq 1}$ in a topological space $\mathbf{S}$, is called *uniformly tight* if for every positive ϵ there exists a compact K_ϵ such that

$$\sup_n \mathrm{P}\{X_n \notin K_\epsilon\} < \epsilon. \tag{3}$$

For a given sequence of random elements $\{(X_n, Z_n)\}_{n\geq 1}$ as in (2), we say that $\{(X_n|Z_n)\}_{n\geq 1}$ is *uniformly tight along* Z_n *in probability* if for every positive ϵ and δ there exists a compact $K_{\epsilon\delta} \subset \mathbf{S}$ such that

$$\sup_n \mathrm{P}\{\mathrm{P}\{X_n \notin K_{\epsilon\delta}|Z_n\} > \epsilon\} < \delta.$$

Henceforth, assume that $\mathbf{S}$ is a normed space with a norm $||f||, f \in \mathbf{S}$. The following proposition shows that for some spaces, the property of weak tightness of a sequence of random elements implies conditionally weak tightness.

Proposition 1 *Let $\{X_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ be defined as in (2). If $\{X_n\}_{n\geq 1}$ is uniformly tight in $\mathbf{S}$, then $\{(X_n|Z_n)\}_{n\geq 1}$ is uniformly tight along $\{Z_n\}_{n\geq 1}$ in probability.*

Theorem 1 *(Continuous Mapping Theorem) Let $h(\cdot)$ be a continuous functional on $\mathbf{S}$, $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random elements in $\mathbf{S}$, and $\{Z_n\}_{n\geq 1}$ be defined as in* (2)*.*

(i) If $X_n \overset{ad}{\leftrightarrow} Y_n$, then $h(X_n) \overset{ad}{\leftrightarrow} h(Y_n)$ as $n \to \infty$;

(ii)If $(X_n|Z_n) \xleftrightarrow{ad(\mathrm{P})} Y_n$, then $(h(X_n) \mid Z_n) \xleftrightarrow{a(\mathrm{P})} h(Y_n)$ as $n \to \infty$.

Remark 1 Let $F_n(\cdot), F_n(\cdot|Z_n)$ and $G_n(\cdot)$ be the distribution functions of $h(X_n)$, $(h(X_n) \mid Z_n)$, and $h(Y_n)$, respectively. Assume that $\{Y_n\}_{n\geq 1}$ is uniformly tight and $\{G_n(\cdot)\}_{n\geq 1}$ is uniformly continuous. It is shown in Belyaev (1996b) that $h(X_n) \overset{ad}{\leftrightarrow} h(Y_n)$ implies $\sup_s \mid F_n(s) - G_n(s) \mid= o(1)$, and if $(h(X_n) \mid Z_n) \xleftrightarrow{ad(\mathrm{P})} h(Y_n)$, then $\sup_s |F_n(s \mid Z_n) - G_n(s) \mid= o_P(1)$ as $n \to \infty$. $\square$

Let $\mathbf{S} = \mathbf{C}^k[0,1]$, $k \geq 0$, with the standard uniform norm

$$||f(\cdot)||_u := \max\{\sup_{[0,1]} |f^{(j)}(t)|,\ j = 0, \ldots, k\}$$

for every $f(\cdot) \in \mathbf{C}^k[0,1]$. The modulus of continuity of a continuous function $f(\cdot)$ is defined by

$$w(f, \delta) := \sup_{|t-s|<\delta,\, t,s\in[0,1]} |f(t) - f(s)|, \quad 0 < \delta \leq 1. \tag{4}$$

The following theorem gives sufficient conditions for conditional weak approaching of sequences of random elements in $\mathbf{C}^k[0,1]$, $k \geq 0$, along a given sequence of random elements. Henceforth, by marginals of $X_n = X_n(\cdot)$, we call all distributions of finite-dimensional collections $X(t_i)$, $i = 0, \ldots, m$, $0 \leq t_0 < \ldots < t_m \leq 1$, for all m.

Theorem 2 *Let* $\{X_n\}_{n\geq 1}$ *and* $\{Y_n\}_{n\geq 1}$ *be sequences of random elements in* $\mathbf{C}^k[0,1]$*, and* $\{Z_n\}_{n\geq 1}$ *be defined as in* (2)*. Assume* $\{X_n\}_{n\geq 1}$ *and* $\{Y_n\}_{n\geq 1}$ *are uniformly tight in* $\mathbf{C}^k[0,1]$*. If marginals of* X_n*, conditionally on* Z_n*, weakly approach marginals* Y_n *along* Z_n *in probability, then* $(X_n|Z_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} Y_n$ *as* $n \to \infty$*.*

The necessary and sufficient conditions of uniform tightness for a sequence of random elements in $\mathbf{C}^k[0,1]$ can be formulated in terms of the continuity modulus (4) (see Wilson (1986), Seleznjev (1986), Rusakov and Seleznjev (1987)).

Denote by $\mathbf{D}[0,1]$ the space of all real-valued right continuous functions with left-hand limits defined on $[0,1]$. Then the following metric generates the Skorohod topology

$$d(x,y) := \inf_{\lambda(\cdot)\in\Lambda} \{\sup_{[0,1]} |x(t) - y(\lambda(t))| + |\lambda(t) - t|\}, \tag{5}$$

where Λ denotes the set of strictly increasing continuous functions $\lambda(\cdot)$ on $[0,1]$, $\lambda(0) = 0$ and $\lambda(1) = 1$. Let for every $f(\cdot) \in \mathbf{D}[0,1]$ and $0 < \delta < 1$,

$$\begin{aligned}\Delta(f,\delta) \;:=\; & \sup_{0\leq t-\delta\leq t_1\leq t\leq t_2\leq t+\delta\leq 1} \{\min(|f(t_2) - f(t)|, |f(t) - f(t_1)|)\}, \\ & + \sup_{0\leq t\leq\delta} |f(t) - f(0)| + \sup_{1-\delta\leq t\leq 1} |f(1) - f(t)|.\end{aligned} \tag{6}$$

Theorem 3 *Let* $\{X_n\}_{n\geq 1}$ *and* $\{Y_n\}_{n\geq 1}$ *be sequences of random elements in* $\mathbf{D}[0,1]$*, and* $\{Z_n\}_{n\geq 1}$ *be defined as in* (2)*. Assume* $\{X_n\}_{n\geq 1}$ *and* $\{Y_n\}_{n\geq 1}$ *are uniformly tight in* $\mathbf{D}[0,1]$*. If marginals of* X_n*, conditionally on* Z_n*, weakly approach marginals* Y_n *along* Z_n *in probability, then* $(X_n|Z_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} Y_n$ *as* $n \to \infty$*.*

Note that the necessary and sufficient conditions of uniform tightness for a sequence of random elements in $\mathbf{D}[0,1]$ can be formulated in terms of the continuity modulus (6) (see Gikhman and Skorohod (1979)). Sufficient conditions can be stated as follows:

Remark 2

(i) Recall, that the sequence of random $\mathbf{D}[0,1]$-elements $\{X_n\}_{n\geq 1}$ is uniformly tight if there exist positive constants α, γ, H_1, H_2, and $\beta > 1$, such that for every $t_1 \leq t \leq t_2$,

$$\mathrm{E}[|X_n(t_2) - X_n(t)|^\alpha |X_n(t) - X_n(t_1)|^\alpha] \leq H_1 |t_2 - t_1|^\beta, \tag{7}$$

and $\mathrm{E}|X_n(0)|^\gamma \leq H_2$ uniformly in n (see Billingsley (1968)).

(ii) For every $\lambda(\cdot) \in \Lambda$ and $f(\cdot), g(\cdot) \in \mathbf{D}[0,1]$,

$$|\sup_{[0,1]} f(t) - \sup_{[0,1]} g(t)| = |\sup_{[0,1]} f(t) - \sup_{[0,1]} g(\lambda(t))| \leq d(f,g).$$

Hence, $\sup_{[0,1]} f(t)$ is a continuous functional in the metric $d(\cdot,\cdot)$ defined in (5). $\square$

3 Resampling for a sequence of random processes

Let $\mathbf{U}_n(t) = \{U_{1n}(t), \ldots, U_{nn}(t)\}$, $t \in [0,1]$, be an array of independent zero-mean random processes, and write $U_{\cdot n}(t) = \sum_{j=1}^n U_{jn}(t)$. For every n, we observe only one realization for each $U_{jn}(t)$, $j = 1, \ldots, n$, on the time interval $[0,1]$. We denote $U_{jn} = U_{jn}(\cdot)$, and let $\mathbf{U}_n = \{U_{1n}, ..., U_{nn}\}$, $U_{\cdot n} = \sum_{j=1}^n U_{jn}$. The observed values of the random element U_{jn} and $\mathbf{U}_n$ will be denoted by u_{jn} and $\mathbf{u}_n$, respectively. The following question will be investigated: whether the distribution of $U_{\cdot n}$ can be evaluated (mimic) using only the data $\mathbf{u}_n$.

We apply to this problem the method suggested for random variables by Belyaev (1995) and (1996b), and developed for vector-valued random variables by Belyaev and Sjöstedt (1996). Given the sample values $\mathbf{U}_n = \mathbf{u}_n$, let $\mathbf{U}_n^\star = \{U_{1n}^\star, \ldots, U_{nn}^\star\}$ be an resampling copy taken from $\mathbf{u}_n$, i.e., $\mathbf{U}_n^\star = \{U_{K_{1n}^\star n}, \ldots, U_{K_{nn}^\star n}\}$, where $K_{1n}^\star, \ldots, K_{nn}^\star$ are independent and identically distributed random variables with the uniform discrete distribution on $\{1, \ldots, n\}$. Denote by $M_{in}^\star$ the number of times that u_{in} was "redrawn" from the original data $\mathbf{u}_n$, $M_{in}^\star = \sum_{j=1}^n I_{\{K_{jn}^\star = i\}}$. Then the vector $(M_{1n}^\star, \ldots, M_{nn}^\star)$ is independent of $\mathbf{U}_n$ and multinomially distributed, $\sum_{i=1}^n M_{in}^\star = n$, $\mathrm{E}M_{in}^\star = 1, \mathrm{Var}(M_{in}^\star) = 1 - 1/n$, and $\mathrm{Cov}(M_{in}^\star, M_{jn}^\star) = -1/n$ if $i \neq j$. We show that, conditionally on $\mathbf{U}_n$, the distribution of the centered resampling sum

$$U_{\cdot n}^{\star 0} := \sum_{j=1}^n U_{jn} - U_{\cdot n}^\star = \sum_{j=1}^n (M_{jn}^\star - 1) U_{jn}$$

weakly approaches in probability the distribution of $U_{\cdot n}$ along $\mathbf{U}_n$.

Theorem 4 *Let $U_{1n}, \ldots, U_{nn}$ be independent zero-mean random elements with values in $\mathbf{C}^k[0,1]$. Assume that there exist $\beta > 1$, positive constants H, and H_i, $i = 0, \ldots, k$, such that*

$$n\mathrm{E}(U_{jn}^{(k)}(t) - U_{jn}^{(k)}(s))^2 \leq H|t-s|^\beta,$$

and $\mathrm{E} \mid \sqrt{n} U_{jn}^{(i)}(t) \mid^{2+\delta} \leq H_i$, $j = 1, \ldots, n$, $i = 0, \ldots, k$, uniformly in n. Then $(U_{\cdot n}^{\star 0} | \mathbf{U}_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} U_{\cdot n}$ as $n \to \infty$.

Remark 3

(i) For the Gaussian case, it is sufficient to claim $\beta > 0$ instead of $\beta > 1$.

(ii) The smoothness of the functional space $\mathbf{C}^k[0,1]$ in Theorem 4 is defined by the choice of investigated functionals. In particular, for supremum, one can take $k = 0$, i.e. take $\mathbf{C}^0[0,1] = \mathbf{C}[0,1]$ the space of continuous functions. □

As a straightforward consequence of Theorem 4, and of the weakly approaching variant of Continuous Mapping Theorem (Theorem 1), we get the following

Corollary 1 *Let $U_{1n}, \ldots, U_{nn}$ be independent zero-mean random $\mathbf{C}[0,1]$-elements and the conditions of Theorem 4 hold with $k = 0$. Then*

$$(\sup_{[0,1]} U^{\star 0}_{\cdot n}(t) | \mathbf{U}_n) \overset{a(\mathrm{P})}{\longleftrightarrow} \sup_{[0,1]} U_{\cdot n}(t).$$

Therefore, the suggested method makes it feasible to evaluate the distribution law of $\sup_{[0,1]} U_{\cdot n}(t)$ based on only one realization of $\mathbf{U}_{\cdot n}$.

Narrowing the class of continuous functions $\mathbf{C}[0,1]$ to the class of k continuously differentiable functions $\mathbf{C}^k[0,1]$, allows us to investigate the weak approaching properties of a wider class of functionals defined on the space of smooth functions. Some integer-valued functionals are continuous in this case in corresponding metrics (see Rusakov and Seleznjev (1987), Seleznjev (1989)). For every $f(\cdot) \in \mathbf{C}^1[0,1]$, we say $f(\cdot)$ has a *non-degenerated crossing of the level u* at a point t, if $f(t) = u$ and $f'(t) \neq 0$. Denote by $\mathbf{C}^1_N[0,1] \subset \mathbf{C}^1[0,1]$ the space of functions with a finite number of non-degenerated crossings for every level u, and write

$$N_u(f) := \#\{t \in (0,1) : f(t) = u, f'(t) \neq 0\}.$$

Then $N_u(\cdot)$ is a continuous functional on $\mathbf{C}^1_N[0,1]$ (see Rusakov and Seleznjev (1987)). Therefore, if $U^{\star 0}_{\cdot n}$ and $U_{\cdot n}$ are random $\mathbf{C}^1_N[0,1]$-elements, then Theorems 1 and 4 give (cf. Corollary 1)

$$(N_u(U^{\star 0}_{\cdot n}) | \mathbf{U}_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} N_u(U_{\cdot n}). \tag{8}$$

Remark 4 Recall that $X(\cdot) \in \mathbf{C}^1[0,1]$ is a $\mathbf{C}^1_N[0,1]$ - valued random element if the one-dimensional density $p_t(x)$ of $X(t)$ is bounded for each $t \in [0,1]$, and $x \in \mathbf{R}^1$ (see Bulinskaya (1961), Cramér and Leadbetter (1967)). For example, this condition holds if $U_{1n}, \ldots, U_{nn}$ are continuously differentiable Gaussian non-degenerated random processes. □

Theorem 5 *Let $U_{1n}, \ldots, U_{nn}$ be independent zero-mean random elements with values in $\mathbf{D}[0,1]$. Assume that there exist $\beta > 1$ and positive constants H_1, H_2, and H_3, such that for every $t_1 \leq t \leq t_2$,*

$$n\mathrm{E}(U_{jn}(t_2) - U_{jn}(t_1))^2 \leq H_1 |t_2 - t_1|^{\beta/2},$$
$$n^2\mathrm{E}[(U_{jn}(t_2) - U_{jn}(t))^2 (U_{jn}(t) - U_{jn}(t_1))^2] \leq H_2 |t_2 - t_1|^{\beta},$$

and $\mathrm{E} \mid \sqrt{n} U_{jn}(t) \mid^{2+\delta} \leq H_3$ uniformly in n. Then, $(U^{\star 0}_{\cdot n} | \mathbf{U}_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} U_{\cdot n}$ as $n \to \infty$.

The following proposition, the Central Limit Resampling Theorem (CLRT) proved in Belyaev and Sjöstedt (1996), will be used in the proofs. Let $\mathbf{N}_k(\boldsymbol{\mu}, \mathbb{C})$ be a normally distributed k-dimensional random variable, with the vector of mathematical expectation $\boldsymbol{\mu}$ and the covariance matrix $\mathbb{C}$.

Proposition 2 (The Central Limit Resampling Theorem)
Let $\mathbb{W}_n = (\mathbf{W}_{1n}, \ldots, \mathbf{W}_{nn})$ be a collection of n independent column vector-valued random variables $\mathbf{W}_{jn} = (W_{1jn}, \ldots, W_{kjn})^T$ with zero means, $\mathrm{E}W_{hjn} = 0$, *$h = 1, \ldots, k$, and covariance matrices $\mathbb{C}_{jn} = \mathrm{E}\mathbf{W}_{jn}\mathbf{W}_{jn}^T$, $j = 1, \ldots, n$, $\mathbb{C}_{\cdot n} = \sum_{j=1}^n \mathbb{C}_{jn}$. Let $\mathbf{W}_{\cdot n}^{\star 0} := \sum_{j=1}^n (M_{jn}^\star - 1)\mathbf{W}_{jn}$, $n = 1, 2, \ldots$. Suppose that for some positive constant H,*

$$\max_{1 \le j \le n,\, 1 \le h \le k} \mathrm{E} \mid \sqrt{n} W_{hjn} \mid^{2+\delta} < H \ \textit{uniformly in } n.$$

Then

$$\mathbf{W}_{\cdot n} \overset{ad}{\leftrightarrow} \mathbf{N}_k(\mathbf{O}, \mathbb{C}_{\cdot n}), \quad \textit{and} \ (\mathbf{W}_{\cdot n}^{\star 0} \mid \mathbb{W}_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} \mathbf{W}_{\cdot n} \ \textit{as } n \to \infty.$$

The following two examples illustrate the performance of resampling methods in imitation of distributions of supremum and level crossings functionals.

Example 1 Let U_{in} be zero-mean independent and identically distributed Gaussian processes $i = 1, \ldots, n = 1, 2, \ldots$, with the common covariance function $C_n(t) = 1/ne^{-t^2}$. Then the conditions of Theorem 4 hold. Hence, for a large number of summands n, the distribution functions of $\sup_{t \in [0,1]} U_{\cdot n}(t)$ and $\sup_{t \in [0,1]} U_{\cdot n}^{\star 0}(t)$ must be close each other (see Remark 1). Let $\bar{F}_n(s) = \mathrm{P}\{V_n > s\}$, $s \ge 0$, be the (theoretical) tail distribution function of $V_n = \sup_{t \in [0,1]} U_{\cdot n}(t)$, and $\hat{\bar{F}}_{nN}(s)$ be the corresponding empirical tail distribution function based on N simulated values of V_n. Note that here $\bar{F}_n(s)$ does not depend on n. For the simulation of realizations u_{in}, the technique developed in Seleznjev (1996) was used. It is well known that $\hat{\bar{F}}_{nN}(s)$ is an unbiased consistent estimator of $\bar{F}_n(s)$, $s > 0$. In Fig 1, we compare $\hat{\bar{F}}_{nN}(s)$ ($n = 50, N = 1000$, theor1 line) with the corresponding empirical estimators found from four independent resampling experiments. In every resampling experiment, we simulate the initial data array of process realizations only once; then the standard resampling procedure (redrawing with replacement) is applied. Further, for every resampling array, we find the value of the sum and its supremum. For the whole sequence of such

Numbers of crossings	*Frequencies of crossings*			
$N_1(\cdot)$	teor1	teor2	resm1	resm2
$N_1 = 0$	0.845	0.770	0.760	0.754
$N_1 = 1$	0.145	0.215	0.216	0.219
$N_1 = 2$	0.010	0.015	0.024	0.027
$N_1 \ge 3$	0.000	0.000	0.000	0.000

Table 1:

resampling supremuma (the number of realizations in a data array $n = 1000$, the

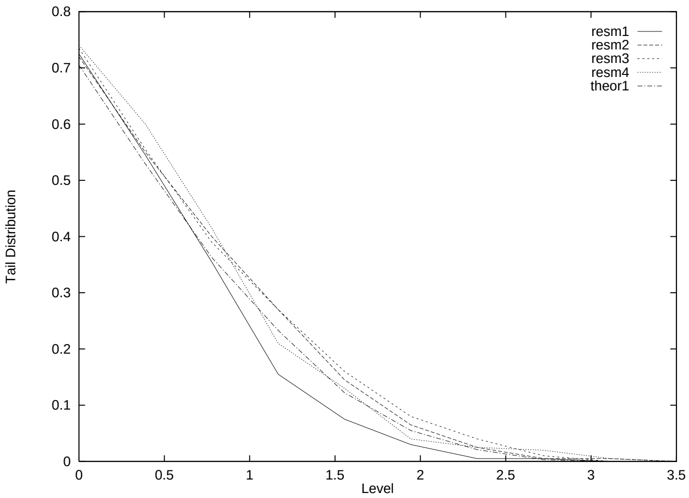

Figure 1: Gaussian processes. Tail distributions for resampling estimates (the number of summands $n = 1000$, the number of simulations $N = 200$, resm1, resm2, resm3, resm4), and theoretical distribution ($n = 50$, $N = 1000$, theor1) of the *sup* functional.

number of resampling arrays $N = 200$), we get the corresponding empirical tail distribution estimate (resm1, resm2, resm3, resm4 lines). Fig 1 demonstrates that the tail distributions of $\sup_{[0,1]} U_{\cdot n}(t)$, and those of centered resampling sum $\sup_{[0,1]} U^{*0}_{\cdot n}(t)$, weakly approach in probability each other along $\{\mathbf{u}_n\}_{n\geq 1}$.

Table 1 shows the approaching of the distribution of the number of crossings of the level 1 by $U_{\cdot n}$, and that of by the centered resampling sum $U^{*0}_{\cdot n}$, where the number of summands $n = 100$. The simulation scheme is the same as for the *sup* functional, but the theoretical distributions are simulated twice with different seed numbers; the number of simulations $N = 1000$. The columns teor1, teor2 show variations of frequencies due to different seed numbers and the numbers of simulations $N = 200$, $N = 1000$, respectively. The number of corresponding resampling arrays (resm1, resm2) $N = 200$.

Example 2 Let U_{jn} be zero-mean independent and identically distributed Shot effect processes (see e.g., Doob (1953)) $i = 1, \ldots,\ n = 1, 2, \ldots,$

$$n^{1/2} U_{jn}(t) = \sum_{i=1}^{\infty} R_{ijn} Q_{ijn} f(t - T_{ijn}),$$

where T_{ijn} are times of "events" of the corresponding Poisson process with an intensity μ; $f(t) = \exp\{-\max(t, 0)\} I_{\{t>0\}}$, R_{ijn} and Q_{ijn} are independent random variables with the uniform distribution on $[-1/2, 1/2]$, and the exponential distribution $F_\eta(x) = 1 - e^{-bx}$, respectively. A Shot effect process $U(t)$ can be represented in the following form

$$U(t) = \int_{-\infty}^{\infty} f(t-s)dV(s),$$

where $V(\cdot)$ is a random process with piecewise-constant realizations between the events. At the moment of an event T_i, the process $V(\cdot)$ increases (or decreases) by R_iQ_i. The random process $V(s)$ has stationary and independent increments, $\mathrm{E}[V(t+s) - V(t)] = 0$, and $\mathrm{E}(V(t+s) - V(t))^2 = \mu\gamma s$, where $\gamma = \mathrm{E}[R_1^2 Q_1^2]$. Therefore, $U(t)$ is a stationary zero-mean random process,

$$\mathrm{E}U(t)^2 = \mu\gamma \int_0^{\infty} f(v)^2 dv,$$
$$\mathrm{E}(U(t+s) - U(t))^2 = \mu\gamma \int_{-\infty}^{\infty} (f(t+s-v) - f(t-v))^2 dv.$$

The increment $U(t+s) - U(t)$ can be written in the following form,

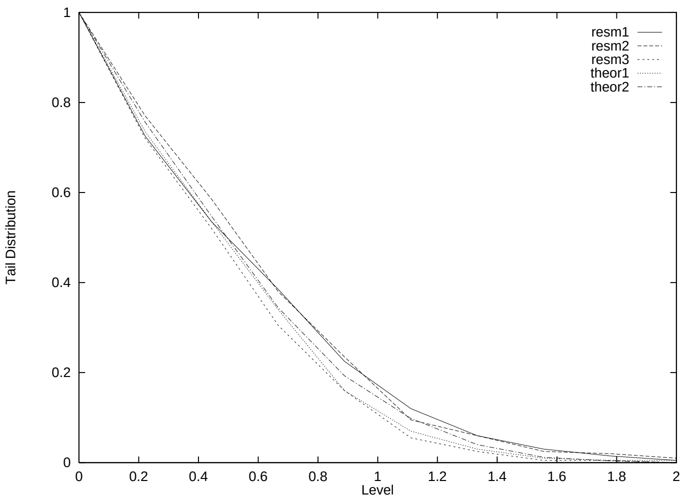

Figure 2: Shot effect processes. The tail distributions for resampling estimates (the number of summands $n = 200$, the number of simulations $N = 200$, resm1, resm2, resm3), and the estimators of theoretical tail distributions ($n = 200$, $N_1 = 200$, theor1 and $N_2 = 1000$, theor2) of the *sup* functional.

$$U(t+s) - U(t) = (e^{-s} - 1)W(t) + W_s(t), \tag{9}$$

where $W(t) = \int_{-\infty}^0 e^v dV(v+t)$ and $W_s(t) = \int_0^s e^{-(s-v)} dV(v+t)$ are independent random processes. By applying (9), we obtain for every $t_1 \le t \le t_2$ that

$$\begin{aligned} &\mathrm{E}(U(t_2) - U(t_1))^2 \le C_1 |t_2 - t_1|, \\ &\mathrm{E}[(U(t_2) - U(t))^2 (U(t) - U(t_1))^2] \le C_2 |t_2 - t_1|^2, \end{aligned} \tag{10}$$

for some positive constants C_1 and C_2. Then, by using Theorem 5, Remark 2(ii) and (10) we get

$$(\sup_{[0,1]} U^{*0}_{\cdot n}(t) | \mathbf{U}_n) \overset{ad(\mathrm{P})}{\longleftrightarrow} \sup_{[0,1]} (U_{\cdot n}(t)).$$

Fig 2 demonstrates that the tail distribution of $\sup_{[0,1]} U_{\cdot n}(t)$, and that of the centered resampling sum $\sup_{[0,1]} U^{*0}_{\cdot n}(t)$, weakly approach each other. The simulation scheme is the same as for the sup functional for Gaussian processes (Example 1). The number of summands is $n = 200$ for theoretical and resampling estimates; the numbers of simulated sums of realizations are $N = 200$ for theor1 line, and $N = 1000$ for theor2 line; the number of resamplings $N = 200$.

4 Proofs

Sequences of random processes approaching each other.

Proof of Proposition 1. The assertion follows by the definition of uniform tightness and Markov's inequality. In fact, let $\epsilon = \bar{\epsilon}\delta$ in (3). Then (3) and Markov's inequality imply that for every $\bar{\epsilon}$ and δ there exists a compact K_ϵ such that

$$\mathrm{P}\{\mathrm{P}\{X_n \notin K_\epsilon | Z_n\} > \bar{\epsilon}\} \le \mathrm{E}[\mathrm{P}\{X_n \notin K_\epsilon | Z_n\}]/\bar{\epsilon} = \mathrm{P}\{X_n \notin K_\epsilon\}/\bar{\epsilon} < \delta.$$

□

Proof of Theorem 1. The proof is similar to that of conventional Continuous Mapping Theorem (see e.g., Billingsley (1968)). For any $f(\cdot) \in \mathbf{C}_b(\mathbf{R}^1)$, the superposition $f(h(\cdot)) \in \mathbf{C}_b(\mathbf{S})$. Hence, by definition for any $f(\cdot) \in \mathbf{C}_b(\mathbf{R}^1)$, we have

$$\mathrm{E}f(h(X_n)) - \mathrm{E}f(h(Y_n)) = o(1),$$

and the assertion (i) follows. Clearly, the assertion (ii) follows by using a similar argument. □

Proof of Theorem 2. For every $f \in \mathbf{C}^k[0,1]$ and $m = 2, 3, \ldots$, we construct a linear approximation function $\bar{s}_m(f)$ based on a standard interpolation spline function of smoothness k, with equidistant interpolation knots (see Nürnberger (1989)). The main idea of the proof is to consider the approximations of X_n and

Y_n, $\bar{s}_m(X_n)$ and $\bar{s}_m(Y_n)$, respectively, instead of X_n and Y_n. Then the asymptotic behaviors of the distributions of $\bar{s}_m(X_n)$ and $\bar{s}_m(Y_n)$ are defined by the marginals of X_n and Y_n. This implies that the assertion in Theorem 2 is correct. Now we come to the detailed presentation of the proof.

For every $f(\cdot) \in \mathbf{C}^k[0,1]$ with given $f(t_i), f^{(1)}(t_i), ..., f^{(k)}(t_i),\ i = 0, ..., m,\ 0 \leq t_0 < ... < t_m \leq 1$, there exists a spline function $s_m(t)$ such that

$$||f - s_m(f)||_u \leq C_k w(f^{(k)}, 1/m), \tag{11}$$

where a positive constant C_k depends only on k (see e.g., Nürnberger (1989)). Define our approximation function $\bar{s}(\cdot)$ in two stages: first, the values of the derivatives of $f(\cdot)$ are approximated pointwise with the given accuracy (11); then the corresponding spline-function ultimately is used with approximated derivative values. The values of derivatives can be estimated by divided diffferences (or by conventional splines, see e.g., Micchelli and Wahba (1981)) defined on a finer (equidistant) trial grid of points. Therefore, we obtain

$$||f - \bar{s}_m(f)||_u \leq \bar{C}_k w(f^{(k)}, 1/m), \tag{12}$$

with some positive constant $\bar{C}_k$. Note that for the most frequently-used case $k \leq 4$, one can omit the first stage and take the conventional interpolation cubic spline with the equidistant knots.

First, we show that for every $c(\cdot) \in \mathbf{C}_b(\mathbf{S})$

$$\mathrm{E}c(Y_n) - \mathrm{E}c(\bar{s}_m(Y_n)) = \mathrm{o}(1) \text{ as } m \to \infty \text{ uniformly in } n. \tag{13}$$

Uniform tightness of Y_n implies that, for every positive ϵ, there exists a compact $K_\epsilon \subset \mathbf{C}^k[0,1]$ such that

$$\mathrm{P}\{Y_n \notin K_\epsilon\} < \epsilon \text{ uniformly in } n. \tag{14}$$

It follows from the uniform continuity of $c(\cdot)$ on the compact K_ϵ, that for every $\eta > 0$, there exists $\delta > 0$ such that

$$|c(f) - c(g)| < \eta \quad \text{whenever } ||f - g||_u < \delta, \quad f, g \in K_\epsilon. \tag{15}$$

Then, the uniform tightness of $\{Y_n\}$ yields for every positive δ

$$\{Y_n \in K_\epsilon\} \subseteq \{w(Y_n^{(k)}, 1/m) \leq \delta\}, \tag{16}$$

for sufficiently large m and all n (see Seleznjev (1986), Wilson (1986), Rusakov and Seleznjev (1987)). Denote by I_1 the indicator of the event $\{Y_n \in K_\epsilon\}$, and write $\bar{I}_1 = 1 - I_1$. We have

$$\mathrm{E}[c(Y_n) - c(\bar{s}_m(Y_n))] = J_1 + J_2,$$

where $J_1 = \mathrm{E}[(c(Y_n) - c(\bar{s}_m(Y_n))I_1]$, and $J_2 = \mathrm{E}[(c(Y_n) - c(\bar{s}_m(Y_n))\bar{I}_1]$. It follows from (14) and the boundness of $c(\cdot)$, $\sup_y \mid c(y) \mid \leq c_2 < \infty$, that

$$|J_2| \leq c_2 \mathrm{P}\{Y_n \notin K_\epsilon\} \leq c_2 \epsilon \quad \text{uniformly in } n. \tag{17}$$

Combining (15), (12), and (16) we get

$$|J_1| \leq \eta \mathrm{P}\{w(Y_n^{(k)}, 1/m) \leq \delta\} \leq \eta, \tag{18}$$

for sufficiently large m and all n. Suppose $\epsilon = \epsilon(m) \to 0$ as $m \to \infty$. Thus, (17) and (18) together give (13).

For $\{(X_n|Z_n)\}_{n\geq 1}$ we use a similar argument. Proposition 1 yields the conditional uniform tightness of $\{(X_n|Z_n)\}_{n\geq 1}$ along Z_n, i.e.,

$$\mathrm{P}\{X_n \notin K_\epsilon | Z_n\} = \mathrm{o}_p(1) \text{ as } \epsilon \to 0 \text{ uniformly in } n. \tag{19}$$

Further, denote by I_2 the indicator of the event $\{X_n \in K_\epsilon\}$ and write $\bar{I}_2 = 1 - I_2$,

$$\mathrm{E}[(c(X_n) - c(\bar{s}_m(X_n)))|Z_n] = \mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 = \mathrm{E}[(c(X_n) - c(\bar{s}_m(X_n)))I_2|Z_n], \quad \mathcal{J}_2 = \mathrm{E}[(c(X_n) - c(\bar{s}_m(X_n)))I_2|Z_n].$$

Clearly, we have again from (19) (cf. (17))

$$\mathcal{J}_2 = \mathrm{o}_p(1) \text{ as } \epsilon \to 0 \text{ uniformly in } n, \tag{20}$$

and also

$$|\mathcal{J}_1| \leq \eta \mathrm{P}\{w(X_n^{(k)}, 1/m) \leq \delta | Z_n\} \leq \eta \qquad \mathrm{P} - \text{a.s.}, \tag{21}$$

for sufficiently large m and all n. Thus, (20) and (21) together give

$$\mathrm{E}[c(X_n)|Z_n] - \mathrm{E}[c(\bar{s}_m(X_n))|Z_n] = \mathrm{o}_p(1) \text{ as } m \to \infty \text{ uniformly in } n. \tag{22}$$

Furthermore, for every $f \in \mathbf{C}^k[0,1]$, denote by $h(f(t_0), \ldots, f(t_m)) = c(\bar{s}_m(f))$. Then h is a continuous and bounded real-valued function, $h \in \mathbf{C}_b(\mathbf{R}^{m+1})$. By assumption, the marginals of X_n, conditionally on Z_n, weakly approach the marginals of Y_n along Z_n in probability, and therefore for every m,

$$\begin{aligned} & \mathrm{E}[c(\bar{s}_m(X_n))|Z_n] - \mathrm{E}[c(\bar{s}_m(Y_n))] \\ = \; & \mathrm{E}[h(X_n(t_0), \ldots, X_n(t_m))|Z_n] - \mathrm{E}[h(Y_n(t_0), \ldots, Y_n(t_m))] \\ = \; & \mathrm{o}_p(1) \text{ as } n \to \infty. \end{aligned} \tag{23}$$

Finally, we have

$$\begin{aligned} & \mathrm{E}[c(X_n)|Z_n] - \mathrm{E}[c(Y_n)] = \mathrm{E}[(c(X_n) - c(\bar{s}_m(X_n)))|Z_n] \\ + \; & (\mathrm{E}[c(\bar{s}_m(X_n))|Z_n] - \mathrm{E}c(\bar{s}_m(Y_n))) + (\mathrm{E}c(Y_n) - \mathrm{E}c(\bar{s}_m(Y_n))), \end{aligned}$$

where the first and the latter summands are small enough for sufficiently large m uniformly in n by (13) and (22). For a given m, the second summand tends to zero in probability by (23). This completes the proof. $\Box$

Proof of Theorem 3. The proof repeats that of Theorem 2. For every $f(\cdot) \in \mathbf{D}[0,1]$, instead of a spline approximation, we use an approximation by piecewise-constant function $h_m(t)$ with the same interpolation knots $t_i = i/m, i = 0, \ldots, m$, and the following inequality (see Gikhman and Skorohod (1979), Sec. 6.5),

$$d(f(\cdot), h_m(f)) \le 1/m + 4\Delta(f(\cdot), 2/m). \tag{24}$$

It is also well known that a sequence of random $\mathbf{D}[0,1]$-elements $\{D_n\}_{n\ge1}$ is uniformly tight in $\mathbf{D}[0,1]$ iff

$$\mathrm{P}\{\Delta(D_n, 1/m) > \delta\} = \mathrm{o}(1); \quad \mathrm{P}\{\sup_{[0,1]} |D_n| > m\} = \mathrm{o}(1) \text{ as } m \to \infty \tag{25}$$

uniformly in n. Then, as in Theorem 2, we use the sequences $h_m(X_n)$ and $h_m(Y_n)$ and the corresponding weak approaching of the marginals along Z_n, instead of the sequences X_n and Y_n. This completes the proof. $\Box$

Resampling for a sequence of random processes

Proof of Theorem 4. First, we verify that $\{U^{*0}_{\cdot n}\}_{n\ge1}$ and $\{U_{\cdot n}\}_{n\ge1}$ are uniformly tight. Then, Theorem 2 and CLRT for random vectors (Proposition 2) imply the assertion.

To simplify the notation, let $V_n := U^{*0}_{\cdot n}$. Random elements $U_{jn}, j = 1, \ldots, n$, are independent, and therefore, by taking the expectation on $M^{\star}_{jn}$ and on $U^{(k)}_{jn}$, we have

$$\begin{aligned}\mathrm{E}(V^{(k)}_n(t) \;-\; V^{(k)}_n(s))^2 &= \mathrm{E}\left(\sum_{j=1}^{n}(M^{\star}_{jn}-1)^2(U^{(k)}_{jn}(t)-U^{(k)}_{jn}(s))^2\right)\\ &= \;(1-1/n)\sum_{j=1}^{n}\mathrm{E}(U^{(k)}_{jn}(t)-U^{(k)}_{jn}(s))^2 \le H|t-s|^{\beta}\end{aligned} \tag{26}$$

uniformly in n. In a similar way, we obtain

$$\mathrm{E}(U^{(k)}_{\cdot n}(t) - U^{(k)}_{\cdot n}(s))^2 \le H|t-s|^{\beta} \text{ uniformly in } n. \tag{27}$$

Further, by assumption we have

$$\begin{aligned}\mathrm{E}V^{(i)}_n(0)^2 &= \mathrm{E}\left(\sum_{j=1}^{n}(M^{\star}_{jn}-1)^2(U^{(i)}_{jn}(0))^2\right) \le C_1,\\ \mathrm{E}(U^{(i)}_{\cdot n}(t))^2 &= \mathrm{E}\sum_{j=1}^{n}(U^{(i)}_{jn}(t))^2 \le C_2\end{aligned} \tag{28}$$

uniformly in n for some positive constants C_1 and C_2, and $i = 0, 1, ..., k$. It follows from (26)-(28), and Theorem 1 in Rusakov and Seleznjev (1987) (see also Seleznjev (1986), Wilson (1986)), that $\{V_n\}$ and $\{U_{\cdot n}\}$ are uniformly tight in $\mathbf{C}^k[0,1]$. It is sufficient to investigate the asymptotic behavior of $(\bar{s}_m(V_n)|\mathbf{U}_n)$ and $\bar{s}_m(U_{\cdot n})$ for a given m along $\mathbf{U}_n$. Since $(M^\star_{1n}, \dots, M^\star_{nn})$ and $\mathbf{U}_n$ are independent, we get

$$\mathcal{L}(\bar{s}_m(V_{\cdot n})|\mathbf{U}_n) = \mathcal{L}(\bar{s}_m(V_{\cdot n})|U_{jn}(t_i), j = 1, \dots, n, i = 1, \dots, m). \qquad (29)$$

Finally, CLRT for random vectors (Proposition 2) yields

$$(\bar{s}_m(V_n)|U_{jn}(t_i), j = 1, \dots, n, i = 1, \dots, m) \overset{ad(\mathrm{P})}{\longleftrightarrow} \bar{s}_m(U_{\cdot n})$$

and the assertion follows by applying the same arguments as in Theorem 2. □

Proof of Theorem 5. The proof is similar to that of Theorem 4. First, we prove that $\{U^{\star 0}_{\cdot n}\}$ and $\{U_{\cdot n}\}$ are uniformly tight in $\mathbf{D}[0,1]$. Denote by the $\Delta_{jn}(t,s) := U_{jn}(t) - U_{jn}(s)$.

The random elements $U_{1n}, \dots, U_{nn}$ are independent. Therefore, we have

$$\mathrm{E}(U^{\star 0}_{\cdot n}(t_2) - U^{\star 0}_{\cdot n}(t))^2(U^{\star 0}_{\cdot n}(t) - U^{\star 0}_{\cdot n}(t_1))^2 = S_{1n}(t_1,t,t_2) + S_{2n}(t_1,t,t_2).$$

where

$$S_{1n}(t_1,t,t_2) = \mathrm{E}\left(\sum_{j=1}^{n}(M^\star_{jn}-1)^2\Delta_{jn}(t_2,t)^2\sum_{i=1}^{n}(M^\star_{in}-1)^2\Delta_{in}(t,t_1)^2\right),$$

$$S_{2n}(t_1,t,t_2) =$$

$$\mathrm{E}\left(\sum_{i\neq j}(M^\star_{jn}-1)^2(M^\star_{in}-1)^2\Delta_{jn}(t_2,t)\Delta_{jn}(t,t_1)\Delta_{in}(t_2,t)\Delta_{in}(t,t_1)\right).$$

Since $\mathrm{E}(M^\star_{jn}-1)^2(M^\star_{in}-1)^2 \le 5/2,\ i \neq j$, and $\mathrm{E}(M^\star_{in}-1)^4 \le 4$, by assumption we obtain

$$\begin{aligned} S_{1n}(t_1,t,t_2) &\le 4n\max_j \mathrm{E}\Delta_{jn}(t_2,t)^2\Delta_{jn}(t,t_1)^2 \\ &+ \frac{5}{2}n(n-1)(\max_j \mathrm{E}\Delta_{jn}(t_2,t)^2)^2 \le C_1|t_2-t_1|^\beta, \end{aligned} \qquad (30)$$

for some positive C_1. In the similar way we find

$$\begin{aligned} S_{2n}(t_1,t,t_2) &\le \frac{5}{2}n(n-1)\max_j(\mathrm{E}|\Delta_{in}(t_2,t)\Delta_{in}(t,t_1)|)^2 \\ &\le \frac{5}{2}n(n-1)\max_j \mathrm{E}\Delta_{jn}(t_2,t)^2\Delta_{jn}(t,t_1)^2 \le C_2|t_2-t_1|^\beta, \end{aligned} \qquad (31)$$

for some positive C_2. Further, we get

$$\mathrm{E}U_{\cdot n}^{\star 0}(0)^2 = \sum_{j=1}^{n} \mathrm{E}(M_{jn}^{\star} - 1)^2 \mathrm{E}U_{jn}(0)^2 \le C_3, \tag{32}$$

for some positive C_3. Now the uniform tightness follows by (30)-(32) and Remark 2(i). For $\{U_{\cdot n}\}_{n\ge 1}$ the argument is similar. Finally, the assertion of the theorem follows from Proposition 2, and from the same arguments as in the proofs of Theorem 2 and Theorem 3. □

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