9.4 Vector bundles

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Definition 9.28. Consider an n-dimensional vector space V over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $G = \mathrm{Aut}(V) = \mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{K})$ acting on V in a natural way. Then $\xi = (E, \pi, B, V, G)$ is a *vector bundle* (topological or smooth).

Theorem 9.29. *Consider vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ with the same typical fiber V and cocycles (transition maps) $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$ and $\Phi'_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$, respectively, for the same cover $\{U_\alpha\}$. These bundles are isomorphic iff there are smooth functions $f_\alpha : U_\alpha \to \mathrm{GL}(V)$ such that*

distinct, not a conjugation

$$\Phi'_{\alpha\beta}(P) = f_\alpha(P)\Phi_{\alpha\beta}(P)(f_\beta(P))^{-1}, \quad P \in U_{\alpha\beta}. \tag{6}$$

bundle charts

$\in V$

Proof. If $f : E \to E'$ is an isomorphism, define $f_\alpha(P)(v) := p_2(\Phi'_\alpha \circ f \circ (\Phi_\alpha)^{-1}(P, v))$. Then

$p_2 : (P, w) \mapsto w$

$$f_\alpha(P)\Phi_{\alpha\beta}(P)(f_\beta(P))^{-1}(v) = p_2(\Phi'_\alpha \circ f \circ (\Phi_\alpha)^{-1})(\Phi_\alpha \circ (\Phi_\beta)^{-1})(P, (\Phi_\beta \circ f^{-1} \circ \Phi'_\beta)(P)v)$$

$$= p_2(\Phi'_\alpha \circ (\Phi'_\beta)^{-1})(P, v) = \Phi'_{\alpha\beta}(P)(v)$$

and we have (6).

If we have (6), define

$$\widetilde{f}_\alpha : U_\alpha \times V \to U_\alpha \times V, \quad (P, v) \mapsto (P, f_\alpha(P)v).$$

Then define locally (for $e \in \pi^{-1}(U_\alpha)$) a bundle map $f : E \to E'$ by

$$f(e) = \left((\Phi'_\alpha)^{-1} \circ \widetilde{f}_\alpha \circ \Phi_\alpha\right)(e).$$

One can verify that f is well defined globally (using (6)) and defines a vector bundle isomorphism. ∎

Problem 9.30. Complete the proof.

Home

Example 9.31. The tangent bundle TM is an example of a vector bundle.

Our main example (generalizing the above one) is the tensor bundle of type (p, q) over M. We consider a slightly general construction, considering not only $E = TM$ as the initial bundle. So we consider a real rank k vector bundle $\xi = (E, \pi, M, \dots)$.

(r, s)

$(0, s)$

Definition 9.32. The total space (as a set) is $T^r_s(\xi) = \sqcup_{P \in M} T^r_s(E_P)$, where $T^r_s(E_P)$ is the k^{r+s}-dimensional real vector space of all (r, s) tensors on the k-dimensional linear space E_P. For each local trivialization (U, Φ) of ξ, $\Phi : \pi^{-1}U \to U \times \mathbb{R}^k$, define the local trivialization

$$\Phi^r_s : \sqcup_{P \in U} T^r_s(E_P) \xrightarrow{E_P} U \times T^r_s(\mathbb{R}^k), \cong \mathbb{R}^{k^{r+s}} \qquad (d\Phi)^{-1} = d(\Phi^{-1})$$

$P_2 \Phi^r_s(\tau)$

$$L_{\Phi^r_s(\tau)}(a^1, \dots, a^r, v_1, \dots, v_s) = L_\tau(F^* a^1, \dots, F^* a^r, d\Phi^{-1} v_1, \dots, d\Phi^{-1} v_s)$$

for any smooth covector fields a^i and vector fields v_j., $\tau \in T^r_s(E_P)$

Home **Problem 9.33.** Verify the details (similarly to the construction of TM).

Remark 9.34. In other words we define smooth sections of $T^r_s(\xi)$ to be such maps $P \mapsto \tau_P \in T^r_s(E_P)$ that $P \mapsto L_{\tau_P}(a^1, \dots, a^r, v_1, \dots, v_s)$ is smooth for any smooth covector fields a^i and vector fields v_j (see Subsection 9.7 for more detail).

rank $(E) = k$

9.5 Principal bundles

Lie group

$(g\cdot f)\cdot h = g(f\cdot h)$

Definition 9.35. If $F = G$ and $\lambda(g)f = gf$, a bundle is called a *principal bundle.*

Problem 9.36. In this case one has a canonical right action of G on E with orbits eG being fibers. Home

Note that the same cocycle can define bundles with distinct fibers. In particular, a $\mathrm{GL}(n,\mathbb{K})$-valued cocycle defines a vector bundle and a principal bundle.

Problem 9.37. (Hopf's bundle) Consider S^{2n-1} as the subset of $\mathbb{C}^n$ given by $S^{2n-1} = \{z \in \mathbb{C}^n : \|z\| = 1\}$, where $z = (z^1, \ldots, z^n)$ and $\|z\| = \sum \overline{z^i} z^i$. Let $S^1 = U(1)$ act on S^{2n-1} by $(a, z) \mapsto az = (az^1, \ldots, az^n)$. The quotient (the space of orbits) is $\mathbb{C}P^{n-1}$. We obtain the *Hopf map* $\pi_n : S^{2n-1} \to \mathbb{C}P^{n-1}$. Prove that this is a principal $U(1)$-bundle (*Hopf bundle*). Class

$\mathbb{C}^1$

Ex: $\pi : E \to M$, $\boxed{G = O(n)} \subset GL(n, \mathbb{R})$

Reduction

$O(n)$: $\mathbb{R}^n \to \mathbb{R}^n$, $S^{n-1} \to S^{n-1}$

R-Vector bundle $\rightsquigarrow$ its spherical bundle.

9.6 Operations on vector bundles

Definition 9.38. The *Whitney sum* $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$ of vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ is defined in the following way. As a set $E_1 \oplus E_2 = \sqcup_{P \in M} (E_1)_P \oplus (E_2)_P$ and for charts $(\Phi_1)_\alpha : (\pi_1)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{k_1}$ and $(\Phi_2)_\alpha : (\pi_2)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^{k_2}$ of local trivializations of π_1 and π_2, respectively we define

$\Phi_2 : \pi_2^{-1}(U) \longrightarrow U \times V_2 \xrightarrow{p_2} V_2$

$$(\Phi_1)_\alpha \oplus (\Phi_2)_\alpha : (v_P, w_P) \mapsto (P, p_2((\Phi_1)_\alpha(v_P)), p_2((\Phi_2)_\alpha(w_P))), \quad v_P \in (E_1)_P, \quad w_P \in (E_2)_P.$$

Home **Problem 9.39.** Verify that this is a structure of a (smooth or topological) vector bundle.

Home **Problem 9.40.** Prove that the Whitney sum can be defined using cocycles in the following way. Suppose that $\{g_{\alpha\beta}\}$ is a cocycle for π_1 and $\{h_{\alpha\beta}\}$ is a cocycle for π_2 for the same cover. Then

$$g_{\alpha\beta} \oplus h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{K}^{k_1} \oplus \mathbb{K}^{k_2}), \quad (g_{\alpha\beta} \oplus h_{\alpha\beta})(P) : (v, w) \mapsto (g_{\alpha\beta}(P)v, h_{\alpha\beta}(P)w)$$

is a cocycle for $\pi_1 \oplus \pi_2$.

Recall that the *tensor product* $V \otimes W$ of linear spaces V and W is the quotient space of the space $V \odot W$ of formal $\mathbb{K}$-linear combinations of elements $\underbrace{v \odot w}_{(v,w)}$ by the subspace generated by elements:

- $(v_1 + v_2) \odot w - v_1 \odot w - v_2 \odot w$,
- $v \odot (w_1 + w_2) - v \odot w_1 - v \odot w_2$,
- $(sv) \odot w - s(v \odot w)$,
- $v \odot (sw) - s(v \odot w)$,

$\Big\}$ generate $R \subset V \odot W$,

$V \otimes W := \underline{V \odot W} / R$

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $s \in \mathbb{K}$. The class of $v \odot w$ is denoted by $v \otimes w$.

Problem 9.41. Let $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ be linear maps of finite-dimensional vector spaces. Then the formula $(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$ defines a well-defined linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$. If f_1 and f_2 are isomorphisms then so is $f_1 \otimes f_2$. Home

If V has a base $e_1, \ldots, e_n$ and W has a base $f_1, \ldots, f_m$, then $V \otimes W$ has the base $e_i \otimes f_j$. The formula $(v \odot \varphi)(w) = \varphi(w)v$, where $v \in V$, $w \in W$, $\varphi \in W^*$, defines an isomorphism $\underline{V \otimes W^* \cong \mathrm{Hom}_{\mathbb{K}}(W, V)}$ (still for finite-dimensional spaces).

Problem 9.42. Verify the details and find the matrix of the operator (for the above bases). Home

$(f_1 \otimes f_2)(v_1 \odot v_2) = f_1(v_1) \odot f_2(v_2)$

$V_1 \odot W_1 \longrightarrow V_2 \odot W_2$

Verify: $f_1 \otimes f_2 : R_1 \longrightarrow R_2 \Rightarrow f_1 \otimes f_2$ is well def.

$V_1 \otimes W_1 = V_1 \odot W_1 / R_1 \longrightarrow V_2 \odot W_2 / R_2 = V_2 \otimes W_2$

Definition 9.43. The *tensor product* bundle $\pi : E_1 \otimes E_2 \to M$ of vector bundles $\pi : E_1 \to M$ and $\pi_2 : E \to M$ with typical fibers V_1 and V_2 has the total space (as a set) $E_1 \otimes E_2 = \sqcup_{P \in M} (E_1)_P \otimes (E_2)_P$. Consider local trivializations Φ_α of E_1 and Ψ_α of E_2 over the same cover $\{U_\alpha\}$. Then the local trivializations for the tensor product are defined as

$$\Phi_\alpha \otimes \Psi_\alpha : (E_1 \otimes E_2)|_{U_\alpha} \to U_\alpha \times (V_1 \otimes V_2),$$

$$e \mapsto (P, [(p_2 \circ \Phi_\alpha) \otimes (p_2 \circ \Psi_\alpha)](e)), \quad e \in (E_1 \otimes E_2)_P = (E_1)_P \otimes (E_2)_P,$$

(isomorphisms by Problem 9.41).

Home **Problem 9.44.** Complete the definition as for TM.

Home **Problem 9.45.** Prove that alternatively the tensor product bundle can be defined by the product cocycle $P \mapsto \Phi_{\alpha\beta}(P) \otimes \Psi_{\alpha\beta}(P)$.

Class **Problem 9.46.** Verify that the tensor product does not depend on the choice of local trivializations, i.e., we obtain isomorphic bundles. Understand the refinement of cocycles.

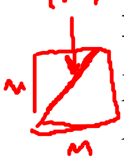
Remark 9.47. This should be done each time when we define some bundle in a similar way, but we do this once.

Definition 9.48. The *pull-back* f^*E of a vector bundle $\pi : E \to M$ by a smooth map $f : N \to M$ has the total space $\sqcup_{Q \in N} E_{f(Q)}$. If $\{(U_\alpha, \Phi_a)\}$ is a bundle atlas (of local trivializations) for E, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, then $\{(U'_\alpha, \Phi'_a)\}$ is a bundle atlas for f^*E, where

$$\underline{U'_\alpha = f^{-1}U_\alpha,} \quad \Phi'_\alpha(e) = \Phi_\alpha(e), \quad e \in (f^*E)_Q = E_{f(Q)}, \quad Q \in f^{-1}U_\alpha.$$

Alternatively the pull-back can be defined with the help of the cocycle $\Phi_{\alpha\beta} \circ f$ for the cover $\{f^{-1}U_\alpha\}$. Evidently this is the same bundle.

$\Delta(M)$



Problem 9.49. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles and let $\Delta : M \to M \times M$ be diagonal map $P \mapsto (P, P)$. Then one can define $\underline{\pi_{E_1 \times E_2} : E_1 \times E_2 \to M \times M}$. Verify that this is a vector bundle. Prove that the Whitney sum $E_1 \oplus E_2$ is naturally isomorphic to the pull-back $\Delta^* \pi_{E_1 \times E_2}$. Home

$\pi_1 \times \pi_2 \neq \pi_1 \oplus \pi_2$

$$\begin{array}{ccc} f^*E & \cdots\cdots & E \\ \downarrow & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

$Q \longrightarrow P = f(Q)$, $E_{f(Q)} \downarrow$

Remark Generally: $N \xrightarrow{f} M$, $\uparrow \pi$ TM

$f^*(TM) \not\cong TN$!!!

Suppose $\dim M = 1$, $\dim N = 2$

If f is a diffeo?

rank $(TN) = \dim N = 2$

rank $(f^*TM) =$ rank $(TM) = 1$

$f^* \sim df^{-1}$ Think about this

$\pi_1 : M \times V \to M$, $\pi_1 = \pi_2$

$\dim M = \dim M = 2$

$\dim(\pi_1 \times \pi_2) = 8$ as a manifold

$\dim(\pi_1 \oplus \pi_2) = 6$

$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$

Definition 9.50. If $\pi : E \to M$ is a vector bundle over M with local trivializations $\{(U_\alpha, \Phi_a)\}$ and transition maps $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$, its *dual bundle* E^* with typical fiber V^* has the total space (as a set) $E^* = \sqcup_{P \in M}(E_P)^*$ and local trivializations $\Phi^*_\alpha : E^*|_{U_\alpha} \to U_\alpha \times V^*$ defined by

$$\overbrace{(p_2(\Phi^*_\alpha(a)))}^{\in V^*}(v) = a((\Phi_\alpha)^{-1}(P, v)), \quad a \in (E^*)_P = (E_P)^*,\ v \in V,\ (\Phi_\alpha)^{-1}(P, v) \in E_P.$$

Home **Problem 9.51.** If we fix a base in V, then $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{K})$. Prove that, for the dual base in V^*, $\Phi^*_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{K})$ is defined by $P \mapsto ((\Phi_{\alpha\beta}(P))^T)^{-1}$.

Class **Problem 9.52.** Prove that $T^r_s(E) \cong (\otimes^r E) \otimes (\otimes^s E^*)$.

$A : V \to V$

$A^* : V^* \to V^*$

9.7 Tensor fields as sections of vector bundles

Denote the linear space of tensor fields of type (p, q) r, s over M by $\mathbf{T}^r_s(M)$.

Suppose that $\tau \in \Gamma(T^r_s(TM))$ is a smooth section and (U, φ) is a chart on M. Define

$\tau(P) \in T^r_s(TM)|_P = T^r_s(T_P M)$

$$T(\tau)^{i_1 \dots i_r}_{j_1 \dots j_s}(P) = L_{\tau(P)}\left(dx^{i_1}, \dots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}\right).$$

PRB.

Theorem 9.53. *The above defined T induces the identification $\Gamma(T^r_s(TM)) \cong \Gamma((\otimes^r TM) \otimes (\otimes^s T^* M)) \cong \mathbf{T}^r_s(M)$.*

Proof. By the definition of $T^r_s(TM)$, the map T is well defined and T is an isomorphism locally . Also the global injectivity is immediate. To prove the global surjectivity one can use a partition of unity. $\square$

Problem 9.54. Complete the argument with a partition of unity. Home

10 Covariant differentiation

$T(0,0)$ $T(0,1)$

$\Downarrow$ $\Downarrow$

$d: f \longmapsto df$

Home **Problem 10.1.** Show that the partial differentiation of components of a tensor field on $\mathbb{R}^n$ is not a tensor operation.

We wish to define on tensor fields on $\mathbb{R}^n$ a tensor operation $\nabla : T(p,q) \to T(p,q+1)$, which coincides in Cartesian coordinates with the partial differentiation. For this purpose we start by an attempt to write down the result of partial differentiation in other coordinates.

Consider first the case of a vector field T^i. Suppose that x^i are Cartesian coordinates in $\mathbb{R}^n$, and $x^{i'}$ is some other coordinate system. Then for the desired ∇ we should have

$$(\nabla T)^i_j = \frac{\partial T^i}{\partial x^j}, \qquad (\nabla T)^{i'}_{j'} = \frac{\partial x^{i'}}{\partial xi}\frac{\partial x^j}{\partial x^{j'}}(\nabla T)^i_j.$$

Then

$$(\nabla T)^{i'}_{j'} = \frac{\partial x^{i'}}{\partial xi}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial}{\partial x^j}\left(\frac{\partial x^i}{\partial x^{k'}}T^{k'}\right) =$$

$$= \frac{\partial x^{i'}}{\partial xi}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^i}{\partial x^{k'}}\frac{\partial T^{k'}}{\partial x^{m'}}\frac{\partial x^{m'}}{\partial x^j} + \frac{\partial x^{i'}}{\partial xi}\frac{\partial x^j}{\partial x^{j'}}T^{k'}\frac{\partial}{\partial x^j}\left(\frac{\partial x^i}{\partial x^{k'}}\right) =$$

$$= \delta^{i'}_{k'}\,\delta^{m'}_{j'}\frac{\partial T^{k'}}{\partial x^{m'}} + T^{k'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}},$$

hence,

$$(\nabla T)^{i'}_{j'} = \frac{\partial T^{i'}}{\partial x^{j'}} + T^{k'}\Gamma^{i'}_{k'j'}, \qquad \Gamma^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^i}\cdot\frac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}}.$$

For a covector field T_i one should have $(\nabla T)_{ij} = \frac{\partial T_i}{dx^j}$ and $(\nabla T)_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} (\nabla T)_{ij}$. Then

$$(\nabla T)_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial}{\partial x^j} \left(\frac{\partial x^{k'}}{\partial x^i} T_{k'} \right) =$$

$$= \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^i} \frac{\partial T_{k'}}{\partial x^{m'}} \frac{\partial x^{m'}}{\partial x^j} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} T_{k'} \frac{\partial}{\partial x^j} \left(\frac{\partial x^{k'}}{\partial x^i} \right) =$$

$$= \delta_{i'}^{k'} \, \delta_{j'}^{m'} \frac{\partial T_{k'}}{\partial x^{m'}} + T_{k'} \frac{\partial^2 x^{k'}}{\partial x^j \partial x^i} \cdot \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}},$$

or

$$(\nabla T)_{i'j'} = \frac{\partial T_{i'}}{\partial x^{j'}} + T_{k'} \bar{\Gamma}_{i'j'}^{k'}, \qquad \bar{\Gamma}_{i'j'}^{k'} = \frac{\partial^2 x^{k'}}{\partial x^j \partial x^i} \cdot \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}.$$

Lemma 10.2. *One has* $\bar{\Gamma}_{i'j'}^{k'} = -\Gamma_{i'j'}^{k'}$.

Proof. Let us differentiate the equality $\frac{\partial x^{i'}}{\partial x^{i''}} \cdot \frac{\partial x^{i''}}{\partial x^{k'}} = \delta_{k'}^{i'}$ in $x^{m'}$:

$$0 = \frac{\partial^2 x^{i''}}{\partial x^{m'} \partial x^{k'}} \cdot \frac{\partial x^{i'}}{\partial x^{i''}} + \frac{\partial x^{i''}}{\partial x^{k'}} \cdot \frac{\partial^2 x^{i'}}{\partial x^{m''} \partial x^{i''}} \cdot \frac{\partial x^{m''}}{\partial x^{m'}} = \Gamma_{m'k'}^{i'} + \bar{\Gamma}_{m'k'}^{i'}. \qquad \square$$

□

Theorem 10.3. *There exists a tensor operation ∇ on $M = \mathbb{R}^n$, defined on a field $T^{i_1 \dots i_p}_{j_1 \dots j_q}$ by*

$$(\nabla T)^{i'_1 \dots i'_p}_{j'_1 \dots j'_q ; m'} = \frac{\partial}{\partial x^{m'}} (T^{i'_1 \dots i'_p}_{j'_1 \dots j'_q}) + \sum_{s=1}^{p} T^{i'_1 \dots i'_{s-1} r' i'_{s+1} \dots i'_p}_{j'_1 \dots j'_q} \Gamma^{i'_s}_{r' m'} - \sum_{s=1}^{q} T^{i'_1 \dots i'_p}_{j'_1 \dots j'_{s-1} r' j'_{s+1} \dots j'_q} \Gamma^{r'}_{j'_s m'},$$

and the functions Γ have the following transformation law

$$\Gamma^{i''}_{j''k''} = \underbrace{\frac{\partial x^{i''}}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^{j''}} \frac{\partial x^{k'}}{\partial x^{k''}} \Gamma^{i'}_{j'k'}}_{(1,2)} + \underbrace{\frac{\partial x^{i''}}{\partial x^{i'}} \frac{\partial^2 x^{i'}}{\partial x^{j''} \partial x^{k''}}}_{\text{not a tensor}}.$$

if $x^{i'}$ – eucl. $\Rightarrow$ $\Gamma^{i'}_{j'k'} = 0$

Proof. The explicit form of ∇ can be found similarly to vector and covector cases (Problem 10.4).

Find the transformation law for Γ.

$$\nabla_{k'} T^{i'} := (\nabla T)^{i'}_{k'} = \frac{\partial T^{i'}}{\partial x^{k'}} + T^{r'} \Gamma^{i'}_{r'k'},$$

$$\nabla_{k''} T^{i''} = \frac{\partial T^{i''}}{\partial x^{k''}} + T^{r''} \Gamma^{i''}_{r''k''} = \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial}{\partial x^{k'}} \left(\frac{\partial x^{i''}}{\partial x^{i'}} T^{i'} \right) + \frac{\partial x^{r''}}{\partial x^{r'}} T^{r'} \Gamma^{i''}_{r''k''} =$$

$$= \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \frac{\partial T^{i'}}{\partial x^{k'}} + T^{i'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{i'}} + T^{r'} \frac{\partial x^{r''}}{\partial x^{r'}} \Gamma^{i''}_{r''k''}.$$

On the other hand,

$$\nabla_{k''} T^{i''} = \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \nabla_{k'} T^{i'} = \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \left(\frac{\partial T^{i'}}{\partial x^{k'}} + T^{r'} \Gamma^{i'}_{r'k'} \right).$$

Hence

$$T^{r'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} \Gamma^{i'}_{r'k'} = T^{r'} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}} + T^{r'} \frac{\partial x^{r''}}{\partial x^{r'}} \Gamma^{i''}_{r''k''}.$$

Since T^i is an arbitrary field,

$$\Gamma^{i''}_{r''k''} = \Gamma^{i'}_{r'k'} \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}} - \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}}.$$

As it was established in the proof of Lemma 10.2,

$$- \quad \frac{\partial x^{r'}}{\partial x^{r''}} \frac{\partial x^{k'}}{\partial x^{k''}} \frac{\partial^2 x^{i''}}{\partial x^{k'} \partial x^{r'}} = \frac{\partial^2 x^{k'}}{\partial x^{r''} \partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{k'}} = \frac{\partial^2 x^{i'}}{\partial x^{r''} \partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{i'}}.$$

$\square$

Home **Problem 10.4.** Find the explicit form of ∇ for general fields.