16.12.2022 **Theorem 15.29.** *Let κ be a connector of a connection on a vector bundle $\pi : E \to M$. Then κ is a vector bundle homomorphism from $d\pi : TE \to TM$ to $\pi : E \to M$ along the map $\pi_{TM} : TM \to M$*

$$\begin{tikzcd} TE \arrow[r, "\kappa"] \arrow[d, "d\pi"'] & E \arrow[d, "\pi"] \\ TM \arrow[r, "\pi_{TM}"] & M. \end{tikzcd} \tag{28}$$

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1) diagram is commutative

2) κ fiberwise linear.

Proof. In the diagram

$$\begin{tikzcd} TE \arrow[rr, "\kappa"] \arrow[dd, "d\pi"'] \arrow[dr] & & E \arrow[dd, "\pi"] \\ & E \arrow[dr, "\pi"] & \\ TM \arrow[rr, "\pi_{TM}"] & & M \end{tikzcd}$$

$$\begin{tikzcd} TE \arrow[r] \arrow[d, "d\pi"'] & E \arrow[d, "\pi"] \\ TM \arrow[r, "\pi_{TM}"] & M \end{tikzcd}$$

$$\begin{tikzcd} TE \arrow[d] \\ TM \end{tikzcd} \quad \left(\begin{tikzcd} TE \arrow[d] \\ M \end{tikzcd}\right) \quad \begin{tikzcd} TE \arrow[d] \\ E \end{tikzcd}$$

the left triangle is commutative by the definition of $d\pi$ and the right one by (27). Thus (28) is commutative.

$$\begin{tikzcd} TE \arrow[r, "\kappa"] \arrow[d] & E \arrow[d, "\pi"] \\ E \arrow[r] & M \end{tikzcd}$$

It remains to verify that κ is linear on fibers. Let $X_p = (d\pi)Z_y$, where $\pi(y) = p$, $Z_y \in T_yE$, $X_p \in T_yM$. Decompose $Z_y = H_y + V_y$, where $H_y \in \mathcal{H}_y$, $V_y \in \mathcal{V}_yE$. Since $\boxed{(d\pi)V_y = 0,}$ we have $X_p = (d\pi)H_y$ and H_y is the horizontal lift $\widetilde{X}_y$ of X_p. Also, $V_y = \mathbf{j}_y w$ for a unique $w \in E_p$ (by Propositions 15.9). Thus $Z_y = \widetilde{X}_y + \mathbf{j}_y w$ and $\kappa(Z_y) = w$ by the definition. By Lemma 15.28 and homogeneity of $\mathcal{H}$ we have

$VE = \pi^* E$

$$(d\mu_r)Z_y = (d\mu_r)\widetilde{X}_y + (d\mu_r)\mathbf{j}_y w = \widetilde{X}_{ry} + \mathbf{j}_{ry} rw.$$

— the second prop. of Ehs. com.

Hence $\kappa((d\mu_r)Z_y) = rw = r\,\kappa(Z_y)$ or $\kappa(r \circledcirc Z_y) = r\,\kappa(Z_y)$ (in the notation of Problem 15.25). Corollary 15.27 completes the proof. $\square$

$$T_yE = \mathcal{V}_yE \oplus \mathcal{H}_y$$
$$Z_y = V_y + H_y$$

$$Z_y \in T_yE \xrightarrow{d\pi} X_p \in T_pM$$

$$y \in E \to p, \quad E \xrightarrow{\pi} M$$

$$\boxed{\operatorname{Ker} \mathcal{K} = \mathcal{H}}$$

"$\mathcal{K}$ is a proj. onto VE along $\mathcal{H}$ along $TM \to M$".

Home **Problem 15.30.** Prove that the addition $\boxplus$ in $TE \to TM$ can be described in the following (similar) form. We have, as above, $Z_y = \widetilde{X}_y + \mathbf{j}_y w$ for some $w \in E_p$ if $(d\pi)Z_y = X_p$ and $\widetilde{X}_y$ is the horizontal lift of X_p. Suppose, that for another vector $U_{y'}$ from the same fiber over X_p we have in the same way $U_{y'} = \widetilde{X}_{y'} + \mathbf{j}_{y'} w'$. Then the sum of these vectors will be given by $\widetilde{X}_{y+y'} + \mathbf{j}_{y+y'}(w + w')$, where $\widetilde{X}_{y+y'}$ is the horizontal lift of X_p to the point $y + y'$.

Class **Problem 15.31.** Using Problem 15.11 and Theorem 15.29 prove that $(\pi_{TE}, \kappa) : TE \to E \oplus E$ is a vector bundle isomorphism along the tangent bundle projection $\pi_T M : TM \to M$, i.e. we have a commutative diagram with fiberwise linear isomorphism in the upper row:

$$\begin{tikzcd} TE \arrow[r, "(\pi_{TE},\kappa)"] \arrow[d, "d\pi"'] & E \oplus E \arrow[d, "\pi\oplus\pi"] \\ TM \arrow[r, "\pi_T M"] & M. \end{tikzcd}$$

Now we introduce the Koszul definition of connection (covariant derivative) for a vector bundle $\pi : E \to M$, which generalizes an affine connection.

Definition 15.32. Let $\pi : E \to M$ and $f : M \to N$ be as above. A *covariant derivative along* f is a map $\nabla^f : TN \times \Gamma_f(E) \to \Gamma_f(E)$ (we write $\nabla^f(v, \sigma) = \underline{\nabla^f_v \sigma}$) having the properties

(i) ∇^f is fiberwise linear in the first argument:

$$\nabla^f_{au+bv}\sigma = a\nabla^f_u\sigma + b\nabla^f_v\sigma,$$

for all $\sigma \in \Gamma_f(E)$, $a, b \in \mathbb{R}$, $u, v \in T_pN$ for some $p \in N$;

(ii) $\nabla^f_u(\sigma_1 + \sigma_2) = \nabla^f_u(\sigma_1) + \nabla^f_u(\sigma_2)$ for any $u \in TN$ and any $\sigma_1, \sigma_2 \in \Gamma_f(E)$;

(iii) for $v \in T_pN$, $h \in C^\infty(N, \mathbb{K})$, and $\sigma \in \Gamma_f(E)$, the Leibniz law is fulfilled: $p \in N$

$$\nabla^f_v(h\sigma)\Big|_p = h(p)\nabla^f_v\sigma + \underline{v(h)}\sigma(p);$$

(iv) for a vector field $p \mapsto v(p)$ from $\mathbb{X}(N)$, the map $p \mapsto \nabla^f_{v(p)}\sigma$ is smooth for all $\sigma \in \Gamma_f(E)$;

(v) if $g : S \to N$ and $f : N \to M$ are smooth, then

$$\nabla^{f\circ g}_u(\sigma \circ g) = \nabla^f_{(dg)u}\sigma,$$

$u \in TS$:

$$\begin{tikzcd}
 & & E \arrow[d, "\pi"] \\
S \arrow[r, "g"'] \arrow[urr, "\sigma\circ g"] & N \arrow[r, "f"'] \arrow[ur, "\sigma"'] & M.
\end{tikzcd}$$

Home **Problem 15.33.** Prove that (ii) and (iii) give the linearity of ∇^f over $\mathbb{K}$ in the second argument .

A related notion (in fact a reduction for $f = \mathrm{Id} : M \to M$) is:

Definition 15.34. Let $\pi : E \to M$ be a smooth $\mathbb{K}$-vector bundle. A *covariant derivative* or *Koszul connection* is a map $\nabla : \mathbb{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ (we write $\nabla(X, s) = \nabla_X s$) having the properties

(i) $\nabla_{fX} s = f \nabla_X s$, for all $s \in \Gamma(M, E)$, $f \in C^\infty(M)$, $X \in \mathbb{X}(M)$;

(ii) $\nabla_{X_1 + X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$ for any $s \in \Gamma(M, E)$, $X_1, X_2 \in \mathbb{X}(M)$;

(iii) $\nabla_X (s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$ for all $s_1, s_2 \in \Gamma(M, E)$, $X \in \mathbb{X}(M)$;

(iv) $\nabla_X (hs) = h \nabla_X s\cancel{s} + X(h)s$ for all $s \in \Gamma(M, E)$, $f \in C^\infty(M)$, $X \in \mathbb{X}(M)$.

Home **Problem 15.35.** Verify that this is a particular case.

Home **Problem 15.36.** Understand that an affine derivative of a vector field along a curve is a particular case of the above definitions.

Ehresmann

Theorem 15.37. *Suppose that $\pi : E \to M$ is a vector bundle with a connection $\mathcal{H}$ and associated connector κ. For any smooth map $f : N \to M$ define the map $\nabla^f : TN \times \Gamma_f(E) \to \Gamma_f(E)$ by the formula*

$$\nabla^f_v \sigma \Big|_p := \kappa(\underbrace{(\overbrace{d\sigma}^{\in TE_{\sigma(p)}})_p v}_{\in E}) \qquad \text{for } v \in T_pN, \quad \sigma \in \Gamma_f(E), \tag{29}$$

$$\sigma : N \to E, \quad N \xrightarrow{f} M, \quad E \xrightarrow{\pi} M$$

For a vector field V on N define $(\nabla^f_V \sigma)(p) := \nabla^f_{V(p)}\sigma$. Then ∇^f satisfies Definition 15.32.

In particular, for $f = \mathrm{Id}_M$ we obtain a Koszul connection.

Conversely, if ∇ is a Koszul connection on $\pi : E \to M$, then we may define an (Ehresmann) connection by

$$\mathcal{H}_y := \{(ds)u - \mathbf{j}_y \nabla_u s | s \in \Gamma(M, E),\ s(\pi(y)) = y,\ u \in T_{\pi(y)}M\}$$

The initial Koszul connection can be restored by the formula $\nabla_v(s) = \kappa((ds)_p v)$, $v \in T_pM$.

$\nabla = \kappa(d\sigma\ v)$

Proof. Since κ and $d\sigma$ are smooth bundle morphisms, the properties (i) and (iv) of Definition 15.32 follow immediately from the definition (29).

If $g : S \to N$ and $f : N \to M$ are smooth and $u \in TS$, then for each $\sigma \in \Gamma_f(E)$ we have

$$\nabla_u^{f\circ g}(\sigma \circ g) = \kappa(d(\sigma \circ g)u) = \kappa(d(\sigma)\underbrace{((dg)u)}_{\uparrow\, TN}) = \nabla^f_{(dg)u}\sigma.$$

This gives (v) of Definition 15.32.

To prove (ii) use the formula for addition in terms of the tangent lift of $\alpha : (u, v) \mapsto u + v$, $u, v \in \cancel{T_pN}$ TE. Consider $\sigma_1, \sigma_2 \in \Gamma_f(E)$, $u \in T_pN$, $u = [\gamma]$ for a smooth curve γ in N with $\gamma(0) = p$. Then

$$
\begin{aligned}
(d\sigma_1)u \boxplus (d\overset{\sigma}{\cancel{s}}_2)u &= (d\alpha)(\overbrace{(d\sigma_1)u}^{u \; TE}, (d\overset{\sigma}{\cancel{s}}_2)u) = \left.\frac{d}{dt}\right|_0 (\sigma_1 \circ \gamma + \sigma_2 \circ \gamma) \\
&= \left.\frac{d}{dt}\right|_0 (\sigma_1 + \sigma_2) \circ \gamma = d(\sigma_1 + \sigma_2)u.
\end{aligned}
$$

Since κ is a bundle homomorphism along π_{TM} we have

$$
\nabla^f_u(\sigma_1 + \sigma_2) = \kappa(d(\sigma_1 + \sigma_2)u) = \kappa((d\sigma_1)u \boxplus (ds_2)u) = \nabla^f_u(\sigma_1) + \nabla^f_u(\sigma_2).
$$

We have obtained (ii) of Definition 15.32.

$\left\{\frac{\partial}{\partial x^i}\right\}$ $\frac{\partial}{\partial t}$ $\frac{d}{dt}$ only 1

Now, as above, let $u \in T_pN$ and $\sigma : N \to E$ is a section along a smooth map $f : N \to M$. We wish to find a formula for $d\mu : T\mathbb{R} \times TE \to TE$, where $\mu : \mathbb{R} \times E \to E$ is the scalar multiplication in the vector bundle $E \to M$. For this purpose consider $(a, y) \in \mathbb{R} \times E$ and $(b \left.\frac{d}{dt}\right|_a, v_y) \in T_a\mathbb{R} \times T_yE$. Let us calculate first in two particular cases. Consider a smooth curve c in E with $c(0) = y$ and $\dot{c}(0) = v_y$, i.e. $v_y = [c]$. Then

$$\begin{aligned}(d\mu)(0_a, v_y) &= \left.\frac{d}{dt}\right|_0 \mu(a, c(t)) = \left.\frac{d}{dt}\right|_0 \mu_a(c(t)) \\ &= (d\mu_a)v_y = a \circledcirc v_y,\end{aligned} \tag{30}$$

where $\circledcirc$ is the scalar multiplication in the vector bundle structure of $TE \to TM$ as described in Problem 15.25.

Now let c be the curve in $\mathbb{R}$ given by $c(t) := a + tb$ so that $c(0) = a$ and $\dot{c}(0) = b \left.\frac{d}{dt}\right|_a$. Then

$$\begin{aligned} (d\mu)\left(b \left.\frac{d}{dt}\right|_a, 0_y \right) &= \left.\frac{d}{dt}\right|_0 \mu(c(t), y) = \left.\frac{d}{dt}\right|_0 ((a+bt)y) \qquad (31) \\ &= \left.\frac{d}{dt}\right|_0 (ay + tby) = \mathbf{j}_{ay}(by). \end{aligned}$$

From (31) and (32) we obtain

$$(d\mu)\left(b \left.\frac{d}{dt}\right|_a, v_y \right) = a \circledcirc v_y + \mathbf{j}_{ay}(by). \qquad (32)$$

Next suppose that $h \in C^\infty(N)$ and c is a curve in N with $c(0) = p$ and $\dot{c}(0) = u \in T_pN$. Then

$$(dh)_p u = \left.\frac{d}{dt}\right|_0 h(c(t)) \left.\frac{\partial}{\partial t}\right|_{h(c(0))} = \left.\frac{\partial h}{\partial x^i}\right|_{c(0)} \left.\frac{dc^i}{dt}\right|_{\cancel{c}(0)} \left.\frac{\partial}{\partial t}\right|_{h(c(0))} = u(h) \left.\frac{\partial}{\partial t}\right|_{h(p)}, \tag{33}$$

where x^i are some coordinates, c is given by $x^i = c^i(t)$, and we write the partial derivative to emphasize that this is a basic vector related to coordinate system t.

To write the next formula we need to introduce the following notation: let $h \times \sigma : N \to \mathbb{R} \times E$ denote the map $(h \times \sigma)(x) = (h(x), \sigma(x))$. Since κ is a bundle morphism, using its definition, (32) and (33) we obtain

$$\begin{aligned}
\nabla^f_u(h\sigma) &= \kappa(d(h\sigma)u) = \kappa(d(\mu \circ (h \times \sigma))u) = \kappa(d(\mu) \circ d(h \times \sigma)(u)) \\
&= \kappa d(\mu) \left(u(h) \left. \frac{\partial}{\partial t} \right|_{h(p)}, (d\sigma)u \right) \\
&= \kappa(h(p) \circledcirc ((d\sigma)u) + \mathbf{j}_{h(p)\sigma(p)}(u(h)\sigma(p))) \\
&= h(p)\kappa((d\sigma)u) + u(h)\sigma(p) = h(p)\nabla^f_u\sigma + u(h)\sigma_p.
\end{aligned}$$

The remaining part to be proved as a problem. $\square$

Problem 15.38. Prove the remaining statements Class

We complete the study of Ehresmann connections by a brief mentioning of the following important case. In the case of a principal smooth G-bundle E over M the Ehresmann connection is supposed to be G-invariant, i.e. the second property (instead of homogeneity) is formulated as

$$\mathcal{H}_{eg} = d(R_g)_e \mathcal{H}_e,$$

where $e \in E$, $g \in G$ and R_g is the right action of G on E (see the definition of a principal bundle).