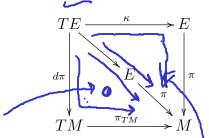
**16.12.2022 Theorem 15.29.** Let  $\kappa$  be a connector of a connection on a vector bundle  $\pi: E \to M$ . Then  $\kappa$  is a vector bundle homomorphism from  $d\pi: TE \to TM$  to  $\pi: E \to M$  along the map  $\pi_{TM}: TM \to M$ 

 $\begin{array}{ccc}
TE & \xrightarrow{\kappa} E \\
\downarrow \pi \\
TM & \xrightarrow{\pi_{TM}} M
\end{array}$ 

1) diagram is commutative

*Proof.* In the diagram

THY THE



2) R fillinge linear. TE (TE) TE

the left triangle is commutative by the definition of  $d\pi$  and the right one by (27). Thus (28) is commutative.

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It remains to verify that  $\kappa$  is linear on fibers. Let  $X_p = (d\pi)Z_y$ , where  $\pi(y) = p$ ,  $Z_y \in T_y E, \ X_p \in T_y M.$  Decompose  $Z_y = H_y + V_y$ , where  $H_y \in \mathcal{H}_y, \ V_y \in \mathcal{V}_y E.$  Since  $(d\pi)V_y = 0$ , we have  $X_p = (d\pi)H_y$  and  $X_p = (d\pi)H_y$  is the horizontal lift  $X_y$  of  $X_p$ . Also,  $X_y = \mathbf{j}_y w$  for a unique  $w \in E_p$  (by Propositions 15.9). Thus  $Z_y = X_y + \mathbf{j}_y w$  and  $\kappa(Z_y) = w$  by the definition. By Lemma 15.28 and homogeneity of  $\mathcal{H}$  we have

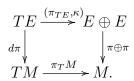
 $(d\mu_r)Z_y=(d\mu_r)\widetilde{X}_y+(d\mu_r)\mathbf{j}_yw=\widetilde{X}_{ry}+\mathbf{j}_{ry}rw.$  Hence  $\kappa((d\mu_r)Z_y)=rw=r\,\kappa(Z_y)$  or  $\kappa(r\odot Z_y)=r\,\kappa(Z_y)$  (in the notation of Problem 13.25). The second problem 13.25 of S . Corollary 15.27 completes the proof.

 $T_{y}E = y_{y}E \oplus \mathcal{H}_{y}$   $Z_{y}E = T_{y}E \oplus \mathcal{H}_{y}$   $Z_{y}E = T_{y}E \oplus \mathcal{H}_{y}$   $Z_{y}E = T_{y}E \oplus \mathcal{H}_{y}$   $Z_{y}E \oplus$ 



Home **Problem 15.30.** Prove that the addition  $\boxplus$  in  $TE \to TM$  can be described in the following (similar) form. We have, as above,  $Z_y = \widetilde{X}_y + \mathbf{j}_y w$  for some  $w \in E_p$  if  $(d\pi)Z_y = X_p$  and  $\widetilde{X}_y$  is the horizontal lift of  $X_p$ . Suppose, that for another vector  $U_{y'}$  from the same fiber over  $X_p$  we have in the same way  $U_{y'} = \widetilde{X}_{y'} + \mathbf{j}_{y'}w'$ . Then the sum of these vectors will be given by  $\widetilde{X}_{y+y'} + \mathbf{j}_{y+y'}(w+w')$ , where  $\widetilde{X}_{y+y'}$  is the horizontal lift of  $X_p$  to the point y + y'.

Class **Problem 15.31.** Using Problem 15.11 and Theorem 15.29 prove that  $(\pi_{TE}, \kappa) : TE \to E \oplus E$  is a vector bundle isomorphism along the tangent bundle projection  $\pi_T M : TM \to M$ , i.e. we have a commutative diagram with fiberwise linear isomorphism in the upper row:



Now we introduce the Koszul definition of connection (covariant derivative) for a vector bundle  $\pi: E \to M$ , which generalizes an affine connection.

**Definition 15.32.** Let  $\pi: E \to M$  and  $f: M \to N$  be as above. A *covariant derivative* along f is a map  $\nabla^f: TN \times \Gamma_f(E) \to \Gamma_f(E)$  (we write  $\nabla^f(v, \sigma) = \nabla^f_v(\sigma)$ ) having the properties

(i)  $\nabla^f$  is fiberwise linear in the first argument:

$$\nabla^f_{au+bv}\sigma = a\nabla^f_u\sigma + b\nabla^f_v\sigma,$$

for all  $\sigma \in \Gamma_f(E)$ ,  $a, b \in \mathbb{R}$ ,  $u, v \in T_pN$  for some  $p \in N$ ;

- (ii)  $\nabla_u^f(\sigma_1 + \sigma_2) = \nabla_u^f(\sigma_1) + \nabla_u^f(\sigma_2)$  for any  $u \in TN$  and any  $\sigma_1, \sigma_2 \in \Gamma_f(E)$ ;
- (iii) for  $v \in T_pN$ ,  $h \in C^{\infty}(N, \mathbb{K})$ , and  $\sigma \in \Gamma_f(E)$ , the Leibniz law is fulfilled:

$$\nabla_v^f(h\sigma) = h(p)\nabla_v^f\sigma + \underbrace{v(h)}\sigma(p);$$

- (iv) for a vector field  $p \mapsto v(p)$  from  $\mathbb{X}(N)$ , the map  $p \mapsto \nabla^f_{v(p)}\sigma$  is smooth for all  $\sigma \in \Gamma_f(E)$ ;
- (v) if  $g: S \to N$  and  $f: N \to M$  are smooth, then

$$\nabla_u^{f \circ g}(\sigma \circ g) = \nabla_{(dg)u}^f \sigma,$$

 $u \in TS$ :

$$S \xrightarrow{\sigma \circ g} X \xrightarrow{\sigma} K$$

$$S \xrightarrow{g} N \xrightarrow{f} M.$$

Home **Problem 15.33.** Prove that (ii) and (iii) give the linearity of  $\nabla^f$  over  $\mathbb{K}$  in the second argument.

A related notion (in fact a reduction for  $f = \text{Id}: M \to M$ ) is:

**Definition 15.34.** Let  $\pi: E \to M$  be a smooth  $\mathbb{K}$ -vector bundle. A covariant derivative or Koszul connection is a map  $\nabla: \mathbb{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$  (we write  $\nabla(X, s) = \nabla_X s$ ) having the properties

- (i)  $\nabla_{fX}s = f\nabla_X s$ , for all  $s \in \Gamma(M, E)$ ,  $f \in C^{\infty}(M)$ ,  $X \in \mathbb{X}(M)$ ;
- (ii)  $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$  for any  $s \in \Gamma(M, E), X_1, X_2 \in \mathbb{X}(M)$ ;
- (iii)  $\nabla_X(s_1+s_2) = \nabla_X s_1 + \nabla_X s_2$  for all  $s_1, s_2 \in \Gamma(M, E), X \in \mathbb{X}(M)$ ;
- (iv)  $\nabla_X(hs) = h\nabla_X ss + X(h)s$  for all  $s \in \Gamma(M, E), f \in C^{\infty}(M), X \in \mathbb{X}(M)$ .

Home **Problem 15.35.** Verify that this is a particular case.

Home **Problem 15.36.** Understand that an affine derivative of a vector field along a curve is a particular case of the above definitions.



**Theorem 15.37.** Suppose that  $\pi: E \to M$  is a vector bundle with a connection  $\mathcal{H}$  and associated connector  $\kappa$ . For any smooth map  $f: N \to M$  define the map  $\nabla^f: TN \times \Gamma_f(E) \to \Gamma_f(E)$  by the formula

$$\nabla_{v}^{f}\sigma := \kappa((d\sigma)_{p}v) \quad \text{for } v \in T_{p}N, \quad \sigma \in \Gamma_{f}(E), \quad \mathcal{I}$$

$$(29)$$

For a vector field V on N define  $(\nabla_V^f \sigma)(p) := \nabla_{V(p)}^f \sigma$ . Then  $\nabla^f$  satisfies Definition 15.32. In particular, for  $f = \operatorname{Id}_M$  we obtain a Koszul connection.

Conversely, if  $\nabla$  is a Koszul connection on  $\pi: E \to M$ , then we may define an (Ehresmann) connection by

$$\mathcal{H}_y := \{ (ds)u - \mathbf{j}_y \nabla_u s | s \in \Gamma(M, E), \ s(\pi(y)) = y, \ u \in T_{\pi(y)}M \}$$

The initial Koszul connection can be restored by the formula  $\nabla_v(s) = \kappa((ds)_p v), \ v \in T_p M$ .

*Proof.* Since  $\kappa$  and  $d\sigma$  are smooth bundle morphisms, the properties (i) and (iv) of Definition 15.32 follow immediately from the definition (29).

If  $g: S \to N$  and  $f: N \to M$  are smooth and  $u \in TS$ , then for each  $\sigma \in \Gamma_f(E)$  we have

$$\nabla_{u}^{f \circ g}(\sigma \circ g) = \kappa(d(\sigma \circ g)u) = \kappa(d(\sigma)((dg)u)) = \nabla_{(dg)u}^{f}\sigma.$$
 Definition 15.32.

This gives (v) of Definition 15.32.

To prove (ii) use the formula for addition in terms of the tangent lift of  $\alpha:(u,v)\mapsto u+v$ ,  $u, v \in T_pN$  Consider  $\sigma_1, \sigma_2 \in \Gamma_f(E), u \in T_pN, u = [\gamma]$  for a smooth curve  $\gamma$  in N with  $\gamma(0) = p$ . Then

then 
$$(d\sigma_1)u \boxplus (d\sigma_2)u = (d\alpha)((d\sigma_1)u, (d\sigma_2)u) = \frac{d}{dt}\Big|_0 (\sigma_1 \circ \gamma + \sigma_2 \circ \gamma)$$
$$= \frac{d}{dt}\Big|_0 (\sigma_1 + \sigma_2) \circ \gamma = d(\sigma_1 + \sigma_2)u.$$

Since 
$$\kappa$$
 is a bundle homomorphism along  $\pi_{TM}$  we have 
$$\nabla_u^f(\sigma_1 + \sigma_2) = \kappa(d(\sigma_1 + \sigma_2)u) = \kappa((d\sigma_1)u \boxplus (ds_2)u) = \nabla_u^f(\sigma_1) + \nabla_u^f(\sigma_2).$$

We have obtained (ii) of Definition 15.32.



Now, as above, let  $u \in T_pN$  and  $\sigma: N \to E$  is a section along a smooth map  $f: N \to M$ . We wish to find a formula for  $d\mu: T\mathbb{R} \times TE \to TE$ , where  $\mu: \mathbb{R} \times E \to E$  is the scalar multiplication in the vector bundle  $E \to M$ . For this purpose consider  $(a, y) \in \mathbb{R} \times E$  and  $(b \frac{d}{dt}|_a, v_y) \in T_a\mathbb{R} \times T_yE$ . Let us calculate first in two particular cases. Consider a smooth curve c in E with c(0) = y and  $\dot{c}(0) = v_y$ , i.e.  $v_y = [c]$ . Then

$$(d\mu)(0_a, v_y) = \frac{d}{dt}\Big|_0 \mu(a, c(t)) = \frac{d}{dt}\Big|_0 \mu_a(c(t))$$

$$= (d\mu_a)v_y = a \odot v_y,$$
(30)

where  $\odot$  is the scalar multiplication in the vector bundle structure of  $TE \to TM$  as described in Problem 15.25.

Now let c be the curve in  $\mathbb R$  given by c(t):=a+tb so that c(0)=a and  $\dot{c}(0)=b\left.\frac{d}{dt}\right|_a$ . Then

$$(d\mu)\left(b\left.\frac{d}{dt}\right|_{a},0_{y}\right) = \left.\frac{d}{dt}\right|_{0}\mu(c(t),y) = \left.\frac{d}{dt}\right|_{0}((a+bt)y)$$

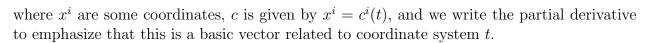
$$= \left.\frac{d}{dt}\right|_{0}(ay+tby) = \mathbf{j}_{ay}(by).$$
(31)

From (31) and (32) we obtain

$$(d\mu)\left(b\left.\frac{d}{dt}\right|_a, v_y\right) = a \otimes v_y + \mathbf{j}_{ay}(by). \tag{32}$$

Next suppose that  $h \in C^{\infty}(N)$  and c is a curve in N with c(0) = p and  $\dot{c}(0) = u \in T_pN$ . Then

$$(dh)_{p}u = \frac{d}{dt}\Big|_{0} h(c(t)) \frac{\partial}{\partial t}\Big|_{h(c(0))} = \frac{\partial h}{\partial x^{i}}\Big|_{c(0)} \frac{dc^{i}}{dt}\Big|_{h(c(0))} \frac{\partial}{\partial t}\Big|_{h(c(0))} = u(h) \frac{\partial}{\partial t}\Big|_{h(p)}, \qquad (33)$$



To write the next formula we need to introduce the following notation: let  $h \times \sigma : N \to \mathbb{R} \times E$  denote the map  $(h \times \sigma)(x) = (h(x), \sigma(x))$ . Since  $\kappa$  is a bundle morphism, using its definition, (32) and (33) we obtain

$$\nabla_{u}^{f}(h\sigma) = \kappa(d(h\sigma)u) = \kappa(d(\mu \circ (h \times \sigma))u) = \kappa(d(\mu) \circ d(h \times \sigma)(u))$$

$$= \kappa d(\mu) \left( u(h) \frac{\partial}{\partial t} \Big|_{h(p)}, (d\sigma)u \right)$$

$$= \kappa(h(p) \odot ((d\sigma)u) + \mathbf{j}_{h(p)\sigma(p)}(u(h)\sigma(p)))$$

$$= h(p)\kappa((d\sigma)u) + u(h)\sigma(p) = h(p)\nabla_{u}^{f}\sigma + u(h)\sigma_{p}.$$

The remaining part to be proved as a problem.

**Problem 15.38.** Prove the remaining statements

Class

We complete the study of Ehresmann connections by a brief mentioning of the following important case. In the case of a principal smooth G-bundle E over M the Ehresmann connection is supposed to be G-invariant, i.e. the second property (instead of homogeneity) is formulated as

$$\mathcal{H}_{eg} = d(R_g)_e \mathcal{H}_e,$$

where  $e \in E$ ,  $g \in G$  and  $R_g$  is the right action of G on E (see the definition of a principal bundle).