General Topology ⟶ Mfd ⟶

⟶ Vector fields, Tensor fields

⟶ Connections, Bundles

⟶ Lie groups. ⟶ de Rham cohomology
integration and diff. of
exterior (differential form).
⟶ K-theory.

Exam in January
Oral

1) Theor.
2) Theor.

Problem.

from a list known

Not using Sources. Only paper and pen.

3 h

Lecture | Probl. sess. | larger.

E-mail
⟶ link to the page of course
⟶ invite to join a Telegram Group. (to be created)
⟶ ... about "office hours:
"consultation" a) in distant
b) on request.

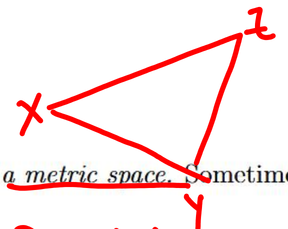
5 excellent
4 good
3 satisf.
2 not passed

1 Some concepts from topology

We start from metric spaces.

Definition 1.1. *A metric* ρ on a set X is a mapping $\rho : X \times X \to [0, \infty)$, restricted to satisfy:

1. $\rho(x, y) = 0 \quad \Leftrightarrow \quad x = y \quad \forall x, y \in X$ (identity axiom);
2. $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$ (symmetry axiom);
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$ (triangle axiom).



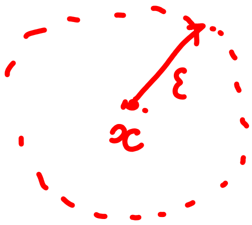
A pair (X, ρ), where X is a set and ρ is a metric on X, is called *a metric space*. Sometimes we write simply X.

A subset $Y \subset X$ is automatically a metric space itself.

$\rho_Y : Y \times Y \to [0, \infty)$

Definition 1.2. *Diameter* of Y is $\operatorname{diam} Y := \sup_{x,y \in Y} \rho(x, y)$. If $\operatorname{diam} Y < \infty$, then Y is *bounded*. A *ball* (ball neighborhood) is

$$B_\varepsilon(x) := \{y \in X \mid \rho(y, x) < \varepsilon\}.$$



The *distance* between $Y \subseteq X$ and $Z \subseteq X$ is

$$\rho(Y, Z) := \inf_{y \in Y, z \in Z} \rho(y, z).$$

Definition 1.3. If $\rho(y, Y) = 0$, then y is an *adherent point* of Y. The *closure* of a subset Y is $\overline{Y}$:={the set of all adherent points of Y}. Evidently, $Y \subseteq \overline{Y}$. A subset Y is *closed*, if $Y = \overline{Y}$.

$\forall \varepsilon > 0$
$\exists y_n \in Y$
$\rho(y_n, y) < \varepsilon$

B $\cdot y_0$

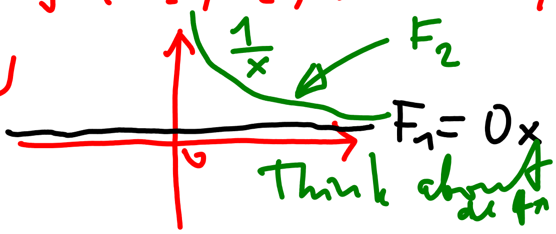
$\rho(B, y_0) = 0$

$\overline{B}$ closed ball $\cdot y_0$

X, F_1, F_2 closed

$F_1 \cap F_2 = \emptyset$

$\rho(F_1, F_2) > 0$ always?

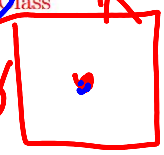


Think about it this.

Definition 1.4. A point x is an *interior point* of a subset Y, if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq Y$ (in particular, $x \in Y$). The *interior* of Y is the set $\operatorname{Int} Y \subseteq Y$ of all its interior points. A subset Y is *open*, if $Y = \operatorname{Int} Y$.

$Y = (0,0)$ in $\mathbb{R}^2$

Problem 1.5. Suppose, X is a metric space. Then $Y \subset X$ is open iff (if and only if) $X \setminus Y$ is closed. In fact, $\operatorname{Int} Y = X \setminus \overline{X \setminus Y}$. Class

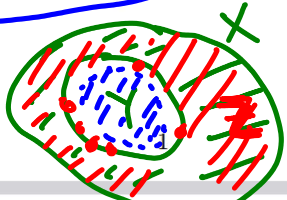


Theorem 1.6. *Suppose, X is a metric space. Then*

$\mathbb{R}^2$ $\operatorname{Int} Y = \emptyset$

1 O *X is open;*

2 O *$\emptyset$ is open;*



$\overline{B}$ closed ball

$\operatorname{Int} \overline{B} = B$

$\operatorname{Int} Y = Y \iff X \setminus \overline{X \setminus Y} = Y \iff \overline{X \setminus Y} = X \setminus Y$

3 O *the union $\bigcup_{\alpha \in A} U_\alpha$ of* ***any*** *collection of open subsets $U_\alpha \subseteq X$ is open;*

4 O *the intersection $\bigcap_{i=1}^{k} U_i$ of a* ***finite*** *collection of open subsets $U_i \subset X$ is open;*

1 C *$\emptyset$ is closed;*

2 C *X is closed;*

3 C *the intersection $\bigcap_{\alpha \in A} F_\alpha$ of* ***any*** *collection of closed subsets $F_\alpha \subset X$ is closed;*

4 C *the union $\bigcup^{k} F_i$ of a* ***finite*** *collection of closed subsets $F_i \subset X$ is closed.*

Ex $U_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$

$-\frac{1}{k} \quad 0 \quad \frac{1}{k}$ $\quad X = \mathbb{R}$

$\bigcap U_k = \{0\}$ is not open

$\operatorname{Int} Y \overset{?}{=} Y \setminus \overline{X \setminus Y}.$

~~Suppose~~ $y_0 \in \operatorname{Int} Y \subseteq Y \iff \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(y_0) \subseteq Y$, $\iff$ any point of X is at a distance $\geq \varepsilon$ from y_0 : $\iff$ $\rho(x, y_0) \geq \varepsilon$, $\forall x \in X \setminus Y \iff y_0$ is not in $\overline{X \setminus Y}$ (and of course $y_0 \in Y$)

$F_\alpha = X \setminus U_\alpha$ open. | $\bigcap_\alpha (X \setminus U_\alpha) = X \setminus \bigcup_\alpha U_\alpha$ verify!

$\operatorname{Int} \varnothing = \varnothing$

X

$\mathbb{R}^1$ $B_\varepsilon^{\mathbb{R}^1}(1)$

0 I 1

I in $\mathbb{R}^1$ I is not open.

I by itself is open.

Proof. Properties 1 O and 2 O are evident. Let us prove 3 O. Suppose, $U := \bigcup_{\alpha\in A} U_\alpha$ and $x \in U$. Then. for some α, we have $x \in U_\alpha$ and $B_{\varepsilon(\alpha)} \subseteq U_\alpha$. Then $B_{\varepsilon(\alpha)} \subseteq U_\alpha \subseteq U$.

Let us prove 4 O. Suppose, $U := \bigcap_{i=1}^{k} U_i$, $x \in U$. Then there are ε_i $(i = 1, \dots, k)$ such that $x \in B_{\varepsilon_i}(x) \subseteq U_i$. Take $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_k\}$. Take $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq U_i$ $\forall i$. Hence, $B_\varepsilon(x) \subset U$.

Finally, by Problem 1.5, k O $\Leftrightarrow$ k C $\forall$ k. □

I

$B_\varepsilon^I(0)$

Home **Problem 1.7.** Show that the finiteness condition is essential.

Home **Problem 1.8.** Prove that $B_\varepsilon(x)$ is open.

Home **Problem 1.9.** Prove that $\operatorname{Int} Y$ is open, i.e., $\operatorname{Int}(\operatorname{Int} Y) = \operatorname{Int} Y$.

Home **Problem 1.10.** Prove that $\overline{Y}$ is closed, i.e., $\overline{\overline{Y}} = \overline{Y}$.

Definition 1.11. A *topology* on a set X is a system τ of its subsets (these subsets are called *open*), restricted to satisfy the following axioms:

1) $X \in \tau$;

2) $\varnothing \in \tau$;

3) if $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha\in A} U_\alpha \in \tau$;

4) if $U_1, \dots, U_k \in \tau$, then $\bigcap_{i=1}^{k} U_i \in \tau$.

Then (X, τ) is called a *topological space*. Any set of the form $F = X \setminus U$, where $U \in \tau$, is called *closed*.

Home **Problem 1.12.** Verify 1 C – 4 C for closed sets in a topological space.

Example 1.13. Any metric space is a topological space.

$a \quad b$ $\tau_0 = \{\varnothing, \{a,b\}\}$ I: this a top.? Axioms are fulfilled.

$a \quad b$ $\tau = \{\varnothing, \{b\}, \{a,b\}\}$.

metric top. $\to$ $\tau_\rho = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

$a \quad r \quad b$ $\varnothing, \{a\}, \{b\}, \{a,b\}$ – list of all subsets.

Metric: $\rho(a,a) = \rho(b,b) = 0$, $\rho(a,b) = r$

$B_r(a) = \{a\} \Rightarrow \{a\}$ is open, sym $\{b\}$ is open.

Class **Problem 1.14.** Find an example of a topological space (X, τ), which is not related to any metric (this is called: topology is not metrizable).

$X = \{a\}$ $\cdot a$ $\exists! \, \tau$ on X. (exists a unique (only one))

Indeed: $\emptyset, \{a\} = X$ all subsets of X by ax. 1, 2. are open $\in \tau$

$\rho(a,a) = 0$.

Definition 1.15. An (open) *neighborhood* of a point $x \in X$ (respectively, of a subset $Y \subseteq X$) in a topological space is any open set, where x (respectively, Y) is contained.

An *adherent point* of $Y \subseteq X$ is a point $x \in X$ such that any its neighborhood has a non-empty intersection with Y. The *closure* of Y is the set $\overline{Y}$ of all adherent points of Y (in particular, $Y \subseteq \overline{Y}$).

A point $x \in Y$ is called an *interior* point of Y, if there exists a neighborhood U of x such that $x \in U \subseteq Y$. The set $\operatorname{Int} Y$ of all interior points of Y is called the *interior* of Y.

To prove

Problem 1.16. $Y \subseteq X$ is closed iff $Y = \overline{Y}$. Home

Problem 1.17. $\overline{Y}$ is closed. Home

Problem 1.18. $Y \subseteq X$ is open iff $Y = \operatorname{Int} Y$. Home

Problem 1.19. $\operatorname{Int} Y$ is open. Home

Definition 1.20. Suppose $Y \subseteq X$, where (X, τ) is a topological space. The system of sets $\tau_1 := \{U \cap Y \mid U \in \tau\}$ is called the *induced* topology (by τ on Y).

Problem 1.21. Verify the axioms for τ_1. Home

Problem 1.22. Suppose that (X, ρ_X) is a metric space. Then one can introduce a topology on $Y \subseteq X$ in two ways: Home

1) ρ_X generates τ_X, which then induces τ_1,
2) ρ_X after the restriction on Y gives ρ_Y, which generates τ_{ρ_Y}.

Prove that $\tau_1 = \tau_{\rho_Y}$.

$\emptyset = Y \cap \emptyset$

$Y = Y \cap X$

$\boxed{(X, \rho_X)}$, $Y \subseteq X$

τ_X a top. on $X \xrightarrow{\text{induced top}} (\tau_X)_1$ on Y

$(Y, \rho_Y) \longrightarrow \tau_{\rho_Y}$

Problem 1.21. Verify the axioms for τ_1. Home

Problem 1.22. Suppose that (X, ρ_X) is a metric space. Then one can introduce a topology on $Y \subseteq X$ in two ways: Home

1) ρ_X generates τ_X, which then induces τ_1,
2) ρ_X after the restriction on Y gives ρ_Y, which generates τ_{ρ_Y}.

Prove that $\tau_1 = \tau_{\rho_Y}$.

Definition 1.23. A subset $Y \subseteq X$ is called *(everywhere) dense*, if $\boxed{\overline{Y} = X}$.

tm $X = \mathbb{R}^n$ $Y = X$. rat., countable!

$\mathbb{Q} \subset \mathbb{R}^1$

$\overline{\mathbb{Q}} = \mathbb{R}^1$ $\frac{1}{3} = \frac{2}{6}$

Problem 1.24. Let $Y_1 \subseteq X$ and $Y_2 \subseteq X$ be dense open sets. Then $Y = Y_1 \cap Y_2$ is a dense open set. Home

$\mathbb{Q}$ $\frac{p}{q}$

Definition 1.25. A map $f : X \to Y$ of topological spaces is called *continuous at a point* $x_0 \in X$, if for any neighborhood of its image $V(f(x_0))$, there exists a neighborhood $U(x_0)$ such that $f(U(x_0)) \subseteq V(f(x_0))$. A map is called *continuous*, if it is continuous at each point.

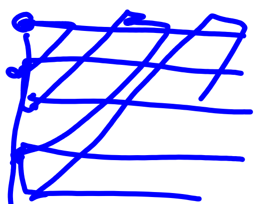
$\mathbb{Z} \times \mathbb{Z}$

Theorem 1.26. *The next properties are equivalent:*

1) *a map $f : X \to Y$ is continuous;*

2) *for any open set $V \subseteq Y$, its full pre-image $f^{-1}(V)$ is open in X;* $\Leftrightarrow$

3) *for any closed set $F \subset Y$ its full pre-image $f^{-1}(F)$ is closed in X.*

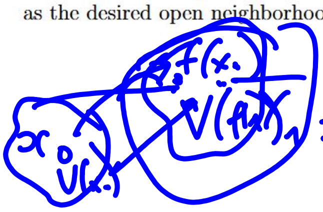


Proof. Since $f^{-1}(Y \backslash V) = f^{-1}(Y) \backslash f^{-1}(V) = X \backslash f^{-1}(V)$, properties 2) and 3) are equivalent.

Suppose, 1) is fulfilled, i.e., f is continuous, and $V \subseteq Y$ is an open set. Then either the pre-image of V is empty, hence open, or there is some point x, i.e., $f(x) \in V$. Then, by definition, for any such x, there exists a neighborhood $U(x)$ such that $f(U(x)) \subseteq V$, i.e., $U(x) \subseteq f^{-1}(V)$. Thus, any point of $f^{-1}(V)$ is interior.

Conversely, suppose 2) is fulfilled. Then, for $V = V(f(x_0))$, one can take $U(x_0) = f^{-1}(V)$ as the desired open neighborhood (see Def. 1.25). □

$\mathbb{Q} \cap I$, $\frac{i}{2^k}$, $\frac{m}{3^s}$

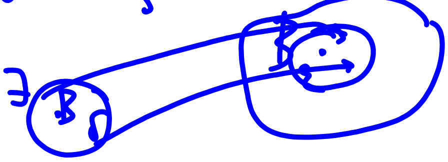


$Y_1 = I$, $\overline{Y}_2 = I$

$Y_1 \cap Y_2 = \emptyset$.

because they are not open.

ε–δ–lang.



$f(U_x) \subset V$

$U_x \subset f^{-1}(V)$

$\underbrace{f^{-1}(V)}_{\text{open}} = \underline{\bigcup U_x}$

---

unif. conv. $f_n \underset{W}{\rightrightarrows} f$ if. $\forall \varepsilon > 0\ \exists N$ s.th.

for $\forall\, n > N$: $|f_n(w) - f(w)| < \varepsilon$, $\forall\, w \in W$

$f_n \in C(X)$. $f \in C(X)$?

$x_0 \in X$, $\forall \varepsilon > 0$, to find a U_{x_0}: s.t. $|f(x) - f(x_0)| < \varepsilon$ for $x \in U_{x_0}$

Home **Problem 1.27.** Suppose, $X = F_1 \cup F_2$, where F_1 and F_2 are closed subsets, and $f : X \to Y$ is a map. Then f is continuous iff $f|_{F_1} : F_1 \to Y$ and $f|_{F_2} : F_2 \to Y$ are continuous.

Class **Problem 1.28.** Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, which is uniformly convergent on X to some function f. Then f is continuous.

Home **Problem 1.29.** Let X and Y be metric spaces. Prove that $f : X \to Y$ is continuous at x_0 as a map of topological spaces iff, for any sequence $\{x_n\}$ with $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

$\tau_X \xrightarrow{id} \tau_Y$

Definition 1.30. A map $f : X \to Y$ is called a *homeomorphism*, if

1) f is a bijection;

2) f and f^{-1} (inverse mapping) are continuous.

$X \xrightarrow{f} Y$, τ_X, τ_Y

Class **Problem 1.31.** Give an example of a continuous bijection, which is not a homeomorphism.

Definition 1.32. A *base of a topology* τ is a system of open sets $\mathcal{B}$ such that any τ–open set is as a union of some of them.

Home **Problem 1.33.** What conditions need to be imposed on an arbitrary system of subsets $\mathcal{B}_1$, to obtain some topology by taking their arbitrary unions?

$\mathcal{B}_1 = \{W\}$

1) $\emptyset \in \mathcal{B}_1$

2) $\bigcup \ldots = X$

(3) $W_1 \cap W_2 \in \mathcal{B}_1$, $W_1, W_2 \in \mathcal{B}_1$

$\forall p \in W_1 \cap W_2$ $\exists W_3 \in \mathcal{B}_1$ s.t. $p \in W_3 \subset W_1 \cap W_2$

Definition 1.34. Suppose that (X, τ_X) and (Y, τ_Y) are topological spaces. Consider in $X \times Y$ the following base of topology:

$$\mathcal{B} := \{V \times W \mid V \in \tau_X,\ W \in \tau_Y\}.$$

The resulting topological space is called the *cartesian product* of X and Y.

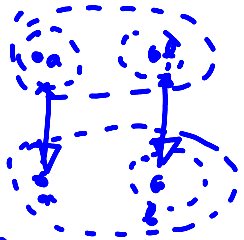
N so large that $|f_N(x) - f(x)| < \varepsilon/3$ $\forall x \in X$

$\exists U_{x_0}$ s.t. $|f_N(x) - f_N(x_0)| < \varepsilon/3$ for $\forall x \in U_{x_0}$

If $x \in U_{x_0}$ then $|f(x) - f(x_0)| =$

$= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \le$

$\le \underbrace{|f(x) - f_N(x)|}_{<\varepsilon/3} + \underbrace{|f_N(x) - f_N(x_0)|}_{<\varepsilon/3} + \underbrace{|f_N(x_0) - f(x_0)|}_{<\varepsilon/3} < \varepsilon.$ □



$f = id$

$\tau_d = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

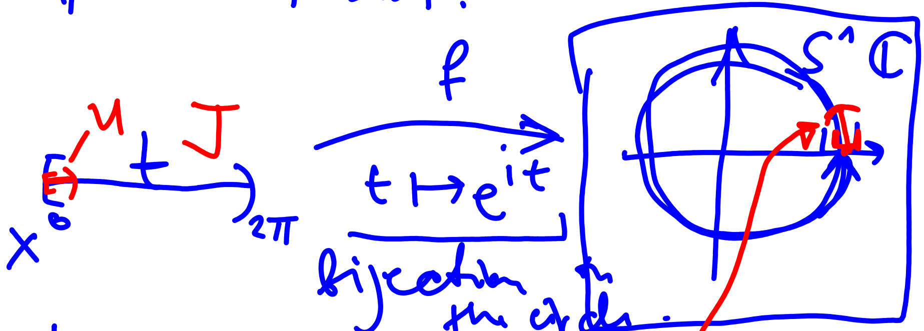
$\tau_1 = \{\emptyset, \{b\}, \{a,b\}\}$.

$f^{-1}(\emptyset), f^{-1}(\{b\}), f^{-1}(\{a,b\}) = \emptyset, \{b\}, \{a,b\}$

So f is cont.

$(f^{-1})^{-1}(\{a\}) = \{a\}$ is not open in τ_1; open in τ_d

f^{-1} is not cont.

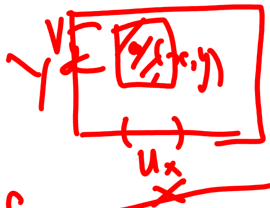


X, U, J, 0, 2π, $t \mapsto e^{it}$ bijection to the circle, $S^1 \subset \mathbb{C}$

f^{-1} is not cont. $(f^{-1})^{-1}(U)$ not open in S^1

Comp $f^{-1} : S^1 \to J$ not cn

Home **Problem 1.35.** Verify (with the help of the previous problem) that $X \times Y$ is really a topological space.



$\mathcal{B}_1 = \{ U_x \times V_y \}$

Home **Problem 1.36.** Prove that $X \times Y$ and $Y \times X$ are homeomorphic.

Home **Problem 1.37.** Prove that $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are homeomorphic.

Home **Problem 1.38.** Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. Define on $X \times Y$ the following distances:

$$\rho_{\max}((x_1, y_1), (x_2, y_2)) := \max\{\rho_X(x_1, x_2),\ \rho_Y(y_1, y_2)\},$$

$$\rho_2((x_1, y_1), (x_2, y_2)) := \sqrt{\rho_X^2(x_1, x_2) + \rho_Y^2(y_1, y_2)},$$

$$\rho_+((x_1, y_1), (x_2, y_2)) := \rho_X(x_1, x_2) + \rho_Y(y_1, y_2).$$

Prove:

1) That these are metrics.
2) That the corresponding topologies on $X \times Y$ coincide.

Class **Problem 1.39.** Prove that (a, b), $[a, b)$ and $[a, b]$ (subsets of real line) are pair-wise non-homeomorphic.

(a,b), $[a,b)$, $[a,b]$

not comp. not comp comp

We need more thing.

a subset of $\mathbb{R}^n$ is compact $\Longleftrightarrow$ it is bounded and closed

$f: (a,b) \cong [a,b)$

$f': (a,b) \setminus \{\underbrace{f^{-1}(a)}_{a_0}\} \cong [a,b) \setminus \{a\} = (a,b)$

$f': (a,a_0) \sqcup (a_0,b) \cong (a,b)$

not connect. connect

Home **Problem 1.35.** Verify (with the help of the previous problem) that $X \times Y$ is really a topological space.

Home **Problem 1.36.** Prove that $X \times Y$ and $Y \times X$ are homeomorphic.

Home **Problem 1.37.** Prove that $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are homeomorphic.

Home **Problem 1.38.** Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. Define on $X \times Y$ the following distances:

$$\rho_{\max}((x_1, y_1), (x_2, y_2)) := \max\{\rho_X(x_1, x_2),\ \rho_Y(y_1, y_2)\},$$

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1.1 Connectedness and arc connectedness

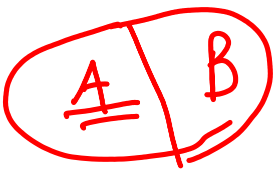
Definition 1.40. A topological space X is called *disconnected*, if one of the following (evidently equivalent to each other) conditions is fulfilled:

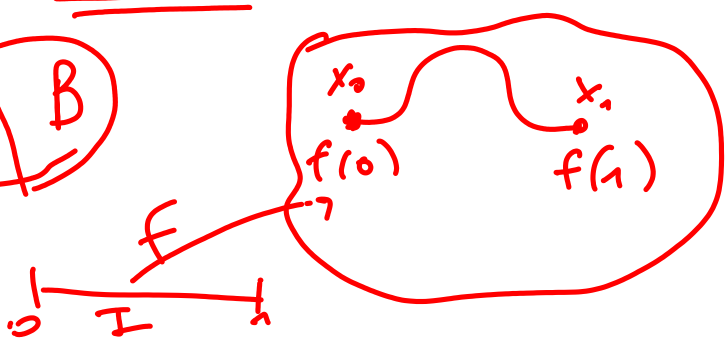
- X is equal to a union of its two non-intersecting non-empty open subsets.
- X has a non-empty subset $A \neq X$, which is open and closed simultaneously. $B = X \setminus A \neq \emptyset$
- X is equal to a union of its two non-intersecting non-empty open and closed simultaneously subsets.

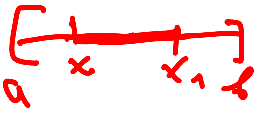
Otherwise X is *connected*.

Definition 1.41. A topological space X is called *arc connected*, if, for any two points $x_0, x_1 \in X$, there exists a continuous map (*path*) $f : [0, 1] \to X$, $f(0) = x_0$, $f(1) = x_1$.

Problem 1.42. Any interval $[a, b] \subset \mathbb{R}$ is connected and arc connected. Class





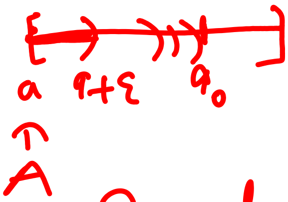


evidently $[a,b]$ is arc connected

Suppose $[a,b]$ is not connected

$[a,b] = A \sqcup B$, $A, B \neq \emptyset$ closed, open

$a \in A$



A is open $\Rightarrow$ for some ε

$[a, a+\varepsilon) \subset A$

Consider all ε, s.t. $[a, a+\varepsilon) \subset A$

sup ε of these ε.

$[a, a_0) \subset A$ and A is closed $\Rightarrow a_0 \in A$,

$[a, a_0] \subset A$, but A is open $\Rightarrow \exists \delta$

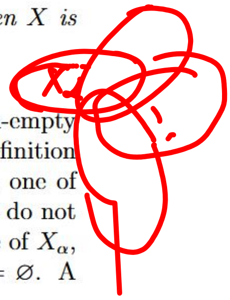
$[a, a_0 + \delta) \subset A$. This contradicts to

$a_0 = \sup$. $\Rightarrow a_0 = b$, $A = [a,b]$

and B is empty.

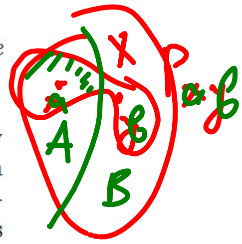
Theorem 1.43. *Suppose,* $X = \bigcup_\alpha X_\alpha$*, each* X_α *is connected, and* $\bigcap_\alpha X_\alpha \neq \varnothing$*. Then* X *is connected.*

Proof. Suppose that X is disconnected, $X = A \cup B$, $A \cap B = \varnothing$, A and B are non-empty closed-open sets. Then, for each α, we have $X_\alpha = (X_\alpha \cap A) \cup (X_\alpha \cap B)$. By the definition of the induced topology, these sets are closed-open in X_α. Since X_α is connected, one of them should be empty. Hence, each X_α belongs entirely either to A, or to B, which do not intersect. Since A and B are non-empty and X is the union of X_α, then at least one of X_α, say X_{α_0} is contained in A and some other, $X_{\alpha_1} \subseteq B$. Then $\bigcap_\alpha X_\alpha \subseteq X_{\alpha_0} \cap X_{\alpha_1} = \varnothing$. A contradiction. □



Theorem 1.44. *Suppose that, for any two points* x *and* y *of a topological space* X*, there exists a connected subset* P_{xy} *such that* $x \in P_{xy}$ *and* $y \in P_{xy}$*. Then* X *is connected.*

Proof. Suppose that X is disconnected: $X = A \cup B$, $A \cap B = \emptyset$, A and B are non-empty closed-open subsets. Then there exist some $a \in A$, $b \in B$ and a corresponding P_{ab}. Then $P_{ab} = (P_{ab} \cap A) \cup (P_{ab} \cap B)$. The subsets $P_{ab} \cap A$ and $P_{ab} \cap B$ are closed-open in P_{ab} and non-empty (the first one contains a, the second one — b). A contradiction with connectedness of P_{ab}. □



Problem 1.45. The image of a connected space under a continuous mapping is connected. Home

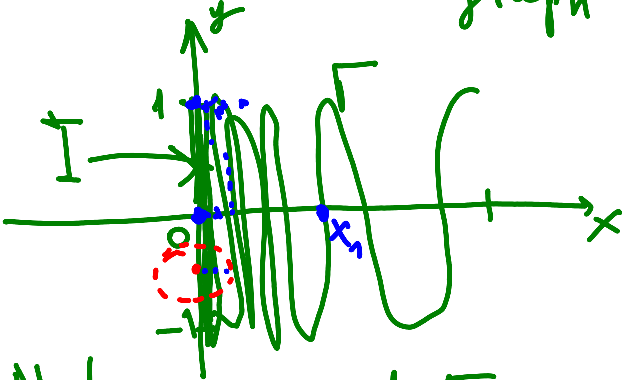
Theorem 1.46. *An arc connected space is connected.*

Proof. By the previous problem, the set $f([0,1])$ is connected, where $f = f_{x_0,x_1}$ is the function from Def. 1.41. Taking $P_{x_0,x_1} := f([0,1])$, apply Theorem 1.44. □

Problem 1.47. Find an example of connected space, which is not arc-connected. Class

$\mathbb{R}^2 \supset X = I \sqcup \Gamma$ — graph of $\sin\frac{1}{x}$

$\dots x > 0$



Note: I and Γ are arc-connected

$\Rightarrow$ connected

So if X is not connected $A = I$, $B = \Gamma$

but I is not open. A contrad.

$\Rightarrow X$ is connected.

$x_0 = (0,0)$, x_1 If X is arc connected.

then $\exists\, f : [0,1] \to X$

$f(0) = x_0$, $f(1) = x_1$

$f(t) = (x(t), y(t))$ $(t \to 0)$

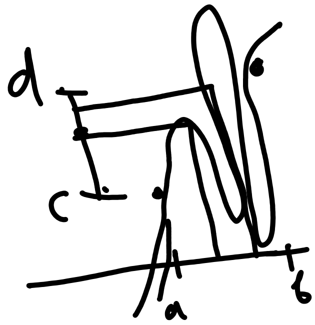
$x(t) \longrightarrow 0$, hence it takes all intermediate values.

so $\exists\, t_k \longrightarrow 0$

$y(t_k) = 1 \not\longrightarrow 0$

$t_k \to 0$

so f cannot be continuous..

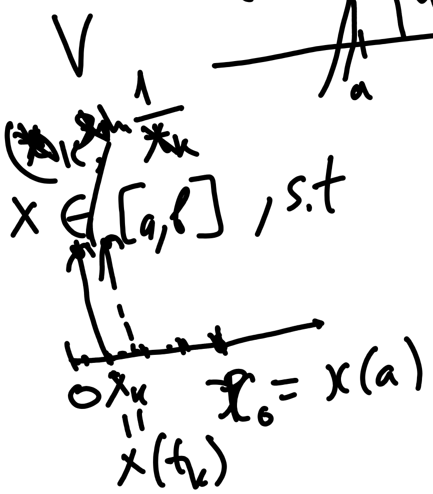


$f : [a,b] \xrightarrow{\text{cont.}} \mathbb{R}$ ~~$[c,d]$~~

$f(a) = c$, $f(b) = d$

$\forall u \in [c,d]$ $\exists$ $x \in [a,b]$, s.t

$f(x) = u$.



$[b, a] \xrightarrow{x(t)} \mathbb{R}$

t