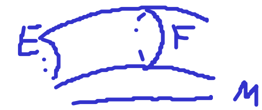
15 Ehresmann and Koszul connections

With the help of ~~this~~ statement we ~~can~~ give the following definition. 12.12.2022



Definition 15.1. Let $\pi : E \to M$ be a smooth fiber bundle with typical fiber F of dimension k. Denote $\mathcal{V}_yE := (d\pi_y)^{-1}(0_p)$, where $\pi(y) = p$. The *vertical bundle* on $\pi : E \to M$ is the real vector bundle $\pi_{\mathcal{V}} : \mathcal{V}E \to E$ with total space

$$\mathcal{V}E := \cancel{\bigcap}\bigcup_{y\in E}\mathcal{V}_yE \subset TE$$

and projection map $\pi_{\mathcal{V}} := \pi_{TE}|_{\mathcal{V}E}$. A vector bundle atlas on $\mathcal{V}E$ is given by charts of the form

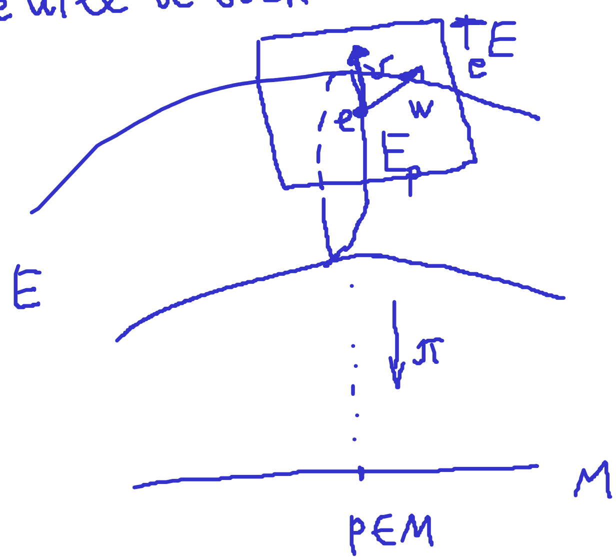
$$"(\pi_{\mathcal{V}}, d\varphi \circ d\Phi)" : \pi_{\mathcal{V}}^{-1}(\pi^{-1}(U) \cap \Phi^{-1}(V)) \to (\pi^{-1}(U) \cap \Phi^{-1}(V)) \times \mathbb{R}^k,$$

where (π, Φ) is a bundle chart on E over U and (V, φ) is a chart in F.

$$"\ldots" \leftrightarrow \Phi : E|_U \to U \times F, \quad U \xleftarrow{p_1} U\times F \xrightarrow{p_2} F, \quad p_2 \circ \Phi$$

Problem 15.2. Verify this. Home

We will be back with another way.



$\vec{v}$ is tangent to E_p

$\vec{w}$ is not...

$\Updownarrow$

$(d\pi)(\vec{v}) = 0$

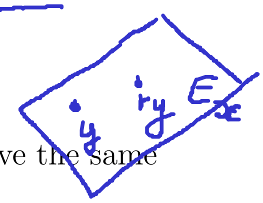
Definition 15.3. A smooth rank k *distribution* on an n-manifold M is a (smooth) rank k vector subbundle ~~$E \to M$~~ of the tangent bundle.

Definition 15.4. A (linear Ehresmann) *connection* on a vector bundle $\pi : E \to M$ is a smooth distribution $\mathcal{H}$ on the total space E such that

1) $\mathcal{H}$ is complementary to the vertical bundle: $TE = \mathcal{H} \oplus \mathcal{V}E$; $\Leftrightarrow$ They are subbundles and for $\forall y$ $(TE)_y = \mathcal{H}_y \oplus (\mathcal{V}E)_y$

2) $\mathcal{H}$ is homogeneous: $d(\mu_r)_y(\mathcal{H}_y) = \mathcal{H}_{ry}$ for all $y \in E$, $r \in \mathbb{R}$, where $\mu_r : E \to E$ is the multiplication map given by $\mu_r : y \to ry$.

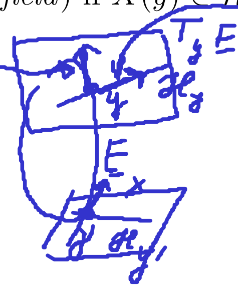
The subbundle $\mathcal{H}$ is called the *horizontal distribution* (or *horizontal subbundle*).



Definition 15.5. For a general bundle (not necessarily a vector bundle), we have the same definition, but only with the property 1).

$E_x \cong F$

Definition 15.6. For $y \in E$, an individual element $w \in T_yE$ is *horizontal* if $w \in \mathcal{H}_y$ and *vertical* if $w \in \mathcal{V}_yE$. A vector field (i.e. a section) $X \in \mathbb{X}(E)$ is said to be a *horizontal vector field* (resp. *vertical vector field*) if $X(y) \in \mathcal{H}_y$ (resp. $X(y) \in \mathcal{V}_yE$) for all $y \in E$.



Problem 15.7. Let $f : N \to M$ be a smooth map and $\pi : E \to M$ a fiber bundle. Prove that the pull-back (Definition 9.48) can be naturally identified with $\{(p, e) \in N \times E \colon f(p) = \pi(e)\}$ Home

f^*E

Problem 15.8. Let $f : N \to M$ be a smooth map and $\pi : E \to M$ a fiber bundle with typical fiber F. Prove that $\mathcal{V}f^*E \to f^*E$ is bundle isomorphic to $\widetilde{f}^*\mathcal{V}E \to f^*E$, where $\widetilde{f} := pr_2|_{f^*E} : f^*E \to E$, $pr_2 : N \times E \to E$ and $f^*E = \{(p, e) \in N \times E \colon f(p) = \pi(e)\}$ (cf. the previous problem). See the diagram: Home

$$\begin{tikzcd}
\mathcal{V}f^*E \arrow[dr, "p"'] & \widetilde{f}^*\mathcal{V}E \arrow[r] \arrow[d] & \mathcal{V}E \arrow[d, "\pi_{\mathcal{V}}"] \\
 & f^*E \arrow[r, "\widetilde{f}"] \arrow[d] & E \arrow[d, "\pi"] \\
 & N \arrow[r, "f"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{V}E \cong \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
E \arrow[r, "\pi"] & M
\end{tikzcd}$$

Proposition 15.9. *The vertical vector bundle $\mathcal{V}E$ is isomorphic to the vector bundle π^*E (as bundles over E). Sometimes they say that $\mathcal{V}E$ is isomorphic to E along π.*

Proof. If $(v, w) \in \pi^*E = \{(p, e) \in E \times E \colon \overset{f(p)}{\pi(p)} = \pi(e)\}$, i.e. $\pi(v) = \pi(w)$, or $v, w \in E_p$ for some p, then $\pi(v + tw)$ is constant in t. Thus we can define a map from $\underline{\pi^*E}$ to $\underline{TE}$ by $(v, w) \mapsto \frac{d}{dt}\big|_0 (v + tw)$. This map evidently maps into $\mathcal{V}E \subset TE$. We obtain a vector bundle isomorphism

$$\lim_{t\to 0} \frac{v-(v+tw)}{t} = \lim \frac{tw}{t}$$

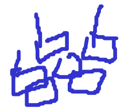
$$\mathbf{j} : \pi^*E \cong \mathcal{V}E, \qquad \mathbf{j} : \underset{\in E_p}{(v, w)} \mapsto \mathbf{j}_v w := \frac{d}{dt}\bigg|_0 (v + tw) = \underset{\in TE_p}{\underline{\underline{w_v}}}.$$

"identification of a vector space F and TF"

□

Problem 15.10. Prove that $\mathbf{j}$ is an isomorphism, i.e. surjective and injective. Home

Problem 15.11. Prove that $\mathcal{H} \cong \pi^*TM$. Home



Class **Problem 15.12.** Let $E \to M$ be a a vector bundle. Suppose that for each $p \in M$ there is a subspace $E'_p \subset E_p$. Then $E' = \cup_{p \in M} E'_p$ is the total space of rank l vector subbundle if and only if for each $p \in M$, there is an open neighborhood U of p on which smooth sections $\sigma_1, \ldots, \sigma_2$ are defined such that for each $q \in U$ the set $\{\sigma_1(q), \ldots \sigma_l(q)\}$ is a basis of E'_q.

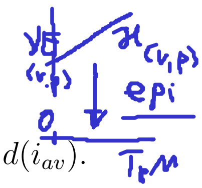
Theorem 15.13. *Every vector bundle admits a connection.*

Proof. For a trivial bundle $pr_1 : M \times V \to M$ and a fixed $v \in V$ define $i_v : M \to M \times V$ by $i_v(p) := (p, v)$. For each $p \in M$, define $\mathcal{H}_{(p,v)} := d(i_v)_p(T_pM)$. Evidently these maps are linear injections smoothly depending on p. Then one can apply the previous problem to obtain that the subspaces $\mathcal{H}_{(p,v)}$ form a subbundle $\mathcal{H}$ of TE. Also,

$\dim \mathcal{H}_{(p,v)} = \dim T_pM$ $\oplus$

$$d(pr_1)(\mathcal{H}_{(p,v)}) = d(pr_1)d(i_v)_p(T_pM) = d(pr_1 \circ i_v)_p(T_pM) = d(\mathrm{Id})_p(T_pM) = T_pM$$

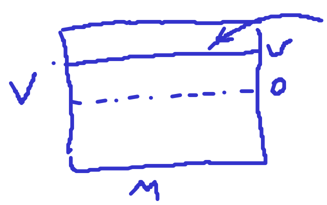
$\Rightarrow TE = \mathcal{V} + \mathcal{H}$



and hence $TE = \mathcal{V} \oplus \mathcal{H}$. For any $a \in \mathbb{R}$ we have $\mu_a \circ i_v = i_{av}$ and $d(\mu_a) \circ d(i_v) = d(i_{av})$. Thus

$\mu_a \circ i_v(p) = \mu_a(p,v) = (p, av) = i_{av}$ dim $\Rightarrow$ iso.

$$d(\mu_a)(\mathcal{H}_{(p,v)}) = d(\mu_a)(d(i_v)(T_pM)) = d(i_{av})(T_pM) = \mathcal{H}_{(p,av)} = \cancel{\mathcal{H}_{(ap,av)}} = \mathcal{H}_{a(p,v)}.$$



graph of i_v

by the first half of the proof

Consider a general vector bundle $\pi : E \to M$ with a trivializing locally finite cover $\{U_\alpha\}$ of M. Choose a connection $\mathcal{H}^\alpha$ on each $\pi^{-1}(U_\alpha)$. Let $\{\rho_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. For each $y \in E$, define

$$L_y : T_{\pi(y)}M \to T_yE, \qquad L_y(v) := \sum_{\{\alpha :\, \pi(y) \in U_\alpha\}} \rho_\alpha(\pi(y)) w_\alpha,$$

where w_α is the unique vector in $\mathcal{H}^\alpha$ such that $(d\pi)w_a = v$. Evidently L_y is linear and $(d\pi)_y \circ L_y = \mathrm{Id}_{T_pM}$. This implies (using Problem 15.12) that $y \mapsto L_y(T_{\pi(y)}M)$ determines a subbundle $\mathcal{H}$ with the property 1). $\square$

Problem 15.14. Verify the property 2). Home

Problem 15.15. Prove the above statement using a Riemannian metric (to be constructed first) and the orthogonal complement. Home

fiberwise

in T_yE $\quad (\mathcal{V}E_y)^\perp =: \mathcal{H}_y$

Definition 15.16. For a smooth fiber bundle $\pi : E \to M$ and a smooth map $f : N \to M$, we call a map $\sigma : N \to E$ a *section of E along f* if $\pi \circ \sigma = f$.

If $\sigma : N \to E$ is a section of E along f, then $\sigma' : N \to f^*E$, $p \mapsto (p, \sigma(p)) \in N \times E$, is a section of the pull-back f^*E.

Home **Problem 15.17.** Prove that all sections of f^*E are of this form.

Definition 15.18. Let $\sigma : N \to E$ be a section of E along a map $f : N \to M$. We say that σ is a *parallel section* if $(d\sigma)v$ is horizontal for all $v \in TN$. If s is a section of E and $\gamma : [a, b] \to E$ is a curve, then we say that s is *parallel along* γ if $s \circ \gamma$ is parallel.

Home **Problem 15.19.** Prove that if s is parallel with respect to the pull-back connection on f^*E, then σ_s is parallel, where $\sigma_s : N \to E$, $\sigma_s(x) = s(x) \in E_{f(x)} = (f^*E)_x$.

$$\begin{tikzcd} f^*E \arrow[d] & & E \arrow[d, "\pi"] \\ N \arrow[u, bend left, "\sigma'"] \arrow[urr, "\sigma"] \arrow[rr, "f"] & & M \end{tikzcd}$$

at each $y \in \pi^{-1}(p) = E_p$ we have a unique horizontal vector $\tilde{v}$ s.t. $\boxed{(d\pi)\tilde{v} = v}$! $v \in T_pM$

Problem 15.20. Let $[0, b]$ be an interval and let $t \in [0, b]$. Suppose that $\pi : E \to [0, b]$ is a vector bundle with some connection. Let $\widetilde{\partial}$ denote the horizontal lift of $\frac{\partial}{\partial t}$. Class

1) For an integral curve $\gamma : [0, a] \to E$ of $\widetilde{\partial}$, show that $\pi \circ \gamma$ is an integral curve of $\frac{\partial}{\partial t}$. Deduce that $\gamma(a) \in E_a$.

2) Prove that for any $t_0 < b$ there exists $\varepsilon = \varepsilon(t_0) > 0$ such that all integral curves of $\widetilde{\partial}$ originating in the fiber E_{t_0} are defined at least on $[t_0, \varepsilon)$.

3) Then 1) and 2) imply that all integral curves of $\widetilde{\partial}$ have domain $[0, b]$.

0 b

The following theorem does not work in the general situation, but for curves this works fortunately.

Theorem 15.21. *Suppose that $\pi : E \to M$ is a vector bundle with a connection $\mathcal{H}$ and $\gamma : [a, b] \to M$ is a smooth curve. Then for each $u \in E_{\gamma(a)}$ there is a unique parallel section $\sigma_{\gamma,u}$ along γ such that $\sigma_{\gamma,u}(a) = u$. Also, the map $P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$, $P_\gamma(u) = \sigma_{\gamma,u}(b)$, is a linear isomorphism.*

$\gamma^* E \to [0, b]$

Proof. One may assume $a = 0$ and apply Problem 15.20 with $\gamma^* E$ instead of E. We obtain an integral curve γ_u of $\widetilde{\partial}$ in $\gamma^* E$ with $\gamma_u(0) = (0, u) \in \gamma^* E$ defined on $[0, b]$. By 1) in Problem 15.20, $pr_1 \circ \gamma_u$ is an integral curve of $\frac{\partial}{\partial t}$ and $pr_1 \circ \gamma_u(t) = t$. Let $\sigma_{\gamma,u} := pr_2 \circ \gamma_u$ on $[0, b]$. Then $\sigma_{\gamma,u}$ is a parallel section of $E \to M$ along γ because $\dot{\gamma}_u$ is horizontal (cf. Problem 15.19). It is unique as an integral curve (Cauchy problem for ODE).

Now prove that the above defined P_γ is linear. First, note that $(r\sigma_{\gamma,u})^{\dot{}} = d(\mu_r) \circ \dot{\sigma}_{\gamma,u}$ is horizontal, because $d(\mu_r)$ preserves $\mathcal{H}$. Then $r\sigma_{\gamma,u}$ is parallel and $P_\gamma(ru) = rP_\gamma(u)$. So, P_γ is homogeneous. Now prove that $P_\gamma = \mathbf{j}_0^{-1} \circ d(P_\gamma) \circ \mathbf{j}_0$ (see the proof of Proposition 15.9 for a similar definition), i.e. a composition of linear maps. For $v_0 \in T_0 E_{\gamma(0)}$, define $\omega(t) = tv$ such that $v_0 = \dot{\omega}(0)$ for an appropriate $v \in E_{\gamma(0)}$. This means that v is v_0 under "an appropriate identification". More precisely,

$$\mathbf{j}_0(v) = \left.\frac{d}{dt}\right|_0 (0 + tv) = v_0, \qquad v = \mathbf{j}_0^{-1}(v_0).$$

By the (third) definition of the tangent map,

$$(dP_\gamma)_0 v_0 = \left.\frac{d}{dt}\right|_0 (P_\gamma \circ \omega).$$

Since $P_\gamma \circ \omega(t) = P_\gamma(tv) = tP_\gamma(v)$ (using the homogeneity proved first), we have

$$(dP_\gamma)_0 v_0 = \mathbf{j}_0(P_\gamma(v)) = \mathbf{j}_0 \circ P_\gamma \circ \mathbf{j}_0^{-1} v_0$$

and $P_\gamma = \mathbf{j}_0^{-1} \circ dP_\gamma \circ \mathbf{j}_0$ is linear.

Finally, evidently P_γ has the inverse P_{γ^-}, where $\gamma^-(t) := \gamma(b-t)$, so it is a linear isomorphism. $\square$

Home **Problem 15.22.** Verify that P_{γ^-} is the inverse to P_γ.

Definition 15.23. The map P_γ from the previous theorem is called *parallel translation* or *parallel transport* along γ from $\gamma(a)$ to $\gamma(b)$. For $t_1, t_2 \in [a, b]$, let $P(\gamma)_{t_1}^{t_2} := P_{\gamma|[t_1,t_2]} : E_{\gamma(t_1)} \to E_{\gamma(t_2)}$ if $t_2 \geq t_1$ and $P(\gamma)_{t_1}^{t_2} := P^{-1}_{\gamma|[t_2,t_1]} : E_{\gamma(t_1)} \to E_{\gamma(t_2)}$ if $t_1 \geq t_2$.

The curve $\sigma_{\gamma,u}$ is a *parallel lift* or *horizontal lift* of the curve γ.

A parallel transport along a piece-wise smooth curve is defined by stages as a composition.

Denote the vector bundle isomorphism from $\mathcal{V}E$ to E along π by $\mathbf{p}$, i.e. $\mathbf{p} : \mathcal{V}E \to E$ is the composition in the upper row of diagram (cf. Proposition 15.9):

$$\begin{tikzcd}
\mathcal{V}E \arrow[r, "\mathbf{j}^{-1}"] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& E \arrow[r, "\pi"] & M.
\end{tikzcd}$$

In the notation of Proposition 15.9 $\mathbf{p} : w_y \mapsto w$ and for each y, it gives the canonical identification of T_yE_p with E_p, and on each fiber, it is the inverse of $\mathbf{j}$. If we have a connection on $\pi : E \to M$, then we have an associated *connector*, which is the map $\kappa : TE \to E$ defined by

$$\kappa(v) := \mathbf{p}(p_{\mathcal{V}}(v)) = \mathbf{j}_y^{-1}(p_{\mathcal{V}}(v)),$$

where $v \in T_yE$ and $p_{\mathcal{V}} : TE = \mathcal{V} \overset{E}{\oplus} \mathcal{H} \to \mathcal{V}$ is the canonical projection. It is a vector bundle homomorphism along $\pi : E \to M$:

$$\begin{tikzcd}
TE \arrow[r, "p_{\mathcal{V}}"] \arrow[rrrr, bend left, "\kappa"] \arrow[drr] & \mathcal{V}E \arrow[r, "\mathbf{j}^{-1}"] \arrow[dr] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& & E \arrow[r, "\pi"] & M.
\end{tikzcd}$$

Problem 15.24. Prove that $d\pi : TE \to TM$ is a vector bundle. In particular, the addition and scalar multiplication on a fiber $(d\pi^{-1})(x)$ of $d\pi : TE \to TM$ are defined by Class

$$u \boxplus v := (d\alpha)(u, v) \text{ for } u, v \in TE \text{ with } (d\pi)u = (d\pi)v = x,$$

$$c \circledcirc v := (d\mu_c)v \text{ for } v \in TE \text{ and } c \in \mathbb{K},$$

where $\alpha(y_1, y_2); = y_1 + y_2$ for $(y_1, y_2) \in E \oplus E$ and $\mu_c y := cy$ for $y \in E$ and $c \in \mathbb{K}$.

Lemma 15.25. *Suppose that $f : \mathbb{R}^K \to \mathbb{R}^k$ is a smooth map such that $f(av) = af(v)$ for all $v \in \mathbb{R}^K$ and $a \in \mathbb{R}$. Then f is linear. Similarly for $\mathbb{C}$.*

Proof. One has $(Df)(0)v = \frac{d}{dt}\big|_{t=0} f(tv) = \frac{d}{dt}\big|_{t=0} tf(v) = f(v)$. Thus $f = (Df)(0)$ and f is linear. Similarly, in the complex case, f is $\mathbb{R}$-linear and by $f(iv) = if(v)$ it is $\mathbb{C}$-linear. $\square$

Applying this lemma to each chart we obtain the following statement. $\mathbb{K}-$

Corollary 15.26. *Suppose that $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M_2$ are $\mathcal{K}$-vector bundles, $\widehat{f} : E_1 \to E_2$ is a fiber bundle morphism over $f : M_1 \to M_2$. If $\widehat{f}$ is homogeneous on each fiber, i.e. $\widehat{f}(av) = a\widehat{f}(v)$ for all $v \in E_1$ and $a \in \mathbb{K}$, then $\widehat{f}$ is linear on fibers, i.e. it is a vector bundle morphism.* ✓

Lemma 15.27. *Let $\mu_r : E \to E$ be multiplication by r. Then for any $p \in M$ and $y, w \in E_p$, we have*

$$(d\mu_r)(\mathbf{j}_y w) = \mathbf{j}_{ry}(rw) = r\mathbf{j}_{ry}w.$$

Proof. Indeed

$$\begin{aligned}(d\mu_r)(\mathbf{j}_y w) &= \left.\frac{d}{dt}\right|_{t=0} \mu_r(y + tw) = \left.\frac{d}{dt}\right|_{t=0} (ry + trw) \\ &= \mathbf{j}_{ry}(rw) = r\mathbf{j}_{ry}w.\end{aligned}$$

$\square$