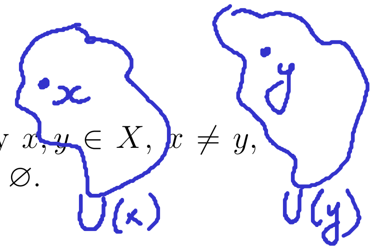
## 1.2 Compact, Hausdorff and normal spaces

12.09.2022




**Definition 1.48.** A topological space $X$ is called *Hausdorff,* if, for any $x, y \in X$, $x \neq y$, there exist their neighborhoods $U(x)$ and $U(y)$ such that $U(x) \cap U(y) = \varnothing$.

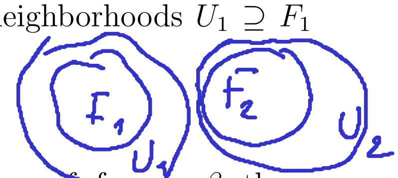
Home **Problem 1.49.** Give an example of non-Hausdorff topological space.

Class **Problem 1.50.** Prove that the cartesian product of Hausdorff spaces is a Hausdorff space.

Home **Problem 1.51.** Prove that in any Hausdorff space each point is a closed set.

more strong

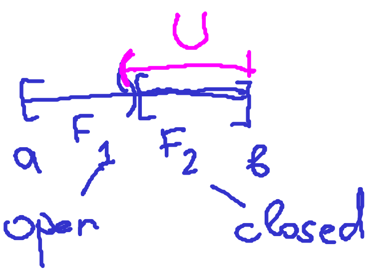
**Definition 1.52.** A topological space $X$ is called *normal,* if it is Hausdorff and, for any two non-intersecting closed sets $F_1$ and $F_2$, there exist their non-intersecting neighborhoods $U_1 \supseteq F_1$ and $U_2 \supseteq F_2$, $U_1 \cap U_2 = \varnothing$.




Class **Problem 1.53.** Verify that any metric space is normal.

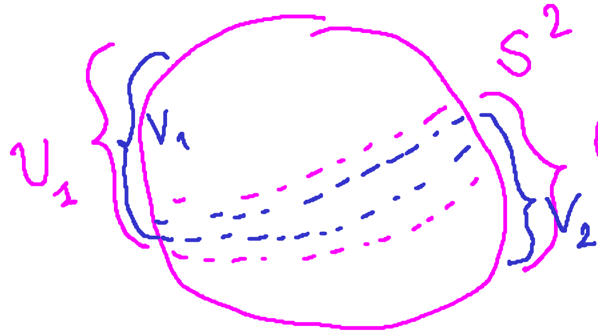
**Definition 1.54.** A cover $\{V_\beta\}_{\beta \in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha \in A}$, if, for any $\beta$, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_\alpha$.

Counterex:




$F_2 \subseteq \underset{\text{open}}{U} \cap F_1 \neq \varnothing$


Not necess.: $|B| > |A|$

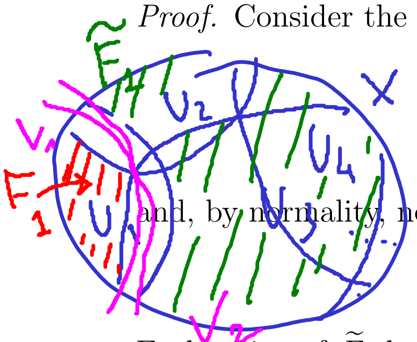


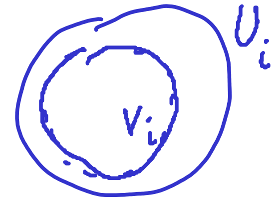

$U_1 \cup U_2 = S^2$

$V_1, V_2$ – refinement

**Theorem 1.55.** *Suppose that $X$ is a normal topological space and $\{U_i\}_{i=1}^N$ is a finite open cover. Then there exists its refinement of the form $\{V_i\}_{i=1}^N$ such that $\overline{V}_i \subseteq U_i$.*

*Proof.* Consider the following closed sets




$$F_1 = \left( X \setminus \bigcup_{i=2}^{N} U_i \right) \subseteq U_1, \qquad \widetilde{F}_1 = X \setminus U_1,$$

by def. of normality

and, by normality, neighborhoods

$$V_1 \supseteq F_1, \qquad \widetilde{V}_1 \supseteq \widetilde{F}_1, \qquad V_1 \cap \widetilde{V}_1 = \varnothing.$$

$U_1, U_2, \ldots U_N$
$\downarrow$
$V_1, U_2, \ldots U_N$
$\downarrow$
$V_2, V_2, U_3, \ldots U_N$
$\downarrow$
$\vdots$
$V_1, V_2, \ldots, V_N$

Each point of $\widetilde{F}_1$ has an open neighborhood $\widetilde{V}_1$, which does not intersect $V_1$. Hence this point can not be an adherent point of $V_1$ and

$$\overline{V}_1 \cap \widetilde{F}_1 = \emptyset, \qquad V_1 \subset \overline{V}_1 \subset (X \setminus \widetilde{F}_1) = U_1.$$

Also, $(V_1, U_2, \ldots, U_N)$ is a cover by the construction of $F_1$. At next steps we replace $U_2$ by $V_2$ and so on. □

$F_1, U_2, \ldots, U_N$ – cover (not open) $\Rightarrow$ $V_1, U_2, \ldots, U_N$ – cover

Home **Problem 1.56.** Let $f : X \to X$ be a continuous self-map of a Hausdorff space. Prove that the set of fixed points $F_f := \{x \in X \mid f(x) = x\}$ is closed.

Home **Problem 1.57.** Prove that $X$ is Hausdorff iff the diagonal $\Delta := \{(x, y) \mid x = y\} \subset X \times X$ is closed in $X \times X$.

Class **Problem 1.58.** Prove that a map $f : X \to Y$, where $Y$ is Hausdorff, is continuous iff its graph $\Gamma_f := \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is closed in $X \times Y$.

**Lemma 1.59. (Uryson's lemma)** *Suppose that $X$ is a normal topological space, $F_0$ and $F_1$ are some closed non-intersecting sets. Then there exists a continuous function $f : X \to [0, 1]$ such that $f|_{F_0} = 0$ and $f|_{F_1} = 1$.*

$\{U, X \setminus F\}$ — open cover.

*Proof.* The normality of $X$ implies that, for any closed $F$ and its open neighborhood $U$, $F \subseteq U$, there exists another neighborhood $V$ such that $F \subseteq V \subset \overline{V} \subset U$ (see the above proof of Theorem 1.55). We will denote this by $V \Subset U$. $V \subset \overline{V} \subset U$

Define $V_q$ for rational $q$ of the form $q = m/2^k$, $m$ odd, by induction over $k$ (i.e., first for 0 and 1, then for 1/2, then for 1/4 and 3/4, then for 1/8, 3/8, 5/8, 7/8 and so on) in such a way that $V_{q_1} \Subset V_{q_2}$ if $q_1 < q_2$. Define $V_0$ and $V_1$ to be open sets $U$ and $V$ from the beginning of the proof, i.e., $F_0 \subseteq V_0$, $F_1 \subseteq X \setminus V_1$, $V_0 \Subset V_1$. Suppose that, by the induction supposition, the sets $V_q$ are defined for $q$ up to $2^k$ as the denominator of $q$. Consider

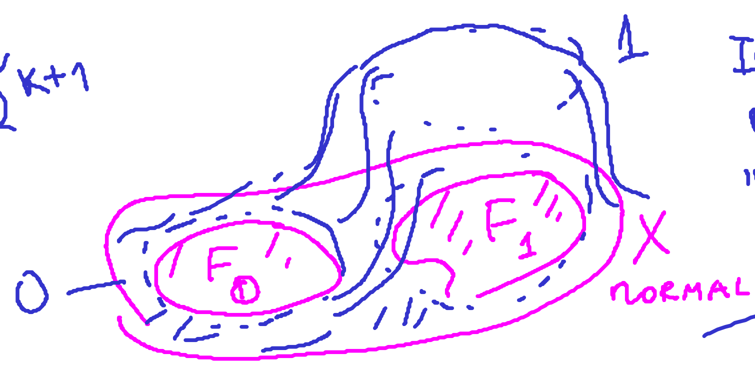
$q \to V_q$, $V$ open

$0, 1$; $\frac{1}{2}$; $\frac{1}{4}$ $\frac{3}{4}$; $\frac{1}{8}$ $\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$ ...

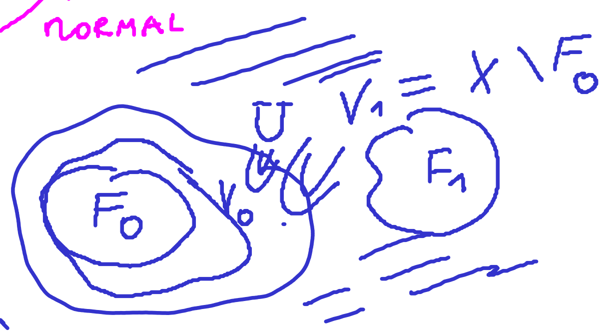
$$F := \overline{V_{\frac{i}{2^k}}}, \qquad U := V_{\frac{i+1}{2^k}},$$

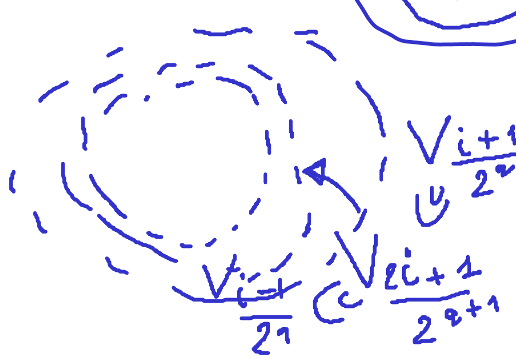
and define $V_{\frac{2i+1}{2^{k+1}}} := V$ (as in the beginning of the proof, for these $F$ and $U$). And so on.

$2^k \downarrow 2^{k+1}$




In any normal space closed non-intersecting sets are "functionally separated"!



The constructed $V_q$ are open and have the following properties:

1) $F_0 \subset V_0$,

2) $V_1 = X \setminus F_1$,

3) if $q_1 < q_2$, then $V_{q_1} \Subset V_{q_2}$.

$q \to \left\{ \frac{m}{2^k} \right\} \subset$ Rational
call $[0,1]$

Define, for any $s \in [0,1]$, the set $V_s$ as $V_s := \bigcup_{q \le s} V_q$. Then $V_s$ is open for any $s$ (as a union of open sets) and satisfies 1) – 3). Indeed, 1) and 2) are evident, and to prove 3), for $s_1 < s_2$,we find $q_1 = m_1/2^k$ and $q_2 = m_2/2^k$ such that $s_1 < q_1 < q_2 < s_2$, where $k$ is sufficiently large. Then $V_{s_1} \subseteq V_{q_1} \Subset V_{q_2} \subseteq V_{s_2}$ and $V_{s_1} \Subset V_{s_2}$.

$V_{q_0} = \bigcup_{q \le q_0} V_q$

Now define $f : X \to [0,1]$ by $f|_{F_0} = 0$ and $f(x) := \sup\{s \mid x \notin V_s\}$. Let us prove that $f$ is continuous. Let $x_0$ and $\varepsilon > 0$ be arbitrary. Let $s_0 = f(x_0)$. Consider

$$U(x_0) := V_{s_0+\frac{\varepsilon}{4}} \setminus \overline{V_{s_0-\frac{\varepsilon}{4}}}.$$

$V_{s_2} = \bigcup_{q \le s_1} V_q \quad s_2 \geq s_1$

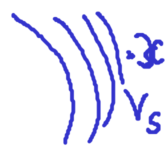
$\cap$

$V_{s_2} = \bigcup_{q \le s_2} V_q$

$0 \quad s_1 \; q_1 \; q_2 \; s_2 \quad 1$

This is an open neighborhood of $x_0$ and, for any $x \in U(x_0)$, one has

$$x \in V_{s_0+\frac{\varepsilon}{4}}, \qquad x \notin \overline{V_{s_0-\frac{\varepsilon}{4}}}.$$




Thus,

$$s_0 - \frac{\varepsilon}{4} \le f(x) \le s_0 + \frac{\varepsilon}{4}, \qquad |f(x) - f(x_0)| \le \frac{\varepsilon}{2} < \varepsilon.$$ □

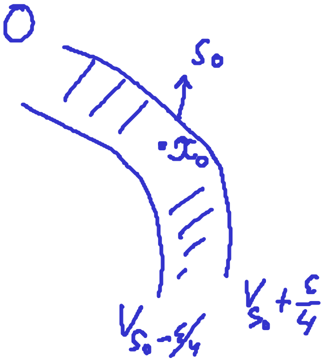
$f : X \to [0,1)$ $\quad f|_{V_0} = 0$ $\quad f(x) = \sup\{s \mid x \notin V_s\}$

$f|_{F_0} = 0$, $\quad f|_{F_1} = 1 \Leftarrow x \notin V_1$, $s \le 1$

It remains: $f$ - cont. ?

$x_0 \in X$, $\quad \varepsilon > 0$

$f(x_0) = s_0$




$U(x_0) = V_{s_0+\frac{\varepsilon}{4}} \setminus \overline{V_{s_0-\frac{\varepsilon}{4}}} =$

$= V_{s_0+\varepsilon/4} \cap \left(X \setminus \overline{V_{s_0-\frac{\varepsilon}{4}}}\right) \Rightarrow U(x_0)$ is open.

open $\qquad$ open

$\forall x \in U(x_0)$

$x \in V_{s_0+\frac{\varepsilon}{4}}$

$x \notin V_{s_0-\frac{\varepsilon}{4}}$ $\Rightarrow s_0 - \frac{\varepsilon}{4} \le f(x) \le s_0 + \frac{\varepsilon}{4}$ $\quad f(x) = \sup\{s \mid x \in V_s\}$

$\Downarrow$

$|f(x) - s_0| \le \frac{\varepsilon}{2} < \varepsilon.$

Home **Problem 1.60.** A closed subset of a closed set is closed in the entire space.

**Problem 1.61.** (Tietze's theorem about extension) [Mishchenko, Fomenko, pp. 78–79]

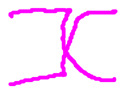
Home Suppose that $X$ is a normal topological space, $F \subset X$ is a closed subset and $f : F \to \mathbf{R}$ is a continuous function. Then $f$ can be extended to a continuous function $g : X \to \mathbf{R}$. If $f$ is bounded, then $g$ can be chosen to be bounded by the same constant.

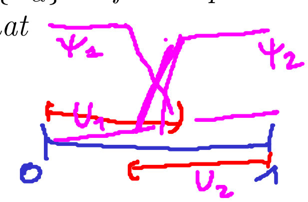
If $f$ is cont. $f^{-1}([0,+\infty)) \cup f^{-1}((-\infty,0))$

**Definition 1.62.** The *support of a function* $f : X \to \mathbf{R}$ is

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}.$$

**Theorem 1.63.** *Suppose that $X$ is a normal topological space and $\{U_\alpha\}$ its finite open cover. Then there exist continuous functions $\psi_\alpha : X \to [0,1] \subset \mathbf{R}$ such that*



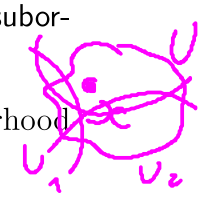



$\psi_1 + \psi_2 \equiv 1$

1) $\operatorname{supp} \psi_\alpha \subset U_\alpha$,

2) $\sum_\alpha \psi_\alpha(x) \equiv 1$.

*This system (not uniquely determined) of functions $\{\psi_\alpha\}$ is called* a partition of unity subordinated to $\{U_\alpha\}$.



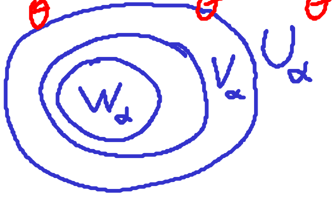

**Remark 1.64.** It is sufficient to ask local finiteness of $\{U_\alpha\}$: every point has a neighborhood such that it intersects only finitely many sets from $\{U_\alpha\}$.

*Proof of theorem.* Using Theorem 1.55 let us find new covers $W_\alpha \Subset V_\alpha \Subset U_\alpha$. By the Uryson lemma we can find continuous functions

$$\theta_\alpha : X \to [0,1], \qquad \theta_\alpha|_{\overline{W_\alpha}} \equiv 1, \qquad \theta_\alpha|_{(X \setminus V_\alpha)} \equiv 0.$$

Thus, $\operatorname{supp} \theta_\alpha \subseteq \overline{V}_\alpha \subseteq U_\alpha$ and $\theta_\alpha|_{W_\alpha} > 0$. Define $\boxed{\theta := \sum_\alpha \theta_\alpha}$. It is a finite sum of continuous functions, hence, itself a continuous function. Since $\{W_\alpha\}$ is a cover and $\theta \geq \theta_\alpha > 0$ on $W_\alpha$, then $\theta > 0$ everywhere. Hence we can define $\psi_\alpha := \frac{\theta_\alpha}{\theta}$. Evidently, 1) and 2) are satisfied. $\square$

$\sum \psi_\alpha = \sum \frac{\theta_\alpha}{\theta} = \frac{\sum \theta_\alpha}{\theta} = \frac{\theta}{\theta} \equiv 1.$




$\overline{W_\alpha} = F_1$ $\qquad$ $X \setminus V_\alpha = F_0$ $\qquad$ URYSON $\Longrightarrow$ $\theta_\alpha$.

$\operatorname{supp} \theta_\alpha \subseteq V_\alpha$ !

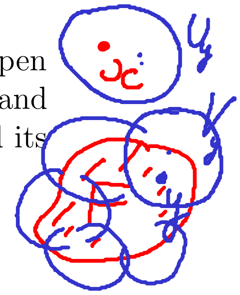
$\theta_\alpha|_{W_\alpha} > 0$

Then

**Definition 1.65.** A topological space $X$ is *compact,* if each its open cover has a finite sub-cover (i.e. there is a finite number of elements, which still cover $X$).

**Problem 1.66.** Prove that any closed interval $[a, b]$ is compact. Class
**Problem 1.67.** Prove that a closed subset of a compact space is compact itself. Home
**Problem 1.68.** Prove that a compact subset of a Hausdorff space is closed. Home

**Theorem 1.69.** *Any compact Hausdorff space is normal.*




I

*Proof.* Let $F \subset X$ be closed and $x \notin F$. Let us prove that there exist non-intersecting open neighborhoods $U(x)$ and $V(F)$. Since $X$ is Hausdorff, for any $y \in F$, there exist $V_y \ni y$ and $U_y \ni x$ such that $V_y \cap U_y = \emptyset$. The neighborhoods $V_y$ form a cover of $F$ and we can find its finite sub-cover $V_{y_1}, \ldots, V_{y_N}$, since $F$ is compact (see Problem 1.67). Define:

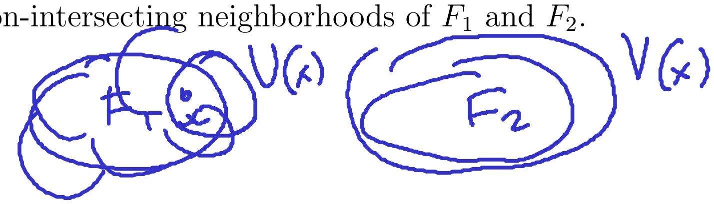
$$V(F) := V_{y_1} \cup \cdots \cup V_{y_N}, \qquad U(x) := \bigcap_{j=1}^{N} U_{y_j}.$$

finite
open

They are as desired.

II

Let now $F_1 \subset X$ and $F_2 \subset X$ be closed. According to the first part of the proof, we can find for each $x \in F_1$ open non-intersecting sets $U(x) \ni x$ and $V(x) \supset F_2$. Then $\{U(x)\}$ is an open cover of $F_1$ and we can find its finite sub-cover $U(x_1), \ldots, U(x_n)$. The sets $\bigcup_{i=1}^{n} U(x_i)$ and $\bigcap_{i=1}^{n} V(x_i)$ are demanded non-intersecting neighborhoods of $F_1$ and $F_2$. $\square$



Home **Problem 1.70.** Prove that a continuous image of a compact is compact.

Class **Problem 1.71.** Let $f : X \to \mathbf{R}^1$ be a continuous function on a compact space $X$. Then $f$ is bounded and reaches its maximal and minimal value.

**Theorem 1.72.** *A continuous bijective mapping of a compact space onto a Hausdorff space is a homeomorphism.*

*Proof.* Let $f : X \to Y$ be a continuous bijection, where $X$ is a compact and $Y$ is Hausdorff. To prove the statement, it is sufficient to prove that the image of any closed subset $F \subset X$ is a closed subset in $Y$. Since $X$ is compact, then $F$ is compact as well (see Problem 1.67). Thus, $f(F)$ is also compact. But $Y$ is Hausdorf. Thus, $f(F)$ is closed (see Problem 1.68). $\square$

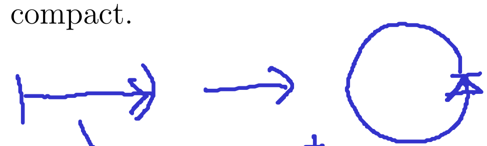
Class **Problem 1.73.** A cartesian product of compact spaces is compact.

$f : X \to Y$,

$f^{-1}$ is cont. ?

$\Updownarrow$

$(f^{-1})^{-1}$(any closed set) is clsd.



is not compact

stands for taking the inverse map

stands for taking ~~the~~ full pre-image.

i.e. $f$(closed set) is closed ?

$f(F)$ clos

F is closed in a compact

$\Rightarrow$ F is compt

$\Rightarrow$ $f(F)$ is a comp. space in a Hausd. space

$\Rightarrow$ $f(F)$ is closed by an lp.

# 2 Manifolds and tangent vectors

has a countable dense set

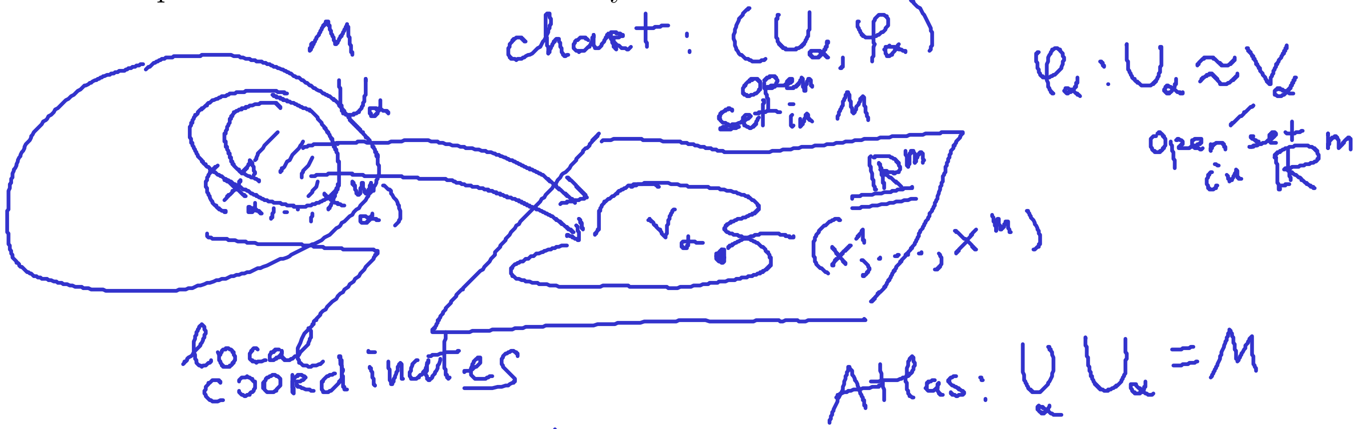
**Definition 2.1.** *A smooth manifold of dimension $m$* is a separable Hausdorff topological space $M$, equipped with a *smooth atlas,* i.e., its open cover $\{U_\alpha\}$ and a collection of homeomorphisms $\varphi_\alpha$, which map $U_\alpha$ onto open subsets $V_\alpha \subset \mathbf{R}^m$ (the dimension $m$ of $M$ is denoted by $\dim M$). They introduce on each $U_\alpha$ *local coordinates*. They are restricted to satisfy the following *compatibility property*: the *change of coordinate maps* (or *overlap maps*, or *transition functions*) $\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ should be smooth as vector-valued functions, defined on an open subset in $\mathbf{R}^m$. A pair $(U_\alpha, \varphi_\alpha)$ is called *a chart.*

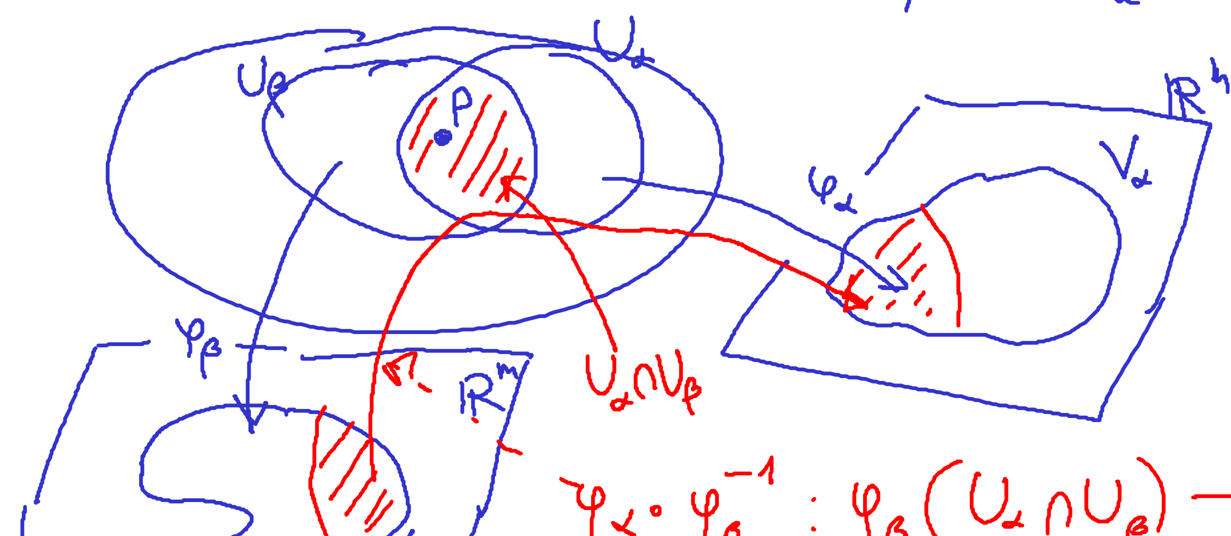
*A smooth structure* is a maximal smooth atlas (not absolutely rigorous definition). These are all charts, that are compatible with all charts of some smooth atlas.

Reminder: a map $f : U \to \mathbf{R}^n$, where $U$ is an open subset of $\mathbf{R}^m$, is called *differentiable* at $u \in U$ iff there is a linear map $Df(u) : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(u+h) - f(u) - Df(u)(h)\|}{\|h\|} = 0.$$

Existence of partial derivatives of coordinate functions at $u$ is not sufficient and existence of continuous partial derivatives is not necessary!!!




$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$

$\mathbb{R}^m$ - open - $\mathbb{R}^m$

We demand: is smooth as

**Remark 2.2.** We have inserted the restriction of the same $m$ for all charts into the definition, but in fact there is a theorem which shows that if we have a homeomorphism $\varphi : U \approx V$, where $U \subseteq \mathbf{R}^n$ and $V \subseteq \mathbf{R}^m$ are some open sets, then $m = n$.

**Remark 2.3.** If we do not demand compatibility, a manifold is called *topological.*

Class **Problem 2.4.** Find an example of a manifold and two non-compatible smooth structures on it, i.e., two smooth atlases $(U_i, \varphi_i)$ and $(V_j, \psi_j)$ such that $\{(U_i, \varphi_i), (V_j, \psi_j)\}$ is not a smooth atlas.

Class **Problem 2.5.** Prove that the sphere $S^n$ and the projective space $\mathbf{R}P^n$ are smooth manifolds.

Home **Problem 2.6.** Are the boundary of a square and 8 smooth manifolds (subspaces of $\mathbf{R}^2$) ?

**Definition 2.7.** A $2n$-dimensional manifold is called *complex analytical,* if all transition functions are complex analytical.

STOP

Home **Problem 2.8.** Prove that $S^2$ is a complex analytical manifold.

**Definition 2.9.** A function $f : M \to \mathbf{R}$ is called *smooth,* if, for any point $P \in M$ and some chart $(U_\alpha, \varphi_\alpha)$ with $P \in U_\alpha$, the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbf{R}$, defined on an open set in $\mathbf{R}^m$, is smooth.