Differential Geometry and Topology

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1 Some concepts from topology

We start from metric spaces. 07.09.2023

Definition 1.1. *A metric* ρ on a set X is a mapping $\rho : X \times X \to [0, \infty)$, restricted to satisfy:

1. $\rho(x, y) = 0 \quad \Leftrightarrow \quad x = y \quad \forall x, y \in X$ (identity axiom);
2. $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$ (symmetry axiom);
3. $\rho(x, z) \le \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$ (triangle axiom).

A pair (X, ρ), where X is a set and ρ is a metric on X, is called *a metric space.* Sometimes we write simply X.

A subset $Y \subset X$ is automatically a metric space itself.

Definition 1.2. *Diameter* of Y is $\operatorname{diam} Y := \sup\limits_{x,y \in Y} \rho(x, y)$. If $\operatorname{diam} Y < \infty$, then Y is *bounded*. A *ball* (ball neighborhood) is

$$B_\varepsilon(x) := \{y \in X \mid \rho(y, x) < \varepsilon\}.$$

The *distance* between $Y \subseteq X$ and $Z \subseteq X$ is

$$\rho(Y, Z) := \inf_{y \in Y, z \in Z} \rho(y, z).$$

Definition 1.3. If $\rho(y, Y) = 0$, then y is an *adherent point* of Y. The *closure* of a subset Y is $\overline{Y}$:={the set of all adherent points of Y}. Evidently, $Y \subseteq \overline{Y}$. A subset Y is *closed*, if $Y = \overline{Y}$.

Definition 1.4. A point x is an *interior point* of a subset Y, if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq Y$ (in particular, $x \in Y$). The *interior* of Y is the set $\operatorname{Int} Y \subseteq Y$ of all its interior points. A subset Y is *open,* if $Y = \operatorname{Int} Y$.

Problem 1.5. Suppose, X is a metric space. Then $Y \subseteq X$ is open iff (if and only if) $X \setminus Y$ is closed. In fact, $\operatorname{Int} Y = X \setminus \overline{X \setminus Y}$. Class

Theorem 1.6. *Suppose, X is a metric space. Then*

1 O *X is open;*

2 O *$\varnothing$ is open;*

3 O *the union* $\bigcup_{\alpha\in A} U_\alpha$ *of* ***any*** *collection of open subsets* $U_\alpha \subseteq X$ *is open;*

4 O *the intersection* $\bigcap_{i=1}^{k} U_i$ *of a* ***finite*** *collection of open subsets* $U_i \subset X$ *is open;*

1 C $\varnothing$ *is closed;*

2 C X *is closed;*

3 C *the intersection* $\bigcap_{\alpha\in A} F_\alpha$ *of* ***any*** *collection of closed subsets* $F_\alpha \subset X$ *is closed;*

4 C *the union* $\bigcup_{i=1}^{k} F_i$ *of a* ***finite*** *collection of closed subsets* $F_i \subset X$ *is closed.*

Proof. Properties 1 O and 2 O are evident. Let us prove 3 O. Suppose, $U := \bigcup_{\alpha\in A} U_\alpha$ and $x \in U$. Then. for some α, we have $x \in U_\alpha$ and $B_{\varepsilon(\alpha)} \subseteq U_\alpha$. Then $B_{\varepsilon(\alpha)} \subseteq U_\alpha \subseteq U$.

Let us prove 4 O. Suppose, $U := \bigcap_{i=1}^{k} U_i$, $x \in U$. Then there are ε_i $(i = 1, \dots, k)$ such that $x \in B_{\varepsilon_i}(x) \subseteq U_i$. Take $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_k\}$. Take $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq U_i$ $\forall i$. Hence, $B_\varepsilon(x) \subset U$.

Finally, by Problem 1.5, k O $\Leftrightarrow$ k C $\forall$ k. $\square$

Home **Problem 1.7.** Show that the finiteness condition is essential.
Home **Problem 1.8.** Prove that $B_\varepsilon(x)$ is open.
Home **Problem 1.9.** Prove that $\operatorname{Int} Y$ is open, i.e., $\operatorname{Int}(\operatorname{Int} Y) = \operatorname{Int} Y$.
Home **Problem 1.10.** Prove that $\overline{Y}$ is closed, i.e., $\overline{\overline{Y}} = \overline{Y}$.

Definition 1.11. A *topology* on a set X is a system τ of its subsets (these subsets are called *open*), restricted to satisfy the following axioms:

1) $X \in \tau$;

2) $\varnothing \in \tau$;

3) if $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha\in A} U_\alpha \in \tau$;

4) if $U_1, \dots, U_k \in \tau$, then $\bigcap_{i=1}^{k} U_i \in \tau$.

Then (X, τ) is called a *topological space*. Any set of the form $F = X \setminus U$, where $U \in \tau$, is called *closed.*

Home **Problem 1.12.** Verify 1 C – 4 C for closed sets in a topological space.

Example 1.13. Any metric space is a topological space.

Class **Problem 1.14.** Find an example of a topological space (X, τ), which is not related to any metric (this is called: topology is not metrizable).

Definition 1.15. An (open) *neighborhood* of a point $x \in X$ (respectively, of a subset $Y \subseteq X$) in a topological space is any open set, where x (respectively, Y) is contained.

An *adherent point* of $Y \subseteq X$ is a point $x \in X$ such that any its neighborhood has a non-empty intersection with Y. The *closure* of Y is the set $\overline{Y}$ of all adherent points of Y (in particular, $Y \subseteq \overline{Y}$).

A point $x \in Y$ is called an *interior* point of Y, if there exists a neighborhood U of x such that $x \in U \subseteq Y$. The set $\operatorname{Int} Y$ of all interior points of Y is called the *interior* of Y.

Problem 1.16. $Y \subseteq X$ is closed iff $Y = \overline{Y}$. Home

Problem 1.17. $\overline{Y}$ is closed. Home

Problem 1.18. $Y \subseteq X$ is open iff $Y = \operatorname{Int} Y$. Home

Problem 1.19. $\operatorname{Int} Y$ is open. Home

Definition 1.20. Suppose $Y \subseteq X$, where (X, τ) is a topological space. The system of sets $\tau_1 := \{U \cap Y \mid U \in \tau\}$ is called the *induced* topology (by τ on Y).

Problem 1.21. Verify the axioms for τ_1. Home

Problem 1.22. Suppose that (X, ρ_X) is a metric space. Then one can introduce a topology on $Y \subseteq X$ in two ways: Home

1) ρ_X generates τ_X, which then induces τ_1,

2) ρ_X after the restriction on Y gives ρ_Y, which generates τ_{ρ_Y}.

Prove that $\tau_1 = \tau_{\rho_Y}$.

Definition 1.23. A subset $Y \subseteq X$ is called *(everywhere) dense,* if $\overline{Y} = X$.

Problem 1.24. Let $Y_1 \subseteq X$ and $Y_2 \subseteq X$ be dense open sets. Then $Y = Y_1 \cap Y_2$ is a dense open set. Home

Definition 1.25. A map $f : X \to Y$ of topological spaces is called *continuous at a point* $x_0 \in X$, if, for any neighborhood of its image $V(f(x_0))$, there exists a neighborhood $U(x_0)$ such that $f(U(x_0)) \subseteq V(f(x_0))$. A map is called *continuous*, if it is continuous at each point.

Theorem 1.26. *The next properties are equivalent:*

1) *a map $f : X \to Y$ is continuous;*

2) *for any open set $V \subseteq Y$, its full pre-image $f^{-1}(V)$ is open in X;*

3) *for any closed set $F \subset Y$ its full pre-image $f^{-1}(F)$ is closed in X.*

Proof. Since $f^{-1}(Y \backslash V) = f^{-1}(Y) \backslash f^{-1}(V) = X \backslash f^{-1}(V)$, properties 2) and 3) are equivalent.

Suppose, 1) is fulfilled, i.e., f is continuous, and $V \subseteq Y$ is an open set. Then either the pre-image of V is empty, hence open, or there is some point x, i.e., $f(x) \in V$. Then, by definition, for any such x, there exists a neighborhood $U(x)$ such that $f(U(x)) \subseteq V$, i.e., $U(x) \subseteq f^{-1}(V)$. Thus, any point of $f^{-1}(V)$ is interior.

Conversely, suppose 2) is fulfilled. Then, for $V = V(f(x_0))$, one can take $U(x_0) = f^{-1}(V)$ as the desired open neighborhood (see Def. 1.25). $\square$

Home **Problem 1.27.** Suppose, $X = F_1 \cup F_2$, where F_1 and F_2 are closed subsets, and $f : X \to Y$ is a map. Then f is continuous iff $f|_{F_1} : F_1 \to Y$ and $f|_{F_2} : F_2 \to Y$ are continuous.

Class **Problem 1.28.** Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, which is uniformly convergent on X to some function f. Then f is continuous.

Home **Problem 1.29.** Let X and Y be metric spaces. Prove that $f : X \to Y$ is continuous at x_0 as a map of topological spaces iff, for any sequence $\{x_n\}$ with $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

Definition 1.30. A map $f : X \to Y$ is called a *homeomorphism,* if

1) f is a bijection;

2) f and f^{-1} (inverse mapping) are continuous.

Class **Problem 1.31.** Give an example of a continuous bijection, which is not a homeomorphism.

Definition 1.32. A *base of a topology* τ is a system of open sets $\mathcal{B}$ such that any τ–open set is as a union of some of them.

Home **Problem 1.33.** What conditions need to be imposed on an arbitrary system of subsets $\mathcal{B}_1$, to obtain some topology by taking their arbitrary unions?

Definition 1.34. Suppose that (X, τ_X) and (Y, τ_Y) are topological spaces. Consider in $X \times Y$ the following base of topology:

$$\mathcal{B} := \{V \times W \mid V \in \tau_X,\ W \in \tau_Y\}.$$

The resulting topological space is called the *cartesian product* of X and Y.

Home **Problem 1.35.** Verify (with the help of the previous problem) that $X \times Y$ is really a topological space.

Home **Problem 1.36.** Prove that $X \times Y$ and $Y \times X$ are homeomorphic.

Home **Problem 1.37.** Prove that $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are homeomorphic.

Home **Problem 1.38.** Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. Define on $X \times Y$ the following distances:

$$\rho_{\max}((x_1, y_1), (x_2, y_2)) := \max\{\rho_X(x_1, x_2),\ \rho_Y(y_1, y_2)\},$$

$$\rho_2((x_1, y_1), (x_2, y_2)) := \sqrt{\rho_X^2(x_1, x_2) + \rho_Y^2(y_1, y_2)},$$

$$\rho_+((x_1, y_1), (x_2, y_2)) := \rho_X(x_1, x_2) + \rho_Y(y_1, y_2).$$

Prove:

1) That these are metrics.

2) That the corresponding topologies on $X \times Y$ coincide.

Class **Problem 1.39.** Prove that (a, b), $[a, b)$ and $[a, b]$ (subsets of real line) are pair-wise non-homeomorphic.

1.1 Connectedness and arc connectedness

Definition 1.40. A topological space X is called *disconnected,* if one of the following (evidently equivalent to each other) conditions is fulfilled:

- X is equal to a union of its two non-intersecting non-empty open subsets.
- X has a non-empty subset $A \neq X$, which is open and closed simultaneously.
- X is equal to a union of its two non-intersecting non-empty open and closed simultaneously subsets.

Otherwise X is *connected.*

Definition 1.41. A topological space X is called *arc connected,* if, for any two points $x_0, x_1 \in X$, there exists a continuous map (*path*) $f : [0,1] \to X$, $f(0) = x_0$, $f(1) = x_1$.

Problem 1.42. Any interval $[a, b] \subset \mathbb{R}$ is connected and arc connected. Class

Theorem 1.43. *Suppose,* $X = \bigcup\limits_{\alpha} X_\alpha$*, each* X_α *is connected, and* $\bigcap\limits_{\alpha} X_\alpha \neq \varnothing$*. Then* X *is connected.*

Proof. Suppose that X is disconnected, $X = A \cup B$, $A \cap B = \varnothing$, A and B are non-empty closed-open sets. Then, for each α, we have $X_\alpha = (X_\alpha \cap A) \cup (X_\alpha \cap B)$. By the definition of the induced topology, these sets are closed-open in X_α. Since X_α is connected, one of them should be empty. Hence, each X_α belongs entirely either to A, or to B, which do not intersect. Since A and B are non-empty and X is the union of X_α, then at least one of X_α, say X_{α_0} is contained in A and some other, $X_{\alpha_1} \subseteq B$. Then $\bigcap\limits_{\alpha} X_\alpha \subseteq X_{\alpha_0} \cap X_{\alpha_1} = \varnothing$. A contradiction. □

Theorem 1.44. *Suppose that, for any two points* x *and* y *of a topological space* X*, there exists a connected subset* P_{xy} *such that* $x \in P_{xy}$ *and* $y \in P_{xy}$*. Then* X *is connected.*

Proof. Suppose that X is disconnected: $X = A \cup B$, $A \cap B = \emptyset$, A and B are non-empty closed-open subsets. Then there exist some $a \in A$, $b \in B$ and a corresponding P_{ab}. Then $P_{ab} = (P_{ab} \cap A) \cup (P_{ab} \cap B)$. The subsets $P_{ab} \cap A$ and $P_{ab} \cap B$ are closed-open in P_{ab} and non-empty (the first one contains a, the second one — b). A contradiction with connectedness of P_{ab}. □

Problem 1.45. The image of a connected space under a continuous mapping is connected. Home

Theorem 1.46. *An arc connected space is connected.*

Proof. By the previous problem, the set $f([0,1])$ is connected, where $f = f_{x_0,x_1}$ is the function from Def. 1.41. Taking $P_{x_0,x_1} := f([0,1])$, apply Theorem 1.44. □

Problem 1.47. Find an example of connected space, which is not arc-connected. Class

1.2 Compact, Hausdorff and normal spaces

14.09.2023

Definition 1.48. A topological space X is called *Hausdorff,* if, for any $x, y \in X$, $x \neq y$, there exist their neighborhoods $U(x)$ and $U(y)$ such that $U(x) \cap U(y) = \varnothing$.

Home **Problem 1.49.** Give an example of non-Hausdorff topological space.

Class **Problem 1.50.** Prove that the Cartesian product of Hausdorff spaces is a Hausdorff space.

Home **Problem 1.51.** Prove that in any Hausdorff space each point is a closed set.

Definition 1.52. A topological space X is called *normal,* if it is Hausdorff and, for any two non-intersecting closed sets F_1 and F_2, there exist their non-intersecting neighborhoods $U_1 \supseteq F_1$ and $U_2 \supseteq F_2$, $U_1 \cap U_2 = \varnothing$.

Class **Problem 1.53.** Verify that any metric space is normal.

Definition 1.54. A cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$, if, for any β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_\alpha$.

Theorem 1.55. *Suppose that X is a normal topological space and $\{U_i\}_{i=1}^N$ is a finite open cover. Then there exists its refinement of the form $\{V_i\}_{i=1}^N$ such that $\overline{V}_i \subseteq U_i$.*

Proof. Consider the following closed sets

$$F_1 = \left(X \setminus \bigcup_{i=2}^N U_i\right) \subseteq U_1, \qquad \widetilde{F}_1 = X \setminus U_1,$$

and, by normality, neighborhoods

$$V_1 \supseteq F_1, \qquad \widetilde{V}_1 \supseteq \widetilde{F}_1, \qquad V_1 \cap \widetilde{V}_1 = \varnothing.$$

Each point of $\widetilde{F}_1$ has an open neighborhood $\widetilde{V}_1$, which does not intersect V_1. Hence this point can not be an adherent point of V_1 and

$$\overline{V}_1 \cap \widetilde{F}_1 = \emptyset, \qquad V_1 \subset \overline{V}_1 \subset (X \setminus \widetilde{F}_1) = U_1.$$

Also, $(V_1, U_2, \ldots, U_N)$ is a cover by the construction of F_1. At next steps we replace U_2 by V_2 and so on. □

Home **Problem 1.56.** Let $f : X \to X$ be a continuous self-map of a Hausdorff space. Prove that the set of fixed points $F_f := \{x \in X \mid f(x) = x\}$ is closed.

Home **Problem 1.57.** Prove that X is Hausdorff iff the diagonal $\Delta := \{(x, y) \mid x = y\} \subset X \times X$ is closed in $X \times X$.

Class **Problem 1.58.** Prove that if a map $f : X \to Y$, where Y is Hausdorff, is continuous, then its graph $\Gamma_f := \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is closed in $X \times Y$.

Lemma 1.59. (Uryson's lemma) *Suppose that X is a normal topological space, F_0 and F_1 are some closed non-intersecting sets. Then there exists a continuous function $f : X \to [0, 1]$ such that $f|_{F_0} = 0$ and $f|_{F_1} = 1$.*

Proof. The normality of X implies that, for any closed F and its open neighborhood U, $F \subseteq U$, there exists another neighborhood V such that $F \subseteq V \subseteq \overline{V} \subseteq U$ (see the above proof of Theorem 1.55). We will denote this by $V \Subset U$.

Define V_q, for rational q of the form $q = m/2^k$, m odd, by induction over k (i.e., first for 0 and 1, then for 1/2, then for 1/4 and 3/4, then for 1/8, 3/8, 5/8, 7/8 and so on) in such a way that $V_{q_1} \Subset V_{q_2}$ if $q_1 < q_2$. For this purpose define V_0 and V_1 to be open sets U and V from the beginning of the proof, i.e., $F_0 \subseteq V_0$, $F_1 \subseteq X \setminus V_1$, $V_0 \Subset V_1$. Suppose that, by the induction supposition, the sets V_q are defined for q up to 2^k as the denominator of q. Consider

$$F := \overline{V_{\frac{i}{2^k}}}, \qquad U := V_{\frac{i+1}{2^k}},$$

and define $V_{\frac{2i+1}{2^{k+1}}} := V$ (as in the beginning of the proof, for these F and U). And so on.

The constructed V_q are open and have the following properties:

1) $F_0 \subset V_0$,

2) $V_1 = X \setminus F_1$,

3) if $q_1 < q_2$, then $V_{q_1} \Subset V_{q_2}$.

Define, for any $s \in [0, 1]$, the set V_s as $V_s := \bigcup_{q \le s} V_q$. Then V_s is open for any s (as a union of open sets) and satisfies 1) – 3). Indeed, 1) and 2) are evident, and to prove 3), for $s_1 < s_2$,we find $q_1 = m_1/2^k$ and $q_2 = m_2/2^k$ such that $s_1 < q_1 < q_2 < s_2$, where k is sufficiently large. Then $V_{s_1} \subseteq V_{q_1} \Subset V_{q_2} \subseteq V_{s_2}$ and $V_{s_1} \Subset V_{s_2}$.

Now define $f : X \to [0, 1]$ by $f|_{F_0} = 0$ and $f(x) := \sup\{s \mid x \notin V_s\}$. Let us prove that f is continuous. Let x_0 and $\varepsilon > 0$ be arbitrary. Let $s_0 = f(x_0)$. Consider

$$U(x_0) := V_{s_0+\frac{\varepsilon}{4}} \setminus \overline{V_{s_0-\frac{\varepsilon}{4}}}.$$

This is an open neighborhood of x_0 and, for any $x \in U(x_0)$, one has

$$x \in V_{s_0+\frac{\varepsilon}{4}}, \qquad x \notin \overline{V_{s_0-\frac{\varepsilon}{4}}}.$$

Thus,

$$s_0 - \frac{\varepsilon}{4} \le f(x) \le s_0 + \frac{\varepsilon}{4}, \qquad |f(x) - f(x_0)| \le \frac{\varepsilon}{2} < \varepsilon.$$

□

Problem 1.60. A closed subset of a closed set is closed in the entire space. Home

Problem 1.61. (Tietze's theorem about extension) [Mishchenko, Fomenko, pp. 78–79] Suppose that X is a normal topological space, $F \subset X$ is a closed subset and $f : F \to \mathbb{R}$ is a continuous function. Then f can be extended to a continuous function $g : X \to \mathbb{R}$. If f is bounded, then g can be chosen to be bounded by the same constant. Home

Definition 1.62. The *support of a function* $f : X \to \mathbb{R}$ is

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Theorem 1.63. *Suppose that X is a normal topological space and $\{U_\alpha\}$ its finite open cover. Then there exist continuous functions $\psi_\alpha : X \to [0, 1] \subset \mathbb{R}$ such that*

1) $\operatorname{supp} \psi_\alpha \subset U_\alpha$,

2) $\sum_\alpha \psi_\alpha(x) \equiv 1$.

This system (not uniquely determined) of functions $\{\psi_\alpha\}$ is called a partition of unity subordinated to $\{U_\alpha\}$.

Remark 1.64. It is sufficient to ask local finiteness of $\{U_\alpha\}$: every point has a neighborhood such that it intersects only finitely many sets from $\{U_\alpha\}$.

Proof of theorem. Using Theorem 1.55 let us find new covers $W_\alpha \Subset V_\alpha \Subset U_\alpha$. By the Uryson lemma we can find continuous functions

$$\theta_\alpha : X \to [0,1], \qquad \theta_\alpha|_{\overline{W_\alpha}} \equiv 1, \qquad \theta_\alpha|_{(X \setminus V_\alpha)} \equiv 0.$$

Thus, $\operatorname{supp} \theta_\alpha \subseteq \overline{V}_\alpha \subseteq U_\alpha$ and $\theta_\alpha|_{W_\alpha} > 0$. Define $\theta := \sum_\alpha \theta_\alpha$. It is a finite sum of continuous functions, hence, itself a continuous function. Since $\{W_\alpha\}$ is a cover and $\theta \geq \theta_\alpha > 0$ on W_α, then $\theta > 0$ everywhere. Hence we can define $\psi_\alpha := \frac{\theta_\alpha}{\theta}$. Evidently, 1) and 2) are satisfied. □

Definition 1.65. A topological space X is *compact,* if each its open cover has a finite sub-cover (i.e. there is a finite number of elements, which still cover X).

Class **Problem 1.66.** Prove that any closed interval $[a,b]$ is compact.

Home **Problem 1.67.** Prove that a closed subset of a compact space is compact itself.

Home **Problem 1.68.** Prove that a compact subset of a Hausdorff space is closed.

Theorem 1.69. *Any compact Hausdorff space is normal.*

Proof. Let $F \subset X$ be closed and $x \notin F$. Let us prove that there exist non-intersecting open neighborhoods $U(x)$ and $V(F)$. Since X is Hausdorff, for any $y \in F$, there exist $V_y \ni y$ and $U_y \ni x$ such that $V_y \cap U_y = \emptyset$. The neighborhoods V_y form a cover of F and we can find its finite sub-cover $V_{y_1}, \dots, V_{y_N}$, since F is compact (see Problem 1.67). Define:

$$V(F) := V_{y_1} \cup \dots \cup V_{y_N}, \qquad U(x) := \bigcap_{j=1}^{N} U_{y_j}.$$

They are as desired.

Let now $F_1 \subset X$ and $F_2 \subset X$ be closed. According to the first part of the proof, we can find for each $x \in F_1$ open non-intersecting sets $U(x) \ni x$ and $V(x) \supset F_2$. Then $\{U(x)\}$ is an open cover of F_1 and we can find its finite sub-cover $U(x_1), \dots, U(x_n)$. The sets $\bigcup_{i=1}^{n} U(x_i)$ and $\bigcap_{i=1}^{n} V(x_i)$ are demanded non-intersecting neighborhoods of F_1 and F_2. □

Home **Problem 1.70.** Prove that a continuous image of a compact is compact.

Class **Problem 1.71.** Let $f : X \to \mathbb{R}^1$ be a continuous function on a compact space X. Then f is bounded and reaches its maximal and minimal value.

Theorem 1.72. *A continuous bijective mapping of a compact space onto a Hausdorff space is a homeomorphism.*

Proof. Let $f : X \to Y$ be a continuous bijection, where X is a compact and Y is Hausdorff. To prove the statement, it is sufficient to prove that the image of any closed subset $F \subset X$ is a closed subset in Y. Since X is compact, then F is compact as well (see Problem 1.67). Thus, $f(F)$ is also compact. But Y is Hausdorf. Thus, $f(F)$ is closed (see Problem 1.68). □

Class **Problem 1.73.** A cartesian product of compact spaces is compact.

2 Manifolds and tangent vectors

Definition 2.1. *A smooth manifold of dimension* m is a second countable (has a countable base) Hausdorff topological space M, equipped with a *smooth atlas,* i.e., its open cover $\{U_\alpha\}$ and a collection of homeomorphisms φ_α, which map U_α onto open subsets $V_\alpha \subset \mathbb{R}^m$ (the dimension m of M is denoted by $\dim M$). They introduce on each U_α *local coordinates*. They are restricted to satisfy the following *compatibility property*: the *change of coordinate maps* (or *overlap maps*, or *transition functions*) $\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ should be smooth as vector-valued functions, defined on an open subset in $\mathbb{R}^m$. A pair $(U_\alpha, \varphi_\alpha)$ is called *a chart*.

A smooth structure is a maximal smooth atlas (not absolutely rigorous definition). These are all charts, that are compatible with all charts of some smooth atlas.

Problem 2.2. Prove that for compact manifolds the replacement of "second countable" by "separable" (has a countable dense set) gives the same concept (in general, "second countable" is a stronger condition). Home

Reminder: a map $f : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^m$, is called *differentiable* at $u \in U$ iff there is a linear map $Df(u) : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\|h\|\to 0} \frac{\|f(u+h) - f(u) - Df(u)(h)\|}{\|h\|} = 0.$$

Existence of partial derivatives of coordinate functions at u is not sufficient and existence of continuous partial derivatives is not necessary!!!

Smooth = sufficiently many times (typically infinitely many) differentiable.

Remark 2.3. We have inserted the restriction of the same m for all charts into the definition, but in fact there is a theorem which shows that if we have a homeomorphism $\varphi : U \approx V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are some open sets, then $m = n$.

Remark 2.4. If we do not demand compatibility, a manifold is called *topological.*

Problem 2.5. Find an example of a manifold and two non-compatible smooth structures on it, i.e., two smooth atlases (U_i, φ_i) and (V_j, ψ_j) such that $\{(U_i, \varphi_i), (V_j, \psi_j)\}$ is not a smooth atlas. Class

Problem 2.6. Prove that the sphere S^n and the projective space $\mathbb{R}P^n$ are smooth manifolds. Class

Problem 2.7. Are the boundary of a square and 8 smooth manifolds (subspaces of $\mathbb{R}^2$) ? Home

Definition 2.8. A $2n$-dimensional manifold is called *complex analytical,* if all transition functions are complex analytical.

Problem 2.9. Prove that S^2 is a complex analytical manifold. Home

Definition 2.10. A function $f : M \to \mathbb{R}$ is called *smooth,* if, for any point $P \in M$ and some chart $(U_\alpha, \varphi_\alpha)$ with $P \in U_\alpha$, the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$, defined on an open set in $\mathbb{R}^m$, is smooth. 21.09.2023

Problem 2.11. Prove that this definition does not depend on the choice of a chart (from the same maximal atlas). Home

Definition 2.12. A continuous mapping $f : M \to N$ of smooth manifolds is called *smooth,* if for any point $P \in M$ and some charts $(U_\alpha, \varphi_\alpha)$, $P \in U_\alpha$, and $(U'_\beta, \varphi'_\beta)$, $f(P) \in U'_\beta$, (these are charts on M and N, respectively) the mapping $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1} : V_\alpha \to V'_\beta \subset \mathbb{R}^n$ defined on an open set in $\mathbb{R}^m$, is smooth, where $\dim M = m$ and $\dim N = n$.

This mapping is called *local* or *coordinate representative maps* for f

Home **Problem 2.13.** Verify that if a mapping is smooth w.r.t. some pair of charts, then it is smooth w.r.t. any other (compatible) pair.

Definition 2.14. A bijective smooth mapping $f : M \to N$ of smooth manifolds is called a *diffeomorphism* , if f^{-1} is smooth.

Home **Problem 2.15.** Verify that the following formulas

$$y^k = \frac{x^k}{\sqrt{\varepsilon^2 - (x^1)^2 - (x^2)^2 - \cdots - (x^n)^2}}, \qquad k = 1, \dots, n,$$

$$x^k = \frac{\varepsilon\, y^k}{\sqrt{1 + (y^1)^2 + (y^2)^2 + \cdots + (y^n)^2}}, \qquad k = 1, \dots, n,$$

define a diffeomorphism $B_\varepsilon(0) \subset \mathbb{R}^n$ and $\mathbb{R}^n$.

Class **Problem 2.16.** Find an example of smooth homeomorphism, which is not a diffeomorphism.

Lemma 2.17. *For any smooth manifold M, there exists an atlas such that all V_α (images of coordinate maps) are open balls (hence by Problem* 2.15*, to the entire space $\mathbf{R}^m$.)*

Proof. Let $(U_\alpha, \varphi_\alpha)$ be an atlas of M. For any $x \in M$, we can choose a chart $U_{\alpha(x)} \ni x$. Choose a small $\varepsilon(x)$ such that $B_{\varepsilon(x)}(\varphi_{\alpha(x)}(x)) \subseteq V_{\alpha(x)} \subseteq \mathbb{R}^m$. Then

$$(\widetilde{U}_x, \widetilde{\varphi}_x), \quad x \in M, \qquad \widetilde{U}_x := \varphi^{-1}_{\alpha(x)}(B_{\varepsilon(x)}(\varphi_{\alpha(x)}(x))), \qquad \widetilde{\varphi}_x := \varphi_{\alpha(x)}|_{\widetilde{U}_x},$$

is the desired atlas. □

Remark 2.18. For any finite atlas of a compact manifold, there exists a subordinated partition of unity, because this manifold is normal as a topological space.

We will suppose all manifolds to be smooth and will call them simply "manifolds".

Theorem 2.19. *For any finite atlas of a compact manifold M, there exists a subordinated smooth partition of unity.*

Proof. Remark that it is sufficient to find a smooth partition of unity for a finite refinement o the initial cover by charts (then we simply take some finite sums of functions as the desired partition).

Second, observe that Lemma 2.17 gives rise to a refinement of the initial atlas (we leave finitely many charts by compactness). Moreover, we can do this for some smaller atlas w.r.t the initial one (as in Theorem 1.55).

Thus, we need to prove the statement for an atlas (W_β, τ_β) such that

$$\tau_\beta(W_\beta) = B_1(0) \subset \mathbb{R}^m, \qquad W^\varepsilon_\beta := \tau_\beta^{-1}(B_{1-\varepsilon}(0)) \text{ is still a cover of } M$$

(these ε's are distinct, but we can take the minimum over this finite set of charts).

Define the following smooth function on $\mathbb{R}^m$:

$$h(x) := \begin{cases} e^{-\dfrac{1}{(1-\varepsilon/2)^2 - \|x\|^2}}, & \text{for } \|x\|^2 < (1-\varepsilon/2)^2, \\ 0, & \text{for } \|x\|^2 \geq (1-\varepsilon/2)^2. \end{cases}$$

Then

$$\operatorname{supp} h = \overline{B_{1-\varepsilon/2}(0)}, \qquad 0 \leq h(x) \leq 1, \qquad h(x) > 0 \text{ on } B_{1-\varepsilon}(0).$$

Define

$$\chi_\beta := \begin{cases} h(\tau_\beta(x)), & \text{for } x \in W_\beta, \\ 0, & \text{for } x \notin W_\beta. \end{cases}$$

Then $\chi_\beta \in C^\infty(M)$, $0 \leq \chi \leq 1$, $\operatorname{supp} \chi_\beta \subset W_\beta$ and $\chi_\beta > 0$ on W_β^ε. Hence, $\psi := \sum_\beta \chi_\beta > 0$ and $\psi_\beta := \chi_\beta/\psi$ is a desired C^∞-partition of unity. □

Problem 2.20. Prove the existence of a subordinated smooth partition of unity for any locally finite atlas of a (non-compact) manifold. [Lee, Thm. 1.72]. Home

Theorem 2.21. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that* $\operatorname{grad} f = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right) \neq \vec{0}$ *at any point of $M = f^{-1}(y_0)$. Then M is a smooth manifold. Some $n-1$ of $x^1, \ldots, x^n$ can be taken as local coordinates (i.e., the corresponding projection is a chart). (Which ones — depends on point.) In particular,* $\dim M = n - 1$.

Proof. Apply the implicit mapping theorem. Namely, suppose that

$$\vec{x}_0 = (x_0^1, \ldots, x_0^n) \in M, \qquad \operatorname{grad} f_{\vec{x}_0} = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)\bigg|_{\vec{x}_0} \neq \vec{0}.$$

Without loss of generality one can assume that $\frac{\partial f}{\partial x^n}\big|_{\vec{x}_0} \neq 0$. By the implicit mapping theorem, there is a neighborhood V of $(x_0^1, \ldots, x_0^{n-1})$ in $\mathbb{R}^{n-1}$, an interval $(x_0^n - \varepsilon, x_0^n + \varepsilon) \in \mathbb{R}^1$ and C^∞-function $g : V \to \mathbb{R}^1$ such that

1. $f(x^1, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})) \equiv 0$ on V,
2. $g(x_0^1, \ldots, x_0^{n-1}) = x_0^n$,
3. $g(x^1, \ldots, x^{n-1}) \in (x_0^n - \varepsilon, x_0^n + \varepsilon)$ for $(x^1, \ldots, x^{n-1}) \in V$,
4. any point $(x^1, \ldots, x^n) \in M \cap (V \times (x_0^n - \varepsilon, x_0^n + \varepsilon))$ is defined by $x^n = g(x^1, \ldots, x^{n-1})$.

Define a chart:

$$U := M \cap (V \times (x_0^n - \varepsilon, x_0^n + \varepsilon)), \quad \varphi : U \to \mathbb{R}^{n-1},\ \varphi(x^1, \ldots, x^n) := (x^1, \ldots, x^{n-1}) \in V.$$

Then, by 1) and 4), the inverse mapping for φ is

$$\varphi^{-1}(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})).$$

Verify that the atlas is smooth. Without loss of generality, suppose that $\vec{x}_0$ is contained in (U, φ) and also in $(\widetilde{U}, \widetilde{\varphi})$, where $\widetilde{\varphi} : (x^1, \ldots, x^n) \mapsto (x^2, \ldots, x^n)$. Then, on $\varphi(U \cap \widetilde{U})$ we have

$$\widetilde{\varphi}\varphi^{-1}(x^1, \ldots, x^{n-1}) = \widetilde{\varphi}(x^1, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})) = (x^2, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})),$$

i.e., a smooth transition function. □

Definition 2.22. (Tensor definition of a tangent vector) *A (tangent) vector* ξ at a point $P \in M$ to a manifold M is a correspondence which, to each chart $(U_\alpha, \varphi_\alpha)$ (i.e., a local coordinate system $(x^1_\alpha, \dots, x^m_\alpha)$ containing P) puts in correspondence an n-tuple of numbers $(\xi^1_\alpha, \dots, \xi^m_\alpha)$. This correspondence is restricted to satisfy the *tensor transformation law*: if to another chart (U_β, φ_β) local coordinate system $(x^1_\beta, \dots, x^m_\beta)$) ξ put in correspondence an n-tuple $(\xi^1_\beta, \dots, \xi^m_\beta)$, then

$$\xi^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha} \xi^j_\alpha, \tag{1}$$

where the summation over repeated up and down indexes j is supposed (the Einstein summation convention).

Class **Problem 2.23.** (a justification of the definition) Suppose that $\gamma : (-1; 1) \to M$ is a smooth mapping and $\gamma(0) = P$. Then the correspondence

$$\xi_\gamma : (x^1, \dots, x^n) \rightsquigarrow \left(\frac{dx^1}{dt}, \dots, \frac{dx^n}{dt} \right) \Bigg|_{t=0}$$

is a vector at P, where, for a local coordinate system $(x^1, \dots, x^n)$, the mapping γ is defined as $(x^1(t), \dots, x^n(t))$.

Home **Problem 2.24.** Any tangent vector at P is uniquely defined by its components for any coordinate system. Moreover, any such n-tuple defines a vector.

Hence, the set of tangent vectors at a point P (*tangent space* $T_P(M)$) is a finite dimensional $\mathbb{R}$-linear space of dimension $\dim M$. The operations do not depend on the choice of local coordinate system by (1).

Definition 2.25. (Definition of tangent vector via curves) Consider two smooth curves $\gamma_1 : (-1, 1) \to M$ and $\gamma_2 : (-1, 1) \to M$ such that

- $\gamma_i(0) = P$
- for some (hence, any) coordinate system $(x^1, \dots, x^m)$ in a neighborhood of P the following holds:

$$\sum_{k=1}^{m} \left[x^k(\gamma_1(t)) - x^k(\gamma_2(t)) \right]^2 = o(t^2), \qquad (t \to 0).$$

Such curves are called (tangentially) *equivalent*: $\gamma_1 \sim \gamma_2$.

All curves satisfying the first condition form non-intersecting equivalence classes called *tangent vectors* to M at point P.

Home **Problem 2.26.** Verify that the above equivalence is really an equivalence relation.

Definition 2.27. (Definition of a tangent vector via differentiation operators) A linear map $D : C^\infty(M) \to \mathbb{R}$, i.e., a linear functional on the space of smooth functions, is called a *differentiation operator* at some point $P \in M$, if

- its values are determined only by values of functions in an arbitrary small neighborhood of P. More precisely, if $f, g \in C^\infty(M)$ satisfy $f \equiv g$ over some neighborhood U of P, then $D(f) = D(g)$ (they say "operator is defined on germs of functions");

- the Leibniz property

$$D(fg) = f(P)D(g) + g(P)D(f) \text{ is fulfilled for any } f, g \in C^\infty(M).$$

Such operator is called a *tangent vector* to M at point P.

Evidently, they form a linear space.

Problem 2.28. Suppose that $(x^1, \dots, x^n)$ is a local coordinate system in a neighborhood of $P \in M$, $P = (x_0^1, \dots, x_0^n)$, and $\xi \in T_P M$ (in the tensor sense) has components ξ^i. Then the mapping Home

$$f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x_0^1, \dots, x_0^n)\xi^i$$

(*directional derivative* w.r.t. ξ) does not depend on the choice of a local coordinate system and defines a differentiation operator.

Theorem 2.29. *These three definitions are equivalent in the sense that the following natural correspondences*

$$\textit{a curve} \;\to\; \textit{its tangent vector in a coordinate system} \to$$

$$\to\; \textit{the directional derivative w.r.t. this vector}$$

gives rise to a bijection of the tangent spaces in three senses (the second map is a linear isomorphism of linear spaces).

Proof. Let us prove the first bijection. Keeping in mind Problem 2.23 we see that to prove that Γ (defined in the problem) is well defined on equivalence classes, it is sufficient to verify in one coordinate system that $\gamma_1 \sim \gamma_2$ imples $\xi_{\gamma_1} = \xi_{\gamma_2}$. Indeed,

$$0 = \lim_{t\to 0} \sum_{k=1}^m \left[\frac{x^k(\gamma_1(t)) - x^k(\gamma_2(t))}{t} \right]^2 =$$

$$= \sum_{k=1}^m \left[\lim_{t\to 0} \frac{(x^k(\gamma_1(t)) - x^k(P)) - (x^k(\gamma_2(t)) - x^k(P))}{t} \right]^2,$$

so $\xi_{\gamma_1} = \xi_{\gamma_2}$. The same calculation shows that two curves are equivalent iff they have the same tangent vector in their intersection point P. Thus, Γ is well defined and injective. Fix a coordinate system x^i in a neighborhood of P. Define a map Δ (may be depending on the choice of coordinates) in the inverse direction by sending a vector ξ with coordinates ξ^i in this system, to a "straight line", i.e. to the following curve: $x^i(t) = x^i(P) + t \cdot \xi^i$. Then $\left.\frac{dx^i}{dt}\right|_{P_0} = \xi^i$ and $\Gamma \circ \Delta = \mathrm{Id}$. Hence, Γ is a surjection. $\square$

Problem 2.30. Prove the second equivalence in the above theorem ([Mishchenko, Fomenko], pp 125–127). Home

Definition 2.31. Suppose that $f : M \to N$ is a smooth map and $P \in M$. The *tangent map* of f at P is a map of tangent spaces $df_P : T_P M \to T_{f(P)} N$, defined in one of the following equivalent ways (corresponding to three ways of defining of a tangent vector). 28.09.2023

First way. Suppose that $(U^M, \varphi^M : U^M \to V^M \subset \mathbb{R}^m)$ is a chart of M in a neighborhood of P, $(U^N, \varphi^N : U^N \to V^N \subset \mathbb{R}^n)$ is a chart of N in a neighborhood of $f(P)$, $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ are the corresponding local coordinate systems. The local representative map of f, namely a map $\varphi^N \circ f \circ (\varphi^M)^{-1} : V^M \to V^N$ can be described as a collection of functions

$$y^1 = f^1(x^1, \ldots, x^m), \ldots, y^n = f^n(x^1, \ldots, x^m).$$

Suppose that $\xi \in T_P M$ puts in correspondence an m-tuple $(\xi^1, \ldots, \xi^m)$ to the system $(x^1, \ldots, x^m)$ (or ξ has coordinates $(\xi^1, \ldots, \xi^m)$ w.r.t. this system). Then we define its image $\eta = (df_P)\xi$ to be a vector with coordinates

$$\eta^j = \frac{\partial f^j}{\partial x^i}\xi^i$$

(assuming the summation) w.r.t. the system $(y^1, \ldots, y^n)$.

Second way. Denote by $[\gamma]$ the equivalence class of a curve γ. Define:

$$(df_P)[\gamma] := [f \circ \gamma].$$

Third way. Consider a differentiation operator ξ at $P \in M$. Then the result of the action of the differentiation operator $(df_P)\xi$ onto $g \in C^\infty(N)$ is given by the following formula

$$((df_P)\xi)(g) := \xi(g \circ f).$$

Home **Problem 2.32.** Verify the equivalence of these three definitions.

Clearly the tangent map is linear.

Definition 2.33. Consider a smooth map $f : M \to N$, $f(P_0) = Q_0$. A point $P_0 \in M$ is called a *regular point* of f if the tangent map

$$df_{P_0} : T_{P_0}M \to T_{Q_0}N$$

is an epimorphism (surjection). A point $Q_0 \in N$ is called a *regular value* of f if any $P \in f^{-1}Q_0$ is a regular point of f.

Theorem 2.34. (Sard's Lemma) (has to be proved in Advanced Calculus course) *Suppose that $f : M \to N$ is a smooth map, M is a compact manifold. Then the set $G \subset N$ of all regular values of f is an open dense set.*

Remark 2.35. For non-compact: $N \setminus G$ has zero measure.

Definition 2.36. A smooth map $f : M \to N$ is called an *immersion,* if, for each point $P \in M$, its tangent map $df_P : T_P M \to T_{f(P)}N$ is a monomorphism (injective linear map). If moreover $f : M \leftrightarrow f(M)$ is a bijection and $f(M)$ is closed in N, then f is called *embedding.*

Home **Problem 2.37.** Give an example of immersion, which is bijective on its image, but is not an embedding.

Definition 2.38. An embedding, which is a homeomorphism on its image is called a *strong embedding.*

Class **Problem 2.39.** For compact manifolds an embedding is always strong.

Definition 2.40. A subset $L \subset M$, $\dim M = m$ is called a *smooth submanifold,* if there exists an atlas $(U_\alpha, \varphi_\alpha)$ of the manifold M such that $\{U_\alpha \cap L\}$ is a smooth atlas of L, where chart mappings are of the form (this is an additional condition for φ_α)

$$\varphi_\alpha|_{U_\alpha \cap L} : U_\alpha \cap L \to V_\alpha \cap \mathbb{R}^l, \qquad \mathbb{R}^l \subset \mathbb{R}^m, \qquad l < m.$$

Such an atlas $(U_\alpha, \varphi_\alpha)$ is called *normal.* Thus, $\dim L = l$, and $(m - l)$ is its *codimension.*

Problem 2.41. Prove that L is closed under the conditions of this definition. Class

Problem 2.42. Suppose that $f : M \to N$ is smooth and $Q_0 \in N$ is a regular value of f. Then $M_{Q_0} := f^{-1}(Q_0)$ is a smooth submanifold $\dim M_{Q_0} = \dim M - \dim N$. As a local coordinates in some neighborhood on M_{Q_0} one can take some $(m - n)$ coordinates of M. Hint: similarly to Theorem 2.21. Home

Problem 2.43. Find an example of embedding such that its image is not a submanifold (and even a manifold). Home

Theorem 2.44. *A subset $A \subset N$ is a submanifold iff it is the image of some manifold M under a strong embedding.*

Proof. If $A \subset N$ is a submanifold, thrn the identical inclusion is a homeomorphism on its image, and by the definition of a submanifold it is am immersion (to calculate its rank use the local representative w.r.t. a normal atlas.)

Conversely, let $f : M \to N$ be a strong embedding. The property of $A = f(M)$ to be a submanifold is local: to observe this, consider an open cover $\{N_i\}$ of A in N and take $A_i = A \cap N_i$. Consider a family of charts $\Psi = \{\psi_i : N_i \to \mathbb{R}^n\}$ of N, which cover A. Let $\Phi = \{\varphi_i : M_i \to \mathbb{R}^m\}_{i \in \Lambda}$ be an atlas of M such that $f(M_i) \subset N_i$ (if necessary, pass to a refinement). More precisely, we take a refinement of N_i such that (conserving the notation) each $M_i = f^{-1}(A \cap N_i)$ is covered by a chart of M.

The localization reduces the situation to the following one: $U := \{V_i\} = \varphi_i(M_i) \subset \mathbb{R}^m$, $f = f_i = \psi_i f \varphi_i^{-1} : U \hookrightarrow \mathbb{R}^n$ is a C^∞-embedding. We need to find locally a diffeomorphism, such that the image of new embedding is contained in $\mathbb{R}^{n-m}$. By the inverse mapping theorem there exist (in a sufficiently small neighborhood) some coordinates $(x^{i_1}, \dots, x^{i_m})$, $1 \le i_1 \le \dots \le i_m \le n$, and a smooth map $g : \mathbb{R}^m_x \to \mathbb{R}^{n-m}_x$ such that the image of $f = f_i$ is the graph of g. Thus we can introduce in $\mathbb{R}^n$ new coordinates:

$$(x^{i_1}, \dots, x^{i_m}, x^{j_1} - (g(x^{i_1}, \dots, x^{i_m}))^{j_1}, \dots, x^{j_{n-m}} - (g(x^{i_1}, \dots, x^{i_m}))^{j_{n-m}}),$$

and obtain that $f(U)$ is just some coordinate plane.

To obtain a normal atlas from these charts (passing from local to global) we need to guarantee that (after a passage to smaller charts, if necessary) N_i contains only $f(M_i)$ and not $f(M_j)$ for $j \neq i$ (we conserve the notation for smaller charts obtained by the inverse mapping theorem). This can be done using the homeomorphism property. Indeed, if any sub-neighborhood of N_i, containing $A \cap N_i$ contains $f(x)$, $x \notin M_i$, this means that $f(M_i)$ is not open in $f(M)$. Hence, f^{-1} is not continuous.

Also, to obtain a normal atlas, we need to add some charts of N which cover the open set $N \setminus f(M)$ (and hence do not intersect $f(M)$). Here we use the condition on $f(M)$ to be closed. $\square$

Problem 2.45. Explain, why the above argument does not work for $8 \subset \mathbb{R}^2$ and $(0, 1) \subset \mathbb{R}^1 \subset \mathbb{R}^2$ (both are images of $(0, 1)$). Class

Remark 2.46. Generally, there are distinct opinions whether $(0,1)\times\{0\}\subset\mathbb{R}^2$ is a submanifold or not. The better answer is "not". Otherwise we need to consider a "normal collection of charts" instead of "normal atlas".

Theorem 2.47. (Very weak Whitney embedding theorem) *Let M be a smooth compact manifold. Then there exists a positive integer K and a strong embedding $f: M\to\mathbb{R}^K$.*

Proof. Suppose that $\{U_\alpha,\varphi_\alpha\}_{\alpha=1}^L$ is a finite atlas of M, $(x_\alpha^1,\dots,x_\alpha^m)$ is a local coordinate system in U_α such that $\varphi_\alpha: U_\alpha\approx B_\alpha=B_1(a_\alpha)\subset\mathbb{R}^m$, where $B_r(b)$ is the ball of radius r centered in b. Take a sufficiently small ε such that $\{U_\alpha^\varepsilon:=\varphi_\alpha^{-1}(B_\alpha^\varepsilon)\}$ still cover M, where $B_\alpha^\varepsilon:=B_{1-\varepsilon}(a_\alpha)$ (this is possible by normality). Now choose

$$f_\alpha\in C^\infty(\mathbb{R}^m),\qquad f_\alpha(\mathbb{R}^m)=[0,1],\qquad f_\alpha(P)=1\Leftrightarrow P\in\overline{B_\alpha^\varepsilon},\qquad \operatorname{supp} f_\alpha\subseteq B_\alpha.$$

Define $g_\alpha^k: M\to\mathbb{R}$, for $k=1,\dots,m$ and $\alpha=1,\dots,L$, by

$$g_\alpha^k(P):=\begin{cases} f_\alpha(\varphi_\alpha(P))x_\alpha^k(P), & \text{for } P\in U_\alpha;\\ 0, & \text{for } P\notin U_\alpha.\end{cases}$$

Then $g_\alpha^k(P)=x_\alpha^k(P)$, when $P\in U_\alpha^\varepsilon$. Thus, $m\cdot L$ functions g_α^k define a C^∞-map

$$g: M\to\mathbb{R}^{m\cdot L}.$$

Define now:

$$\varphi: M\to\mathbb{R}^K=\mathbb{R}^{m\cdot L+L},\qquad \varphi(P):=(\underbrace{\ g(P)\ }_{m\cdot L\text{ functions}};\underbrace{f_\alpha(\varphi_\alpha(P))}_{L\text{ functions}}).$$

Then $\operatorname{rk} d\varphi\geq\operatorname{rk} dg$. If $P\in U_\alpha^\varepsilon$, then

$$\operatorname{rk} dg|_P\geq\operatorname{rk}\left(\frac{\partial g_\alpha^k(P)}{\partial x_\alpha^j}\right)\geq\operatorname{rk}\left(\frac{\partial x_\alpha^k(P)}{\partial x_\alpha^j}\right)=m.$$

Since evidently $\operatorname{rk} d\varphi\leq m$, we have $\operatorname{rk} d\varphi\equiv m$. Thus, φ is an immersion.

Now prove that φ is injective, i.e. it is a bijection onto its image. Let $P\neq Q$. Then one can find α such that $P\in U_\alpha^\varepsilon$. Hence, $f_\alpha(\varphi_\alpha(P))=1$. If in this situation $f_\alpha(\varphi_\alpha(Q))<1$, then we are done. If $f_\alpha(\varphi_\alpha(Q))=1$, then $Q\in U_\alpha^\varepsilon$, and $g_\alpha^k(P)=x_\alpha^k(P)$, $g_\alpha^k(Q)=x_\alpha^k(Q)$. Since $P\neq Q$, there exists some coordinate with $x_\alpha^{k_0}(P)\neq x_\alpha^{k_0}(Q)$. Thus, $g_\alpha^{k_0}(P)\neq g_\alpha^{k_0}(Q)$ and $\varphi(P)\neq\varphi(Q)$.

Since M is compact and $\varphi(M)\subseteq\mathbb{R}^K$ is Hausdorff, by Theorem 1.72, φ is a homeomorphism onto its image. Also, the image is closed (as a compact set in a Hausdorff space). So, φ is a strong embedding. □

We formulate without proofs:

Theorem 2.48. (Weak Whitney theorem) *In the previous theorem one can take a not necessary compact manifold and $K=2\cdot\dim M+1$.*

Theorem 2.49. (Strong Whitney theorem) *In the previous theorem one can take $K=2\cdot\dim M$.*

3 Tangent bundle

Definition 3.1. Let $\dim M = m$. Define the *tangent bundle* $N = TM$ of M. As a set, N is formed by all couples (P, ξ), where $P \in M$ and $\xi \in T_P M$, i.e. ξ is a tangent vector at P. Topology and a structure of a smooth manifold are defined by some bijective maps of some subsets of N onto some open subsets of $\mathbb{R}^{2m}$. These maps are declared to be homeomorphisms and charts (hence, $\dim N = 2m$). Namely, if (U, φ) is some chart of M, then the corresponding subset of N is the set of all couples (P, ξ) with $P \in U$, and the corresponding map Φ to $\mathbb{R}^{2m}$ is defined as 05.10.2023

$$\Phi(P, \xi) = (x^1, \ldots, x^m; \xi^1, \ldots, \xi^m),$$

where

$$\varphi(P) = (x^1, \ldots, x^m), \quad \xi = \xi^1 \frac{\partial}{\partial x^1} + \cdots + \xi^m \frac{\partial}{\partial x^m},$$

i. e. ξ as a tangent vector (the first definition) puts in correspondence the collection ξ^i to the coordinate system $(x^1, \ldots, x^m)$ (or has coordinates ξ^i w.r.t. it). Then the local coordinate changes are the same as on M (for the first m coordinates) and with the help of the Jacobi matrix of the appropriate change (for the last m coordinates). In particular, the transition functions are smooth.

Problem 3.2. Check the details explicitly. Class

Problem 3.3. If M is a C^k-manifold, then $T_* M$ is a C^{k-1}-manifold. Class

4 Manifolds with boundary

Introduce the following notation:

$$\mathbb{R}^n_+ \subset \mathbb{R}^n, \qquad \mathbb{R}^n_+ := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\},$$
$$\mathbb{R}^{n-1}_0 := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n = 0\}.$$

We will say that a continuous function $f : \mathbb{R}^n_+ \to \mathbb{R}^1$ is *differentiable* in the following situation. For interior points ($x^n > 0$) we will conserve the usual notion. For boundary points ($\vec{x}_0 \in \mathbb{R}^{n-1}_0$, or $x^n = 0$) we will demand the property:

$$f(\vec{x}) = f(\vec{x}_0) + \sum_{i=1}^{n} f_i \cdot (x^i - x^i_0) + o(\vec{x} - \vec{x}_0), \qquad \lim_{\substack{\vec{x} \to \vec{x}_0 \\ x^n \geq 0}} \frac{o(\vec{x} - \vec{x}_0)}{\|\vec{x} - \vec{x}_0\|} = 0.$$

Then $f_i = \frac{\partial f}{\partial x^i}(\vec{x}_0)$, $(i = 1, 2, \ldots, n-1)$, and

$$f_n = \lim_{h \to +0} \frac{f(x^1_0, \ldots, x^{n-1}_0, x^n_0 + h) - f(x^1_0, \ldots, x^{n-1}_0, x^n_0)}{h} \tag{2}$$

(one-side partial derivative).

Definition 4.1. A second countable Hausdorff topological space M is called a *manifold with boundary,* if there exists its open cover $\{U_\alpha\}$ and coordinate homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n_+$, where $V_\alpha \subset \mathbb{R}^n_+$ are open subsets, such that the transition maps

$$\varphi_\beta \varphi_\alpha^{-1} : V_{\alpha\beta} = \varphi_a(U_\alpha \cap U_\beta) \to V_{\beta\alpha} = \varphi_b(U_\alpha \cap U_\beta)$$

are smooth in the above sense.

We call a point $P \in M$ *interior point,* if $x^n_\alpha(P) > 0$ and *boundary point,* if $x^n_\alpha(P) = 0$.

Home **Problem 4.2.** Is the notion of an interior point of a manifold related to the notion of an interior point from topology?

Lemma 4.3. *The definitions of boundary and interior points do not depend on the choice of (compatible) charts.*

Proof. Suppose the opposite: in a neighborhood of $P \in M$ two charts induce local coordinate systems $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ from $\mathbb{R}^n_{+,x}$ and $\mathbb{R}^n_{+,y}$, and we have $x^n(P) > 0$, but $y^n(P) = 0$. For these charts we have the corresponding coordinate homeomorphisms of a (maybe smaller) neighborhood $U \ni P$ onto $V \subset \mathbb{R}^n_x$ and $\tilde{V} \subset \mathbb{R}^n_{+,y}$, respectively (taking the intersection we can suppose that both homeomorphisms are defined on the same neighborhood). We have the corresponding transition map, i.e. a smooth homeomorphism $\varphi : V \to \tilde{V}$, $y^k = \varphi^k(x^1, \dots, x^n)$, satisfying

1. $y^n = \varphi^n(x^1, \dots, x^n) \geq 0$,
2. $y^n(P) = \varphi^n(x^1_0, \dots, x^n_0) = 0$.

Thus, $y^n = \varphi^n$ has its minimum at $(x^1_0, \dots, x^n_0)$. Since V is open in $\mathbb{R}^n_x$, the point $(x^1_0, \dots, x^n_0)$ is interior and the necessary conditions of a local extreme:

$$\left.\frac{\partial \varphi^n}{\partial x^i}\right|_{(x^1_0, \dots, x^n_0)} = 0, \qquad (i = 1, \dots, n).$$

But then $\det \left\| \left.\frac{\partial \varphi^j}{\partial x^i}\right|_{(x^1_0, \dots, x^n_0)} \right\| = 0$ and a smooth inverse does not exist, because for the above definition of one-side partial derivative (2) the differentiation rule still works (multiplication of Jacobi matrices). ∎

Definition 4.4. We call the *boundary* ∂M of a manifold with boundary M the set of all its boundary points.

Theorem 4.5. *The boundary of a manifold of dimension m is a manifold of dimension $m-1$.*

Proof. Take restrictions of charts to the boundary. ∎

Home **Problem 4.6.** Check all axioms.

5 Orientation

Definition 5.1. A manifold is called *oriented,* if an atlas is chosen such that all transition mappings have positive Jacobians. If it is possible to find such atlas on a manifold M, then M is called *orientable.*

Class **Problem 5.2.** A path *changing the orientation* is a closed path ($\gamma(0) = \gamma(1)$) such that there exists a collection of charts $U_1, \dots, U_k$, which cover this path, each chart intersects only with its two neighboring charts, the intersections are connected, and all Jacobians of transition maps are positive, except for one. Prove that a manifold is not orientable iff there exists a changing the orientation path for it.

Problem 5.3. A *local orientation* is a choice of orientation (i.e., a basis) in each tangent space. A local orientation is *locally constant,* if, for each connected chart U the standard basis ∂_i defines a local orientation (over this chart), which is either the same as the local orientation in all points, or is the opposite to it in all points. Prove that a (connected) manifold is orientable iff it has a locally constant local orientation. Class

Problem 5.4. A connected orientable manifold can be oriented exactly in two ways. Class

Problem 5.5. Prove that spheres S^n, for any n , and the torus T^2 are orientable. Home

Problem 5.6. Prove that any complex analytical manifold is orientable (as a real manifold). Home

Problem 5.7. Prove that a Möbius strip and the projective plane $\mathbb{R}P^2$ are non-orientable manifolds. Class

Theorem 5.8. *The boundary ∂M of an orientable manifold M is an orientable manifold.*

Proof. Suppose that an atlas $\{U_\alpha, (x_\alpha^1, \dots, x_\alpha^n)\}$ $(x_\alpha^n \geq 0)$ defines an orientation of M, i.e., $\det \left\| \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right\|_{i,j=1}^n > 0$. On ∂M one can take an atlas of the form $W_\alpha = U_\alpha \cap \partial M$ with local coordinates $(x_\alpha^1, \dots, x_\alpha^{n-1})$. Let us prove that it gives an orientation on ∂M, i.e., for any $P \in W_\alpha \cap W_\beta$, we have $\det \left\| \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right\|_{i,j=1}^{n-1} > 0$. Since on $W_\alpha \cap W_\beta$ we have $x_\alpha^n = x_\beta^n \equiv 0$, then $\frac{\partial x_\alpha^n}{\partial x_\beta^i} \equiv 0$, $i = 1, \dots, n-1$. Thus, we have at P:

$$0 < \det \left\| \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right\|_{i,j=1}^n = \det \left\| \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right\|_{i,j=1}^{n-1} \cdot \frac{\partial x_\alpha^n}{\partial x_\beta^n}. \tag{3}$$

Also at P:

$$\begin{aligned} \frac{\partial x_\alpha^n}{\partial x_\beta^n} &= \lim_{h \to +0} \frac{x_\alpha^n(x_\beta^1(P), \dots, x_\beta^n(P) + h) - x_\alpha^n(x_\beta^1(P), \dots, x_\beta^n(P))}{h} = \\ &= \lim_{h \to +0} \frac{x_\alpha^n(x_\beta^1(P), \dots, x_\beta^n(P) + h)}{h}. \end{aligned}$$

Since the expression under lim is positive, the limit is non-negative. Also, by (3), it is non-zero, hence it is positive: $\left. \frac{\partial x_\alpha^n}{\partial x_\beta^n} \right|_P > 0$. Now (3) implies $\det \left\| \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right\|_{i,j=1}^{n-1} > 0$. □

Example 5.9. The inverse is false: the Möbius strip is not orientable, while its boundary S^1 is orientable.

Definition 5.10. If M is oriented, we will call *canonical* the orientation constructed in the above proof.

6 Riemannian metric

Definition 6.1. A *Riemannian metric* on a manifold M is the correspondence g, which associates with each local coordinate system $(x_\alpha^1, \dots, x_\alpha^m)$ on U_α a collection of m^2 smooth functions $g_{ij}^\alpha : U_\alpha \to \mathbb{R}$ restricted to satisfy:

1) at each point $x \in U$ the matrix $\|g_{ij}\|$ is symmetric (non-degenerated) positively definite;

2) the tensor law is fulfilled: the functions g^{β}_{kl}, associated with a coordinate system $(x^1_{\beta}, \dots, x^m_{\beta})$, satisfy at each point of the intersection of coordinate neighborhoods $U_{\alpha} \cap U_{\beta}$ one has

$$g^{\beta}_{kl} = g^{\alpha}_{ij} \frac{\partial x^i_{\alpha}}{\partial x^k_{\beta}} \frac{\partial x^j_{\alpha}}{\partial x^l_{\beta}}$$

(with summation over the repeated indexes).

12.10.2022 The couple (M, g) is called a *Riemannian manifold.*

Home **Problem 6.2.** It is sufficient to verify the first condition at each point $P \in M$ only for one chart.

Definition 6.3. For our study of tensors it is convenient to introduce the following notation developing the Einstein one. We will denote the local coordinate systems by (U, φ), (U', φ'), (U'', φ''), etc. and the corresponding coordinates by $(x^1, \dots, x^m)$, $(x^{1'}, \dots, x^{m'})$, $(x^{1''}, \dots, x^{m''})$ etc. So, roughly speaking, $x^{i'}$ is in fact $x'^{i'}$. Also, as above, a summation over repeated indexes is supposed. In this notation the tensor transform laws for a vector and for a Riemannian metric will take the form:

$$\xi^{i'} = \xi^i \frac{\partial x^{i'}}{\partial x^i}, \qquad g_{i'j'} = g_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}.$$

Lemma 6.4. *A Riemannian metric g induces an inner product of (tangent) vectors $\vec{\xi}, \vec{\eta} \in T_P M$ by the equality*

$$\langle \vec{\xi}, \vec{\eta} \rangle := g(\vec{\xi}, \vec{\eta}) := g_{ij} \xi^i \eta^j.$$

Proof. Everything is evident, except for independence on local coordinates (i.e., that the product is well defined): $g_{ij} \xi^i \eta^j = g_{i'j'} \xi^{i'} \eta^{j'}$. This can be done directly dy the definition of a Riemannian metric and by the first definition of a tangent vector. □

Home **Problem 6.5.** Do this verification in full detail.

Definition 6.6. A *bilinear form* is a Riemannian metric without condition 1).

Home **Problem 6.7.** Prove the equivalence of definitions of a bilinear form at a point via the tensor law and as a form on the tangent space (in the linear-algebraic sense).

Definition 6.8. Suppose that $f : N \to M$ is a smooth map and g is a bilinear form (on tangent vectors to) M. Define the value of its *pull-back* or *inverse image* $f^* g$ on vectors $\vec{\xi}, \vec{\eta} \in T_P N$ by

$$(f^* g)(\vec{\xi}, \vec{\eta}) := g((df_P)\vec{\xi}, (df_P)\vec{\eta}).$$

In coordinates one can define the pull-back as follows. Suppose that $(x^1, \dots, x^n)$ are some coordinates in a neighborhood of P, $(y^1, \dots, y^m)$ are some coordinates in a neighborhood of $f(P)$, and $(f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n))$ is the corresponding coordinate form (a local representative map) of f. Then (in coordinates $(x^1, \dots, x^n)$)

$$(f^* g)_{ij} := g_{kl} \frac{\partial f^k}{\partial x^i} \frac{\partial f^l}{\partial x^j}.$$

Problem 6.9. Verify that these two definitions are equivalent. Home

Problem 6.10. Prove that if $i : N \to M$ is an immersion and g is a Riemannian metric on M, then i^*g is a Riemannian metric on N. Why this fails to be true for a general smooth map? Class

Definition 6.11. Let $i : N \hookrightarrow M$ be an inclusion of a submanifold N into a Riemannian manifold (M, g). Then i^*g is called the *induced* Riemannian metric on the submanifold N.

Theorem 6.12. *Each compact manifold M can be equipped with a Riemannian metric.*

Proof. Let $F : M \to \mathbb{R}^p$ be an embedding from Theorem 2.47. Then $F^*g_{\mathbb{R}^p}$ is a Riemannian metric on M. □

Problem 6.13. Prove this theorem directly with the help of a partition of unity (without a usage of an embedding). Home

7 Lie groups, matrix groups

Definition 7.1. A smooth manifold G is called a *Lie group* if it is a group such that the multiplication map $\mu : G \times G \to G$, $(g, h) \mapsto gh$, and the inverse map (*inversion*) $\mathrm{inv} : G \to G$, $\mathrm{inv}(g) = g^{-1}$, are C^∞ maps.

Example 7.2. The group $\mathrm{GL}(n, \mathbb{R})$ of all invertible real $n \times n$ matrices is a Lie group (*general linear group*). Indeed, a global chart on $\mathrm{GL}(n, \mathbb{R})$ is given by the n^2 functions x^i_j, where $x^i_j(A)$ is ij-th entry (or matrix element) of A. Multiplication is clearly smooth. For the inversion map one has $A^{-1} = \mathrm{adj}(A)/\det(A)$, where $\mathrm{adj}(A)$ is the adjoint matrix (whose entries are the cofactors). Thus, A^{-1} depends smoothly on the entries of A. Similarly, the group $\mathrm{GL}(n, \mathbb{C})$ of all invertible complex $n \times n$ matrices is a Lie group.

Problem 7.3. Let H be an open subgroup of G. Prove that H is closed. *Hint:* prove that the cosets gH, $g \in G$, are open. Deduce that the complement $G \setminus H$ is also open and hence H is closed. Home

Theorem 7.4. *If G is a connected Lie group and U is a neighborhood of the identity element e, then U generates the group (every element of G is a (finite) product of elements of U).*

Proof. We will prove that even the smaller neighborhood $V := \mathrm{inv}(U) \cap U$ generates G, where V is symmetric ($\mathrm{inv}(V) = V$). For any open W_1 and W_2 in G, the set $W_1W_2 = \{w_1w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}$ is an open set being a union of the open sets $\cup_{g \in W_1} gW_2$. In particular, the inductively defined sets $V^n = VV^{n-1}$, $n = 1, 2, \ldots$, are open. We have

$$e \in V \subseteq V^2 \subseteq \cdots \subseteq V^n \subseteq \cdots .$$

Evidently each V^n is symmetric and so also is the union $V^\infty := \cup_{n=1}^\infty V^n$. Also, V^∞ is closed under multiplication. Thus V^∞ is an open subgroup. Hence, it is also closed (Problem 7.3). Since G is connected, $G = B^\infty$. □

We will need the following intuitively clear statement.

Lemma 7.5. *Suppose that L is a submanifold of M, K is a submanifold of N, $f : M \to N$ is a smooth map such that $f(L) \subseteq K$. Then $f : L \to K$ is smooth.*

Proof. This can be easily verified in normal atlases. □

Home **Problem 7.6.** Do this.

Lemma 7.7. *If H is an abstract subgroup of a Lie group G that is also a manifold and has a cover by normal charts, then H is a closed Lie subgroup.*

Proof. The multiplication and the inversion on H are smooth by Lemma 7.5. It remains to prove that H is closed. Let $g_0 \in \overline{H}$ be arbitrary. Suppose that (U, φ) is a normal chart and $e \in U$, where e is the unity element. Define $\delta : G \times G \to G$ to be $\delta(g_1, g_2) = {g_1}^{-1} g_2$ and choose an open set V such that $e \in V \subset \overline{V} \subset U$. By continuity of the map δ we can find an open neighborhood O of e such that $O \times O \subset \delta^{-1}(V)$. Now if $\{h_i\}$ is a sequence in H converging to $g_0 \in \overline{H}$, then $g_0^{-1} h_i \to e$ and $g_0^{-1} h_i \in O$ for all sufficiently large i. Since $h_j^{-1} h_i = (g_0^{-1} h_j)^{-1} g_0^{-1} h_i$, we have $h_j^{-1} h_i \in V$ for sufficiently large i, j. For any sufficiently large fixed j, we have

$$\lim_{i \to \infty} h_j^{-1} h_i = h_j^{-1} g_0 \in \overline{V} \subset U.$$

Since (U, φ) is a normal chart, $U \cap H$ is closed in U. Thus since each $h_j^{-1} h_i$ is in $U \cap H$, we have $h_j^{-1} g_0 \in U \cap H \subset H$ for all sufficiently large j. Hence, g_0 in H and we are done. □

Definition 7.8. Let $O(n) \subset M(n, \mathbb{R})$ be the *orthogonal (matrix) group*:

$$O(n) = \{A \in M(n, \mathbb{R}) \colon A^T A = I\},$$

where $I = e$ is the unity matrix.

Let $U(n) \subset M(n, \mathbb{C})$ be the *unitary (matrix) group*:

$$U(n) = \{A \in M(n, \mathbb{C}) \colon \overline{A}^T A = I\}.$$

Let $SL(n, \mathbb{K}) \subset M(n, \mathbb{K})$ be the *special linear group*:

$$SL(n, \mathbb{K}) = \{A \in M(n, \mathbb{K}) \colon \det(A) = 1\}.$$

We define the *special orthogonal* and the *special unitary* (matrix) groups as

$$SO(n) = O(n) \cap SL(n, \mathbb{R}), \qquad SU(n) = U(n) \cap SL(n, \mathbb{C}).$$

Consider $\mathbb{K}^{2n}$ and a non-degenerate skew-symmetric $\mathbb{K}$-bilinear form, having the canonical form in the standard base:

$$(v, w)_{Sp} = \sum_{i=1}^{n} v^i w^{n+i} - \sum_{j=1}^{n} v^{n+j} w^j.$$

Then the *symplectic (matrix) groups* are given by

$$Sp(2n, \mathbb{K}) := \{A \in M(2n, \mathbb{K}) \colon (Av, Aw)_{Sp} = (v, w)_{Sp}\}.$$

Remark 7.9. One can prove that $Sp(2n, \mathbb{K}) \subset SL(2n, \mathbb{K})$, but this is not so easy (see e.g. `https://homepages.wmich.edu/~mackey/detsymp.pdf`).

Problem 7.10. Prove that $A \in Sp(2n, \mathbb{K})$ iff $A^T J A = J$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Home

Problem 7.11. Prove that the matrix groups from Definition 7.8 are Lie groups and closed Lie subgroups of $GL(n, \mathbb{K})$. Use Lemma 7.7 and Example 7.2. Class Home

Example 7.12. Another important example is abelian Lie groups. One can prove that any connected compact abelian group is n-torus $\mathbf{T}^n$. It also "comes from matrix groups" in the sense that

$$\mathbf{T}^n \cong S^1 \times \cdots \times S^1, \qquad S^1 \cong U(1).$$

Problem 7.13. Prove that a direct product of Lie groups is a Lie group. Home

Since our matrix groups G are realized as submanifolds of the full matrix algebra $i : G \hookrightarrow M(n, \mathbb{K}) \cong \mathbb{K}^{n^2}$ (i.e., as surfaces), we have a natural inclusion of tangent space $T_P G \subset T_{i(P)} M(n, \mathbb{K}) \cong \mathbb{K}^{n^2}$. In this sense one should understand the following problems.

Problem 7.14. Prove that the conditions in right column define $T_e G$ for the corresponding G in left column: Class Home

G	Conditions
$O(n)$	$A^T = -A$
$SO(n)$	$A^T = -A$
$U(n)$	$\overline{A}^T = -A$
$Sp(2n, \mathbb{K})$	$JA^T J = A$

Remark 7.15. In fact a choice of a base gives rise to an isomorphism between the algebra of linear mappings $V \to V$, where V is a $\mathbb{K}$-vector space of dimension n, and the algebra $M(n, \mathbb{K})$. So the above Lie groups (and some other) can be considered in a more general setting (see Ch. 5 of [Lee]).

8 Tensors: first definitions and properties

Definition 8.1. A *tensor field* of type (p, q) on a manifold M of dimension n is a correspondence, which to each coordinate system $(x) = (x^1, \ldots, x^n)$ on an open set U puts in correspondence a system of n^{p+q} smooth functions $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ on U, called *components,* such that for any two coordinate systems (x) and (x') the components on $U \cap U'$ satisfy *the tensor law* 19.10.2023

$$T^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q} = T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \cdot \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}}.$$

Definition 8.2. Consider two tensor fields T and S of type (p, q). Their *sum* $T + S$ is defined by

$$(T + S)^{i_1 \ldots i_p}_{j_1 \ldots j_q} := T^{i_1 \ldots i_p}_{j_1 \ldots j_q} + S^{i_1 \ldots i_p}_{j_1 \ldots j_q}.$$

Lemma 8.3. *$T + S$ is a tensor of type (p, q).*

Problem 8.4. Verify this. Home

Definition 8.5. If $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is a tensor field on M and $f \in C^\infty(M)$, then evidently the *product* of function and tensor $f \cdot T : (x^1, \ldots, x^n) \rightsquigarrow f \cdot T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is a tensor field of type (p, q).

Class **Problem 8.6.** Prove that any tensor of type $(1,1)$, which is invariant under orthogonal coordinate changes, is a scaling of δ^i_j (i.e. is equal to $\lambda\delta^i_j$).

Home **Problem 8.7.** Prove that any tensor with $p+q=3$ invariant w.r.t. any coordinate changes is equal to 0.

Class **Problem 8.8.** Prove that a tensor field of type $(1,1)$ gives a linear operator in each point.

Home **Problem 8.9.** Prove that C^i_i, $C^i_j C^j_i$, $C^i_j C^j_k C^k_i$, can be expressed in terms of coefficients of the polynomial $\det(C-\lambda E)$.

Class **Problem 8.10.** For a smooth function f, $\operatorname{grad} f$ is a tensor of type $(0,1)$.

Definition 8.11. A tensor field of type $(0,1)$ is called a *covector field.*

By a problem above $dx^i = \operatorname{grad} x^i$ is a covector (over a coordinate neighborhood).

Home **Problem 8.12.** Covectors are linear functionals on vectors (at each point).

Home **Problem 8.13.** The bases $\{\frac{\partial}{\partial x^i}\}$ in $T_P M$ and $\{dx^j\}$ in $T^*_P M$ are dual to each other.

Consider a $C^\infty(M)$-linear map $L(v_1,\dots,v_q;a^1,\dots,a^p)$ which arguments are q vector and p covector fields, and taking values in $C^\infty(M)$. Consider the following correspondences

$$T\mapsto L_T,\qquad L_T(v_1,\dots,v_q;a^1,\dots,a^p) := T^{i_1\dots i_p}_{j_1\dots j_q} v_1^{j_1}\dots v_q^{j_q}\cdot a^1_{i_1}\dots a^p_{i_p},$$

and

$$L\mapsto T_L,\qquad T_L:(x^1,\dots,x^n)\rightsquigarrow (T_L)^{i_1\dots i_p}_{j_1\dots j_q} := L\left(\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_q}};\ dx^{i_1},\dots,dx^{i_p}\right).$$

Class **Problem 8.14.**

1. Explain, how it is possible to substitute locally defined fields at the place of globally defined.
2. L_T is a multilinear function and does not depend on the choice of coordinate system.
3. T_L satisfies (p,q)-tensor law.
4. These maps are inverse to each other.

Definition 8.15. A field S of type (p,q) is obtained from a field T of type (p,q) by a *transposition of upper* (similarly – for lower) *indexes* with numbers (positions) a and b, if $S^{i_1\dots i_a\dots i_b\dots i_p}_{j_1\dots j_q} = T^{i_1\dots i_b\dots i_a\dots i_p}_{j_1\dots j_q}$.

The result is a tensor field. This is evident if we consider multilinear maps.

Home **Problem 8.16.** Show by example that a transposition of an upper and a lower indexes is not a tensor operation. Consider the case of a tensor of type $(1,1)$ (linear operator). Conclude in particular that the property of a matrix of an operator to be symmetric $C^i_j = C^j_i$ depends on coordinate system.

Definition 8.17. A *contraction of a tensor* T of type (p,q) in the upper index number a and the lower index number b is a tensor S of type $(p-1,q-1)$, defined by

$$S^{i_1\dots i_{p-1}}_{j_1\dots j_{q-1}} := \sum_i T^{i_1\dots i_{a-1} i i_a\dots i_{p-1}}_{j_1\dots j_{b-1} i j_b\dots j_{q-1}}.$$

This is really a tensor field of type $(p-1, q-1)$, because

$$L_S(v_1, \dots, v_{q-1}; a^1, \dots, a^{p-1}) =$$

$$= \sum_i L_T \left(v_1, \dots, v_{a-1}, \frac{\partial}{\partial x^i}, v_a, \dots, v_{q-1}; a^1, \dots, a^{b-1}, dx^i, a^b, \dots, a^{p-1} \right),$$

and

$$\sum_i \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} = 1,$$

hence the right-hand side does not depend on the choice of coordinate system.

Example 8.18. A contraction C_i^i of a tensor of type $(1,1)$ is the trace of a linear operator.

Definition 8.19. The *tensor product* $T \otimes S$ of a tensor field T of type (p,q) and a tensor field S of type (r,t) is a tensor field of type $(p+r, q+t)$, defined by

$$(T \otimes S)^{i_1,\dots,i_{p+r}}_{j_1,\dots,j_{q+t}} := T^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot S^{i_{p+1},\dots,i_{p+r}}_{j_{q+1},\dots,j_{q+t}}.$$

The corresponding multilinear map $L_{T\otimes S}$ is simply the product of L_T and L_S. Hence, it is a multilinear map (for appropriate variables). Thus, $T \otimes S$ is really a tensor field.

Problem 8.20. Suppose that a tensor field X is of type $(1,0)$ and W is of type $(0,1)$. Find the rank of $X \otimes W$. Home

Problem 8.21. Prove that locally, for any coordinate system, one has the following presentation Class

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}.$$

The coefficients are determined uniquely.

Definition 8.22. A tensor field b_{ij} of type $(0,2)$ is *non-degenerate* (or *non-singular*), if $\det \|b_{ij}\| \neq 0$.

Problem 8.23. Verify that this condition does not depend on coordinate system. Home

Problem 8.24. Prove that the components of its inverse matrix b^{jk} (i.e., $b^{jk} b_{ki} = \delta_i^j$), form a tensor of type $(2,0)$. Home

Definition 8.25. The operation of *index raising* of a tensor T of type (p,q) with the help of b is the composition of tensor product with b and contraction. The result S is a tensor of type $(p+1, q-1)$. For example, for the first index:

$$S^{i_1 \dots i_{p+1}}_{j_1,\dots,j_{q-1}} := b^{i_1 i} T^{i_2 \dots i_{p+1}}_{i\, j_1,\dots,j_{q-1}}.$$

Similarly one can define the *index lowering*:

$$S^{i_1 \dots i_{p-1}}_{j_1,\dots,j_{q+1}} := b_{j_1 i} T^{i\, i_1 \dots i_{p-1}}_{j_2,\dots,j_{q+1}}.$$

Definition 8.26. Define the *symmetrization* of a tensor field T of type $(0,q)$ as

$$\mathrm{Sym}\,(T)_{j_1,\dots,j_q} = T_{(j_1,\dots,j_q)} = \frac{1}{q!} \sum_{\sigma \in S_q} T_{j_{\sigma(1)},\dots,j_{\sigma(q)}},$$

and the *antisymmetrization* as

$$\mathrm{Alt}\,(T)_{j_1,\dots,j_q} = T_{[j_1,\dots,j_q]} = \frac{1}{q!} \sum_{\sigma \in S_q} (-1)^\sigma\, T_{j_{\sigma(1)},\dots,j_{\sigma(q)}}.$$

Evidently these maps are tensor operations (as a compositions of tensor operations). The result of the symmetrization (resp., antisymmetrization) is a *symmetric* (resp., *alternating*) tensor field of the same type, i.e. its components do not change under a transposition of two neighboring indices (resp., change the sign under a transposition of two neighboring indices).

Home **Problem 8.27.** Prove that the antisymmetrization is a linear map, which is a projection onto the subspace of alternating tensors and all symmetric tensors belong to its kernel.

Lemma 8.28. *An alternating tensor field $T_{i_1\dots i_n}$ on M, $\dim M = n$ (i.e., a field of maximal degree) is defined by only one its component (essential) $T_{12\dots n}$. The other non-zero components differ from it by a sign ± 1. More precisely,*

$$T_{i_1\dots i_n} = T_{\sigma(12\dots n)} = (-1)^\sigma T_{12\dots n}.$$

The essential component of T at a point in some other coordinate system is obtained by multiplication by the Jacobian of the appropriate coordinate change.

Proof. The first statement follows from the definition. The second one:

$$T_{1'\dots n'} = T_{i_1\dots i_n}\cdot\frac{\partial x^{i_1}}{\partial x^{1'}}\cdots\frac{\partial x^{i_n}}{\partial x^{n'}} = \left(\sum_\sigma (-1)^\sigma \frac{\partial x^{\sigma(1)}}{\partial x^{1'}}\cdots\frac{\partial x^{\sigma(n)}}{\partial x^{n'}}\right) T_{12\dots n} = \det\left\|\frac{\partial x^i}{\partial x^{i'}}\right\|\cdot T_{12\dots n}.$$

□

Definition 8.29. Define the *exterior product* (or *wedge product*) $R = T\wedge P$ of two alternating tensors $T_{i_1\dots i_k}$ and $P_{i_1\dots i_q}$ by formula

$$R_{i_1\dots i_{k+q}} = const\cdot T_{[i_1\dots i_k}P_{i_{k+1}\dots i_{k+q}]} = \frac{1}{k!\,q!}\sum_{\sigma\in S_{k+q}}(-1)^\sigma T_{\sigma(i_1\dots i_k}P_{i_{k+1}\dots i_{k+q})}.$$

Up to scaling this is a composition of tensor product and antisymmetrization.

For alternating tensors of type $(0,q)$ one can use the language of differential forms. We have by the definition of exterior product (for any putting of brackets)

$$dx^{i_1}\wedge\dots\wedge dx^{i_q} = \sum_{\sigma\in S_q}(-1)^\sigma dx^{\sigma(i_1}\otimes\dots\otimes dx^{i_q)}.$$

Home **Problem 8.30.** Verify this (first solve the next problem).

Home **Problem 8.31.** Prove the associativity of the exterior product.

Class **Problem 8.32.** Prove that the exterior products $dx^{i_1}\wedge\dots\wedge dx^{i_q}$, $i_1 < i_2 < \dots < i_q$ form a base of the space of alternating tensors of type $(0,q)$ (at a point). Find the dimension of this space.

Home **Problem 8.33.** Find the dimension of the space of symmetric tensors of type $(0,q)$ (at a point). Using Problems 8.32 and 8.27 study whether the space of all tensors of type $(0,q)$ (at a point) is a direct sum of symmetric and alternating tensors.

Then the decomposition of an alternating tensor w.r.t. the above base is:

$$\begin{aligned} T &= T_{i_1\dots i_q}dx^{i_1}\otimes\dots\otimes dx^{i_q} = \sum_{i_1<\dots<i_q}\sum_{\sigma\in S_q} T_{\sigma(i_1)\dots\sigma(i_q)}dx^{\sigma(i_1)}\otimes\dots\otimes dx^{\sigma(i_q)} = \\ &= \textstyle\sum_{i_1<\dots<i_q}\displaystyle\sum_{\sigma\in S_q}(-1)^\sigma T_{i_1\dots i_q}dx^{\sigma(i_1)}\otimes\dots\otimes dx^{\sigma(i_q)} = \sum_{i_1<\dots<i_q} T_{i_1\dots i_q}dx^{i_1}\wedge\dots\wedge dx^{i_q}. \quad (4)\end{aligned}$$

This is called a representation of an alternating tensor as a *differential form*. Since the above products form a base, the decomposition (4) is unique.

Problem 8.34. Verify that the exterior product of differential forms can be found in the following way: multiply the expressions and then order the differentials (keeping in mind sign changes). Home

Problem 8.35. (a corollary of Lemma 8.28) The expression $\sqrt{\det \|g_{ij}\|}dx^1 \wedge \ldots \wedge dx^n$ is a tensor w.r.t. coordinate changes with positive Jacobian, where g_{ij} is a Riemannian metric. Class

This tensor is called a *volume form*. Later we will introduce the concept of integration and will calculate the volume of a Riemannian manifold using its volume form.

Problem 8.36. Represent the trace of a matrix as a result of tensor operations. Home

Problem 8.37. Represent the determinant of a matrix as a result of tensor operations. Home

Problem 8.38. Find the type of tensors formed by coefficients of Class

1. vector product,
2. mixed (triple) product

of vectors in $\mathbb{R}^3$. Prove that these tensors are obtained from each other by index raising and lowering.

9 Fiber bundles

9.1 General definitions

First, consider the case of topological spaces. 26.10.2023

Definition 9.1. A (locally trivial) *fiber bundle* is a 5-tuple $\xi = (E, B, p, F, G)$, where E, B, F are topological spaces, $p : E \to B$ is a continuous surjection, G is a topological group being a subgroup of $\mathrm{Homeo}(F)$ (homeomorphism group as an abstract group), such that there is an open cover U_α of B and homeomorphisms $\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ restricted to satisfy

1) the diagram

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\Phi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \times F \arrow[dl, "p_1"] \\ & U_\alpha & \end{tikzcd}$$

is commutative, where p_1 is the projection on the first factor (this implies that each *fiber* $E_b = p^{-1}(b)$ is homeomorphic to F);

2) over an intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have by 1) the commutative diagram

$$\begin{tikzcd} U_{\alpha\beta} \times F \arrow[rr, "\Phi_\alpha \circ (\Phi_\beta)^{-1}"] \arrow[dr, "p_1"'] & & U_{\alpha\beta} \times F \arrow[dl, "p_1"] \\ & U_{\alpha\beta} & \end{tikzcd}$$

which gives rise to a map

$$\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Homeo}(F), \qquad \Phi_{\alpha\beta}(P)(f) = p_2(\Phi_\alpha \circ (\Phi_\beta)^{-1}(P, f)), \tag{5}$$

and the condition is: $\Phi_{\alpha\beta}(P) \in G \subseteq \mathrm{Homeo}(F)$ for each $P \in U_{\alpha\beta}$;

3) $\Phi_{\alpha\beta} : U_{\alpha\beta} \to G$ is continuous.

In this situation E is the *total space*, B is the *base*, F is the *typical fiber*, G is the *structure group*, and p is the *projection* of ξ. The couple (U_α, Φ_α) is called a *local trivialization*.

Remark 9.2. Sometimes it is more convenient (cf. the tangent bundle) to have local trivializations "of second type": they are defined by two homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$ and $\Phi_\alpha : p^{-1}(U_\alpha) \to V_\alpha \times F$ in such a way that the diagram

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[d, "p"'] & V_\alpha \times F \arrow[d, "p_1"] \\ U_\alpha \arrow[r, "\varphi_\alpha"] & V_\alpha. \end{tikzcd}$$

commutes. If $V_\alpha = U_\alpha$ and $\varphi_\alpha = \mathrm{Id}$ we obtain the above definition.

At the first glance this seems a distinct definition, but this is not the case:

Home **Problem 9.3.** Reformulate in detail the items of the above definition to the case of "second type". Using another trivializations, namely

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[d, "p"'] & V_\alpha \times F \arrow[r, "\varphi_\alpha^{-1}\times \mathrm{Id}"] \arrow[d, "p_1"] & U_\alpha \times F \arrow[d, "p_1"] \\ U_\alpha \arrow[r, "\varphi_\alpha"] & V_\alpha \arrow[r, "(\varphi_\alpha)^{-1}"] & U_\alpha, \end{tikzcd}$$

prove that the two definitions are equivalent.

Definition 9.4. For a *smooth fiber bundle* we require in addition: E, B, F are smooth manifolds, $G \subseteq \mathrm{Diffeo}(F)$ (diffeomorphism group) and all mappings are smooth.

Home **Problem 9.5.** Suppose that we do not require E to be a smooth manifold in the previous definition. Nevertheless it will be automatically smooth if other conditions are fulfilled (cf. the construction of tangent bundle).

Example 9.6. The simplest examples are given by *trivial bundles* $E = F \times B \to B$, in particular, $B \to B$, $F = pt$.

Definition 9.7. Let $\xi_1 = (E_1, M_1, \pi_1, F, G)$ and $\xi_2 = (E_2, M_2, \pi_2, F, G)$ be two smooth fiber bundles with local trivializations $\{(U_\alpha, F_\alpha)\}$ and $\{(\widetilde{U}_\beta, \widetilde{\Phi}_\beta)\}$ respectively. A pair $(\widehat{h}, h)$ is a *bundle morphism* along h if

1) $h : M_1 \to M_2$, $\widehat{h} : E_1 \to E_2$ are smooth maps;

2) $\widehat{h}$ maps $(E_1)_P$ to $(E_2)_{h(P)}$, i.e., the following diagram is commutative:

$$\begin{tikzcd} E_1 \arrow[r, "\widehat{h}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "h"] & M_2. \end{tikzcd}$$

3) if $U_\alpha \cap h^{-1}(\widetilde{U}_\beta) \neq \varnothing$, there exists a smooth map $h_{\alpha\beta} : U_\alpha \cap h^{-1}(\widetilde{U}_\beta) \to G$ such that for each $P \in U_\alpha \cap h^{-1}(\widetilde{U}_\beta)$ one has

$$\left(\widetilde{\Phi}_\beta \circ \widehat{h} \circ \Phi_\alpha^{-1}\right)(P, f) = (h(P), h_{\alpha\beta}(P)f) \text{ for all } f \in F.$$

(this implies that $\widehat{h} : (E_1)_P \to (E_2)_{h(P)}$ is a diffeomorphism)

The notions of an *identity* morphism and an *inverse* morphism are evident. An invertible morphism is an *isomorphism*.

Definition 9.8. A *smooth section*of a smooth bundle$\xi = (E, M, p, F, G)$ is a smooth map $s : M \to E$ such that $p \circ s = \mathrm{Id}_M$. The set of all smooth sections is denoted by $\Gamma(\xi)$ or $\Gamma^\infty(\xi)$.

For a topological fiber bundle one defines a *continuous section* in the same way. A *local section* is defined only on an open set U.

Remark 9.9. Sometimes the set of sections is empty (see Problem 9.26 below).

9.2 Cocycle approach

Evidently one has:

Lemma 9.10. *The above defined functions $\Phi_{\alpha\beta} : U_{\alpha\beta} \to G$ have the following properties (called* COCYCLE PROPERTIES*):*

$$\begin{aligned} \Phi_{\alpha\alpha}(P) &= e \in G \textit{ for } P \in U_\alpha, \\ \Phi_{\alpha\beta}(P) &= (\Phi_{\beta\alpha}(P))^{-1} \textit{ for } P \in U_\alpha \cap U_\beta, \\ \Phi_{\alpha\beta}(P)\Phi_{\beta\gamma}(P)\Phi_{\gamma\alpha}(P) &= e \textit{ for } P \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Definition 9.11. An open cover U_α of a topological space X (resp., a manifold M) and a system of continuous (smooth in the case of manifolds) functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$, where G is a topological group (a Lie group in the case of manifolds) acting effectively on a topological space F by homeomorphisms (respectively, on a smooth manifold M by diffeomorphisms) is called a *cocycle* if it has the properties from Lemma 9.10, i.e.,

$$\begin{aligned} g_{\alpha\alpha}(P) &= e \in G \text{ for } P \in U_\alpha, \\ g_{\alpha\beta}(P) &= (g_{\beta\alpha}(P))^{-1} \text{ for } P \in U_\alpha \cap U_\beta, \\ g_{\alpha\beta}(P)g_{\beta\gamma}(P)g_{\gamma\alpha}(P) &= e \text{ for } P \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Here by an *action* we call a group homomorphism $\lambda : G \to \mathrm{Homeo}(F)$ such that the map $G \times F \to F$, $(g, f) \mapsto \lambda(g)(f)$ is continuous. The action is *effective* if $\operatorname{Ker} \lambda = \{e\}$, i.e., λ is a monomorphism. So, in most part of situations we can think about G as about a subgroup of $\mathrm{Homeo}(G)$. Similarly, one defines in the smooth case.

Definition 9.12. If $p : X \to X/\sim$ is a surjective map, where $\sim$ is an equivalence relation on a topological space X, then the *quotient topology* on $Y = X/\sim$ is defined as follows. A subset $U \subseteq Y$ is open iff $p^{-1}(U)$ is open in X. Roughly speaking this is the maximal topology such that p is continuous.

Theorem 9.13. *Suppose that M and F are smooth manifolds, G is a Lie group, λ is an action of G on F, U_α is an open cover of M, $g_{\alpha\beta} : U_{\alpha\beta} \to G$ is a cocycle. Then there exists a fiber bundle ξ over M with typical fiber F and structure group G such that for some local trivialization atlas $\{(U_\alpha,)\}$ one has over $U_{\alpha\beta} = U_\alpha \cap U_\beta$*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(P, f) = (P, \lambda(g_{\alpha\beta}(P))(f)).$$

Proof. On the disjoint union $\Sigma := \sqcup_\alpha \{\alpha\} \times U_\alpha \times F$ define an equivalence relation (Problem 9.14) by

$$\{\alpha\} \times U_\alpha \times F \ni (\alpha, P, f) \sim (\beta, P', f') \in \{\beta\} \times U_\beta \times F \Leftrightarrow P = P' \text{ and } f = \lambda(g_{\alpha\beta}(P))(f').$$

Take $E := \Sigma/\sim$ with the quotient topology and the natural projection $\Pi : \Sigma \to E$. Let $\pi : E \to M$ be the projection induced by $(\alpha, P, f) \mapsto P$ (Problem 9.15). Since

$$\Pi^{-1}\Pi(\{\alpha\} \times U_\alpha \times F) = \sqcup_\beta \{\beta\} \times (U_\alpha \cap U_\beta) \times F$$

is open, the sets $\Pi(\{\alpha\} \times U_\alpha \times F) = \pi^{-1}(U_\alpha)$ are open and one can define local trivializations in a natural way:

$$\Phi_\alpha[(\alpha, P, f)] := (P, f) \text{ for } [(\alpha, P, f)] \in \pi^{-1}(U_\alpha).$$

We need to verify that the map is well defined: if $(\alpha, P, f) \sim (\alpha, Q, f')$ then $P = Q$ (this follows immediately from the definition of $\sim$) and $f = f'$ (this follows from the first cocycle property $f = \lambda(g_{\alpha\alpha}(P))(f') = f'$). In fact this means that Π is injective on each $\{\alpha\} \times U_\alpha \times F$. Let us find transition functions. Suppose that $P \in U_\alpha \cap U_\beta$. Then $\Phi_\beta^{-1}(P, f) = [(\beta, P, f)]$. Since $P \in U_\alpha \cap U_\beta$, then $[(\beta, P, f)] = [(\alpha, Q, f')]$. By the definition of $\sim$, $Q = P$ and $f' = \lambda(g_{\alpha\beta}(P))(f)$. Hence, $\Phi_\alpha \circ \Phi_\beta^{-1}(P, f) = (P, \lambda(g_{\alpha\beta}(P))(f))$. It remains to verify some details (Problems 9.16, 9.17). $\square$

Home **Problem 9.14.** Using the cocycle properties prove that this is an equivalence relation (axioms of identity, reflexivity and transitivity)

Home **Problem 9.15.** Prove that π is well defined.

Home **Problem 9.16.** Prove that E is second countable and Hausdorff.

Home **Problem 9.17.** Prove that all the necessary maps in the proof are smooth. Then E is a manifold by Problem 9.5.

Home **Problem 9.18.** Formulate and prove a similar theorem for the topological case.

Remark 9.19. We will not discuss the conditions for two cocycles to determine isomorphic fiber bundles in the general case.

Class **Problem 9.20.** Consider the Möbius band E_M as the following quotient space of $\mathbb{R} \times (-1, 1)$:

$$E_M = (\mathbb{R} \times (-1, 1))/\sim, \text{ where } (x, t) \sim (x + 2\pi n, (-1)^n t), \quad n \in \mathbb{Z}.$$

For

$$S^1 = \mathbb{R}/\approx, \text{ where } x \approx x + 2\pi n, \quad n \in \mathbb{Z},$$

define $\pi : E_M \to S^1$ by $\pi([x,t]) = [x]$. Prove that this is a fiber bundle. Find an appropriate cocycle with $G = \mathbb{Z}_2$, $F = (-1,1)$.

Problem 9.21. Using the same cocycle on S^1 and $\lambda : \mathbb{Z}_2 \to \mathrm{Diffeo}(S^1)$, $\lambda(-1)(z) = -z$ (as complex numbers) take $F = S^1$ and obtain a fiber bundle (twisted torus). Prove that it is not isomorphic to the trivial bundle $S^1 \times S^1 \to S^1$ as a bundle with structure group $\mathbb{Z}_2$, but isomorphic to the trivial bundle as a bundle with structure group $U(1) = S^1$. Class

9.3 Coverings

Definition 9.22. In some sense the most simple case is that of discrete F (typically, finite or countable). These fiber bundles are called *coverings*.

Problem 9.23. Prove that $\pi : \mathbb{R} \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(t) = e^{2\pi i t}$, is a covering with $F = \mathbb{Z}$. Class

Problem 9.24. Prove that $\pi : S^1 \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(z) = z^2$, is a covering with $F = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Class

Problem 9.25. Find appropriate cocycles for these two examples. Home

Problem 9.26. In the above examples there is no sections. Home

Remark 9.27. Let us note without proving that each path $\gamma(t)$ in the base B of a covering has a unique (up to the choice of starting point) *covering path* $\widetilde{\gamma}(t)$ in E such that $p\widetilde{\gamma}(t) = \gamma(t)$ at any t. This is not a section! (cf. Problem 9.26). Think about this.

9.4 Vector bundles

02.11.2023

Definition 9.28. Consider an n-dimensional vector space V over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $G = \mathrm{Aut}(V) = \mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{K})$ acting on V in a natural way. Then $\xi = (E, \pi, B, V, G)$ is a *vector bundle* (topological or smooth).

Theorem 9.29. *Consider vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ with the same typical fiber V and cocycles (transition maps) $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$ and $\Phi'_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$, respectively, for the same cover $\{U_\alpha\}$. These bundles are isomorphic iff there are smooth functions $f_\alpha : U_\alpha \to \mathrm{GL}(V)$ such that*

$$\Phi'_{\alpha\beta}(P) = f_\alpha(P)\Phi_{\alpha\beta}(P)(f_\beta(P))^{-1}, \quad P \in U_{\alpha\beta}. \tag{6}$$

Proof. If $f : E \to E'$ is an isomorphism, define $f_\alpha(P)(v) := p_2(\Phi'_\alpha \circ f \circ (\Phi_\alpha)^{-1}(P,v))$. Then

$$f_\alpha(P)\Phi_{\alpha\beta}(P)(f_\beta(P))^{-1}(v) = p_2(\Phi'_\alpha \circ f \circ (\Phi_\alpha)^{-1})(\Phi_\alpha \circ (\Phi_\beta)^{-1})(P, (\Phi_\beta \circ f^{-1} \circ (\Phi'_\beta)^{-1})(P)v)$$
$$= p_2(\Phi'_\alpha \circ (\Phi'_\beta)^{-1})(P,v) = \Phi'_{\alpha\beta}(P)(v)$$

and we have (6).

If we have (6), define

$$\widetilde{f}_\alpha : U_\alpha \times V \to U_\alpha \times V, \quad (P,v) \mapsto (P, f_\alpha(P)v).$$

Then define locally (for $e \in \pi^{-1}(U_\alpha)$) a bundle map $f : E \to E'$ by

$$f(e) = \left((\Phi'_\alpha)^{-1} \circ \widetilde{f}_\alpha \circ \Phi_\alpha\right)(e).$$

One can verify that f is well defined globally (using (6)) and defines a vector bundle isomorphism. □

Home **Problem 9.30.** Complete the proof.

Example 9.31. The tangent bundle TM is an example of a vector bundle.

Our main example (generalizing the above one) is the tensor bundle of type (p, q) over M. We consider a slightly general construction, considering not only $E = TM$ as the initial bundle. So we consider a real rank k vector bundle $\xi = (E, \pi, M, \dots)$.

Definition 9.32. The total space (as a set) is $T^r_s(\xi) = \sqcup_{P\in M} T^r_s(E_P)$, where $T^r_s(E_P)$ is the k^{r+s}-dimensional real vector space of all (r, s) tensors on the k-dimensional linear space E_P. For each local trivialization (U, Φ) of ξ, $\Phi : \pi^{-1}U \to U \times \mathbb{R}^k$, define the local trivialization

$$\Phi^r_s : \sqcup_{P\in U} T^r_s(E_P) \to U \times T^r_s(\mathbb{R}^k),$$

$$L_{p_2\Phi^r_s(\tau)}(a^1, \dots, a^r, v_1, \dots, v_s) = L_\tau(\Phi^* a^1, \dots, \Phi^* a^r, d\Phi^{-1} v_1, \dots, d\Phi^{-1} v_s)$$

for any smooth covector fields a^i and vector fields v_j on $\Phi(U)$.

Home **Problem 9.33.** Verify the details (similarly to the construction of TM).

Remark 9.34. In other words we define smooth sections of $T^r_s(\xi)$ to be such maps $P \mapsto \tau_P \in T^r_s(E_P)$ that $P \mapsto L_{\tau_P}(a^1, \dots, a^r, v_1, \dots, v_s)$ is smooth for any smooth covector fields a^i and vector fields v_j (see Subsection 9.7 for more detail).

9.5 Principal bundles

Definition 9.35. If $F = G$ and $\lambda(g)f = gf$, a bundle is called a *principal bundle*.

Home **Problem 9.36.** In this case one has a canonical right action of G on E with orbits eG being fibers.

Note that the same cocycle can define bundles with distinct fibers. In particular, a $\mathrm{GL}(n, \mathbb{K})$-valued cocycle defines a vector bundle and a principal bundle.

Class **Problem 9.37.** (Hopf's bundle) Consider S^{2n-1} as the subset of $\mathbb{C}^n$ given by $S^{2n-1} = \{z \in \mathbb{C}^n : \|z\| = 1\}$, where $z = (z^1, \dots, z^n)$ and $\|z\| = \sum \overline{z^i} z^i$. Let $S^1 = U(1)$ act on S^{2n-1} by $(a, z) \mapsto az = (az^1, \dots, az^n)$. The quotient (the space of orbits) is $\mathbb{C}P^{n-1}$. We obtain the *Hopf map* $\pi_n : S^{2n-1} \to \mathbb{C}P^{n-1}$. Prove that this is a principal $U(1)$-bundle (*Hopf bundle*).

9.6 Operations on vector bundles

Definition 9.38. The *Whitney sum* $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$ of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is defined in the following way. As a set $E_1 \oplus E_2 = \sqcup_{P\in M}(E_1)_P \oplus (E_2)_P$ and for charts $(\Phi_1)_\alpha : (\pi_1)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{k_1}$ and $(\Phi_2)_\alpha : (\pi_2)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{k_2}$ of local trivializations of π_1 and π_2, respectively we define

$$(\Phi_1)_\alpha \oplus (\Phi_2)_\alpha : (v_P, w_P) \mapsto (P, p_2((\Phi_1)_\alpha(v_P)), p_2((\Phi_2)_\alpha(w_P))), \quad v_P \in (E_1)_P, \quad w_P \in (E_2)_P.$$

Home **Problem 9.39.** Verify that this is a structure of a (smooth or topological) vector bundle.

Home **Problem 9.40.** Prove that the Whitney sum can be defined using cocycles in the following way. Suppose that $\{g_{\alpha\beta}\}$ is a cocycle for π_1 and $\{h_{\alpha\beta}\}$ is a cocycle for π_2 for the same cover. Then

$$g_{\alpha\beta} \oplus h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{K}^{k_1} \oplus \mathbb{K}^{k_2}), \quad (g_{\alpha\beta} \oplus h_{\alpha\beta})(P) : (v, w) \mapsto (g_{\alpha\beta}(P)v, h_{\alpha\beta}(P)w)$$

is a cocycle for $\pi_1 \oplus \pi_2$.

Recall that the *tensor product* $V \otimes W$ of linear spaces V and W is the quotient space of the space $V \odot W$ of formal $\mathbb{K}$-linear combinations of elements $v \odot w$ by the subspace generated by elements:

- $(v_1 + v_2) \odot w - v_1 \odot w - v_2 \odot w$,
- $v \odot (w_1 + w_2) - v \odot w_1 - v \odot w_2$,
- $(sv) \odot w - s(v \odot w)$,
- $v \odot (sw) - s(v \odot w)$,

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $s \in \mathbb{K}$. The class of $v \odot w$ is denoted by $v \otimes w$.

Problem 9.41. Let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ be linear maps of finite-dimensional vector spaces. Then the formula $(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$ defines a well-defined linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$. If f_1 and f_2 are isomorphisms then so is $f_1 \otimes f_2$. Home

If V has a base $e_1, \dots, e_n$ and W has a base $f_1, \dots, f_m$, then $V \otimes W$ has the base $e_i \otimes f_j$. The formula $(v \odot \varphi)(w) = \varphi(w)v$, where $v \in V$, $w \in W$, $\varphi \in W^*$, defines an isomorphism $V \otimes W^* \cong \mathrm{Hom}_{\mathbb{K}}(W, V)$ (still for finite-dimensional spaces).

Problem 9.42. Verify the details and find the matrix of the operator (for the above bases). Home

Definition 9.43. The *tensor product* bundle $\pi : E_1 \otimes E_2 \to M$ of vector bundles $\pi : E_1 \to M$ and $\pi_2 : E \to M$ with typical fibers V_1 and V_2 has the total space (as a set) $E_1 \otimes E_2 = \sqcup_{P \in M} (E_1)_P \otimes (E_2)_P$. Consider local trivializations Φ_α of E_1 and Ψ_α of E_2 over the same cover $\{U_\alpha\}$. Then the local trivializations for the tensor product are defined as

$$\Phi_\alpha \otimes \Psi_\alpha : (E_1 \otimes E_2)|_{U_\alpha} \to U_\alpha \times (V_1 \otimes V_2),$$

$$e \mapsto (P, [(p_2 \circ \Phi_\alpha) \otimes (p_2 \circ \Psi_\alpha)](e)), \quad e \in (E_1 \otimes E_2)_P = (E_1)_P \otimes (E_2)_P,$$

(isomorphisms by Problem 9.41).

Problem 9.44. Complete the definition as for TM. Home

Problem 9.45. Prove that alternatively the tensor product bundle can be defined by the product cocycle $P \mapsto \Phi_{\alpha\beta}(P) \otimes \Psi_{\alpha\beta}(P)$. Home

Problem 9.46. Verify that the tensor product does not depend on the choice of local trivializations, i.e., we obtain isomorphic bundles. Understand the refinement of cocycles. Class

Remark 9.47. This should be done each time when we define some bundle in a similar way, but we do this once.

Definition 9.48. The *pull-back* f^*E of a vector bundle $\pi : E \to M$ by a smooth map $f : N \to M$ has the total space $\sqcup_{Q \in N} E_{f(Q)}$. If $\{(U_\alpha, \Phi_a)\}$ is a bundle atlas (of local trivializations) for E, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, then $\{(U'_\alpha, \Phi'_a)\}$ is a bundle atlas for f^*E, where

$$U'_\alpha = f^{-1}U_\alpha, \quad \Phi'_\alpha(e) = \Phi_\alpha(e), \quad e \in (f^*E)_Q = E_{f(Q)}, \quad Q \in f^{-1}U_\alpha.$$

Alternatively the pull-back can be defined with the help of the cocycle $\Phi_{\alpha\beta} \circ f$ for the cover $\{f^{-1}U_\alpha\}$. Evidently this is the same bundle.

Home **Problem 9.49.** Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles and let $\Delta : M \to M \times M$ be diagonal map $P \mapsto (P,P)$. Then one can define $\pi_{E_1 \times E_2} : E_1 \times E_2 \to M \times M$. Verify that this is a vector bundle. Prove that the Whitney sum $E_1 \oplus E_2$ is naturally isomorphic to the pull-back $\Delta^* \pi_{E_1 \times E_2}$.

Definition 9.50. If $\pi : E \to M$ is a vector bundle over M with local trivializations $\{(U_\alpha, \Phi_a)\}$ and transition maps $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$, its *dual bundle* E^* with typical fiber V^* has the total space (as a set) $E^* = \sqcup_{P \in M} (E_P)^*$ and local trivializations $\Phi^*_\alpha : E^*|_{U_\alpha} \to U_\alpha \times V^*$ defined by

$$(p_2(\Phi^*_\alpha(a)))(v) = a((\Phi_\alpha)^{-1}(P,v)), \quad a \in (E^*)_P = (E_P)^*,\ v \in V,\ (\Phi_\alpha)^{-1}(P,v) \in E_P.$$

Home **Problem 9.51.** If we fix a base in V, then $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{K})$. Prove that, for the dual base in V^*, $\Phi^*_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{K})$ is defined by $P \mapsto ((\Phi_{\alpha\beta}(P))^T)^{-1}$.

Class **Problem 9.52.** Prove that $T^r_s(E) \cong (\otimes^r E) \otimes (\otimes^s E^*)$.

9.7 Tensor fields as sections of vector bundles

Denote the linear space of tensor fields of type (r,s) over M by $\mathbf{T}^r_s(M)$.

Suppose that $\tau \in \Gamma(T^r_s(TM))$ is a smooth section and (U, φ) is a chart on M. Define

$$T(\tau)^{i_1 \dots i_r}_{j_1 \dots j_s}(P) = L_{\tau(P)}\left(dx^{i_1}, \dots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}} \right).$$

Theorem 9.53. *The above defined T induces the identification $\Gamma(T^r_s(TM)) \cong \Gamma((\otimes^r TM) \otimes (\otimes^s T^* M)) \cong \mathbf{T}^r_s(M)$.*

Proof. By the definition of $T^r_s(TM)$, the map T is well defined and T is an isomorphism locally . Also the global injectivity is immediate. To prove the global surjectivity one can use a partition of unity. □

Home **Problem 9.54.** Complete the argument with a partition of unity.

10 Covariant differentiation

Home **Problem 10.1.** Show that the partial differentiation of components of a tensor field on $\mathbb{R}^n$ is not a tensor operation.

We wish to define on tensor fields on $\mathbb{R}^n$ a tensor operation $\nabla : T(p,q) \to T(p,q+1)$, which coincides in Cartesian coordinates with the partial differentiation. For this purpose we start by an attempt to write down the result of partial differentiation in other coordinates.

Consider first the case of a vector field T^i. Suppose that x^i are Cartesian coordinates in $\mathbb{R}^n$, and $x^{i'}$ is some other coordinate system. Then for the desired ∇ we should have

$$(\nabla T)^i_j = \frac{\partial T^i}{\partial x^j}, \qquad (\nabla T)^{i'}_{j'} = \frac{\partial x^{i'}}{\partial xi} \frac{\partial x^j}{\partial x^{j'}} (\nabla T)^i_j.$$

Then

$$(\nabla T)^{i'}_{j'} = \frac{\partial x^{i'}}{\partial xi} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial}{\partial x^j} \left(\frac{\partial x^i}{\partial x^{k'}} T^{k'} \right) =$$

$$= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^i}{\partial x^{k'}}\frac{\partial T^{k'}}{\partial x^{m'}}\frac{\partial x^{m'}}{\partial x^j} + \frac{\partial x^{i'}}{\partial x i}\frac{\partial x^j}{\partial x^{j'}} T^{k'}\frac{\partial}{\partial x^j}\left(\frac{\partial x^i}{\partial x^{k'}}\right) =$$

$$= \delta^{i'}_{k'}\delta^{m'}_{j'}\frac{\partial T^{k'}}{\partial x^{m'}} + T^{k'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}},$$

hence,

$$(\nabla T)^{i'}_{j'} = \frac{\partial T^{i'}}{\partial x^{j'}} + T^{k'}\Gamma^{i'}_{k'j'}, \qquad \Gamma^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^i}\cdot\frac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}}.$$

For a covector field T_i one should have $(\nabla T)_{ij} = \frac{\partial T_i}{dx^j}$ and $(\nabla T)_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}(\nabla T)_{ij}$. Then

$$(\nabla T)_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial}{\partial x^j}\left(\frac{\partial x^{k'}}{\partial x^i}T_{k'}\right) =$$

$$= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^i}\frac{\partial T_{k'}}{\partial x^{m'}}\frac{\partial x^{m'}}{\partial x^j} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}T_{k'}\frac{\partial}{\partial x^j}\left(\frac{\partial x^{k'}}{\partial x^i}\right) =$$

$$= \delta^{k'}_{i'}\delta^{m'}_{j'}\frac{\partial T_{k'}}{\partial x^{m'}} + T_{k'}\frac{\partial^2 x^{k'}}{\partial x^j\partial x^i}\cdot\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}},$$

or

$$(\nabla T)_{i'j'} = \frac{\partial T_{i'}}{\partial x^{j'}} + T_{k'}\bar\Gamma^{k'}_{i'j'}, \qquad \bar\Gamma^{k'}_{i'j'} = \frac{\partial^2 x^{k'}}{\partial x^j\partial x^i}\cdot\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}.$$

Lemma 10.2. *One has* $\bar\Gamma^{k'}_{i'j'} = -\Gamma^{k'}_{i'j'}$.

Proof. Let us differentiate the equality $\frac{\partial x^{i'}}{\partial x^{i''}}\cdot\frac{\partial x^{i''}}{\partial x^{k'}} = \delta^{i'}_{k'}$ in $x^{m'}$:

$$0 = \frac{\partial^2 x^{i''}}{\partial x^{m'}\partial x^{k'}}\cdot\frac{\partial x^{i'}}{\partial x^{i''}} + \frac{\partial x^{i''}}{\partial x^{k'}}\cdot\frac{\partial^2 x^{i'}}{\partial x^{m''}\partial x^{i''}}\cdot\frac{\partial x^{m''}}{\partial x^{m'}} = \Gamma^{i'}_{m'k'} + \bar\Gamma^{i'}_{m'k'}. \qquad \square$$

$\square$

Theorem 10.3. *There exists a tensor operation ∇ on $M = \mathbb{R}^n$, defined on a field $T^{i_1\dots i_p}_{j_1\dots j_q}$ by*

$$(\nabla T)^{i'_1\dots i'_p}_{j'_1\dots j'_q;m'} = \frac{\partial}{\partial x^{m'}}(T^{i'_1\dots i'_p}_{j'_1\dots j'_q}) + \sum_{s=1}^{p} T^{i'_1\dots i'_{s-1}r'i'_{s+1}\dots i'_p}_{j'_1\dots j'_q}\Gamma^{i'_s}_{r'm'} - \sum_{s=1}^{q} T^{i'_1\dots i'_p}_{j'_1\dots j'_{s-1}r'j'_{s+1}\dots j'_q}\Gamma^{r'}_{j'_s m'},$$

and the functions Γ have the following transformation law

$$\Gamma^{i''}_{j''k''} = \frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^{k'}}{\partial x^{k''}}\Gamma^{i'}_{j'k'} + \frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial^2 x^{i'}}{\partial x^{j''}\partial x^{k''}}.$$

Proof. The explicit form of ∇ can be found similarly to vector and covector cases (Problem 10.4).

Find the transformation law for Γ.

$$\nabla_{k'}T^{i'} := (\nabla T)^{i'}_{k'} = \frac{\partial T^{i'}}{\partial x^{k'}} + T^{r'}\Gamma^{i'}_{r'k'},$$

$$\nabla_{k''}T^{i''} = \frac{\partial T^{i''}}{\partial x^{k''}} + T^{r''}\Gamma^{i''}_{r''k''} = \frac{\partial x^{k'}}{\partial x^{k''}}\frac{\partial}{\partial x^{k'}}\left(\frac{\partial x^{i''}}{\partial x^{i'}}T^{i'}\right) + \frac{\partial x^{r''}}{\partial x^{r'}}T^{r'}\Gamma^{i''}_{r''k''} =$$

$$= \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \frac{\partial T^{i'}}{\partial x^{k'}} + T^{i'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{i'}} + T^{r'} \frac{\partial x^{r''}}{\partial x^{r'}} \Gamma^{i''}_{r''k''}.$$

On the other hand,

$$\nabla_{k''} T^{i''} = \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \nabla_{k'} T^{i'} = \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \left(\frac{\partial T^{i'}}{\partial x^{k'}} + T^{r'} \Gamma^{i'}_{r'k'} \right).$$

Hence

$$T^{r'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \Gamma^{i'}_{r'k'} = T^{r'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}} + T^{r'} \frac{\partial x^{r''}}{\partial x^{r'}} \Gamma^{i''}_{r''k''}.$$

Since T^i is an arbitrary field,

$$\Gamma^{i''}_{r''k''} = \Gamma^{i'}_{r'k'} \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} - \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}}.$$

As it was established in the proof of Lemma 10.2,

$$- \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}} = \frac{\partial^2 x^{k'}}{\partial x^{r''} \partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{k'}} = \frac{\partial^2 x^{i'}}{\partial x^{r''} \partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}}.$$

□

Home **Problem 10.4.** Find the explicit form of ∇ for general fields.

09.11.2023 **Definition 10.5.** An *operation* ∇ *of covariant differentiaition* (or *affine connection*) ∇ is defined on a manifold M, if, for each chart, a collection of smooth functions Γ^i_{jk}, $i, j, k = 1, \dots, \dim M$, such that for distinct charts we have equality

$$\Gamma^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^i_{jk} + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}}.$$

Then the action of ∇ on a tensor field is defined by

$$(\nabla T)^{i_1 \dots i_p}_{j_1 \dots j_q; m} = \frac{\partial}{\partial x^m} (T^{i_1 \dots i_p}_{j_1 \dots j_q}) + \sum_{s=1}^{p} T^{i_1 \dots i_{s-1} r i_{s+1} \dots i_p}_{j_1 \dots j_q} \Gamma^{i_s}_{rm} - \sum_{s=1}^{q} T^{i_1 \dots i_p}_{j_1 \dots j_{s-1} r j_{s+1} \dots j_q} \Gamma^r_{j_s m},$$

Remark 10.6. As one can see from the above calculations considered "in the inverse direction", ∇ is a tensor operation.

Remark 10.7. The existence of a connection will follow from the existence of a Riemannian connection (a theorem below).

Definition 10.8. The *torsion tensor* of an affine connection Γ^i_{jk} is the tensor, determined in each coordinate system by the equality $\Omega^i_{jk} := \Gamma^i_{jk} - \Gamma^i_{kj}$.

Lemma 10.9. *Ω is really a tensor field of type* $(1, 2)$.

Home **Problem 10.10.** Verify this.

Class **Problem 10.11.** Proof that ∇ commutes with contraction.

Definition 10.12. A connection Γ is called *symmetric,* if $\Omega = 0$.

Lemma 10.13. *A connection ∇ has the properties:*

1) *the operation ∇ is linear over $\mathbb{R}$;*

2) *the operation ∇ is a tensor operation;*

3) *the covariant derivative of a function (i.e., of a tensor of tupe $(0,0)$) coincides with its gradient: $\nabla_k f = \frac{\partial f}{\partial x^k}$;*

4) *the operation ∇ on a vector and on a covector field has the form:*

$$\nabla_k T^i = \frac{\partial T^i}{\partial x^k} + T^j \Gamma^i_{jk},$$

$$\nabla_k T_i = \frac{\partial T_i}{\partial x^k} - T_j \Gamma^j_{ik};$$

5) *for arbitrary tensor fields T and S one has the Leibniz formula:*

$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S).$$

Proof. All the properties are evident except of 5). Verify it, for instance, for vector fields:

$$\begin{aligned}
\nabla_k (T^i S^j) &= \frac{\partial}{\partial x^k}(T^i S^j) + T^r S^j \Gamma^i_{rk} + T^i S^r \Gamma^j_{rk} = \\
&= (\frac{\partial}{\partial x^k} T^i) S^j + T^i \frac{\partial}{\partial x^k}(S^j) + T^r S^j \Gamma^i_{rk} + T^i S^r \Gamma^j_{rk} = \\
&= (\frac{\partial T^i}{\partial x^k} + T^r \Gamma^i_{rk}) S^j + T^i (\frac{\partial S^j}{\partial x^k} + P^r \Gamma^j_{rk}) = \\
&= (\nabla_k T^i) S^j + T^i (\nabla_k S^j).
\end{aligned}$$

□

Problem 10.14. Do the calculation for arbitrary fields. Home

Theorem 10.15. *The above properties* 1) – 5) *uniquely define the covariant differentiation. More precisely, one can find in a unique way functions Γ^i_{jk}, which satisfy the transformation law from the definition of a connection, and the action of ∇ on arbitrary field will be given by the formula from the same definition.*

Proof. Denote $e_i := \frac{\partial}{\partial x^i}$ and $e^j = d\,x^j$. Then Γ^i_{jk} can be determined uniquely from

$$\nabla_k e_i = \Gamma^j_{ik} e_j, \qquad \nabla_k e^i = -\Gamma^i_{jk} e^j. \tag{7}$$

Remark that while obtaining the transformation law of Γ^i_{jk} in Theorem 10.3, we used only the relation as in item 4). Thus, the same calculation gives now the desired transformation law.

It remains to obtain the formula of differentiation of arbitrary fields. Do this for a field of type $(1,1)$. Locally we have

$$T = T^i_j \, e_i \otimes e^j.$$

Then

$$\nabla_k T^l_m = (\nabla T)^l_{m;k} = (\nabla (T^i_j \, e_i \otimes e^j))^l_{m;k} =$$

$$= \left((\nabla T^i_j) \otimes e_i \otimes e^j + T^i_j (\nabla e_i) \otimes e^j + T^i_j e_i \otimes (\nabla e^j)\right)^l_{m;k} =$$

$$= \frac{\partial T^l_m}{\partial x^k} + \left(T^i_j (\Gamma^r_{ik} e_r) \otimes e^j\right)^l_m - \left(T^i_j e_i \otimes (\Gamma^j_{rk} e^r)\right)^l_m =$$

$$= \frac{\partial T^l_m}{\partial x^k} + T^i_m \Gamma^l_{ik} - T^l_j \Gamma^j_{mk}.$$

□

Home **Problem 10.16.** Do the calculation in the general case.

Definition 10.17. An affine symmetric connection ∇ on a Riemannian manifold (M, g) is called *Riemannian* (or *metric compatible*, or *Levi-Civita connection*) if $\nabla g = 0$.

Class **Problem 10.18.** Prove that in this case ∇ commutes with the operations of rising and lowering of indexes.

Theorem 10.19. *On any Riemannian manifold (M, g) there exists a unique Levi-Civita connection. Its coefficients (Christoffel symbols) are*

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{ir} \left(\frac{\partial g_{kr}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^r} \right). \tag{8}$$

Proof. Prove that the Christoffel symbols of a Levi-Civita connection should satisfy (8). Then the uniqueness will be proved. We have by the definition that

$$0 = \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - g_{rj} \Gamma^r_{ik} - g_{ir} \Gamma^r_{jk}.$$

Using the lowering of the first index $\Gamma_{ijk} := g_{ir} \Gamma^r_{jk}$ and cyclic permutation we obtain:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{jik} + \Gamma_{ijk},$$

$$\frac{\partial g_{ki}}{\partial x^j} = \Gamma_{ikj} + \Gamma_{kij},$$

$$\frac{\partial g_{jk}}{\partial x^i} = \Gamma_{kji} + \Gamma_{jki}.$$

Add the first two equalities to each other and subtract the third one. Keeping in mind the symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$, we obtain

$$\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ki}}{\partial x^j} = \Gamma_{jik} + \Gamma_{ijk} + \Gamma_{ikj} + \Gamma_{kij} - \Gamma_{kji} - \Gamma_{jki} =$$

$$= \Gamma_{jki} + \Gamma_{ijk} + \Gamma_{ijk} + \Gamma_{kji} - \Gamma_{kji} - \Gamma_{jki} = 2\Gamma_{ijk} = 2 g_{ir} \Gamma^r_{jk}.$$

Multiplying by the inverse matrix for g_{ij}, we arrive to

$$\Gamma^r_{jk} = \frac{1}{2} g^{ir} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^i} \right).$$

To prove the existence, simply define the coefficients bu the formulas (8). □

Problem 10.20. Verify that this is a connection, i.e., verify the transformation law for the above Γ^r_{jk}. Home

Definition 10.21. A coordinate system is *Euclidean w.r.t. a metric,* if g_{ij} in this system are constant (hence, in some other coordinate system are δ_{ij} (in the entire neighborhood!)).

A coordinate system is *Euclidean w.r.t. a connection,* if in it one has $\Gamma^i_{jk} \equiv 0$.

Problem 10.22. Prove the equivalence of these two properties for the Levi-Civita connection. Home

11 Parallel transport and geodesics

The parallel transport is a way to compare (tangent) vectors in distinct points. E.g., on plane "the Euclidean coordinates of vectors should be constant"= their partial derivatives vanish. In the general situation it is natural to require the vanishing of its covariant derivative. But (for more complicated manifolds) this is too restrictive. We arrive to the requirement: components of a field are covariant constant "along a curve" = "parallel transport along a curve". The result may depend on the choice of a curve connecting two points. Let us pass to precise definitions.

Let a manifold M be equipped with an affine connection ∇. Suppose that two points P and Q of M are connected by a smooth curve $\gamma : [0,1] \to M$, $\gamma(0) = P$, $\gamma(1) = Q$. On this curve we have the *velocity field* ξ along γ (use the third definition of a tangent vector).

Definition 11.1. The *covariant derivative of a tensor field T of type (p,q) along a curve γ* is a tensor field $\nabla_{\dot\gamma}(T)$, defined as the contraction of the tensor product of the velocity field with the covariant derivative of T:

$$(\nabla_{\dot\gamma}(T))^{i_1,\dots,i_p}_{j_1,\dots,j_q} := \xi^k \nabla_k T^{i_1,\dots,i_p}_{j_1,\dots,j_q}.$$

Of course this is not a "field on manifold" as in our initial definition, because it is defined only at points of the curve.

Definition 11.2. A vector field T is called *parallel along γ with respect ∇,* if $\nabla_{\dot\gamma}(T) \equiv 0$.

Rewrite these equations in local coordinates $(x^1,\dots,x^n)$. If

$$\gamma(t) = (x^1(t),\dots,x^n(t)), \qquad \xi^k = \frac{dx^k(t)}{dt},$$

the equations will take the form:

$$\xi^k \nabla_k T^i = \frac{dx^k(t)}{dt}\left(\frac{\partial T^i}{\partial x^k} + T^r \Gamma^i_{rk}\right) = 0,$$

$$\frac{dx^k(t)}{dt}\frac{\partial T^i}{\partial x^k} + T^r\Gamma^i_{rk}\frac{dx^k(t)}{dt} = \frac{dT^i}{dt} + T^r\Gamma^i_{rk}\frac{dx^k(t)}{dt} = 0.$$

Definition 11.3. The last equality is called the *parallel transport equation of a vector along a curve.*

The problem of parallel transport is as follows. Given a smooth curve γ, connecting points P and Q of a manifold M equipped with a connection ∇, and a vector $v \in T_P M$. Find a vector $w \in T_Q M$, such that there is a covariant constant vector field $V(t)$ with $V(0) = v$ and $V(1) = w$. The problem can be solved consequently for pieces of γ lying in one coordinate neighborhood, we may assume without loss of generality, that the entire curve lies in one coordinate neighborhood.

We arrived to a problem of solving of a system of ordinary differential equations of the first order for functions $V^i(t)$ with the initial value $V^i(0) = v^i$ (Cauchy problem). The system has a derivatives-free right side. Hence, a solution of this problem exists, is unique and extendable up to Q, i.e., $t = 1$.

Definition 11.4. The vector $w = V(1) \in T_Q M$ is called *parallel to $v \in T_P M$ along γ.*

Lemma 11.5. *Let (M, g) be a Rimannian manifold. A symmetric affine connection ∇ on M is a Levi-Civita connection if and only if the corresponding parallel transport conserves the inner product of vectors w.r.t. g.*

Proof. Suppose, ∇ is a Levi-Civita connection, $\langle ., . \rangle$ the inner product defined by g, $V(t)$ and $W(t)$ are vector fields satisfying the parallel transport equations along $\gamma : [0, 1] \to M$. We need to show that $\frac{d}{dt}\langle V(t), W(t)\rangle \equiv 0$. Indeed, for $\mathbf{S}$ denoting contraction,

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}\langle V(t), W(t)\rangle = \xi^k \nabla_k \left(g_{ij} V^i W^j\right) =$$

$$= \xi^k \nabla_k (\mathbf{SS}(g \otimes V \otimes W)) = \xi^k (\mathbf{SS}\nabla_k (g \otimes V \otimes W)) =$$

$$= \mathbf{SS}(\xi^k \nabla_k g \otimes V \otimes W + g \otimes \xi^k \nabla_k V \otimes W + g \otimes V \otimes \xi^k \nabla_k W) = 0.$$

Conversely, if this equality is true for any parallel vector fields along any curve, then for arbitrary vectors ξ, V and W one has

$$\mathbf{SS}(\xi^k \nabla_k g \otimes V \otimes W) = \xi^k V^i W^j \nabla_k g_{ij} = 0.$$

Taking the basic vectors we arrive to $\nabla_k g_{ij} = 0$. □

Remark 11.6. The parallel transport can be defined for piece-wise smooth curves as the composition of transports over smooth parts.

Definition 11.7. A curve γ on a manifold M equipped with an affine connection ∇ is called a *geodesic,* if its velocity field is parallel along γ, i.e., $\nabla_{\dot\gamma}(\dot\gamma) = 0$.

In some local coordinates $(x^1, \ldots, x^n)$ we obtain the following equations:

$$\frac{dx^k}{dt}\left(\nabla_k \xi^i\right) = 0, \qquad i = 1, \ldots, n,$$

where $\xi^i = \frac{dx^i}{dt}$. Hence,

$$\frac{dx^k}{dt}\left(\frac{\partial}{\partial x^k}\xi^i + \Gamma^i_{rk}\xi^r\right) = 0,$$

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{rk}\frac{dx^r}{dt}\frac{dx^k}{dt} = 0, \qquad i = 1, \ldots, n. \tag{9}$$

Lemma 11.8. *Suppose that $P \in M$, $v \in T_PM$. Then there exists locally a unique geodesic $\gamma(t)$ such that $\gamma(0) = P$ and $\dot\gamma(0) = v$. It depends smoothly on this initial data.*

Proof. In local coordinates in a neighborhood of P the problem of finding of the desired geodesic becomes a problem of solving of the Cauchy problem for the appropriate system of n ordinary differential equations of the second order, resolved with respect to the highest derivative. From an ODE course we know that this solution locally exists, is unique and depends smoothly on the initial data. □

Problem 11.9. The velocity field of a geodesic of a Levi-Civita connection has constant length (i.e. its parametrization is a scaling of the arc length one). Home

Problem 11.10. If two geodesics are tangent to each other in some point (with the same velocity), then they coincide. Home

Problem 11.11. A parallel transport of a vector v along a geodesic conserves the angle between v and the curve (i.e., the velocity vector). Home

Lemma 11.12. (geometric meaning of Christoffel symbols) *For basic vector fields $e_i := \frac{\partial}{\partial x^i}$ of a coordinate system one has $\nabla_{e_i}(e_j) = \Gamma^r_{ji} e_r$ (an expansion of a vector w.r.t. this base). Equivalently the result of an infinitely small parallel transport of the frame $\{e_\alpha\}$ in the i^{th} direction has the coefficients $\Gamma^\alpha_{\beta i}$ in the initial base.*

Proof. By definition

$$(\nabla_{e_i}(e_j))^k = (e_i)^s\, (\nabla_s(e_j))^k = \delta^s_i \left(\frac{\partial (e_j)^k}{\partial x^s} + \Gamma^k_{rs}(e_j)^r \right) =$$

$$= \delta^s_i \left(\frac{\partial (\delta^k_j)}{\partial x^s} + \Gamma^k_{rs}\delta^r_j \right) = \delta^s_i\, \left(\Gamma^k_{rs}\delta^r_j \right) = \Gamma^k_{ji}.$$

□

Problem 11.13. Describe geometrically the parallel transport for the Levi-Civita connection on a surface in $\mathbb{R}^3$ (projection). Class

Problem 11.14. Deduce that a curve on a surface in $\mathbb{R}^3$ is a geodesics iff its normal (the second derivative for the natural parametrization = parametrization by the arc length) is orthogonal to tangent plane. Home

Problem 11.15. Find geodesics on the standard sphere S^2 (without direct calculation). Class

Problem 11.16. Find geodesics on the standard sphere S^2 (direct calculation). Home

Problem 11.17. Find geodesics on the pseudosphere = the upper half-plane with coordinates (x, y) and the metric $ds^2 = \frac{dx^2 - dy^2}{x^2}$. Home

Problem 11.18. Prove that if two surfaces in $\mathbb{R}^3$ are tangent to each other (tangent planes coincide) along a curve then two respective parallel transports along this curve coincide. Class

Problem 11.19. Find the rotation angle for the parallel transport of a vector along the circle being the base of the standard round cone. *Hint:* the cone is locally isometric to the plane. Class

Problem 11.20. Find the rotation angle for the parallel transport of a vector along the circle being a parallel of the standard sphere. *Hint:* use the previous two problems. Home

16.11.2023 **Theorem 11.21.** *Let (M, g) be a Riemannian manifold. For any point $P_0 \in M$, there exist a neighborhood U and a number $\varepsilon > 0$ such that any two points of U are connected by a unique (up to a scaling of its parameter) geodesic of length less then ε. This geodesic depends smoothly on its ends.*

Proof. By Lemma 11.8 one can define, for some neighborhood V of $(P_0, 0)$ in the tangent bundle TM of the form

$$V = \{(P, v) \in TM \mid P \in U, \ \|v\| < \varepsilon\}$$

(where U is some neighborhood of P_0), a smooth map

$$E : V \to M \times M, \qquad (P, v) \mapsto (P, \exp_P(v)),$$

where $\exp_P$ maps a vector v to the point $\gamma(1)$ of a unique geodesic starting in P in the direction v (i.e. $\dot\gamma(0) = v$). Since the existence theorem is local, only geodesics with small v (solutions of the corresponding Cauchy problem for the system of ODE) are proved to be extendable till $t = 1$.

Calculate the Jacobian of E in $(P_0, 0)$. For this purpose, along with the coordinates $(x^1, \dots, x^n; v^1, \dots, v^n)$ in a neighborhood of $(P_0, 0)$ in TM, where $v = v^i \frac{\partial}{\partial x^i}$, consider coordinates $(x_1^1, \dots, x_1^n; x_2^1, \dots, x_2^n)$ in $U \times U \subset M \times M$. For the tangent map dE one has:

$$\frac{\partial x_1^i}{\partial x^j} = \delta_j^i, \qquad \frac{\partial x_1^i}{\partial v^j} = 0, \qquad d_{P_0} \exp_{P_0}([v \cdot t]) = \left.\frac{d\gamma_v}{dt}\right|_0 = v$$

according to the second definition of a tangent vector. Thus, the Jacobi matrix $d_{P_0}E$ is equal to $\begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$, where I is the identity matrix and the Jacobian is equal to 1.Hence, by the implicit mapping theorem, the map E maps diffeomorphicaly some neighborhood V' of the point $(P_0, 0) \in TM$ onto a neighborhood W' of the point (P_0, P_0) in $M \times M$. Passing to some smaller neighborhoods if necessary, one can assume that $W' = U' \times U'$ and U' is a subset of a ball of diameter ε w.r.t. g (the lower bound of lengths of curves connecting its center P_0 with any its point is less then $\varepsilon/2$). Then U' is the desired neighborhood of P_0. Indeed, let P and Q be two arbitrary points of U'. Consider a geodesic γ starting from the point P' in the direction of v, where $(P', v) = E^{-1}(P, Q)$. Then, by the definition of E, we have $P' = P$ and $\gamma(1) = Q$. Thus, the points P and Q are connected by the geodesic γ and γ smoothly depends on P and Q. Find its length. As it is proved above, the parameter of a geodesic can differ from the arc length only by a scaling, which is equal to $\|v\|$ for the case under consideration. Then the length of γ from 0 to 1 is equal to $1 \cdot \|v\| < \varepsilon$. It remains to verify the uniqueness. Suppose, that P and Q are connected by a geodesic of length less then ε. Then it is a solution of the appropriate initial value problem and is unique, because the length of its velocity vector at 0 is less then $\varepsilon \cdot t$, where $\gamma(t) = Q$ (otherwise E is not a bijection). □

Home **Problem 11.22.** Prove that in coordinates determined by exp, all Γ_{jk}^i vanish in P_0.

12 Differentiation and integration of differential forms

Consider some symmetric affine connection ∇ on a manifold M (for example, the Levi-Civita connection for some Riemannian metric) and a differential form ω of degree k, i.e., an alternating (antisymmetric) tensor field of type $(0,k)$. Denote the space of such forms by $\Omega^k(M)$. Then one can define the *exterior derivative* or *gradient* $d\omega$ of the form ω by the following formula

$$d\omega := \pm\frac{(k+1)!}{k!} Alt\nabla\omega,$$

or, in local coordinates,

$$(d\omega)_{j_1\dots j_{k+1}} = \pm\frac{1}{k!}\sum_{\sigma\in S_{k+1}}(-1)^\sigma\nabla_{\sigma(j_{k+1})}\omega_{\sigma(j_1)\dots\sigma(j_k)}.$$

where we denote $\sigma(j_k) := j_{\sigma(k)}$ and $\pm$ is chosen to have

$$\pm(-1)^\sigma = \operatorname{sgn}\begin{pmatrix} 1\dots k, k+1 \\ \sigma(k+1)\sigma(1)\dots\sigma(k)\end{pmatrix},$$

i.e., $\pm = (-1)^k$. By the definition of ∇, $d\omega$ is a differential form of degree $k+1$.

Lemma 12.1. *The gradient $d\omega$ does not depend on the choice of a symmetric connection. Namely,*

$$(d\omega)_{j_1\dots j_{k+1}} = \sum_{s=1}^{k+1}(-1)^{s+1}\frac{\partial\omega_{j_1\dots j_{s-1}j_{s+1}\dots j_{k+1}}}{\partial x^{j_s}}.$$

Proof. By the definition of ∇,

$$(d\omega)_{j_1\dots j_{k+1}} =$$

$$= \frac{(-1)^k}{k!}\sum_{\sigma\in S_{k+1}}(-1)^\sigma\left[\frac{\partial\omega_{\sigma(j_1)\dots\sigma(j_k)}}{\partial x^{\sigma(j_{k+1})}} - \sum_{r=1}^{k}\omega_{\sigma(j_1)\dots\sigma(j_{r-1})\alpha\sigma(j_{r+1})\dots\sigma(j_k)}\Gamma^\alpha_{\sigma(j_r)\sigma(j_{k+1})}\right] =$$

$$= \frac{(-1)^k}{k!}\sum_{\sigma\in S_{k+1}}(-1)^\sigma\frac{\partial\omega_{\sigma(j_1)\dots\sigma(j_k)}}{\partial x^{\sigma(j_{k+1})}} -$$

$$-\frac{(-1)^k}{k!}\sum_{\text{over even }\sigma\in S_{k+1}}(-1)^\sigma\sum_{r=1}^{k}\left[\Gamma^\alpha_{\sigma(j_r)\sigma(j_{k+1})} - \Gamma^\alpha_{\sigma(j_{k+1})\sigma(j_r)}\right]\omega_{\sigma(j_1)\dots\sigma(j_{r-1})\alpha\sigma(j_{r+1})\dots\sigma(j_k)} =$$

(since ∇ is symmetric)

$$= \frac{(-1)^k}{k!}\sum_{\sigma\in S_{k+1}}(-1)^\sigma\frac{\partial\omega_{\sigma(j_1)\dots\sigma(j_k)}}{\partial x^{\sigma(j_{k+1})}} =$$

$$= \frac{(-1)^k}{k!}\sum_{s=1}^{k+1}\sum_{\tau\in S_k}\operatorname{sgn}\begin{pmatrix} 1\dots k+1 \\ \tau(1)\dots\tau(s-1)\tau(s+1)\dots\tau(k+1)s\end{pmatrix}\frac{\partial\omega_{\tau(j_1)\dots\tau(j_{s-1})\tau(j_{s+1})\dots\tau(j_{k+1})}}{\partial x^{j_s}} =$$

$$= \frac{1}{k!}\sum_{s=1}^{k+1}\sum_{\tau\in S_k}(-1)^{s-1}(-1)^\tau\frac{\partial\omega_{\tau(j_1)\dots\tau(j_{s-1})\tau(j_{s+1})\dots\tau(j_{k+1})}}{\partial x^{j_s}} =$$

(since ω is alternating)

$$= \frac{1}{k!}\sum_{s=1}^{k+1}\sum_{\tau\in S_k}(-1)^{s-1}(-1)^\tau(-1)^\tau\frac{\partial\omega_{j_1\dots j_{s-1},j_{s+1}\dots j_{k+1}}}{\partial x^{j_s}} =$$

$$= \frac{1}{k!} \cdot k! \sum_{s=1}^{k+1} (-1)^{s+1} \frac{\partial \omega_{j_1 \ldots j_{s-1}, j_{s+1} \ldots j_{k+1}}}{\partial x^{j_s}}.$$

$\square$

Home **Problem 12.2.** The exterior derivative of a differential form can be obtained by "direct differentiation". Namely, prove that for

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

one has (keeping in mind that, for a function f, $\nabla f = df$ and a tensor of type $(0,1)$ is always (anti)symmetric)

$$d\omega = \sum_{i_1 < \cdots < i_k} d(\omega_{i_1 \ldots i_k}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_{i_1 < \cdots < i_k} \sum_{i_0} \frac{\partial(\omega_{i_1 \ldots i_k})}{\partial x^{i_0}} dx^{i_0} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Theorem 12.3. *Let $\omega_{(1)}$ and $\omega_{(2)}$ be differential forms of degrees p and q respectively. Then*

$$d(\omega_{(1)} \wedge \omega_{(2)}) = d\omega_{(1)} \wedge \omega_{(2)} + (-1)^p \omega_{(1)} \wedge d\omega_{(2)}.$$

Proof. Since the both sides of the desired equality are linear in ω, it is sufficient to verify it in one chart for forms

$$\omega_{(1)} = f\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \qquad \omega_{(2)} = g\, dx^{j_1} \wedge \ldots \wedge dx^{j_q}.$$

Then by Problem 12.2

$$d(\omega_{(1)} \wedge \omega_{(2)}) = d(fg\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}) =$$

$$= \frac{\partial f}{\partial x^k} g\, dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} + f\, \frac{\partial g}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} =$$

$$= \left(\frac{\partial f}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \right) \wedge \left(g\, dx^{j_1} \wedge \ldots \wedge dx^{j_q} \right) +$$

$$+(-1)^p \left(f\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \right) \wedge \left(\frac{\partial g}{\partial x^k}\, dx^k \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} \right) =$$

$$= d\omega_{(1)} \wedge \omega_{(2)} + (-1)^p \omega_{(1)} \wedge d\omega_{(2)}.$$

$\square$

Theorem 12.4. *For any differential form ω one has $d(d\omega) = 0$.*

Proof. Once again it is sufficient to verify this for a form $\omega = f\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$. Moreover, if the theorem is proved for $\omega_{(1)}$ and $\omega_{(2)}$, then it is true for its exterior product. Indeed,

$$dd(\omega_{(1)} \wedge \omega_{(2)}) = d(d\omega_{(1)} \wedge \omega_{(2)} + (-1)^p \omega_{(1)} \wedge d\omega_{(2)}) =$$

$$= dd\omega_{(1)} \wedge \omega_{(2)} + (-1)^{p+1} d\omega_{(1)} \wedge d\omega_{(2)} + (-1)^p d\omega_{(1)} \wedge d\omega_{(2)} + (-1)^{p+p} \omega_{(1)} \wedge dd\omega_{(2)} = 0.$$

It remains to verify the statement for f and dx^i. One has

$$d(df) = d\left(\frac{\partial f}{\partial x^k}\, dx^k\right) = \frac{\partial^2 f}{\partial x^i\, \partial x^k}\, dx^i \wedge dx^k = \sum_{i<k} \left(\frac{\partial^2 f}{\partial x^i\, \partial x^k} - \frac{\partial^2 f}{\partial x^k\, \partial x^i}\right) dx^i \wedge dx^k = 0.$$

For dx^i, apply the last calculation to $f = x^i$:

$$dd(dx^i) = d(ddx^i) = d(0) = 0.$$

□

Definition 12.5. A differential form ω is *closed,* if $d\omega = 0$, i.e., $\omega \in \operatorname{Ker} d$. A differential form ω is *exact,* if $\omega = d\omega_1$ for some ω_1, i.e., $\omega \in \operatorname{Im} d$.

By the previous lemma, the linear map d has the property $\operatorname{Im} d \subset \operatorname{Ker} d$. So, if

$$Z^k(M) := \operatorname{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))$$

is the space of closed k-forms and

$$B^k(M) := \operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$$

is the space of exact k-forms, then $B^k(M) \subseteq Z^k(M)$ and one can define the *de Rham cohomology of degree* k as the quotient linear space $H^k(M) = Z^k(M)/B^k(M)$.

Immediately from the definition one has the following statement.

Theorem 12.6. 1) *Let $\Omega \in \Omega^k(M)$. Consider the equation:*

$$d\omega = \Omega. \tag{10}$$

It has a solution iff Ω is closed and the cohomology class $[\Omega] = 0 \in H^k(M)$ ($\Leftrightarrow$ Ω is exact).

2) *Any two ω_1 solutions ω_2 of* (10) *differ by a closed form: $d(\omega_1 - \omega_2) = 0$. The set of all solutions is the coset of the subspace $Z^{k-1}(M)$ containing any solution ω.*

3) *The space $Z^k(M)$ is isomorphic to the direct sum of $B^k(M)$ and $H^k(M)$.* □

As a particular case (up to Problem 12.8) of the pull-back of a form, one can define the pull-back of a differential form:

Definition 12.7. Let $f : M \to N$ be a smooth map of smooth manifolds and $\omega \in \Omega^k(N)$ be a differential form.The *pull-back* or the *inverse image* $f^*\omega$ of this form is the following multilinear map of vector fields on M:

$$f^*\omega(\vec{v}_1, \dots, \vec{v}_k) := \omega(d_P f(\vec{v}_1), \dots, d_P f(\vec{v}_k)), \qquad \vec{v}_i \in T_P M.$$

Problem 12.8. Verify that the obtained form is a differential form (i.e., antisymmetric). Home

Lemma 12.9. *Suppose that $(x^1, \dots, x^m)$ is a local coordinate system in a neighborhood of $P \in M$ and $(y^1, \dots, y^n)$ is a local coordinate system in a neighborhood of $f(P) \in N$, so the corresponding local representative map of $f: M \to N$ is defined by some functions*

$$y^1 = f^1(x^1, \dots, x^m), \dots, y^n = f^n(x^1, \dots, x^m),$$

and a form $\omega \in \Omega^k(N)$ has locally the expansion

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n) dy^{i_1} \wedge \dots \wedge dy^{i_k}.$$

Then the pull-back of ω has locally the form

$$f^*(\omega) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(f^1(x^1, \dots, x^m), \dots, f^n(x^1, \dots, x^m)) \times$$

$$\times df^{i_1}(x^1, \dots, x^m) \wedge \dots \wedge df^{i_k}(x^1, \dots, x^m). \tag{11}$$

Proof. One has

$$f^*(\omega)(\vec{v}_1, \dots, \vec{v}_k) = \omega(d_P f(\vec{v}_1), \dots, d_P f(\vec{v}_k)) =$$

$$= \left(\sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n) dy^{i_1} \wedge \dots \wedge dy^{i_k} \right) (d_P f(\vec{v}_1), \dots, d_P f(\vec{v}_k)) =$$

$$= \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n)\, k!\, Alt^{[i_1, \dots, i_k]} \left\{ dy^{i_1}(d_P f(\vec{v}_1)) \dots dy^{i_k}(d_P f(\vec{v}_k)) \right\} =$$

$$= \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n)\, k!\, Alt^{[i_1, \dots, i_k]} \left\{ \frac{\partial f^{i_1}}{\partial x^{j_1}} (\vec{v}_1)^{j_1} \dots \frac{\partial f^{i_k}}{\partial x^{j_k}} (\vec{v}_1)^{j_k} \right\} =$$

$$= \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n)\, k!\, Alt^{[i_1, \dots, i_k]} \left\{ df^{i_1}(\vec{v}_1) \dots df^{i_k}(\vec{v}_1) \right\} =$$

$$= \left(\sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(y^1, \dots, y^n) df^{i_1} \wedge \dots \wedge df^{i_k} \right) (\vec{v}_1, \dots, \vec{v}_k).$$

□

Theorem 12.10. *The operation of pull-back has the following properties:*

1) *for $f: M \to N$ and $g: N \to K$ one has $(gf)^* = f^* g^*$;*

2) *$f^* d_N = d_M f^*$, where d_N and d_M are the exterior derivatives on N and M, respectively;*

3) *$f^*(\operatorname{Ker} d_N) \subseteq \operatorname{Ker} d_M$ and $f^*(\operatorname{Im} d_N) \subseteq \operatorname{Im} d_M$, hence f^* gives rise to a map of cohomologies*

$$f^* : H^k(N) \to H^k(M).$$

Proof. The first equality is an immediate consequence of Lemma 12.9. To prove the second one, note that by the same lemma, Theorems 12.4 and 12.3, and Problem 12.2 we have

$$f^*(d\omega) = f^*\left(d\left(\sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}(y^1,\dots,y^n)dy^{i_1}\wedge\dots\wedge dy^{i_k}\right)\right) =$$

$$= f^*\left(\sum_{i_1<\dots<i_k}\sum_{s=1}^{n}\frac{\partial\omega_{i_1\dots i_k}}{\partial y^s}(y^1,\dots,y^n)dy^s\wedge dy^{i_1}\wedge\dots\wedge dy^{i_k}\right) =$$

$$= \sum_{i_1<\dots<i_k}\sum_{s=1}^{n}\frac{\partial\omega_{i_1\dots i_k}}{\partial y^s}(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m))df^s\wedge df^{i_1}\wedge\dots\wedge df^{i_k} =$$

$$= \sum_{i_1<\dots<i_k} d(\omega_{i_1\dots i_k}(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m))\wedge df^{i_1}\wedge\dots\wedge df^{i_k} =$$

$$= d\left(\sum_{i_1<\dots<i_k}\omega_{i_1\dots i_k}(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m))df^{i_1}\wedge\dots\wedge df^{i_k}\right) = df^*(\omega).$$

To prove the third relations, note (using the second one) that, if $d_N\omega = 0$, then $d_M f^*\omega = f^*d_N\omega = 0$. Similarly, if $\omega_{(1)} = d_N\omega$, then

$$f^*(\omega_{(1)}) = f^*d_N\omega = d_M f^*\omega.$$

□

Problem 12.11. Prove that cohomologies of diffeomorphic manifolds coincide. Home

Definition 12.12. A differential form Ω of degree k on $M\times I$ *does not depend on dt*, if its value on any system of vectors of the form $(\frac{\partial}{dt},\vec v_1,\dots,\vec v_{k-1})$ is 0.

Lemma 12.13. *Locally this is equivalent to the following: in the local expansion of Ω w.r.t. the basis $dx^{i_1}\wedge\dots\wedge dx^{i_k}$ there is no summands containing dt.*

Proof. By the definition of action of form on vectors. □

Problem 12.14. Write down this in detail. Home

Lemma 12.15. *Any differential form Ω on $M\times I$ can be represented in the form $\Omega = \Omega_{(1)} + \Omega_{(2)}\wedge dt$, where $\Omega_{(1)}$ and $\Omega_{(2)}$ do not depend on dt. This representation is unique.*

Proof. Suppose that this lemma is proved for forms supported in one chart. Then consider a partition of unity $\{\varphi_\alpha\}$ on M and the corresponding "cylindrical" partition of unity $\varphi'_\alpha(x,t) = \varphi_a(x)$ on $M\times I$. Then

$$\Omega = \sum_\alpha \varphi'_\alpha\Omega = \sum_\alpha(\Omega_{(1,\alpha)} + \Omega_{(2,\alpha)}\wedge dt) = \left(\sum_\alpha\Omega_{(1,\alpha)}\right) + \left(\sum_\alpha\Omega_{(2,\alpha)}\right)\wedge dt$$

is the desired representation. In turn, in one chart it is sufficient to group terms without dt and terms with dt, and move dt on the last position (for the second group terms). We keep in mind here Lemma 12.13.

The uniqueness also may be verified in one chart. Indeed, if $\omega = \Omega_1' + \Omega_2' \wedge dt = \Omega_1 + \Omega_2 \wedge dt$ and $\psi_\alpha \Omega_1' = \psi_\alpha \Omega_1$, $\psi_\alpha \Omega_2' = \psi_\alpha \Omega_2$ for each function ψ_α from a partition of unity, then summarizing we obtain $\Omega_1' = \Omega_1$ and $\Omega_2' = \Omega_2$. In turn, over one chart Ω_1 and Ω_2 by Lemma 12.13 can be determined only in the above way (grouping terms), because the ordered products of dx^i form a basis. □

Lemma 12.16. *Suppose that smooth maps f_0 and f_1 from a manifold M to a manifold N are homotopic to each other, i.e., there exists a smooth map F such that*

$$F : M \times I \to N, \qquad F(P,0) = f_0(P), \quad F(P,1) = f_1(P) \quad \forall P \in M.$$

Then there exists a linear map $D : \Omega^*(N) \to \Omega^{*-1}(M)$ such that for any ω one has

$$(f_0^* - f_1^*)(\omega) = \pm(d_M D - D d_N)(\omega). \tag{12}$$

Proof. For any ω on N, decompose $F^*(\omega) = \Omega_1 + \Omega_2 \wedge dt$ according to the previous lemma. Define

$$D(\omega) := \int_0^1 \Omega_2(t) dt. \tag{13}$$

This is the integration of coefficients in t as in parameter (evidently the result does not depend on coordinate system). Then D is well defined because of the uniqueness in the previous lemma. Since $f_0^* = \varphi_0^* F^*$, $f_1^* = \varphi_1^* F^*$, where

$$\varphi_0 : M \to M \times I, \quad \varphi_0(P) = (P,0), \qquad \varphi_1 : M \to M \times I, \quad \varphi_1(P) = (P,1),$$

we have

$$f_0^*(\omega) = \Omega_1(0), \qquad f_1^*(\omega) = \Omega_1(1) \tag{14}$$

(we substitute in $F^*\Omega$: $dt = 0$ and $t = 0$ or $t = 1$). Also,

$$F^* d_N \omega = d_{M\times I} F^* \omega = d_{M\times I}(\Omega_1 + \Omega_2 \wedge dt) = d_M \Omega_1 \pm \frac{\partial}{\partial t}\Omega_1(t) \wedge dt + d_M \Omega_2 \wedge dt$$

and

$$D d_N(\omega) = \int_0^1 \left(\pm \frac{\partial}{\partial t}\Omega_1(t) + d_M \Omega_2(t) \right) dt = \pm(\Omega_1(1) - \Omega_1(0)) + d_M \int_0^1 \Omega_2(t)\, dt. \tag{15}$$

In the same time

$$d_M D(\omega) = d_M \int_0^1 \Omega_2(t) dt. \tag{16}$$

From (14), (15) and (16) we obtain (12). □

Theorem 12.17. *Suppose that smooth maps f_0 and f_1 from M to N are homotopic to each other. Then $f_0^* = f_1^*$ in cohomology.*

Proof. Let a closed from ω on N represent a cohomology class $[\omega]$. In particular, $d_N \omega = 0$. For the map D from the previous lemma, we have

$$(f_0^* - f_1^*)(\omega) = \pm(d_M D - D d_N)(\omega) = d_M(D\omega).$$

This is 0 in the cohomology of M. □

Problem 12.18. Find the de Rham cohomology of manifolds:

1. Interval (a, b). Home
2. Circle S^1. Class
3. Eucledean space $\mathbb{R}^n$. Home
4. Sphere S^2. Class
5. The plane $\mathbb{R}^2$ without one point. *Hint:* using homotopies reduce to case 2). Home
6. The plane $\mathbb{R}^2$ without two points. Class

Problem 12.19. Prove the *Poincare lemma*: any closed form on any manifold is locally exact. *Hint:* reduce to the third case above.

Definition 12.20. Suppose that M is a smooth oriented manifold, $\dim M = n$, and $\omega \in \Omega^n(M)$ is a form of maximal degree with a compact support in one chart (U, φ_α) with coordinates $(x^1_\alpha, \dots, x^n_\alpha)$. (We assume here and below by default a chart from an **orienting atlas.**) Define the *integral* of ω over U by the formula 23.11.2023

$$\int\limits_U \omega := \int\limits_{\varphi_\alpha(U)\subset\mathbb{R}^n} \omega^\alpha_{12\dots n}\, dx^1_\alpha \dots dx^n_\alpha. \tag{17}$$

Lemma 12.21. *This integral is well defined, i.e., the right-hand side of* (17) *does not depend on the choice of local coordinates in* U.

Proof. Suppose that (U, φ_β) is another chart with the same U and local coordinates $(x^1_\beta, \dots, x^n_\beta)$. Since both charts have the same orientation, the rule of changing of variables in a multiple integral and Lemma 8.28 give

$$\int\limits_{\varphi_\beta(U)\subset\mathbb{R}^n} \omega^\beta_{12\dots n}\, dx^1_\beta \dots dx^n_\beta = \int\limits_{\varphi_\alpha(U)\subset\mathbb{R}^n} \omega^\beta_{12\dots n} \cdot \left|\det \left\| \frac{\partial x^i_\beta}{\partial x^j_\alpha} \right\| \right| dx^1_\alpha \dots dx^n_\alpha =$$

$$= \int\limits_{\varphi_\alpha(U)\subset\mathbb{R}^n} \omega^\beta_{12\dots n} \cdot \det \left\| \frac{\partial x^i_\beta}{\partial x^j_\alpha} \right\| dx^1_\alpha \dots dx^n_\alpha = \int\limits_{\varphi_\alpha(U)\subset\mathbb{R}^n} \omega^\alpha_{12\dots n}\, dx^1_\alpha \dots dx^n_\alpha.$$

□

Problem 12.22. Suppose that $K \subseteq M$ is a compact set and $\{U_\alpha\}$ is a locally finite open cover of M. Then $K \cap U_\alpha \neq \varnothing$ only for finitely many α. Home

Definition 12.23. Suppose that M is a smooth oriented manifold, $\dim M = n$, and $\omega \in \Omega^n(M)$ is a form of maximal degree with a compact support. For a locally finite atlas $\{(U_\alpha, \varphi_\alpha)\}$ and its subordinated partition of unity ψ_α, define the *integral* by

$$\int\limits_M \omega = I(M, \omega, \{(U_\alpha, \varphi_\alpha, \psi_\alpha)\}) := \sum_\alpha \int\limits_{U_\alpha} \psi_\alpha \omega. \tag{18}$$

By Problem 12.22, the sum is in fact finite.

Lemma 12.24. *This integral is well defined, i.e., the value does not depend on the choice of* $\{(U_\alpha, \varphi_\alpha, \psi_\alpha)\}$.

Proof. If we have two distinct atlases, then take their union, and for each of them take zero functions on the added sets to complete the corresponding partition of unity. Evidently, in each of these two cases, the right-hand side of (18) will not change. So the proof is reduced to a a verification of

$$I(M, \omega, \{(U_\alpha, \varphi_\alpha, \psi_\alpha)\}) = I(M, \omega, \{(U_\alpha, \varphi'_\alpha, \psi'_\alpha)\}).$$

The independence of each summand on the choice of coordinates, i.e., φ_α, was proved in the previous lemma. So we need to prove that

$$I(M, \omega, \{(U_\alpha, \varphi_\alpha, \psi_\alpha)\}) = I(M, \omega, \{(U_\alpha, \varphi_\alpha, \psi'_\alpha)\}).$$

Define $\gamma_i := \psi_{\alpha_i} - \psi'_{\alpha_i}$, $i = 1, \dots, N$, (because, for a fixed form, by Problem 12.22, the sum is in fact finite). Then

$$\sum_{i=1}^{k} \gamma_\alpha = 0, \qquad k = N. \tag{19}$$

The proof is reduced to a verification (under the supposition of (19)) of

$$\sum_{i=1}^{k} \int\limits_{U_{\alpha_i}} \gamma_i \omega = 0, \qquad k = N. \tag{20}$$

We will prove it by induction over k. For $k = 1$ the statement is evident. Suppose that for $k = 1, \dots, N-1$ and arbitrary $\gamma_i : M \to \mathbb{R}_+$ with $\operatorname{supp} \gamma_i \subset U_{\alpha_i}$ the equality (19) implies (20). Find a continuous function $\chi : M \to [0,1]$ which is equal to 1 on $\operatorname{supp} \gamma_{\alpha_N} \subset \alpha_N$ and $\operatorname{supp} \chi \subset U_{\alpha_N}$. It exists because M is normal. Then

$$\chi \gamma_N \equiv \gamma_N, \qquad \gamma_N = -\sum_{i=1}^{N-1} \gamma_i = -\sum_{i=1}^{N-1} \chi \gamma_i, \qquad \operatorname{supp}(\chi \gamma_i) \subseteq (U_N \cap U_{\alpha_i}).$$

Hence,

$$\sum_{i=1}^{N} \int\limits_{U_{\alpha_i}} \gamma_i \omega = \int\limits_{U_N} \gamma_N \omega + \sum_{i=1}^{N-1} \int\limits_{U_{\alpha_i}} \gamma_i \omega = -\sum_{i=1}^{N-1} \int\limits_{U_{\alpha_i}} \chi \gamma_i \omega + \sum_{i=1}^{N-1} \int\limits_{U_{\alpha_i}} \gamma_i \omega =$$

$$= \sum_{i=1}^{N-1} \int\limits_{U_{\alpha_i}} (\gamma_i - \chi \gamma_i) \omega. \tag{21}$$

Since

$$\sum_{i=1}^{N-1} (\gamma_i - \chi \gamma_i) = \sum_{i=1}^{N-1} \gamma_i - \chi \sum_{i=1}^{N-1} \gamma_i = \sum_{i=1}^{N-1} \gamma_i + \chi \gamma_N = \sum_{i=1}^{N-1} \gamma_i + \gamma_N = \sum_{i=1}^{N} \gamma_i = 0,$$

we can apply to (21) the induction supposition. $\square$

Evidently we have:

Proposition 12.25. *The integral gives rise to an $\mathbb{R}$-linear map*

$$\Omega^n_{comp}(M, \mathrm{Or}) \to \mathbb{R}.$$

Problem 12.26. Prove that the change of orientation changes the sign of an integral but not its absolute value. Home

Definition 12.27. In particular, we can define the *volume of a compact oriented Riemannian manifold* as the absolute value of the integral of the volume form.

Problem 12.28. Prove that (under some reasonable restrictions) an integral of a form can be calculated by integration of restrictions of the form to some sets each of which lies in a chart and then summation of the results. Home

Theorem 12.29. (General Stokes Formula). *Consider a smooth oriented manifold M with boundary ∂M, $\dim M = n$, and a compactly supported differential form $\omega \in O^{n-1}(M)$. Consider the orientation of ∂M introduced in the proof of Theorem 5.8. Then*

$$(-1)^n \int_M d\omega = \int_{\partial M} \omega \qquad \left(= \int_{\partial M} j^*\omega \right), \tag{22}$$

where $j : \partial M \to M$ is the inclusion of the boundary.

Proof. As before, we may consider an atlas with charts with $V_\alpha = \varphi_\alpha(U_\alpha) = \mathbb{R}^n_+$ or $\mathbb{R}^n$. Both sides of (22) are linear in ω. Hence, it is sufficient to verify the equality for a form compactly supported in one chart (using a partition of unity) . Moreover, it is sufficient to verify for forms (using the expansion w.r.t. a local base)

$$\omega = f(x^1, \ldots, x^n)\, dx^1 \wedge \ldots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n, \qquad d\omega = (-1)^{k-1} \frac{\partial f}{\partial x^k} dx^1 \wedge \ldots \wedge dx^n,$$

where $f : \mathbb{R}^n_+ \to \mathbb{R}$ is a smooth compactly supported function (for the case of $\mathbb{R}^n_+$). We have $x^n \geq 0$ and ∂M is characterized by $x^n = 0$. Consider first the case of $k \leq n-1$, i.e., $k \neq n$. Locally the inclusion of the boundary has the form:

$$j : \partial M \to M, \qquad j(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, 0),$$

and $dx^n = 0$. Hence $j^*\omega = 0$ (see also (11)). For the right-hand side of (22) we have

$$\int\limits_{\mathbb{R}^n_+} d\omega = \int\limits_{\mathbb{R}^n_+} (-1)^{k-1} \frac{\partial f}{\partial x^k}\, dx^1 \ldots dx^n =$$

$$= (-1)^{k-1} \int\limits_{\mathbb{R}^{n-1}_+} \left\{ \int\limits_{-\infty}^{+\infty} \frac{\partial f}{\partial x^k} dx^k \right\} dx^1 \ldots dx^{k-1} dx^{k+1} dx^n =$$

$$= (-1)^{k-1} \int\limits_{\mathbb{R}^{n-1}_+} \Big\{ f(x^1, \ldots, x^{k-1}, +\infty, x^{k+1}, \ldots, x^n) -$$

$$- f(x^1, \ldots, x^{k-1}, -\infty, x^{k+1}, \ldots, x^n) \Big\}\, dx^1 \ldots dx^{k-1} dx^{k+1} dx^n =$$

$$= (-1)^{k-1} \int\limits_{\mathbb{R}^{n-1}_+} \Big\{0 - 0\Big\}\, dx^1 \dots dx^{k-1} dx^{k+1} dx^n = 0$$

(the above passage from the multiple integral to the iterated (Fubini's theorem) is correct because of compactness of the support and the vanishing "at infinity" by the same reason).

Consider now the case of $k = n$. We have

$$\int\limits_{\mathbb{R}^n_+} d\omega = \int\limits_{\mathbb{R}^n_+} (-1)^{n-1} \frac{\partial f}{\partial x^n}\, dx^1 \dots dx^n =$$

$$= (-1)^{n-1} \int\limits_{\mathbb{R}^{n-1}_0} \left\{ \int\limits_0^{+\infty} \frac{\partial f}{\partial x^n} dx^n \right\} dx^1 \dots dx^{n-1} =$$

$$= (-1)^{n-1} \int\limits_{\mathbb{R}^{n-1}_0} \Big\{ f(x^1, \dots, x^{n-1}, +\infty) - f(x^1, \dots, x^{n-1}, 0) \Big\}\, dx^1 \dots dx^{n-1} =$$

$$= (-1)^n \int\limits_{\mathbb{R}^{n-1}_0} f(x^1, \dots, x^{n-1}, 0)\, dx^1 \dots dx^{n-1} = (-1)^n \int\limits_{\mathbb{R}^{n-1}_0} \varphi^* \omega$$

(with the same usage of compactness as above).

In the case of $\mathbb{R}^n$ we have that the chart does not intersect ∂M and $j^*\omega = 0$. So the right-hand side of (22) vanishes. The left-hand side of (22) vanishes by the same calculation, as in the case $k < n$ above. □

Class **Problem 12.30.** The general Stokes formula implies Green's formula from vector calculus

$$\oint_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy.$$

Home **Problem 12.31.** The general Stokes formula implies divergence (Gauss–Ostrogradsky) theorem from vector calculus

$$\oiint_{\partial V} P(x,y,z)dy \wedge dz + Q(x,y,z)dz \wedge dx + R(x,y,z)dx \wedge dy =$$
$$= \iiint_V \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z} \right)$$

Home **Problem 12.32.** The general Stokes formula implies the classical Stokes formula from vector calculus: for a piece Σ of a surface,

$$\oint_{\partial \Sigma} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$
$$= \iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

Problem 12.33. Denote $\vec{F} = (P, Q, R) = P\vec{i} + Q\vec{j} + R\vec{k}$.

a) Let $d\vec{r} = (dx, dy, dz)$. Understand why Home

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma Pdx + Qdy + Rdz.$$

b) Let $\vec{n}$ be the unit normal field on a surface Σ. Understand why Class

$$\iint_\Sigma \vec{F} \cdot \vec{n}\, dS = \iint_\Sigma Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

Problem 12.34. Obtain from the general Stokes formula the vector calculus formulas:

a) classical Stokes: Class

$$\iint_\Sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, dS = \iint_\Sigma (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{r},$$

b) Gauss-Ostrogradsky: Home

$$\iiint_V \operatorname{div} \vec{F}\, dx\, dy\, dz = \iiint_V \vec{\nabla} \cdot \vec{F}\, dx\, dy\, dz = \oiint_{\partial V} \vec{F} \cdot \vec{n}\, dS.$$

13 Riemann Curvature Tensor

We will consider **symmetric** connections. Consider locally in coordinates $(x^1, \ldots, x^n)$ the action of $\nabla_k\nabla_l - \nabla_l\nabla_k$ on a vector field T^i (so the result is a tensor field of type (1,2)). We have

$$\nabla_l T^i = \frac{\partial T^i}{\partial x^l} + T^r \Gamma^i_{rl},$$

$$\nabla_k\nabla_l T^i = \frac{\partial^2 T^i}{\partial x^k\, \partial x^l} + \frac{\partial T^r}{\partial x^k}\Gamma^i_{rl} + T^r \frac{\partial \Gamma^i_{rl}}{\partial x^k} + \Gamma^i_{sk}\left(\frac{\partial T^s}{\partial x^l} + T^r\Gamma^s_{rl}\right) - \Gamma^s_{lk}\left(\frac{\partial T^i}{\partial x^s} + T^r \Gamma^i_{rs}\right),$$

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)\, T^i =$$

$$= T^r\left(\frac{\partial \Gamma^i_{rl}}{\partial x^k} - \frac{\partial \Gamma^i_{rk}}{\partial x^l}\right) + \frac{\partial T^r}{\partial x^k}\Gamma^i_{rl} - \frac{\partial T^r}{\partial x^l}\Gamma^i_{rk} + \frac{\partial T^s}{\partial x^l}\Gamma^i_{sk} - \frac{\partial T^s}{\partial x^k}\Gamma^i_{sl} + T^r\Gamma^i_{sk}\Gamma^s_{rl} - T^r\Gamma^i_{sl}\Gamma^s_{rk} =$$

$$= T^r\left(\frac{\partial \Gamma^i_{rl}}{\partial x^k} - \frac{\partial \Gamma^i_{rk}}{\partial x^l} + \Gamma^i_{sk}\Gamma^s_{rl} - \Gamma^i_{sl}\Gamma^s_{rk}\right).$$

Denote

$$R^i_{q,kl} := \frac{\partial \Gamma^i_{ql}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^l} + \Gamma^i_{sk}\Gamma^s_{ql} - \Gamma^i_{sl}\Gamma^s_{qk}, \tag{23}$$

and obtain that

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)\, T^i = T^q\, R^i_{q,kl}.$$

Lemma 13.1. *Functions $R^i_{q,kl}$ form a tensor of type $(1,3)$.*

Proof. For any vector field T, the functions $(\nabla_k\nabla_l - \nabla_l\nabla_k)\, T^i$, i.e., $T^q\, R^i_{q,kl}$, form a tensor field of type $(1,2)$. Since $R^i_{q,kl} = (e_q)^s\, R^i_{s,kl}$, we have

$$R^{i'}_{q',k'l'} = (e_{q'})^{s'}\, R^{i'}_{s',k'l'} = (e_{q'})^s\, R^i_{s,kl}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^{i'}}{\partial x^i} = (e_{q'})^{s'}\frac{\partial x^s}{\partial x^{s'}}\, R^i_{s,kl}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^{i'}}{\partial x^i} =$$

$$= \delta^{s'}_{q'}\frac{\partial x^s}{\partial x^{s'}}\, R^i_{s,kl}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^{i'}}{\partial x^i} = R^i_{s,kl}\frac{\partial x^s}{\partial x^{q'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^{i'}}{\partial x^i} = R^i_{q,kl}\frac{\partial x^q}{\partial x^{q'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^{i'}}{\partial x^i}.$$

□

Definition 13.2. The tensor $R^i_{q,kl}$ is called the *Riemann curvature tensor* of a symmetric connection ∇.

Pass to the invariant definition of R.

Definition 13.3. Recall that the *commutator* of vector fields X and Y is the vector field

$$[X,Y]^k := X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i}.$$

For any symmetric connection,

$$\nabla_X Y^k - \nabla_Y X^k = X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma^k_{ji} \right) - Y^i \left(\frac{\partial X^k}{\partial x^i} + X^j \Gamma^k_{ji} \right) = [X,Y]^k, \tag{24}$$

in particular, the operation is a tensor one (the result is a vector field).

Definition 13.4. Define the *curvature operator* by

$$R(X,Y)Z := \nabla_X \nabla_Y (Z) - \nabla_Y \nabla_X (Z) - \nabla_{[X,Y]}(Z).$$

It maps a triple of vector fields X, Y and Z to some fourth vector field. The notation $R(X,Y)Z$, not $R(X,Y,Z)$, reflects the roles of variables.

Theorem 13.5. *The map R is 3-linear over functions. Thus, it defines a tensor field of type $(1,3)$.*

Proof. If T is a 3-linear map of vector fields valued in vector fields, then the map

$$\widetilde{T}(X,Y,Z;\omega) := \omega(T(X,Y,Z))$$

will be 4-linear map of 3 vector and 1 covector field arguments valued in functions, i.e., a tensor field of type $(1,3)$.

3-linearity at a point (i.e., over $\mathbb{R}$) is evident. It remains to verify linearity for functions, i.e., $R(X,Y)(fZ) = f \cdot R(X,Y)Z$ and two similar identities (Problem 13.6). □

Class **Problem 13.6.** Verify the linearity of $R(X,Y)(Z)$ for functions.

Lemma 13.7. *The definitions are equivalent.*

Proof. For local basic vector fields $e_i = \frac{\partial}{\partial x^i}$ we have

$$R(e_i,e_j)Z^k = \nabla_{e_i}\nabla_{e_j} Z^k - \nabla_{e_j}\nabla_{e_i} Z^k + \nabla_{[e_i,e_j]} Z^k = \nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k,$$

because $\nabla_{e_i} Z^k = (e_i)^m \nabla_m Z^k = \delta^m_i \nabla_m Z^k = \nabla_i Z^k$ and hence

$$[e_i,e_j] = \nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} = \nabla_i e_j - \nabla_j e_i = \Gamma^l_{ji} e_l - \Gamma^l_{ij} e_l = 0, \tag{25}$$

using (24). Linearity completes the proof. □

30.11.2023 **Theorem 13.8.** $\ast$ (symmetries of the Riemann curvature tensor)

1) *anti-symmetric in X and Y: $R(X,Y)Z + R(Y,X)Z = 0$, or $R^i_{j,kl} + R^i_{j,lk} = 0$;*

2) *Jacobi identity:* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, *or* $R^i_{j,kl} + R^i_{k,lj} + R^i_{l,kj} = 0$;

3) *for any Levi-Civita connection* $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$, *or* $R_{ij,kl} + R_{ji,kl} = 0$, *where* $R_{ij,kl} = g_{ir}R^r_{j,kl}$;

4) *for any Levi-Civita connection* $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$, *or* $R_{ij,kl} = R_{kl,ij}$.

The proof of this statement can be found in [Lee, Theorem 13.19].

We proceed with Levi-Civita connections.

Problem 13.9. For any Levi-Civita connection one has Home

$$R_{iqkl} = g_{ir}R^r_{qkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^q \partial x^k} + \frac{\partial^2 g_{qk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^q \partial x^l} - \frac{\partial^2 g_{ql}}{\partial x^i \partial x^k}\right) + g_{mp}(\Gamma^m_{qk}\Gamma^p_{il} - \Gamma^m_{ql}\Gamma^p_{ik}).$$

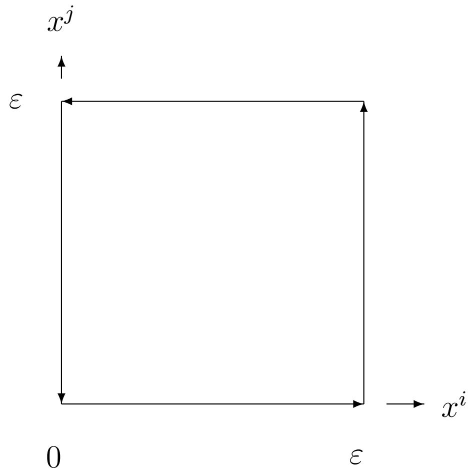
Definition 13.10. A Riemannian manifold (M,g) is *flat*, if the curvature tensor is identically zero.

Theorem 13.11. *A manifold is flat iff it is locally euclidean in metric ($g_{ij} = const$) or connection ($\Gamma^i_{jk} = 0$) sense.*

Problem 13.12. Prove this. In one direction this follows from Problem 13.9. For the other direction, see Theorem 13.18 in [Lee]. Home

Keeping in mind the definition, the following statement about the geometric meaning of the Riemann curvature tensor is not surprising:

Problem 13.13. Let $(x^1, \ldots, x^n)$ be some coordinates in a neighborhood of $P \in M$, where (M, ∇) is a manifold equipped with a symmetric connection (not necessary Levi-Civita), $x^i(P) = 0, \quad \forall\, i$. Suppose that $\xi \in T_PM$ is an arbitrary vector and $\xi_\varepsilon = \xi_\varepsilon(i,j)$ is the result of its parallel transport around coordinate square in x^i, x^j with sides of length ε (i.e., formed by segments of four coordinate curves in the x^i, x^j-plane - see figure). Home



Then

$$\lim_{\varepsilon \to 0} \frac{\xi^k_\varepsilon - \xi^k}{\varepsilon^2} = R^k_{l,ij} \xi^l.$$

(see Theorem 5.11 in `http://math.uchicago.edu/~may/REU2016/REUPapers/Wan.pdf` or Theorem 12.47 in [Lee])

This observation immediately implies the "if" direction of the following statement:

Theorem 13.14. *A Riemannian manifold is flat if and only if results of parallel transport along two homotopic curves are the same (equivalently, the result of parallel transport along a contractible loop is the same as the initial vector).*

Proof. To prove the "only if" consider two homotopic curves γ_0, $\gamma_1 : (-\varepsilon, 1+\varepsilon) \to M$ (we need an extension to an open interval because the direct product with $[0,1]$ should be a manifold) with the properties $\gamma_0(0) = \gamma_1(0) = P_0$, $\gamma_0(1) = \gamma_1(1) = P_1$, such that a homotopy $G : (-\varepsilon, 1+\varepsilon) \times [0,1] \to M$ satisfies this for each t (we suppose $s \in (-\varepsilon, 1+\varepsilon)$ and $t \in [0,1]$). Consider the vector field $\xi_t(s)$ being the velocity field of $G(s,t)$ for fixed t (in particular, $\xi_0(s)$ and $\xi_1(s)$ are the velocity fields of γ_0 and γ_1), and the vector field $\eta_s(t)$ being the velocity field of $G(s,t)$ for fixed s. For a given $v \in T_{P_0}M$, define the vector field $v_s(t)$, where $v_s(t)$ is the result of the parallel transport of v along $\gamma_t(s) = G(s,t)$ for fixed t to the point with parameter s. (Note, that in the definition of a parallel transport we have not asked the regularity of a curve (non-vanishing of the velocity) but only its smoothness) Then the field $v_s(t)$ is parallel along $G(s,t)$ for fixed s.

Indeed,

$$\nabla_{\xi_t(s)} \nabla_{\eta_s(t)} v^i_s(t) - \nabla_{\eta_s(t)} \nabla_{\xi_t(s)} v^i_s(t) - \nabla_{[\xi_t(s), \eta_s(t)]} v^i_s(t) = R^i_{j,kl} v^j_s(t) \xi^k_t(s) \eta^l_s(t).$$

By the definition of $v_s(t)$, the second summand in the l.h.s. vanishes. By the supposition, the r.h.s. vanishes too. The third summand in the l.h.s. vanishes by the following argument: if $G(t,s) = (x^1(t,s), \ldots, x^n(t,s))$, then

$$[\xi_t(s), \eta_s(t)]^k = \xi_t(s)^j \frac{\partial \eta_s(t)^k}{\partial x^j} - \eta_s(t)^j \frac{\partial \xi_t(s)^k}{\partial x^j} =$$

$$= \frac{\partial x^j}{\partial s} \frac{\partial}{\partial x^j} \left(\frac{\partial x^k}{\partial t} \right) - \frac{\partial x^j}{\partial t} \frac{\partial}{\partial x^j} \left(\frac{\partial x^k}{\partial s} \right) = \frac{\partial^2 x^k}{\partial s \partial t} - \frac{\partial^2 x^k}{\partial t \partial s} = 0.$$

Thus, the field $\nabla_{\eta_s(t)} v_s(t)$ is parallel along $\gamma_t(s)$ and vanishes for $s=0$ (since $v_0(t)\equiv v$). Hence, $\nabla_{\eta_s(t)} v_s(t)=0$ for any s, in particular, for $s=1$.

Then, since $G(1,t)\equiv P_1$, we have $\eta_1(t)\equiv 0$ and

$$0=\nabla_{\eta_1(t)} v_1^i(t)=\frac{d}{dt}v_1^i(t)+\Gamma^i_{mk}\eta_1^m(t)v_1^k(t)=\frac{d}{dt}v_1^i(t),$$

i.e., v_1 does not depend on t. □

14 Lie algebra of a Lie group

Definition 14.1. Denote by $\mathbb{X}(G)$ the space of vector fields on G. A vector field $X\in\mathbb{X}(G)$ is called *left invariant* iff $(L_g)_*X=X$ for all $g\in G$, where $(L_g)_*X=(d(L_g))\circ X\circ L_g^{-1}$ and $L_g: G\to G$ is the left translation. So the definition can be reformulate as $(dL_g)_x X_x = X_{gx}$. So $X\in\mathbb{X}(G)$ is left invariant iff the following diagram commutes for every $g\in G$:

$$\begin{tikzcd} TG \arrow[r, "d(L_g)"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_g"] & G \arrow[u, "X"'] \end{tikzcd}.$$

Similarly, for right translations. The (evidently linear) space of left invariant vector fields will be denoted by $\mathbb{X}^L(G)$ and of right invariant vector fields will be denoted by $\mathbb{X}^R(G)$.

Lemma 14.2. *Suppose, $f: M\to N$ is a smooth map. Then $(df)[X,Y]=[(df)X,(df)Y]$.*

Proof.

$$\begin{aligned}(df)[X,Y]_{f(P)}(g)=[X,Y]_P(g\circ f)=X_p(Y(g\circ f))-Y_p(X(g\circ f))=\\ =X_p((df)Y(g)\circ f)-Y_p((df)X(g)\circ f)=\\ =(df)X_{f(p)}((df)Y(g))-(df)Y_{f(p)}((df)X(g))=[(df)X,(df)Y]_{f(P)}(g).\end{aligned}$$

□

From Lemma 14.2 we obtain:

Lemma 14.3. *$\mathbb{X}^L(G)$ is closed under the Lie bracket operation.*

Definition 14.4. For a vector $v\in T_eG$, define a smooth left (resp. right) invariant vector field L^v (resp. R^v) such that $L^v(e)=v$ (resp. $R^v(e)=v$) by

$$L^v(g)=d(L_g)_e v,\qquad R^v(g)=d(R_g)_e v. \tag{26}$$

Problem 14.5. Show that $v\mapsto L^v$ (resp. $v\mapsto R^v$) gives a linear isomorphism $T_eG\cong\mathbb{X}^L(G)$ (resp., $T_eG\cong\mathbb{X}^R(G)$). Home

Definition 14.6. A vector space (a) over a field $\mathbb{K}$ is called *Lie algebra* if it is equipped with a bilinear map $(a)\times(a)\to(a)$ denoted $(v,w)\mapsto[v,w]$ such that

$$[v,w]=-[w,v]$$

and such that we have the *Jacobi identity*

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$

for all $x,y,z\in(a)$.

Evidently,

Proposition 14.7. *If G is a Lie group of dimension n, then $\mathbb{X}^L(G)$ is an n-dimensional Lie algebra for the Lie bracket of vector fields.*

The above isomorphism transfers the Lie algebra structure to T_eG.

Proposition 14.8. *For a fixed $A \in \mathrm{GL}(V)$, the map $L_A : \mathrm{GL}(V) \to \mathrm{GL}(V)$ given by $B \mapsto A \circ B$ has tangent map given by $(B, X) \mapsto (A \circ B, A \circ X)$, where $(B, X) \in \mathrm{GL}(V) \times L(V, V) \cong T(\mathrm{GL}(V))$.*

Also, the left invariant vector field $\widetilde{X}$ corresponding to $X \in L(V, V)$ has the form $\widetilde{X}(A) = (A, AX)$.

Proof. In local coordinates (which are global here) the tangent map is defined by multiplication by the Jacobi matrix, which is

$$\frac{\partial (AX)^\nu_\mu}{\partial X^\rho_\sigma} = A^\tau_\mu \delta^\rho_\tau \delta^\nu_\sigma = A^\rho_\mu \delta^\nu_\sigma, \qquad \frac{\partial (AX)^\nu_\mu}{\partial X^\rho_\sigma} V^\sigma_\rho = A^\rho_\mu \delta^\nu_\sigma V^\sigma_\rho = A^\rho_\mu V^\nu_\rho = (AV)^\nu_\mu.$$

The second statement is now evident, because (X, AX) is a left-invariant (by the first statement) with X at e, and such a field is unique. □

The exponential map and related topics were discussed in detail in the course on Lie groups and Lie algebras, so we omit this topic here.

14.1 The Maurer-Cartan form

Definition 14.9. Define $\mathfrak{g}$-valued 1-forms (i.e. smooth fiber-wise $\mathbb{R}$-linear maps $TG \to \mathfrak{g}$) ω_G and ω_G^{right} by

$$\omega_G(X_g) = d(L_{g^{-1}})_g X_g, \qquad \omega_G^{\text{right}}(X_g) = d(R_{g^{-1}})_g X_g,$$

where $X_g \in T_gG$ is the value of a vector field X at $g \in G$. These forms are called the *left Maurer-Cartan form* and *right Maurer-Cartan form* respectively.

Home **Problem 14.10.** Explain the smoothness.

Theorem 14.11. *The tangent bundle of a Lie group is trivial. More specifically, the maps*

$$\mathrm{triv}_L : TG \to G \times \mathfrak{g}, \qquad \mathrm{triv}_L(v_g) = (g, \omega_G(v_g)), \qquad v_g \in T_gG,$$

$$\mathrm{triv}_R : TG \to G \times \mathfrak{g}, \qquad \mathrm{triv}_R(v_g) = (g, \omega_G^{\text{right}}(v_g)), \qquad v_g \in T_gG,$$

give two examples of trivializations of TG.

Proof. Evidently we have smooth bundle maps and they are invertible with

$$\mathrm{triv}_L^{-1}(g, v) = L^v(g), \qquad \mathrm{triv}_R^{-1}(g, v) = R^v(g).$$

Indeed, by (26)

$$\mathrm{triv}_L(L^v(g)) = (g, d(L_{g^{-1}})_g d(L_g)_e v) = (g, v), \qquad L^{\omega_G(v_g)}(g) = d(L_g)_e d(L_{g^{-1}})_g (v_g) = v_g,$$

and similarly for triv_R. □

Problem 14.12. Complete the remaining details. Home

Theorem 14.13. *For any $v \in \mathfrak{g}$, $g \in G$ one has*

$$\mathrm{triv}_R \circ \mathrm{triv}_L^{-1}(g, v) = (g, \mathrm{Ad}_g(v)),$$

where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{Ad}_g(v) = d(R_{g^{-1}} L_g)v$.

Proof. $\mathrm{triv}_R \circ \mathrm{triv}_L^{-1}(g, v) = (g, d(R_{g^{-1}})d(L_g)v) = (g, \mathrm{Ad}_g(v))$. □

Consider the Maurer-Cartan form in the case of matrix groups. Suppose that G is a Lie subgroup of $\mathrm{GL}(n)$ and consider the coordinate functions x^i_j on $\mathrm{GL}(n)$ defined by $x^i_j(A) = a^i_j$, where $A = \|a^i_j\|$. We have the associated 1- forms dx^i_j. Restrict both the functions x^i_j and the forms dx^i_j to G, denoting these restrictions by the same symbols. Then the (left) Maurer-Cartan form can be expressed as

$$\omega_G = \|x^i_j\|^{-1} \|dx^i_j\|.$$

Indeed, $v_g \in T_g G \subseteq T_g \,\mathrm{GL}(n)$ has the expansion

$$v_g = \sum_{i,j} v^i_j \frac{\partial}{\partial x^i_j}\bigg|_g .$$

Then, by Proposition 14.8

$$\|x^i_j\|^{-1}\|dx^i_j\|(v_g) = \|g^i_j\|^{-1}\|v^i_j\| = d(L_{g^{-1}})_g v_g = \omega_G(v_g),$$

where $g = \|g^i_j\|$.
Problem 14.14. Find the explicit form of the Maurer-Cartan form of $G = SO(2)$. Class

15 Ehresmann and Koszul connections

07.12.2023

Definition 15.1. Let $\pi : E \to M$ be a smooth vector bundle with typical fiber F of dimension k. Denote $\mathcal{V}_y E := (d\pi_y)^{-1}(0_p)$, where $\pi(y) = p$. The *vertical bundle* on $\pi : E \to M$ is the real vector bundle $\pi_{\mathcal{V}} : \mathcal{V}E \to E$ with total space

$$\mathcal{V}E := \sqcup_{y \in E} \mathcal{V}_y E \subset TE$$

and projection map $\pi_{\mathcal{V}} := \pi_{TE}|_{\mathcal{V}E}$. A vector bundle atlas on $\mathcal{V}E$ is given by charts of the form

$$(\pi_{\mathcal{V}}, d\varphi \circ d\Phi) : \pi_{\mathcal{V}}^{-1}(\pi^{-1}(U) \cap \Phi^{-1}(V)) \to (\pi^{-1}(U) \cap \Phi^{-1}(V)) \times \mathbb{R}^k,$$

where (π, Φ) is a bundle chart on E over U and (V, φ) is a chart in F.

Problem 15.2. Verify this. Home

Definition 15.3. A smooth rank k *distribution* on an n-manifold M is a (smooth) rank k vector subbundle of the tangent bundle.

Definition 15.4. A (linear Ehresmann) *connection* on a vector bundle $\pi : E \to M$ is a smooth distribution $\mathcal{H}$ on the total space E such that

1) $\mathcal{H}$ is complementary to the vertical bundle: $TE = \mathcal{H} \oplus \mathcal{V}E$;

2) $\mathcal{H}$ is homogeneous: $d(\mu_r)_y(\mathcal{H}_y) = \mathcal{H}_{ry}$ for all $y \in E$, $r \in \mathbb{R}$, where $\mu_r : E \to E$ is the multiplication map given by $\mu_r : y \to ry$.

The subbundle $\mathcal{H}$ is called the *horizontal distribution* (or *horizontal subbundle*).

Definition 15.5. For a general bundle (not necessarily a vector bundle), we have the same definition, but only with the property 1).

Definition 15.6. For $y \in E$, an individual element $w \in T_yE$ is *horizontal* if $w \in \mathcal{H}_y$ and *vertical* if $w \in \mathcal{V}_yE$. A vector field (i.e. a section) $X \in \mathbb{X}(E) = \Gamma(TE)$ is said to be a *horizontal vector field* (resp. *vertical vector field*) if $X(y) \in \mathcal{H}_y$ (resp. $X(y) \in \mathcal{V}_yE$) for all $y \in E$.

Home **Problem 15.7.** Let $f : N \to M$ be a smooth map and $\pi : E \to M$ a fiber bundle. Prove that the pull-back f^*E (Definition 9.48) can be naturally identified with $\{(p, e) \in N \times E \colon f(p) = \pi(e)\}$

Home **Problem 15.8.** Let $f : N \to M$ be a smooth map and $\pi : E \to M$ a fiber bundle with typical fiber F. Prove that $\mathcal{V}f^*E \to f^*E$ is bundle isomorphic to $\widetilde{f}^*\mathcal{V}E \to f^*E$, where $\widetilde{f} := pr_2|_{f^*E} : f^*E \to E$, $pr_2 : N \times E \to E$ and $f^*E = \{(p, e) \in N \times E \colon f(p) = \pi(e)\}$ (cf. the previous problem). See the diagram:

$$\begin{tikzcd}
\mathcal{V}f^*E \arrow[dr, "p"'] & \widetilde{f}^*\mathcal{V}E \arrow[r] \arrow[d] & \mathcal{V}E \arrow[d, "\pi_{\mathcal{V}}"] \\
& f^*E \arrow[r, "\widetilde{f}"] \arrow[d] & E \arrow[d, "\pi"] \\
& N \arrow[r, "f"] & M
\end{tikzcd}$$

Proposition 15.9. *For a vector bundle E, the vertical vector bundle $\mathcal{V}E$ is isomorphic to the vector bundle π^*E (as bundles over E). Sometimes they say that $\mathcal{V}E$ is isomorphic to E along π.*

Proof. If $(v, w) \in \pi^*E = \{(p, e) \in E \times E \colon \pi(p) = \pi(e)\}$, i.e. $\pi(v) = \pi(w)$, or $v, w \in E_p$ for some p, then $\pi(v + tw)$ is constant in t. Thus we can define a map from π^*E to TE by $(v, w) \mapsto \frac{d}{dt}\big|_0 (v + tw)$. This map evidently maps into $\mathcal{V}E \subset TE$. We obtain a vector bundle isomorphism

$$\mathbf{j} : \pi^*E \cong \mathcal{V}E, \qquad \mathbf{j} : (v, w) \mapsto \mathbf{j}_v w := \frac{d}{dt}\bigg|_0 (v + tw) = w_v.$$

□

Home **Problem 15.10.** Prove that $\mathbf{j}$ is an isomorphism, i.e. surjective and injective.

Home **Problem 15.11.** Prove that $\mathcal{H} \cong \pi^*TM$.

Class **Problem 15.12.** Let $E \to M$ be a a vector bundle. Suppose that for each $p \in M$ there is a subspace $E'_p \subset E_p$. Then $E' = \cup_{p \in M} E'_p$ is the total space of rank l vector subbundle if and only if for each $p \in M$, there is an open neighborhood U of p on which smooth sections $\sigma_1, \ldots, \sigma_2$ are defined such that for each $q \in U$ the set $\{\sigma_1(q), \ldots \sigma_l(q)\}$ is a basis of E'_q.

Theorem 15.13. *Every vector bundle admits a connection.*

Proof. For a trivial bundle $pr_1 : M \times V \to M$ and a fixed $v \in V$ define $i_v : M \to M \times V$ by $i_v(p) := (p, v)$. For each $p \in M$, define $\mathcal{H}_{(p,v)} := d(i_v)_p(T_pM)$. Evidently these maps are linear injections smoothly depending on p. Then one can apply the previous problem to obtain that the subspaces $\mathcal{H}_{(p,v)}$ form a subbundle $\mathcal{H}$ of TE. Also,

$$d(pr_1)(\mathcal{H}_{(p,v)}) = d(pr_1)d(i_v)_p(T_pM) = d(pr_1 \circ i_v)_p(T_pM) = d(\mathrm{Id})_p(T_pM) = T_pM$$

and hence $TE = \mathcal{V} \oplus \mathcal{H}$. For any $a \in \mathbb{R}$ we have $\mu_a \circ i_v = i_{av}$ and $d(\mu_a) \circ d(i_v) = d(i_{av})$. Thus

$$d(\mu_a)(\mathcal{H}_{(p,v)}) = d(\mu_a)(d(i_v)(T_pM)) = d(i_{av})(T_pM) = \mathcal{H}_{(p,av)} = \mathcal{H}_{a(p,v)}.$$

Consider a general vector bundle $\pi : E \to M$ with a trivializing locally finite cover $\{U_\alpha\}$ of M. Choose a connection $\mathcal{H}^\alpha$ on each $\pi^{-1}(U_\alpha)$. Let $\{\rho_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. For each $y \in E$, define

$$L_y : T_{\pi(y)}M \to T_yE, \qquad L_y(v) := \sum_{\{\alpha :\, \pi(y) \in U_\alpha\}} \rho_\alpha(\pi(y))w_\alpha,$$

where w_α is the unique vector in $\mathcal{H}^\alpha$ such that $(d\pi)w_a = v$. Evidently L_y is linear and $(d\pi)_y \circ L_y = \mathrm{Id}_{T_pM}$. This implies (using Problem 15.12) that $y \mapsto L_y(T_{\pi(y)}M)$ determines a subbundle $\mathcal{H}$ with the property 1). $\square$

Problem 15.14. Verify the property 2). Home
Problem 15.15. Prove the above statement using a Riemannian metric (to be constructed first) and the orthogonal complement. Home

Definition 15.16. For a smooth fiber bundle $\pi : E \to M$ and a smooth map $f : N \to M$, we call a map $\sigma : N \to E$ a *section of E along f* if $\pi \circ \sigma = f$. The set of these sections is denoted $\Gamma_f(E)$.

If $\sigma : N \to E$ is a section of E along f, then $\sigma' : N \to f^*E$, $p \mapsto (p, \sigma(p)) \in N \times E$, is a section of the pull-back f^*E.
Problem 15.17. Prove that all sections of f^*E are of this form. Home

Definition 15.18. Let $\sigma : N \to E$ be a section of E along a map $f : N \to M$. We say that σ is a *parallel section* if $(d\sigma)v$ is horizontal for all $v \in TN$. If s is a section of E and $\gamma : [a, b] \to M$ is a curve, then we say that s is *parallel along* γ if $s \circ \gamma$ is parallel.

Proposition 15.19. *Suppose that $\mathcal{H}$ is a connection on $\pi : E \to M$, $f : N \to M$ is a smooth map, $\widetilde{f} = pr_2|_{f^*E} : f^*E \to E$. Then $f^*\mathcal{H} = (d\widetilde{f})^{-1}\mathcal{H}$ is a distribution, which defines a connection on $f^*E \to N$ (the* ***pull-back connection****):*

$$\begin{tikzcd}
f^*\mathcal{H} \arrow[r, hook] & Tf^*E \arrow[r, "d\widetilde{f}"] \arrow[d] & TE \arrow[d, "d\pi"] \\
& TN \arrow[r, "df"] & TM
\end{tikzcd}$$

(see also Problem 15.25 below).

Proof. By the definition of $\widetilde{f}$, we have $(f^*\mathcal{H})_{(q,y)} = (d\widetilde{f}_{(q,y)})^{-1}\mathcal{H}_y$, where $(q,y) \in f^*E$.

Note that the natural bundle isomorphism $(d(pr_1), d(pr_2)) : T(N \times E) \cong TN \times TE$ maps $T(f^*E)$ to $\{(v,w) \in TN \times TE \colon (df)v = (d\pi)w\}$. Indeed, a class of curve $(\gamma_1, \gamma_2) : I \to N \times E$ in $TN \times TE$ defines a vector $(v,w) \in Tf^*E$ iff $(\gamma_1(t), \gamma_2(t)) \in f^*E$ for any t, i.e. $f \circ \gamma_1(t) = \pi \circ \gamma_2(t)$, or equivalently $(df)v = (d\pi)w$. Also, by the definitions, under this isomorphism $(\mathcal{V}f^*E)_{(q,y)}$ corresponds to $\{0_q\} \times \mathcal{V}_yE$ and $(f^*\mathcal{H})_{(q,y)}$ corresponds to $\{(v,w) \in T_qN \times \mathcal{H}_y \colon (df)v = (d\pi)w\}$.

By Problem 15.8, $d\widetilde{f}$ is an isomorphism of vertical distributions. (This also follows from the above identification.) Then $f^*\mathcal{H} = (d\widetilde{f})^{-1}\mathcal{H}$ is a smooth family of subspces $(f^*\mathcal{H})_{(q,y)}$ complementary to $(\mathcal{V}f^*E)_{(q,y)}$. Hence, this is a distribution (by Problem 15.12) and this distribution is complementary to $\mathcal{V}f^*E$. It remains to verify that the distribution is homogeneous. The multiplication μ_a^* on $f^*E \subset N \times E$ is defined as $\mu_a^*(q,y) = (q, \mu_a y)$. Then $(d\mu_a^*)_{(q,y)}(v,w) = (v, (d\mu_a)w)$. Hence, by the above description of $(f^*\mathcal{H})_{(q,y)}$ and the homogeneity of $\mathcal{H}$, we obtain the homogeneity of $f^*\mathcal{H}$, $\square$

Home **Problem 15.20.** Prove that if s is parallel with respect to the pull-back connection on f^*E, then σ_s is parallel, where $\sigma_s : N \to E$, $\sigma_s(x) = s(x) \in E_{f(x)} = (f^*E)_x$.

Class **Problem 15.21.** Let $[0,b]$ be an interval and let $t \in [0,b]$. Suppose that $\pi : E \to [0,b]$ is a vector bundle with some connection. Let $\widetilde{\partial}$ denote the horizontal lift of $\frac{\partial}{\partial t}$.

1) For an integral curve $\gamma : [0,a] \to E$ of $\widetilde{\partial}$, show that $\pi \circ \gamma$ is an integral curve of $\frac{\partial}{\partial t}$. Deduce that $\gamma(a) \in E_a$.

2) Prove that for any $t_0 < b$ there exists $\varepsilon = \varepsilon(t_0) > 0$ such that all integral curves of $\widetilde{\partial}$ originating in the fiber E_{t_0} are defined at least on $[t_0, \varepsilon)$.

3) Then 1) and 2) imply that all integral curves of $\widetilde{\partial}$ have domain $[0,b]$.

The following theorem does not work in the general situation, but for curves this works fortunately.

Theorem 15.22. *Suppose that $\pi : E \to M$ is a vector bundle with a connection $\mathcal{H}$ and $\gamma : [a,b] \to M$ is a smooth curve. Then for each $u \in E_{\gamma(a)}$ there is a unique parallel section $\sigma_{\gamma,u}$ along γ such that $\sigma_{\gamma,u}(a) = u$. Also, the map $P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$, $P_\gamma(u) = \sigma_{\gamma,u}(b)$, is a linear isomorphism.*

Proof. One may assume $a = 0$ and apply Problem 15.21 with γ^*E instead of E and $\gamma^*\mathcal{H}$ instead of $\mathcal{H}$. We obtain an integral curve γ_u of $\widetilde{\partial}$ (an $\gamma^*\mathcal{H}$-horizontal lift of $\frac{\partial}{\partial t}$) in γ^*E with $\gamma_u(0) = (0,u) \in \gamma^*E$ defined on $[0,b]$. By 1) in Problem 15.21, $pr_1 \circ \gamma_u$ is an integral curve of $\frac{\partial}{\partial t}$ and $pr_1 \circ \gamma_u(t) = t$. Let $\sigma_{\gamma,u} := pr_2 \circ \gamma_u$ on $[0,b]$. Then $\sigma_{\gamma,u}$ is a parallel section of $E \to M$ along γ because $\dot{\gamma}_u$ is horizontal (see Problem 15.20 and the identification in Proposition 15.19). It is unique as an integral curve (Cauchy problem for ODE).

Now prove that the above defined P_γ is linear. First, note that $(r\sigma_{\gamma,u})^{\dot{}} = d(\mu_r) \circ \dot{\sigma}_{\gamma,u}$ is horizontal, because $d(\mu_r)$ preserves $\mathcal{H}$. Then $r\sigma_{\gamma,u}$ is parallel and $P_\gamma(ru) = rP_\gamma(u)$. So, P_γ is homogeneous. Now prove that $P_\gamma = \mathbf{j}_0^{-1} \circ d(P_\gamma) \circ \mathbf{j}_0$ (see the proof of Proposition 15.9 for a similar definition), i.e. a composition of linear maps. For $v_0 \in T_0E_{\gamma(0)}$, define $\omega(t) = tv$ such that $v_0 = \dot{\omega}(0)$ for an appropriate $v \in E_{\gamma(0)}$. This means that v is v_0 under "an appropriate identification". More precisely,

$$\mathbf{j}_0(v) = \left.\frac{d}{dt}\right|_0 (0 + tv) = v_0, \qquad v = \mathbf{j}_0^{-1}(v_0).$$

By the (third) definition of the tangent map,

$$(dP_\gamma)_0 v_0 = \frac{d}{dt}\bigg|_0 (P_\gamma \circ \omega).$$

Since $P_\gamma \circ \omega(t) = P_\gamma(tv) = tP_\gamma(v)$ (using the homogeneity proved first), we have

$$(dP_\gamma)_0 v_0 = \mathbf{j}_0(P_\gamma(v)) = \mathbf{j}_0 \circ P_\gamma \circ \mathbf{j}_0^{-1} v_0$$

and $P_\gamma = \mathbf{j}_0^{-1} \circ dP_\gamma \circ \mathbf{j}_0$ is linear.

Finally, evidently P_γ has the inverse P_{γ^-}, where $\gamma^-(t) := \gamma(b-t)$, so it is a linear isomorphism. $\square$

Problem 15.23. Verify that P_{γ^-} is the inverse to P_γ. Home

Definition 15.24. The map P_γ from the previous theorem is called *parallel translation* or *parallel transport* along γ from $\gamma(a)$ to $\gamma(b)$. For $t_1, t_2 \in [a,b]$, let $P(\gamma)_{t_1}^{t_2} := P_{\gamma|[t_1,t_2]} : E_{\gamma(t_1)} \to E_{\gamma(t_2)}$ if $t_2 \geq t_1$ and $P(\gamma)_{t_1}^{t_2} := P^{-1}_{\gamma|[t_2,t_1]} : E_{\gamma(t_1)} \to E_{\gamma(t_2)}$ if $t_1 \geq t_2$.

The curve $\sigma_{\gamma,u}$ is a *parallel lift* or *horizontal lift* of the curve γ.

A parallel transport along a piece-wise smooth curve is defined by stages as a composition.

Denote the vector bundle isomorphism from $\mathcal{V}E$ to E along π by $\mathbf{p}$, i.e. $\mathbf{p} : \mathcal{V}E \to E$ is the composition in the upper row of diagram (cf. Proposition 15.9): 14.12.2023

$$\begin{tikzcd} \mathcal{V}E \arrow[r, "\mathbf{j}^{-1}"] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ & E \arrow[r, "\pi"] & M. \end{tikzcd}$$

In the notation of Proposition 15.9 $\mathbf{p} : w_y \mapsto w$ and for each y, it gives the canonical identification of T_yE_p with E_p, and on each fiber, it is the inverse of $\mathbf{j}$. If we have a connection on $\pi : E \to M$, then we have an associated *connector*, which is the map $\kappa : TE \to E$ defined by

$$\kappa(v) := \mathbf{p}(p_\mathcal{V}(v)) = \mathbf{j}_y^{-1}(p_\mathcal{V}(v)),$$

where $v \in T_yE$ and $p_\mathcal{V} : TE = \mathcal{V}E \oplus \mathcal{H} \to \mathcal{V}$ is the canonical projection. It is a vector bundle homomorphism along $\pi : E \to M$:

$$\begin{tikzcd} TE \arrow[r, "p_\mathcal{V}"] \arrow[rrr, bend left, "\kappa"] \arrow[drr] & \mathcal{V}E \arrow[r, "\mathbf{j}^{-1}"] \arrow[dr] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ & & E \arrow[r, "\pi"] & M. \end{tikzcd} \tag{27}$$

Problem 15.25. Prove that $d\pi : TE \to TM$ is a vector bundle. In particular, the addition and scalar multiplication on a fiber $(d\pi^{-1})(x)$ of $d\pi : TE \to TM$ are defined by Class

$$u \boxplus v := (d\alpha)(u,v) \text{ for } u, v \in TE \text{ with } (d\pi)u = (d\pi)v = x,$$

$$c \circledcirc v := (d\mu_c)v \text{ for } v \in TE \text{ and } c \in \mathbb{K},$$

where $\alpha(y_1,y_2) := y_1 + y_2$ for $(y_1,y_2) \in E \oplus E$ and $\mu_c y := cy$ for $y \in E$ and $c \in \mathbb{K}$.

Lemma 15.26. *Suppose that $f : \mathbb{R}^K \to \mathbb{R}^k$ is a smooth map such that $f(av) = af(v)$ for all $v \in \mathbb{R}^K$ and $a \in \mathbb{R}$. Then f is linear. Similarly for $\mathbb{C}$.*

Proof. One has $(Df)(0)v = \frac{d}{dt}\big|_{t=0} f(tv) = \frac{d}{dt}\big|_{t=0} tf(v) = f(v)$. Thus $f = (Df)(0)$ and f is linear. Similarly, in the complex case, f is $\mathbb{R}$-linear and by $f(iv) = if(v)$ it is $\mathbb{C}$-linear. $\square$

Applying this lemma to each chart we obtain the following statement.

Corollary 15.27. *Suppose that $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M_2$ are $\mathbb{K}$-vector bundles, $\widehat{f} : E_1 \to E_2$ is a fiber bundle morphism over $f : M_1 \to M_2$. If $\widehat{f}$ is homogeneous on each fiber, i.e. $\widehat{f}(av) = a\widehat{f}(v)$ for all $v \in E_1$ and $a \in \mathbb{K}$, then $\widehat{f}$ is linear on fibers, i.e. it is a vector bundle morphism.*

Lemma 15.28. *Let $\mu_r : E \to E$ be multiplication by r. Then for any $p \in M$ and $y, w \in E_p$, we have*

$$(d\mu_r)(\mathbf{j}_y w) = \mathbf{j}_{ry}(rw) = r\mathbf{j}_{ry}w.$$

Proof. Indeed

$$\begin{aligned}(d\mu_r)(\mathbf{j}_y w) &= \frac{d}{dt}\bigg|_{t=0} \mu_r(y + tw) = \frac{d}{dt}\bigg|_{t=0} (ry + trw) \\ &= \mathbf{j}_{ry}(rw) = r\mathbf{j}_{ry}w.\end{aligned}$$

$\square$

Theorem 15.29. *Let κ be a connector of a connection on a vector bundle $\pi : E \to M$. Then κ is a vector bundle homomorphism from $d\pi : TE \to TM$ to $\pi : E \to M$ along the map $\pi_{TM} : TM \to M$*

$$\begin{tikzcd} TE \arrow[r, "\kappa"] \arrow[d, "d\pi"'] & E \arrow[d, "\pi"] \\ TM \arrow[r, "\pi_{TM}"] & M. \end{tikzcd} \tag{28}$$

Proof. In the diagram

$$\begin{tikzcd} TE \arrow[rr, "\kappa"] \arrow[dd, "d\pi"'] \arrow[dr] & & E \arrow[dd, "\pi"] \\ & E \arrow[dr, "\pi"] & \\ TM \arrow[rr, "\pi_{TM}"] & & M \end{tikzcd}$$

the left triangle is commutative by the definition of $d\pi$ and the right one by (27). Thus (28) is commutative. It remains to verify that κ is linear on fibers. Let $X_p = (d\pi)Z_y$, where $\pi(y) = p$, $Z_y \in T_yE$, $X_p \in T_pM$. Decompose $Z_y = H_y + V_y$, where $H_y \in \mathcal{H}_y$, $V_y \in \mathcal{V}_yE$. Since $(d\pi)V_y = 0$, we have $X_p = (d\pi)H_y$ and H_y is the horizontal lift $\widetilde{X}_y$ of X_p. Also, $V_y = \mathbf{j}_y w$ for a unique $w \in E_p$ (by Propositions 15.9). Thus $Z_y = \widetilde{X}_y + \mathbf{j}_y w$ and $\kappa(Z_y) = w$ by the definition. By Lemma 15.28 and homogeneity of $\mathcal{H}$ we have

$$(d\mu_r)Z_y = (d\mu_r)\widetilde{X}_y + (d\mu_r)\mathbf{j}_y w = \widetilde{X}_{ry} + \mathbf{j}_{ry}rw.$$

Hence $\kappa((d\mu_r)Z_y) = rw = r\,\kappa(Z_y)$ or $\kappa(r \odot Z_y) = r\,\kappa(Z_y)$ (in the notation of Problem 15.25). Corollary 15.27 completes the proof. $\square$

Problem 15.30. Prove that the addition $\boxplus$ in $TE \to TM$ can be described in the following (similar) form. We have, as above, $Z_y = \widetilde{X}_y + \mathbf{j}_y w$ for some $w \in E_p$ if $(d\pi)Z_y = X_p$ and $\widetilde{X}_y$ is the horizontal lift of X_p. Suppose, that for another vector $U_{y'}$ from the same fiber over X_p we have in the same way $U_{y'} = \widetilde{X}_{y'} + \mathbf{j}_{y'}w'$. Then the sum of these vectors will be given by $\widetilde{X}_{y+y'} + \mathbf{j}_{y+y'}(w + w')$, where $\widetilde{X}_{y+y'}$ is the horizontal lift of X_p to the point $y + y'$. Home

Problem 15.31. Using Problem 15.11 and Theorem 15.29 prove that $(\pi_{TE}, \kappa) : TE \to E \oplus E$ is a vector bundle isomorphism along the tangent bundle projection $\pi_T M : TM \to M$, i.e. we have a commutative diagram with fiberwise linear isomorphism in the upper row: Class

$$\begin{tikzcd}
TE \arrow[r, "(\pi_{TE},\kappa)"] \arrow[d, "d\pi"'] & E \oplus E \arrow[d, "\pi\oplus\pi"] \\
TM \arrow[r, "\pi_T M"] & M.
\end{tikzcd}$$

Now we introduce the Koszul definition of connection (covariant derivative) for a vector bundle $\pi : E \to M$, which generalizes an affine connection.

Definition 15.32. Let $\pi : E \to M$ and $f : N \to M$ be as above. A *covariant derivative along* f is a map $\nabla^f : TN \times \Gamma_f(E) \to \Gamma_f(E)$ (we write $\nabla^f(v, \sigma) = \nabla^f_v \sigma$) having the properties

(i) ∇^f is fiberwise linear in the first argument:
$$\nabla^f_{au+bv}\sigma = a\nabla^f_u\sigma + b\nabla^f_v\sigma,$$
for all $\sigma \in \Gamma_f(E)$, $a, b \in \mathbb{R}$, $u, v \in T_pN$ for some $p \in N$;

(ii) $\nabla^f_u(\sigma_1 + \sigma_2) = \nabla^f_u(\sigma_1) + \nabla^f_u(\sigma_2)$ for any $u \in TN$ and any $\sigma_1, \sigma_2 \in \Gamma_f(E)$;

(iii) for $v \in T_pN$, $h \in C^\infty(N, \mathbb{K})$, and $\sigma \in \Gamma_f(E)$, the Leibniz law is fulfilled:
$$\nabla^f_v(h\sigma)|_p = h(p)\nabla^f_v\sigma + v(h)\sigma(p);$$

(iv) for a vector field $p \mapsto v(p)$ from $\mathbb{X}(N)$, the map $p \mapsto \nabla^f_{v(p)}\sigma$ is smooth for all $\sigma \in \Gamma_f(E)$;

(v) if $g : S \to N$ and $f : N \to M$ are smooth, then
$$\nabla^{f\circ g}_u(\sigma \circ g) = \nabla^f_{(dg)u}\sigma,$$
$u \in TS$:

$$\begin{tikzcd}
 & & E \arrow[d, "\pi"] \\
S \arrow[r, "g"'] \arrow[rru, "\sigma\circ g"] & N \arrow[r, "f"'] \arrow[ru, "\sigma"'] & M.
\end{tikzcd}$$

Problem 15.33. Prove that (ii) and (iii) give the linearity of ∇^f over $\mathbb{K}$ in the second argument . Home

A related notion (in fact a reduction for $f = \mathrm{Id} : M \to M$) is:

Definition 15.34. Let $\pi : E \to M$ be a smooth $\mathbb{K}$-vector bundle. A *covariant derivative* or *Koszul connection* is a map $\nabla : \mathbb{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ (we write $\nabla(X, s) = \nabla_X s$) having the properties

(i) $\nabla_{fX}s = f\nabla_X s$, for all $s \in \Gamma(M,E)$, $f \in C^\infty(M)$, $X \in \mathbb{X}(M)$;

(ii) $\nabla_{X_1+X_2}s = \nabla_{X_1}s + \nabla_{X_2}s$ for any $s \in \Gamma(M,E)$, $X_1, X_2 \in \mathbb{X}(M)$;

(iii) $\nabla_X(s_1+s_2) = \nabla_X s_1 + \nabla_X s_2$ for all $s_1, s_2 \in \Gamma(M,E)$, $X \in \mathbb{X}(M)$;

(iv) $\nabla_X(hs) = h\nabla_X s + X(h)s$ for all $s \in \Gamma(M,E)$, $f \in C^\infty(M)$, $X \in \mathbb{X}(M)$.

Home **Problem 15.35.** Verify that this is a particular case.

Home **Problem 15.36.** Understand that an affine derivative of a vector field along a curve is a particular case of the above definitions.

Theorem 15.37. *Suppose that $\pi : E \to M$ is a vector bundle with a connection $\mathcal{H}$ and associated connector κ. For any smooth map $f : N \to M$ define the map $\nabla^f : TN \times \Gamma_f(E) \to \Gamma_f(E)$ by the formula*

$$\nabla^f_v \sigma|_p := \kappa((d\sigma)_p v) \qquad \text{for } v \in T_pN, \quad \sigma \in \Gamma_f(E), \tag{29}$$

For a vector field V on N define $(\nabla^f_V \sigma)(p) := \nabla^f_{V(p)}\sigma$. Then ∇^f satisfies Definition 15.32.

In particular, for $f = \mathrm{Id}_M$ we obtain a Koszul connection.

Conversely, if ∇ is a Koszul connection on $\pi : E \to M$, then we may define an (Ehresmann) connection by

$$\mathcal{H}_y := \{(ds)u - \mathbf{j}_y \nabla_u s | s \in \Gamma(M,E),\ s(\pi(y)) = y,\ u \in T_{\pi(y)}M\}$$

The initial Koszul connection can be restored by the formula $\nabla_v(s) = \kappa((ds)_p v)$, $v \in T_pM$.

Proof. Since κ and $d\sigma$ are smooth bundle morphisms, the properties (i) and (iv) of Definition 15.32 follow immediately from the definition (29).

If $g : S \to N$ and $f : N \to M$ are smooth and $u \in TS$, then for each $\sigma \in \Gamma_f(E)$ we have

$$\nabla^{f\circ g}_u(\sigma \circ g) = \kappa(d(\sigma \circ g)u) = \kappa(d(\sigma)((dg)u)) = \nabla^f_{(dg)u}\sigma.$$

This gives (v) of Definition 15.32.

To prove (ii) use the formula for addition in terms of the tangent lift of $\alpha : (u,v) \mapsto u+v$, $u, v \in E$. Consider $\sigma_1, \sigma_2 \in \Gamma_f(E)$, $u \in T_pN$, $u = [\gamma]$ for a smooth curve γ in N with $\gamma(0) = p$. Then

$$\begin{aligned}(d\sigma_1)u \boxplus (d\sigma_2)u &= (d\alpha)((d\sigma_1)u, (d\sigma_2)u) = \left.\frac{d}{dt}\right|_0 (\sigma_1 \circ \gamma + \sigma_2 \circ \gamma) \\ &= \left.\frac{d}{dt}\right|_0 (\sigma_1 + \sigma_2) \circ \gamma = d(\sigma_1 + \sigma_2)u.\end{aligned}$$

Since κ is a bundle homomorphism along π_{TM} we have

$$\nabla^f_u(\sigma_1 + \sigma_2) = \kappa(d(\sigma_1 + \sigma_2)u) = \kappa((d\sigma_1)u \boxplus (ds_2)u) = \nabla^f_u(\sigma_1) + \nabla^f_u(\sigma_2).$$

We have obtained (ii) of Definition 15.32.

Now, as above, let $u \in T_pN$ and $\sigma : N \to E$ is a section along a smooth map $f : N \to M$. We wish to find a formula for $d\mu : T\mathbb{R} \times TE \to TE$, where $\mu : \mathbb{R} \times E \to E$ is the scalar multiplication in the vector bundle $E \to M$. For this purpose consider $(a, y) \in \mathbb{R} \times E$ and

$(b \left.\frac{d}{dt}\right|_a, v_y) \in T_a\mathbb{R} \times T_yE$. Let us calculate first in two particular cases. Consider a smooth curve c in E with $c(0) = y$ and $\dot{c}(0) = v_y$, i.e. $v_y = [c]$. Then

$$\begin{aligned}(d\mu)(0_a, v_y) &= \left.\frac{d}{dt}\right|_0 \mu(a, c(t)) = \left.\frac{d}{dt}\right|_0 \mu_a(c(t)) \\ &= (d\mu_a)v_y = a \circledcirc v_y,\end{aligned} \tag{30}$$

where $\circledcirc$ is the scalar multiplication in the vector bundle structure of $TE \to TM$ as described in Problem 15.25. Now let c be the curve in $\mathbb{R}$ given by $c(t) := a + tb$ so that $c(0) = a$ and $\dot{c}(0) = b \left.\frac{d}{dt}\right|_a$. Then

$$\begin{aligned}(d\mu)\left(b \left.\frac{d}{dt}\right|_a, 0_y\right) &= \left.\frac{d}{dt}\right|_0 \mu(c(t), y) = \left.\frac{d}{dt}\right|_0 ((a + bt)y) \\ &= \left.\frac{d}{dt}\right|_0 (ay + tby) = \mathbf{j}_{ay}(by).\end{aligned} \tag{31}$$

From (31) and (32) we obtain

$$(d\mu)\left(b \left.\frac{d}{dt}\right|_a, v_y\right) = a \circledcirc v_y + \mathbf{j}_{ay}(by). \tag{32}$$

Next suppose that $h \in C^\infty(N)$ and c is a curve in N with $c(0) = p$ and $\dot{c}(0) = u \in T_pN$. Then

$$(dh)_p u = \left.\frac{d}{dt}\right|_0 h(c(t)) \left.\frac{\partial}{\partial t}\right|_{h(c(0))} = \left.\frac{\partial h}{\partial x^i}\right|_{c(0)} \left.\frac{dc^i}{dt}\right|_0 \left.\frac{\partial}{\partial t}\right|_{h(c(0))} = u(h) \left.\frac{\partial}{\partial t}\right|_{h(p)}, \tag{33}$$

where x^i are some coordinates, c is given by $x^i = c^i(t)$, and we write the partial derivative to emphasize that this is a basic vector related to coordinate system t. To write the next formula we need to introduce the following notation: let $h \times \sigma : N \to \mathbb{R} \times E$ denote the map $(h \times \sigma)(x) = (h(x), \sigma(x))$. Since κ is a bundle morphism, using its definition, (32) and (33) we obtain

$$\begin{aligned}\nabla^f_u(h\sigma) &= \kappa(d(h\sigma)u) = \kappa(d(\mu \circ (h \times \sigma))u) = \kappa(d(\mu) \circ d(h \times \sigma)(u)) \\ &= \kappa d(\mu)\left(u(h) \left.\frac{\partial}{\partial t}\right|_{h(p)}, (d\sigma)u\right) \\ &= \kappa(h(p) \circledcirc ((d\sigma)u) + \mathbf{j}_{h(p)\sigma(p)}(u(h)\sigma(p))) \\ &= h(p)\kappa((d\sigma)u) + u(h)\sigma(p) = h(p)\nabla^f_u\sigma + u(h)\sigma_p.\end{aligned}$$

The remaining part to be proved as a problem. □

Problem 15.38. Prove the remaining statements Class

We complete the study of Ehresmann connections by a brief mentioning of the following important case. In the case of a principal smooth G-bundle E over M the Ehresmann connection is supposed to be G-invariant, i.e. the second property (instead of homogeneity) is formulated as

$$\mathcal{H}_{eg} = d(R_g)_e\mathcal{H}_e,$$

where $e \in E$, $g \in G$ and R_g is the right action of G on E (see the definition of a principal bundle).

16 Basic K-theory

21.12.2023 We will say (one of equivalent definitions) that a space X is *paracompact* if it is Hausdorff and every open cover has a partition of unity subordinate to the cover, a collection of continuous maps $\varphi_\beta : X \to [0,1]$ each having support contained in some set of the open cover, and such that $\sum_\beta \varphi_b = 1$ with only finitely many of the φ_β 's nonzero near each point of X.

Definition 16.1. An *inner product* on a vector bundle $p : E \to B$ is a map $\langle\,,\,\rangle : E \oplus E \to \mathbb{K}$ which restricts in each fiber to an inner product, a positive definite symmetric bilinear form for $\mathbb{K} = \mathbb{R}$ and Hermitian form for $\mathbb{K} = \mathbb{C}$.

Proposition 16.2. *An inner product exists for a vector bundle $p : E \to B$ if B is compact Hausdorff or more generally paracompact.*

Proof. Let U_α be an open cover of B for which there exist local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$. These can be used to pull back the standard inner product in $\mathbb{K}^n$ to an inner product $\langle\,,:\rangle_\alpha$ on $p^{-1}(U_\alpha)$. An inner product on all of E is then obtained by setting $\langle v, w\rangle = \sum_\alpha \varphi_\alpha(x)\langle v, w\rangle_\alpha$, where φ_α is a partition of unity subordinated to $\{U_\alpha\}$ and $x = p(v) = p(w)$. □

Proposition 16.3. *If $E \to B$ is a vector bundle over a paracompact base B and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp \cong E$.*

Proof. Choose an inner product on E and let $E_0^\perp$ be the subspace of E which in each fiber consists of all vectors orthogonal to vectors in E_0. If the natural projection $E_0^\perp \to B$ is a vector bundle, then $E_0 \oplus E_0^\perp$ is isomorphic to E via the map $(v, w) \mapsto v + w$.

To prove that $E_0^\perp \to B$ is a vector bundle, note that this is a local property and we may assume that E is the product $B \times \mathbb{K}^n$. Since E_0 is a vector bundle, for $m := \dim E$, find m independent local sections $s_i : b \mapsto s_i(b))$ in a neighborhood $U(b_0)$ of arbitrary point $b_0 \in B$. Consider a base $s_1(b_0), \dots, s_m(b_0), v_{m+1}, \dots, v_n$ of $\mathbb{K}^n$ and constant sections $s_i : b \mapsto v_i$, $i = m+1, \dots, n$. Then the sections $s_1, \dots, s_m, s_{m+1}, \dots, s_n$ are still independent over some (maybe smaller) neighborhood $U'(b_0) \subseteq U(b_0)$ (consider the continuity of the determinant). Apply the Gram-Schmidt orthogonalization process to these sections in each fiber, using the given inner product, to obtain new sections s_i'. The explicit formulas for the Gram-Schmidt process show that the s_i' 's are continuous, and the first m of them are a basis for E_0 in each fiber over $U'(b_0)$. The sections s_i' define a local trivialization $h : p^{-1}(U'(b_0)) \to U'(b_0) \times \mathbb{K}^n$ by the formula $h(b, s_i'(b)) = (b, e_i)$, where $\{e_i\}$ is the canonical base of $\mathbb{K}^n$. The map h takes E_0 to $U'(b_0) \times \mathbb{K}^m$ and $E_0^\perp$ to $U'(b_0) \times \mathbb{K}^{n-m}$, so $h|_{E_0^\perp}$ is a local trivialization of $E_0^\perp$ over $U'(b_0)$ (see also Problem 15.12). □

Proposition 16.4. *For each vector bundle $p : E \to B$ over a compact Hausdorff space B there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is a trivial bundle.*

Proof. Each point $x \in B$ has an open trivializing neighborhood U_x. By Urysohn's Lemma there is a map $\varphi_x : B \to [0,1]$ with $\varphi(x) = 1$ and $\operatorname{supp} \varphi_x \subset U_x$. The sets $V_x = \varphi_x^{-1}(0,1]$, $x \in B$, form an open cover of B. By compactness this cover has a finite subcover. Let the corresponding V_x 's and φ_x 's be relabeled V_i and φ_i, $i = 1, \dots m$. In particular, $V_i \subset U_{x(i)}$ for some $x(i)$. Define $g_i : E \to \mathbb{K}^n$ by $g_i(v) = \varphi_i(p(v))(\pi_i h_i(v))$, where h_i is the restriction of a local trivialization over $U_{x(i)}$, $h_i : p^{-1}(V_i) \to V_i \times \mathbb{K}^n$, and π_i is the projection $\pi_i : V_i \times \mathbb{K}^n \to \mathbb{K}^n$. Since g_i is a linear injection of each fiber over V_i, then

$$f : E \to B \times \mathbb{K}^N, \quad N = mn, \quad f(e) = (p(e), g_1(e), \ldots, g_m(e)),$$

is an injective morphism of vector bundles. By Proposition 16.3 there is a complementary subbundle E' such that $E \oplus E'$ is isomorphic to $B \times \mathbb{K}^N$. □

Definition 16.5. Denote the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over B by $\mathrm{Vect}^n_{\mathbb{K}}(B)$.

Problem 16.6. Let $f : X \to Y$ be a continuous map. Prove that the pull-back $E \mapsto f^*E$ gives a map $f^* : \mathrm{Vect}^n_{\mathbb{K}}(Y) \to \mathrm{Vect}^n_{\mathbb{K}}(X)$, i.e. isomorphic bundles have isomorphic pull-backs. Home

Problem 16.7. Verify that the operation $\oplus$ of Whitney's sum gives an abelian semi-group structure on $\mathrm{Vect}^n_{\mathbb{K}}(B)$. (Semi-group is a set with operation satisfying all axioms of group except of the existence of inverse) So, you need to verify that Home

1) if $E \cong G$ and $E' \cong G'$ then $E \oplus E' \cong G \oplus G'$ (operation is well-defined);

2) $E \oplus E' \cong E' \oplus E$ (operation is abelian);

3) $0_B \oplus E \cong E$, where $0_B = B \times \{0\}$ is 0-dimensional trivial bundle (existence of unity);

4) $(E \oplus E') \oplus E'' \cong E \oplus (E' \oplus E'')$ (associativity).

Problem 16.8. Prove that $f^* : \mathrm{Vect}^n_{\mathbb{K}}(Y) \to \mathrm{Vect}^n_{\mathbb{K}}(X)$ is a homomorphism of semi-groups, i.e. $f^*(E \oplus E') \cong f^*E \oplus f^*E'$. Home

Theorem 16.9. *Given a vector bundle $p : E \to B$ and homotopic maps $f_0, f_1 : A \to B$, then the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is compact Hausdorff or more generally paracompact.*

Immediately we obtain:

Corollary 16.10. *For homotopic maps $f_0, f_1 : A \to B$ of paracompact spaces $f_0^* = f_1^* : \mathrm{Vect}^n_{\mathbb{K}}(B) \to \mathrm{Vect}^n_{\mathbb{K}}(A)$.*

Corollary 16.11. *For a homotopy equivalence $f : A \to B$ of paracompact spaces $f^* : \mathrm{Vect}^n_{\mathbb{K}}(B) \to \mathrm{Vect}^n_{\mathbb{K}}(A)$ is an isomorphism of semigroups.*

We obtain Theorem 16.9 immediately from the following statement.

Proposition 16.12. *The restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.*

We need two preliminary facts.

Lemma 16.13. *A vector bundle $p : E \to X \times [a, b]$ is trivial if its restrictions over $X \times [a, c]$ and $X \times [c, b]$ are both trivial for some $c \in (a, b)$.*

Proof. Denote these restrictions by $E_1 = p^{-1}(X \times [a, c])$ and $E_2 = p^{-1}(X \times [c, b])$ and by $h_1 : E_1 \to X \times [a, c] \times \mathbb{K}^n$ and $h_2 : E_2 \to X \times [c, b] \times \mathbb{K}^n$ the corresponding isomorphisms. These isomorphisms may not agree on $p^{-1}(X \times \{c\})$, but they can be made to agree by replacing h_2 by its composition with the "cylindrical" isomorphism $X \times [c, b] \times \mathbb{K}^n \to X \times [c, b] \times \mathbb{K}^n$ which on each slice $X \times \{t\} \times \mathbb{K}^n$ is given by

$$h_1 h_2^{-1}|_{X \times \{c\} \times \mathbb{K}^n} X \times \{c\} \times \mathbb{K}^n \to X \times \{c\} \times \mathbb{K}^n.$$

Since h_1 and h_2 agree on $E_1 \cap E_2$, they define a trivialization of E (see Problem 1.27). □

Lemma 16.14. *For a vector bundle $p : E \to X \times I$, there exists an open cover $\{U_\alpha\}$ of X so that each restriction $p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial.*

Proof. For each $x \in X$ and $t \in I$, we can find open neighborhoods U_t of x and $\varepsilon_t > 0$ such that the bundle is trivial over $V_t = U_t \times (t - \varepsilon_t, t + \varepsilon_t)$. This is an open cover of the compact set $\{x\} \times I$ homeomorphic to I. Hence we can find a finite subcover $V_i = V_{t_i}$ $(i = 1, \ldots, s)$. Then for an appropriate partition $0 = t_0 < t_1 < \cdots < t_k = 1$ and $U_x := \cap_i U_{t_i}$, the bundle is trivial over each $U_x \times [t_j, t_{j+1}]$. Thus by Lemma 16.13, it is trivial over $U_x \times I$ and U_x is the desired cover. □

Proof of Proposition 16.12. Suppose that X is compact Hausdorff and choose its compact subcover $\{U_i\}$, $i = 1, \ldots, m$, of the cover constructed in Lemma 16.14. So E is trivial over each $U_i \times I$. Choose a partition of unity $\{\varphi_i\}$ subordinated to $\{U_i\}$. For $i \geq 0$, let $\psi_i := \varphi_1 + \cdots + \varphi_i$. So, $\psi_0 = 0$ and $\psi_m = 1$. Let X_i be the graph of ψ_i:

$$X_i = \{(x, t) \in X \times I \colon t = \psi_i(x)\}$$

and let $p_i : E_i \to X_i$ be the restriction of E over X_i. Since E is trivial over $U_i \times I$, the natural projection homeomorphism $X_i \to X_{i-1}$ lifts to a homeomorphism $\omega_i : E_i \to E_{i-1}$ which is the identity outside $p^{-1}(U_i \times I)$ and which takes each fiber of E_i isomorphically onto the corresponding fiber of E_{i-1}. Namely, on points in $p^{-1}(U_i \times I) \cong U_i \times I \times \mathbb{K}^n$ we define $\omega_i(x, \psi_i(x), v) = (x, \psi_{i-1} x, v)$. The composition $\omega = \omega_1 \omega_2 \cdots \omega_m$ is then an isomorphism from the restriction of E over $X \times \{1\}$ to the restriction over $X \times \{0\}$.

The paracompact case we leave as a problem. □

Home **Problem 16.15.** Similarly to the compact case, prove the paracompact one.

It is convenient to use a slightly broader definition of vector bundle which allows the fibers of a vector bundle $p : E \to X$ to have different dimensions. The existence of local trivializations implies that the dimensions of fibers are locally constant over X, but if X is not connected the dimensions of fibers may be distinct over distinct components.

Denote the trivial n-dimensional bundle by $\varepsilon^n \to X$.

In the remaining part of the lecture we deal only with *compact* Hausdorff base spaces.

Definition 16.16. Two vector bundles E_1 and E_2 over X are *stably isomorphic* ($E_1 \approx_s E_2$) if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$ for some n.

We write $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^m \cong E_2 \oplus \varepsilon^n$ for some m and n.

Evidently, $\approx_s$ and $\sim$ are equivalence relations on $\mathrm{Vect}_{\mathbb{K}}(X)$ (isomorphism classes without restrictions on dimensions).

Home **Problem 16.17.** Verify that $\mathrm{Vect}_{\mathbb{K}}(X)/\approx_s$ and $\mathrm{Vect}_{\mathbb{K}}(X)/\sim$ are abelian semigroups.

Theorem 16.18. *If X is compact Hausdorff, then the set* $\mathrm{Vect}_{\mathbb{K}}(X)/\sim$ *of* $\sim$*-equivalence classes of vector bundles over X forms an abelian group with respect to* $\oplus$*.*

Proof. We need to prove only the existence of inverses, i.e. that for each vector bundle $\pi : E \to X$ there is a bundle $E' \to X$ such that $E \oplus E' \equiv \varepsilon^m$ for some m. If all the fibers of E have the same dimension, this is Proposition 16.4. In the general case let $X_i = \{x \in X\colon \dim(\pi^1(x)) = i\}$ (disjoint open sets in X). Their number is finite by compactness. So first we add to E a bundle E' over each X_i as above to obtain ε^{m_i}, and then a bundle E'' which is trivial of suitable dimension over each X_i to obtain ε^m over entire X. □

Definition 16.19. This group is called the *reduced K-group* and is denoted $\widetilde{K}_{\mathbb{K}}(X)$.

Theorem 16.20. *Let $(S,+)$ be an (abelian) semigroup with the unit element 0_S. Consider the set S^2 of formal differences $s_1 - s_2$ (or equivalently, couples (s_1,s_2)), $s_1,s_2 \in S$ with the equivalence relation $s_1 - s_1' = s_2 - s_2'$ iff $s_1 + s_2' = s_2 + s_1'$ and the addition*

$$(s_1 - s_1') + (s_2 - s_2') = (s_1 + s_2) - (s_1' + s_2').$$

The quotient set with this addition is then an abelian group called the Grothendieck group of S and denoted $G(S)$. If S has the cancellation property ($s_1 + s_2 = s_1 + s_3$ implies $s_2 = s_3$), the map $s \mapsto s - 0_S$ is an injective homomorphism of semigroups.

Proof. First, note that the addition is well defined on the quotient (i.e. respects the equivalence relation). Indeed, if $s_1 - s_1'$ is equivalent to $t_1 - t_1'$ and $s_2 - s_2'$ is equivalent to $t_2 - t_2'$, i.e. $s_1 + t_1' = t_1 + s_1'$ and $s_2 + t_2' = t_2 + s_2'$ then

$$(s_1 - s_1') + (s_2 - s_2') = (s_1 + s_2) - (s_1' + s_2'), \qquad (t_1 - t_1') + (t_2 - t_2') = (t_1 + t_2) - (t_1' + t_2'),$$

$$(s_1 + s_2) + (t_1' + t_2') = (s_1 + t_1') + (s_2 + t_2') = (t_1 + s_1') + (t_2 + s_2') = (t_1 + t_2) + (s_1' + s_2'),$$
$$(s_1 + s_2) - (s_1' + s_2') = (t_1 + t_2) - (t_1' + t_2').$$

Similarly one can prove that the class of $0_S - 0_S$ is the unity, the inverse to $s_1 - s_1'$ is $s_1' - s_1$ and other axioms.

Since $(s - 0_S) + (t - 0_S) = (s + t) - (0_S + 0_S) = (s + t) - 0_S$, the map $s \mapsto s - 0_S$ is a homomorphism (this doe not require the cancellation property). Now suppose that we have this property and $s - 0_S = t - 0_S$, i.e. $s + 0_S = t + 0_S$, $s = t$. So the map is injective. □

Problem 16.21. Complete the proof. Home
Problem 16.22. Find $G(\mathbb{N})$, $\mathbb{N} = \{0, 1, 2, \dots\}$. Home

Lemma 16.23. *We have the cancellation property for* $\mathrm{Vect}_{\mathbb{K}}(X)/\approx_s$*.*

Proof. If $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ (i.e. $E_1 \oplus E_2 \oplus \varepsilon^m \cong E_1 \oplus E_3 \oplus \varepsilon^m$ for some m), choose a bundle E_1' such that $E_1 \oplus E_1' \cong \varepsilon^n$ for some n (Proposition 16.4). Then $\varepsilon^{n+m} \oplus E_2 \cong \varepsilon^{n+m} \oplus E_3$ and $E_2 \approx_s E_3$. □

Problem 16.24. Prove that generally $\mathrm{Vect}_{\mathbb{K}}(X)$ has no cancellation property. *Hint:* consider a hypersurface with non-trivial tangent bundle and its sum with the normal bundle. Home

(Roughly speaking the cancellation property is fulfilled for bundles of large rank w.r.t. dimension of base.)

Definition 16.25. The *K-group of X* is defined as $K(X) = G(\mathrm{Vect}_{\mathbb{K}}(X)/\approx_s)$.

Home **Problem 16.26.** Prove that $G(\mathrm{Vect}_{\mathbb{K}}(X)) \cong G(\mathrm{Vect}_{\mathbb{K}}(X)/\approx_s)$. So one can define $K_{\mathbb{K}}(X)$ without using of $\approx_s$.

Theorem 16.27. *If X and Y are homotopy equivalent then $K_{\mathbb{K}}(X) \cong K_{\mathbb{K}}(Y)$*

Proof. Quite similarly to Corollary 16.11 one obtains in this case that $\mathrm{Vect}_{\mathbb{K}}(X) \cong \mathrm{Vect}_{\mathbb{K}}(Y)$ as semigroups. Then $G(\mathrm{Vect}_{\mathbb{K}}(X)) \cong G(\mathrm{Vect}_{\mathbb{K}}(Y))$, hence $K_{\mathbb{K}}(X) \cong K_{\mathbb{K}}(Y)$ by Problem 16.26. $\square$

There is a natural homomorphism $K_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(X)$ sending $E - \varepsilon^n$ to the class of E. This is well-defined since if $E - \varepsilon^n = E' - \varepsilon^m$ in $K_{\mathbb{K}}(X)$, then $E \oplus \varepsilon^m \cong E' \oplus \varepsilon^n$ i.e. $E \sim E'$. This map $K_{\mathbb{K}}(X) \to \widetilde{K}_{\mathbb{K}}(X)$ is obviously surjective, and its kernel consists of elements $E - \varepsilon^n$ with $E \sim \varepsilon^0$, hence $E \oplus \varepsilon^m \cong e^n$, $E \approx_s \varepsilon^{n-m}$. So the kernel in $K_{\mathbb{K}}(X)$ consists of the elements of the form $\varepsilon^n - \varepsilon^m$ and is isomorphic to $\mathbb{Z}$. The restriction of vector bundles to a basepoint $x_0 \in X$ defines a homomorphism $\gamma : K_{\mathbb{K}}(X) \to K_{\mathbb{K}}(x_0) \cong \mathbb{Z}$ (cf. Problem 16.22) which restricts to an isomorphism on the subgroup $\{\varepsilon^n - \varepsilon^m\}$. Thus we have a splitting $K_{\mathbb{K}}(X) \cong \mathrm{Ker}\,\gamma \oplus \mathbb{Z} \cong \widetilde{K}_{\mathbb{K}}(X) \oplus \mathbb{Z}$, depending on the choice of x_0.