

GEOMETRY OF REIDEMEISTER CLASSES AND TWISTED BURNSIDE THEOREM

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Dedicated to the memory of Professor Yuri Solovyov

ABSTRACT. The purpose of the present mostly expository paper (based mainly on [16, 17, 39, 15, 11]) is to present the current state of the following conjecture of A. Fel'shtyn and R. Hill [12], which is a generalization of the classical Burnside theorem.

Let G be a countable discrete group, ϕ one of its automorphisms, $R(\phi)$ the number of ϕ -conjugacy (or twisted conjugacy) classes, and $S(\phi) = \#\text{Fix}(\widehat{\phi})$ the number of ϕ -invariant equivalence classes of irreducible unitary representations. If one of $R(\phi)$ and $S(\phi)$ is finite, then it is equal to the other.

This conjecture plays an important role in the theory of twisted conjugacy classes (see [25], [10]) and has very important consequences in Dynamics, while its proof needs rather sophisticated results from Functional and Non-commutative Harmonic Analysis.

First we prove this conjecture for finitely generated groups of type I and discuss its applications.

After that we discuss an important example of an automorphism of a type II₁ group which disproves the original formulation of the conjecture.

Then we prove a version of the conjecture for a wide class of groups, including almost polycyclic groups (in particular, finitely generated groups of polynomial growth). In this formulation the role of an appropriate dual object plays the finite-dimensional part of the unitary dual. Some counter-examples are discussed.

Then we begin a discussion of the general case (which also needs new definition of the dual object) and prove the weak twisted Burnside theorem for general countable discrete groups. For this purpose we prove a non-commutative version of Riesz-Markov-Kakutani representation theorem.

Finally we explain why the Reidemeister numbers are always infinite for Baumslag-Solitar groups.

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1. INTRODUCTION AND FORMULATION OF RESULTS

Definition 1.1. Let G be a countable discrete group and $\phi : G \rightarrow G$ an endomorphism. Two elements $x, x' \in G$ are said to be ϕ -conjugate or *twisted conjugate*, iff there exists $g \in G$ with

$$x' = gx\phi(g^{-1}).$$

We shall write $\{x\}_\phi$ for the ϕ -conjugacy or *twisted conjugacy* class of the element $x \in G$. The number of ϕ -conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ -conjugacy classes are the usual conjugacy classes in the group G .

If G is a finite group, then the classical Burnside theorem (see e.g. [26, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of G . Let \widehat{G} be the *unitary dual* of G , i.e. the set of equivalence classes of unitary irreducible representations of G .

Remark 1.2. If $\phi : G \rightarrow G$ is an epimorphism, it induces a map $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$, $\widehat{\phi}(\rho) = \rho \circ \phi$ (because a representation is irreducible if and only if the scalar operators in the space of representation are the only ones which commute with all operators of the representation). This is not the case for a general endomorphism ϕ , because $\rho\phi$ can be reducible for an irreducible representation ρ , and $\widehat{\phi}$ can be defined only as a multi-valued map. But nevertheless we can define the set of fixed points $\text{Fix } \widehat{\phi}$ of $\widehat{\phi}$ on \widehat{G} .

Therefore, by the Burnside's theorem, if ϕ is the identity automorphism of any finite group G , then we have $R(\phi) = \#\text{Fix}(\widehat{\phi})$.

To formulate the main theorem of the first part of the paper for the case of a general endomorphism we first need an appropriate definition of the $\text{Fix}(\widehat{\phi})$.

Definition 1.3. Let $\text{Rep}(G)$ be the space of equivalence classes of finite dimensional unitary representations of G . Then the corresponding map $\widehat{\phi}_R : \text{Rep}(G) \rightarrow \text{Rep}(G)$ is defined in the same way as above: $\widehat{\phi}_R(\rho) = \rho \circ \phi$.

Let us denote by $\text{Fix}(\widehat{\phi})$ the set of points $\rho \in \widehat{G} \subset \text{Rep}(G)$ such that $\widehat{\phi}_R(\rho) = \rho$.

Theorem 1.4. *Let G be a finitely generated discrete group of type I, ϕ one of its endomorphism, $R(\phi)$ the number of ϕ -conjugacy classes, and $S(\phi) = \#\text{Fix}(\widehat{\phi})$ the number of $\widehat{\phi}$ -invariant equivalence classes of irreducible unitary representations. If one of $R(\phi)$ and $S(\phi)$ is finite, then it is equal to the other.*

Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square-free.} \end{cases}$$

Theorem 1.5 (Congruences for the Reidemeister numbers). *Let $\phi : G \rightarrow G$ be an endomorphism of a countable discrete group G such that all numbers $R(\phi^n)$ are finite and let H be a subgroup of G with the properties*

$$\phi(H) \subset H,$$

$$\forall x \in G \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.$$

If the pair (H, ϕ^n) satisfies the conditions of Theorem 1.4 for any $n \in \mathbb{N}$, then one has for all n ,

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \pmod{n}.$$

These theorems were proved previously in a special case of Abelian finitely generated plus finite group [12, 13].

For groups of type II_1 the situation is much more complicated. We discuss in detail the case of a semi-direct product of the action of \mathbb{Z} on $\mathbb{Z} \oplus \mathbb{Z}$ by a hyperbolic automorphism with finite Reidemeister number (four to be precise) and the number of fixed points of $\widehat{\phi}$ on \widehat{G} equal or greater than five [17]. The origin of this phenomenon lies in bad separation properties of \widehat{G} for general discrete groups. A more deep study leads to the following general theorem.

Theorem 1.6 (weak twisted Burnside theorem, [39]). *The number $R_*(\phi)$ of Reidemeister classes related to twisted invariant functions on G from the Fourier-Stieltjes algebra $B(G)$ is equal to the number $S_*(\phi)$ of generalized fixed points of $\widehat{\phi}$ on the Glimm spectrum of G , i. e. on the complete regularization of \widehat{G} , if one of $R_*(\phi)$ and $S_*(\phi)$ is finite.*

The argument goes along the following line. The well-known Riesz(-Markov-Kakutani) theorem identifies the space of linear functionals on algebra $A = C(X)$ and the space of regular measures on X . To prove the weak twisted Burnside theorem we first obtain a generalization of this theorem to the case of non-commutative C^* -algebra A via Dauns-Hofmann sectional representation theorem. The corresponding measures on Glimm spectrum are functional-valued. In extreme situation this theorem is tautological, but for group C^* -algebras of discrete groups in many cases one obtains some new tool for counting twisted conjugacy classes.

Keeping in mind that for hyperbolic groups $R(\phi)$ is always infinite while in the "opposite" case the twisted Burnside theorem is proved, we can formulate the following conjecture, which is in fact a program of further actions.

Conjecture 1.7. *There exists a class of groups \mathcal{G} such that*

- *for any group $G \notin \mathcal{G}$ and any automorphism $\phi : G \rightarrow G$ the Reidemeister number $R(\phi)$ is always infinite,*

- for any group $G \in \mathcal{G}$ there exists a subset of ideals $\mathcal{M} \subset \text{Prim } C^*(G)$ such that its points are separated and $R(\phi)$ coincides with the number of fixed points of $\widehat{\phi}$ on \mathcal{M} supposing one of these numbers to be finite.

One of candidates for \mathcal{M} is the set of maximal ideals (let us remind that the Glimm spectrum is in fact the space of maximal ideals of the center of $C^*(G)$, cf. [4, 39]). The strategy of proof will be based on the weak theorem 1.6.

We also consider a formulation of the main conjecture with counting only finite-dimensional fixed points on the unitary dual. In [15] we prove this version for a very large class of groups.

Theorem 1.8. *For a wide class of groups, which includes almost polycyclic groups (in particular, finitely generated groups of polynomial growth) the Reidemeister number $R(\phi)$ coincides with the number $S_f(\phi)$ of finite-dimensional $\widehat{\phi}$ -fixed points on the unitary dual, if one of these numbers is finite.*

This version of the main conjecture (with the consideration of only finite-dimensional representations) has some counter-examples. One of them, coming from D. Osin's group [32] we discuss after the mentioned theorem in Section 11. Also we propose there some possible formulation of the conjecture appropriate for residually finite groups.

Let us remark, that in some cases the weak twisted Burnside theorem easily implies the twisted Burnside theorem (in particular, in the form with finite-dimensional representations). For example, we will show directly that $R(\phi) = R_*(\phi)$ in Abelian case. On the other hand, the unitary dual coincides with the Glimm spectrum. Slightly more complicated argument is valid for some more general groups covered by Theorem 1.8, in particular, the Heisenberg group. More precisely, it is possible to extract from the first part of the proof of Theorem 1.8 (see below Section 11) that $R(\phi) = R_*(\phi)$. Moreover, characteristic functions of Reidemeister classes are related to finite dimensional representations, which are maximal and Hausdorff separated. Hence, the finite-dimensional fixed points form a part of generalized fixed points on Glimm spectrum and $S_*(\phi) \geq S_f(\phi)$. In fact, we have an equality here, because if there exists a functional coming from some other generalized fixed point on Glimm spectrum, it cannot be a coefficient of a finite-dimensional representation because Glimm spectrum is Hausdorff separated while all finite-dimensional (generalized) fixed points are already counted in $S_f(\phi)$. The details concerning the relation between separateness and linear independence are contained in Section 11.

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [25, 10]), in Selberg theory (see, eg. [37, 1]), and Algebraic Geometry (see, e.g. [22]).

The congruences give some necessary conditions for the realization problem for Reidemeister numbers in topological dynamics. The relations with Selberg theory will be presented in a forthcoming paper.

Let us remark that it is known that the Reidemeister number of an endomorphism of a finitely generated Abelian group is finite iff 1 is not in the spectrum of the restriction of this endomorphism to the free part of the group (see, e.g. [25]). The Reidemeister number is infinite for any automorphism of a non-elementary Gromov hyperbolic group [18].

To make the presentation more detailed and transparent we start from a new approach (E.T.) for Abelian (Section 2) and compact (Section 3) groups. Only after that we develop this approach and prove the main theorem for finitely generated groups of type I [16] in Section 5. A discussion of some examples leading to conjectures is the subject of Section 6. In Section 7 we present the mentioned important example for type II_1 groups [17]. After that we pass to the demonstration of the weak twisted Burnside theorem in Section 10. For this purpose we present in Section 9 a non-commutative version of Riesz-Markov-Kakutani theorem [39] (some necessary information on operator fields is collected in Section 8). In Section 11 we discuss almost polycyclic groups and related matter. Section 12 is devoted to the discussion of Baumslag-Solitar groups [11].

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2. ABELIAN CASE

Let ϕ be an automorphism of an Abelian group G .

Lemma 2.1. *The twisted conjugacy class H of e is a subgroup. The other ones are cosets gH .*

Proof. The first statement follows from the equalities

$$h\phi(h^{-1})g\phi(g^{-1}) = gh\phi((gh)^{-1}), \quad (h\phi(h^{-1}))^{-1} = \phi(h)h^{-1} = h^{-1}\phi(h).$$

For the second statement suppose $a \sim b$, i.e. $b = ha\phi(h^{-1})$. Then

$$gb = gha\phi(h^{-1}) = h(ga)\phi(h^{-1}), \quad gb \sim ga.$$

□

Lemma 2.2. *Suppose, $u_1, u_2 \in G$, χ_H is the characteristic function of H as a set. Then*

$$\chi_H(u_1u_2^{-1}) = \begin{cases} 1, & \text{if } u_1, u_2 \text{ are in one coset,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose, $u_1 \in g_1H$, $u_2 \in g_2H$, hence, $u_1 = g_1h_1$, $u_2 = g_2h_2$. Then

$$u_1u_2^{-1} = g_1h_1h_2^{-1}g_2^{-1} \in g_1g_2^{-1}H.$$

Thus, $\chi_H(u_1u_2^{-1}) = 1$ if and only if $g_1g_2^{-1} \in H$ and u_1 and u_2 are in the same class. Otherwise it is 0. □

The following Lemma is well known.

Lemma 2.3. *For any subgroup H the function χ_H is of positive type.*

Proof. Let us take arbitrary elements u_1, u_2, \dots, u_n of G . Let us reenumerate them in such a way that some first are in g_1H , the next ones are in g_2H , and so on, till g_mH , where g_jH are different cosets. By the previous Lemma the matrix $\|p_{it}\| := \|\chi_H(u_i u_t^{-1})\|$ is block-diagonal with square blocks formed by units. These blocks, and consequently the whole matrix are positively semi-defined. \square

Lemma 2.4. *In the Abelian case characteristic functions of twisted conjugacy classes belong to the Fourier-Stieltjes algebra $B(G) = (C^*(G))^*$.*

Proof. By Lemma 2.1 in this case the characteristic functions of twisted conjugacy classes are the shifts of the characteristic function of the class H of e . Hence, by Corollary (2.19) of [7], these characteristic functions are in $B(G)$. \square

Let us remark that there exists a natural isomorphism (Fourier transform)

$$u \mapsto \widehat{u}, \quad C^*(G) = C_r^*(G) \cong C(\widehat{G}), \quad \widehat{g}(\rho) := \rho(g),$$

(this is a number because irreducible representations of an Abelian group are 1-dimensional). In fact, it is better to look (for what follows) at an algebra $C(\widehat{G})$ as an algebra of continuous sections of a bundle of 1-dimensional matrix algebras. over \widehat{G} .

Our characteristic functions, being in $B(G) = (C^*(G))^*$ in this case, are mapped to the functionals on $C(\widehat{G})$ which, by the Riesz-Markov-Kakutani theorem, are measures on \widehat{G} . Which of these measures are invariant under the induced (twisted) action of G ? Let us remark, that an invariant non-trivial functional gives rise to at least one invariant space – its kernel.

Let us remark, that convolution under the Fourier transform becomes point-wise multiplication. More precisely, the twisted action, for example, is defined as

$$g[f](\rho) = \rho(g)f(\rho)\rho(\phi(g^{-1})), \quad \rho \in \widehat{G}, \quad g \in G, \quad f \in C(\widehat{G}).$$

There are 2 possibilities for the twisted action of G on the representation algebra $A_\rho \cong \mathbb{C}$: 1) the linear span of the orbit of $1 \in A_\rho$ is equal to all A_ρ , 2) and the opposite case (the action is trivial).

The second case means that the space of intertwining operators between A_ρ and $A_{\widehat{\phi}\rho}$ equals \mathbb{C} , and ρ is a fixed point of the action $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$. In the first case this is the opposite situation.

If we have a finite number of such fixed points, then the space of twisted invariant measures is just the space of measures concentrated in these points. Indeed, let us describe the action of G on measures in more detail.

Lemma 2.5. *For any Borel set E one has $g[\mu](E) = \int_E g[1] d\mu$.*

Proof. The restriction of measure to any Borel set commutes with the action of G , since the last is point wise on $C(\widehat{G})$. For any Borel set E one has

$$g[\mu](E) = \int_E 1 dg[\mu] = \int_E g[1] d\mu.$$

\square

Hence, if μ is twisted invariant, then for any Borel set E and any $g \in G$ one has

$$\int_E (1 - g[1]) d\mu = 0.$$

Lemma 2.6. *Suppose, $f \in C(X)$, where X is a compact Hausdorff space, and μ is a regular Borel measure on X , i.e. a functional on $C(X)$. Suppose, for any Borel set $E \subset X$ one has $\int_E f d\mu = 0$. Then $\mu(h) = 0$ for any $h \in C(X)$ such that $f(x) = 0$ implies $h(x) = 0$. I.e. μ is concentrated off the interior of $\text{supp } f$.*

Proof. Since the functions of the form fh are dense in the space of the referred to above h 's, it is sufficient to verify the statement for fh . Let us choose an arbitrary $\varepsilon > 0$ and a simple function $h' = \sum_{i=1}^n a_i \chi_{E_i}$ such that $|\mu(fh') - \mu(fh)| < \varepsilon$. Then

$$\mu(fh') = \sum_{i=1}^n \int_{E_i} a_i f d\mu = \sum_{i=1}^n a_i \int_{E_i} f d\mu = 0.$$

Since ε is an arbitrary one, we are done. \square

Applying this lemma to a twisted invariant measure μ and $f = 1 - g[1]$ we obtain that μ is concentrated at our finite number of fixed points of $\widehat{\phi}$, because outside of them $f \neq 0$.

If we have an infinite number of fixed points, then the space is infinite-dimensional (we have an infinite number of measures concentrated in finite number of points, each time different) and Reidemeister number is infinite as well. So, we are done.

3. COMPACT CASE

Let G be a compact Hausdorff group, hence \widehat{G} is a discrete space. Then $C^*(G) = \oplus M_i$, where M_i are the matrix algebras of irreducible representations. The infinite sum is in the following sense:

$$C^*(G) = \{f_i\}, i \in \{1, 2, 3, \dots\} = \widehat{G}, f_i \in M_i, \|f_i\| \rightarrow 0 (i \rightarrow \infty).$$

When G is finite and \widehat{G} is finite this is exactly Peter-Weyl theorem.

A characteristic function of a twisted class is a functional on $C^*(G)$. For a finite group it is evident, for a general compact group it is necessary to verify only the measurability of the twisted class with the respect to Haar measure, i.e. that twisted class is Borel. For a compact G , the twisted conjugacy classes being orbits of twisted action are compact and hence closed.

Under the identification it passes to a sequence $\{\varphi_i\}$, where φ_i is a functional on M_i (the properties of convergence can be formulated, but they play no role at the moment). The conditions of invariance are the following: for each $\rho_i \in \widehat{G}$ one has $g[\varphi_i] = \varphi_i$, i.e. for any $a \in M_i$ and any $g \in G$ one has $\varphi_i(\rho_i(g)a\rho_i(\phi(g^{-1}))) = \varphi_i(a)$.

Let us recall the following well-known fact.

Lemma 3.1. *Each functional on matrix algebra has form $a \mapsto \text{Tr}(ab)$ for a fixed matrix b .*

Proof. One has $\dim(M(n, \mathbb{C}))' = \dim(M(n, \mathbb{C})) = n \times n$ and looking at matrices as at operators in V , $\dim V = n$, with base e_i , one can remark that functionals $a \mapsto \langle ae_i, e_j \rangle$, $i, j = 1, \dots, n$, are linearly independent. Hence, any functional takes form

$$a \mapsto \sum_{i,j} b_j^i \langle ae_i, e_j \rangle = \sum_{i,j} b_j^i a_i^j = \text{Tr}(ba), \quad b := \|b_j^i\|.$$

□

Now we can study invariant functionals:

$$\begin{aligned} \text{Tr}(b\rho_i(g)a\rho_i(\phi(g^{-1}))) &= \text{Tr}(ba), & \forall a, g, \\ \text{Tr}((b - \rho_i(\phi(g^{-1}))b\rho_i(g))a) &= 0, & \forall a, g, \end{aligned}$$

hence,

$$b - \rho_i(\phi(g^{-1}))b\rho_i(g) = 0, \quad \forall g.$$

Since ρ_i is irreducible, the dimension of the space of such b is 1 if ρ_i is a fixed point of $\widehat{\phi}$ and 0 in the opposite case by the Schur lemma. So, we are done.

Remark 3.2. In fact we are only interested in finite discrete case. Indeed, for a compact G , the twisted conjugacy classes being orbits of twisted action are compact and hence closed. If there is a finite number of them, then they are open as well. Hence, the situation is more or less reduced to a discrete group: quotient by the component of unity.

4. EXTENSIONS AND REIDEMEISTER CLASSES

Let us denote by $\bar{\tau}_g : G \rightarrow G$ the automorphism $\tau_g(\tilde{g}) = g\tilde{g}g^{-1}$ for $g \in G$. Its restriction on a normal subgroup we will denote by τ_g as well.

Lemma 4.1. $\{g\}_\phi k = \{gk\}_{\tau_{k^{-1}} \circ \phi}$.

Proof. Let $g' = fg\phi(f^{-1})$ be ϕ -conjugate to g . Then

$$g'k = fg\phi(f^{-1})k = fgk k^{-1} \phi(f^{-1})k = f(gk)(\tau_{k^{-1}} \circ \phi)(f^{-1}).$$

Conversely, if g' is $\tau_{k^{-1}} \circ \phi$ -conjugate to g , then

$$g'k^{-1} = fg(\tau_{k^{-1}} \circ \phi)(f^{-1})k^{-1} = fgk^{-1} \phi(f^{-1}).$$

Hence a shift maps ϕ -conjugacy classes onto classes related to another automorphism. □

Corollary 4.2. $R(\phi) = R(\tau_g \circ \phi)$.

Consider a group extension respecting homomorphism ϕ :

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0 \\ & & \phi' \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0, \end{array}$$

where H is a normal subgroup of G . The argument below, especially related the role of fixed points, has a partial intersection with [19, 20].

First of all let us notice that the Reidemeister classes of ϕ in G are mapped epimorphically on classes of $\bar{\phi}$ in G/H . Indeed,

$$(2) \quad p(\tilde{g})p(g)\bar{\phi}(p(\tilde{g}^{-1})) = p(\tilde{g}g\phi(\tilde{g}^{-1})).$$

Suppose, $R(\phi) < \infty$. Then the previous remark implies $R(\bar{\phi}) < \infty$. Consider a class $D = \{h\}_{\tau_g\phi'}$, where $\tau_g(h) := ghg^{-1}$, $g \in G$, $h \in H$. The corresponding equivalence relation is

$$(3) \quad h \sim \tilde{h}hg\phi'(\tilde{h}^{-1})g^{-1}.$$

Since H is normal, the automorphism $\tau_g : H \rightarrow H$ is well defined. We will denote by D the image iD as well. By (3) the shift Dg is a subset of Hg is characterized by

$$(4) \quad hg \sim \tilde{h}(hg)\phi'(\tilde{h}^{-1}).$$

Hence it is a subset of $\{hg\}_\phi \cap Hg$ and the partition $Hg = \cup(\{h\}_{\tau_g\phi'})g$ is a subpartition of $Hg = \cup(Hg \cap \{hg\}_\phi)$.

We need the following statements.

Lemma 4.3. *Suppose, the extension (1) satisfies the following conditions:*

- (1) $\#\text{Fix}\bar{\phi} = k < \infty$,
- (2) $R(\phi) < \infty$.

Then

$$(5) \quad R(\phi') \leq k \cdot (R(\phi) - R(\bar{\phi}) + 1).$$

If G/H is abelian, let g_i be some elements with $p(g_i)$ being representatives of all different $\bar{\phi}$ -conjugacy classes, $i = 1, \dots, R(\bar{\phi})$. Then

$$(6) \quad \sum_{i=1}^{R(\bar{\phi})} R(\tau_{g_i}\phi') \leq k \cdot R(\phi).$$

Proof. Consider classes $\{z\}_{\phi'}$, $z \in G$, i.e. the classes of relation $z \sim hz\phi'(h^{-1})$, $h \in H$. The group G acts on them by $z \mapsto gz\phi(g^{-1})$. Indeed,

$$\begin{aligned} g[\tilde{h}h\phi(\tilde{h}^{-1})]\phi(g^{-1}) &= (g\tilde{h}g^{-1})(gh\phi(g^{-1}))(\phi(g)\phi(\tilde{h}^{-1})\phi(g^{-1})) \\ &= (g\tilde{h}g^{-1})(gh\phi(g^{-1}))\phi(g\tilde{h}g^{-1}) \in \{gh\phi(g^{-1})\}_{\phi'}, \end{aligned}$$

because H is normal and $g\tilde{h}g^{-1} \in H$. Due to invertibility, this action of G transposes classes $\{z\}_{\phi'}$ inside one class $\{g\}_\phi$. Hence, the number d of classes $\{h\}_{\phi'}$ inside $\{h\}_\phi \cap H$ does not exceed the number of $g \in G$ such that $p(g)\bar{\phi}(p(g^{-1})) = \bar{e}$. Since two elements g and gh in one H -coset induce the same permutation of classes $\{h\}_{\phi'}$, the mentioned number d does not exceed the number of $z \in G/H$ such that $z\bar{\phi}(z^{-1}) = \bar{e}$, i.e. $d \leq k$. This implies (5).

Now we discuss ϕ -classes over $\bar{\phi}$ -classes other than $\{\bar{e}\}_{\bar{\phi}}$ for an abelian G/H . An estimation analogous to the above one leads to the number of $z \in G/H$ such that $zz_0\bar{\phi}(z^{-1}) = z_0$ for some fixed z_0 . But for an Abelian G/H they form the same group $\text{Fix}(\bar{\phi})$. This together with the description (4) of shifts of D at the beginning of the present Section implies (6). \square

Lemma 4.4. *Suppose, in the extension (1) the group H is abelian. Then $\#\text{Fix}(\phi) \leq \#\text{Fix}(\phi') \cdot \#\text{Fix}(\bar{\phi})$.*

Proof. Let $s : G/H \rightarrow G$ be a section of p . If $s(z)h$ is a fixed point of ϕ then

$$(7) \quad (s(z))^{-1}\phi(s(z)) = h\phi'(h^{-1}).$$

Hence, $z \in \text{Fix}(\bar{\phi})$ and left hand side takes $k := \#\text{Fix}(\bar{\phi})$ values h_1, \dots, h_k . Let us estimate the number of $s(z)h$ for a fixed z such that $(s(z))^{-1}\phi(s(z)) = h_i$. These h have to satisfy (7). Since H is abelian, if one has

$$h_i = h\phi'(h^{-1}) = \tilde{h}\phi'(\tilde{h}^{-1}),$$

then $h^{-1}\tilde{h} \in \text{Fix}(\phi')$ and we are done. \square

Theorem 4.5. *Let A be a finitely generated Abelian group, $\psi : A \rightarrow A$ its automorphism with $R(\psi) < \infty$. Then $\#\text{Fix}(\psi) < \infty$.*

Moreover, $R(\psi) \geq \#\text{Fix}(\psi)$.

Proof. Let T be the torsion subgroup. It is finite and characteristic. We obtain the extension $T \rightarrow A \rightarrow A/T$ respecting ϕ . Since $A/T \cong \mathbb{Z}^k$, $\text{Fix}(\bar{\psi} : A/T \rightarrow A/T) = \bar{e}$, by [25],[10, Sect. 2.3.1]. Hence, by Lemma 4.4, $\#\text{Fix}(\psi) \leq \#\text{Fix}(\psi')$, $\psi' : T \rightarrow T$. For any finite abelian group T one clearly has $\#\text{Fix}(\psi') = R(\psi')$ by Theorem 11.3 (cf. [10, p. 7]). Finally, $R(\psi') \leq R(\psi)$ by (6). \square

Lemma 4.6. *Suppose, $|G/H| = N < \infty$. Then $R(\tau_g\phi') \leq NR(\phi)$. More precisely, the mentioned subpartition is not more than in N parts.*

Proof. Consider the following action of G on itself: $x \mapsto gx\phi(g^{-1})$. Then its orbits are exactly classes $\{x\}_\phi$. Moreover it maps classes (4) onto each other. Indeed,

$$\tilde{g}h(hg)\phi'(\tilde{h}^{-1})\phi(\tilde{g}^{-1}) = \hat{h}\tilde{g}(hg)\phi(\tilde{g}^{-1})\phi'(\hat{h}^{-1})$$

using normality of the H . This map is invertible ($\tilde{g} \leftrightarrow \tilde{g}^{-1}$), hence bijection. Moreover, \tilde{g} and $\hat{g}\tilde{h}$, for any $\hat{h} \in H$, act in the same way. Or in the other words, H is in the stabilizer of this permutation of classes (4). Hence, the cardinality of any orbit $\leq N$. \square

Hence, for any finite G/H the number of classes of the form (4) is finite: it is $\leq NR(\phi)$.

Lemma 4.7. *Suppose, H satisfies the following property: for any automorphism of H with finite Reidemeister number the characteristic functions of Reidemeister classes of ϕ are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of H . Then the characteristic functions of classes (4) are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of G .*

Proof. Let $\rho_1, \rho_2, \dots, \rho_k$ be the above irreducible representations of H , ρ its direct sum acting on V , and π the regular (finite dimensional) representation of G/H . Let $\rho_1^I, \dots, \rho_k^I, \rho^I$ be the corresponding induced representations of G . Let us remind that in this simple situation the representation ρ^I is defined as a representation of G in the space $l_2(G/H, V) \cong \bigoplus_1^{|G/H|} V$ defined by the formula

$$[\rho^I(g)f](x) = \rho(s(x)g(s(xg))^{-1})f(xg), \quad f \in l_2(G/H, V), \quad x \in G/H,$$

for some fixed section $s : G/H \rightarrow G$ of the canonical projection $G \rightarrow G/H$.

Let the characteristic function of D be represented under the form $\chi_D(h) = \langle \rho(h)\xi, \eta \rangle$. Let $\xi^I \in L^2(G/H, V)$ be defined by the formulas $\xi^I(\bar{e}) = \xi \in V$, $\xi^I(\bar{g}) = 0$ if $\bar{g} \neq \bar{e}$. Define similarly η^I . Then for $h \in i(H)$ we have

$$\rho^I(h)\xi^I(\bar{g}) = \rho(s(\bar{g})hs(\bar{g}h)^{-1})\xi(\bar{g}h) = \rho(hs(\bar{g})s(\bar{g})^{-1})\xi(\bar{g}) = \begin{cases} \rho(h)\xi, & \text{if } \bar{g} = \bar{e}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\langle \rho^I(h)\xi^I, \eta^I \rangle|_{i(H)}$ is the characteristic function of $i(D)$. Let $u, v \in L^2(G/H)$ be such vectors that $\langle \pi(\bar{g})u, v \rangle$ is the characteristic function of \bar{e} . Then

$$\langle (\rho^I \otimes \pi)(\xi^I \otimes u, \eta^I \otimes v) \rangle$$

is the characteristic function of $i(D)$. Other characteristic functions of classes (4) are shifts of this one. Hence, they are matrix elements of the representation $\rho^I \otimes \pi$. It is finite dimensional. Hence it can be decomposed in a finite direct sum of irreducible representations. \square

Corollary 4.8 (of previous two lemmata). *Under the assumptions of the previous lemma, the characteristic functions of Reidemeister classes of ϕ are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of G .*

5. THE CASE OF GROUPS OF TYPE I

Theorem 5.1. *Let G be a discrete group of type I. Then*

- [6, 3.1.4, 4.1.11] *The dual space \widehat{G} is a T_1 -topological space.*
- [38] *Any irreducible representation of G is finite-dimensional.*

Remark 5.2. In fact a discrete group G is of type I if and only if it has a normal, Abelian subgroup M of finite index. The dimension of any irreducible representation of G is at most $[G : M]$ [38].

Suppose $R = R(\phi) < \infty$, and let $F \subset L^\infty(G)$ be the R -dimensional space of all twisted-invariant functionals on $L^1(G)$. Let $K \subset L^1(G)$ be the intersection of kernels of functionals from F . Then K is a linear subspace of $L^1(G)$ of codimension R . For each $\rho \in \widehat{G}$ let us denote by K_ρ the image $\rho(K)$. This is a subspace of a (finite-dimensional) full matrix algebra. Let cd_ρ be its codimension.

Let us introduce the following set

$$\widehat{G}_F = \{\rho \in \widehat{G} \mid \text{cd}_\rho \neq 0\}.$$

Lemma 5.3. *One has $\text{cd}_\rho \neq 0$ if and only if ρ is a fixed point of $\widehat{\phi}$. In this case $\text{cd}_\rho = 1$.*

Proof. Suppose, $\text{cd}_\rho \neq 0$ and let us choose a functional φ_ρ on the (finite-dimensional full matrix) algebra $\rho(L^1(G))$ such that $K_\rho \subset \text{Ker } \varphi_\rho$. Then for the corresponding functional $\varphi_\rho^* = \varphi_\rho \circ \rho$ on $L^1(G)$ one has $K \subset \text{Ker } \varphi_\rho^*$. Hence, $\varphi_\rho^* \in F$ and is twisted-invariant, as well as φ_ρ . Then we argue as in the case of compact group (text after Lemma 3.1).

Conversely, if ρ is a fixed point of $\widehat{\phi}$, it gives rise to a (unique up to scaling) non-trivial twisted-invariant functional φ_ρ . Let $x = \rho(a)$ be any element in $\rho(L^1(G))$ such that $\varphi_\rho(x) \neq 0$. Then $x \notin K_\rho$, because $\varphi_\rho^*(a) = \varphi_\rho(x) \neq 0$, while φ_ρ^* is a twisted-invariant functional on $L^1(G)$. So, $\text{cd}_\rho \neq 0$.

The uniqueness (up to scaling) of the intertwining operator implies the uniqueness of the corresponding twisted-invariant functional. Hence, $\text{cd}_\rho = 1$. \square

Hence,

$$(8) \quad \widehat{G}_F = \text{Fix}(\widehat{\phi}).$$

From the property $\text{cd}_\rho = 1$ one obtains for this (unique up to scaling) functional φ_ρ :

$$(9) \quad \text{Ker } \varphi_\rho = K_\rho.$$

Lemma 5.4. *$R = \#\widehat{G}_F$, in particular, the set \widehat{G}_F is finite.*

Proof. First of all we remark that since G is finitely generated almost Abelian (cf. Remark 5.2) there is a normal Abelian subgroup H of finite index invariant under ϕ . Hence we can apply Lemma 4.8 to G, H, ϕ . So there is a finite collection of irreducible representations of G such that any twisted-invariant functional is a linear combination of matrix elements of them, i.e. linear combination of functionals on them. If one of them gives a non-trivial contribution, it has to be a twisted-invariant functional on the corresponding matrix algebra. Hence, by the argument above, these representations belong to \widehat{G}_F , and the appropriate functional is unique up to scaling. Hence, $R \leq S$.

Then we use T_1 -separation property. More precisely, suppose some points ρ_1, \dots, ρ_s belong to \widehat{G}_F . Let us choose some twisted-invariant functionals $\varphi_i = \varphi_{\rho_i}$ corresponding to these points as it was described (i.e. choose some scaling). Assume that $\|\varphi_i\| = 1$, $\varphi_i(x_i) = 1$, $x_i \in \rho_i(L^1(G))$. If we can find $a_i \in L^1(G)$ such that $\varphi_i(\rho_i(a_i)) = \varphi_i^*(a_i)$ is sufficiently large and $\rho_j(a_i)$, $i \neq j$, are sufficiently small (in fact it is sufficient $\rho_j(a_i)$ to be close enough to $K_j := K_{\rho_j}$), then $\varphi_j^*(a_i)$ are small for $i \neq j$, and φ_i^* are linear independent and hence, $s < R$. This would imply $S := \#\widehat{G}_F \leq R$ is finite. Hence, $R = S$.

So, the problem is reduced to the search of the above a_i . Let $d = \max_{i=1, \dots, s} \dim \rho_i$. For each i let $c_i := \|b_i\|$, where x_i is the unitary equivalence of ρ_i and $\widehat{\phi}\rho_i$ and $x_i = \rho_i(b_i)$.

Let $c := \max_{i=1, \dots, s} c_i$ and $\varepsilon := \frac{1}{2 \cdot s^2 \cdot d \cdot c}$.

One can find a positive element $a'_i \in L^1(G)$ such that $\|\rho_i(a'_i)\| \geq 1$ and $\|\rho_j(a'_i)\| < \varepsilon$ for $j \neq i$. Indeed, ρ_i can be separated from one point, and hence from the finite number of points: ρ_j , $j \neq i$. Hence, one can find an element v_i such that $\|\rho_i(v_i)\| > 1$, $\|\rho_j(v_i)\| < 1$ for $j \neq i$ [6, Lemma 3.3.3]. The same is true for the positive element $u_i = v_i^* v_i$. (Due to density we do not distinguish elements of L^1 and C^*). Now for a sufficiently large n the element $a'_i := (u_i)^n$ has the desired properties.

Let us take $a_i := a'_i b_i^*$. Then

$$(10) \quad \begin{aligned} \varphi_i^*(a_i) &= \text{Tr}(x_i \rho_i(a_i)) = \text{Tr}(x_i \rho_i(a'_i) \rho_i(b_i)^*) = \text{Tr}(x_i \rho_i(a'_i) x_i^*) \\ &= \text{Tr}(x_i \rho_i(a'_i) (x_i)^{-1}) = \text{Tr}(\rho_i(a'_i)) \geq \frac{1}{\dim \rho_i} \geq \frac{1}{d}. \end{aligned}$$

For $j \neq i$ one has

$$(11) \quad \|\varphi_j^*(a_i)\| = \|\varphi_j(\rho_j(a'_i b_i^*))\| \leq c_i \cdot \varepsilon.$$

Then the $s \times s$ matrix $\Phi = \varphi_j^*(a_i)$ can be decomposed into the sum of the diagonal matrix Δ and off-diagonal Σ . By (10) one has $\Delta \geq \frac{1}{d}$. By (11) one has

$$\|\Sigma\| \leq s^2 \cdot c_i \cdot \varepsilon \leq s^2 \cdot c \cdot \frac{1}{2 \cdot s^2 \cdot d \cdot c} = \frac{1}{2d}.$$

Hence, Φ is non-degenerate and we are done. \square

Lemma 5.4 together with (8) completes the proof of Theorem 1.4 for automorphisms.

We need the following additional observations for the proof of Theorem 1.4 for a general endomorphism (in which (3) is false for infinite-dimensional representations).

- Lemma 5.5.** (1) *If ϕ is an epimorphism, then \widehat{G} is $\widehat{\phi}_R$ -invariant.*
 (2) *For any ϕ the set $\text{Rep}(G) \setminus \widehat{G}$ is $\widehat{\phi}_R$ -invariant.*
 (3) *The dimension of the space of intertwining operators between $\rho \in \widehat{G}$ and $\widehat{\phi}_R(\rho)$ is equal to 1 if and only if $\rho \in \text{Fix}(\widehat{\phi})$. Otherwise it is 0.*

Proof. (1) and (2): This follows from the characterization of irreducible representation as that one for which the centralizer of $\rho(G)$ consists exactly of scalar operators.

(3) Let us decompose $\widehat{\phi}_R(\rho)$ into irreducible ones. Since $\dim H_\rho = \dim H_{\widehat{\phi}(\rho)}$ one has only 2 possibilities: ρ does not appear in $\widehat{\phi}(\rho)$ and the intertwining number is 0, otherwise $\widehat{\phi}_R(\rho)$ is equivalent to ρ . In this case $\rho \in \text{Fix}(\widehat{\phi})$. \square

The proof of Theorem 1.4 can be now repeated for the general endomorphism with the new definition of $\text{Fix}(\widehat{\phi})$. The item (3) supplies us with the necessary property.

6. EXAMPLES AND THEIR DISCUSSION

The natural candidate for the dual object to be used instead of \widehat{G} in the case when the different notions of the dual do not coincide (i.e. for groups more general than type I one groups) is the so-called quasi-dual \widehat{G} , i.e. the set of quasi-equivalence classes of factor-representations (see, e.g. [6]). This is a usual object when we need a sort of canonical decomposition for regular representation or group C^* -algebra. More precisely, one needs the support \widehat{G}_p of the Plancherel measure.

Unfortunately the following example shows that this is not the case.

Example 6.1. Let G be a non-elementary Gromov hyperbolic group. As it was shown by Fel'shtyn [18] with the help of geometrical methods, for any automorphism ϕ of G the Reidemeister number $R(\phi)$ is infinite. In particular this is true for free group in two generators F_2 . But the support $(\widehat{F_2})_p$ consists of one point (i.e. regular representation is factorial).

The next hope was to exclude from this dual object the II_1 -points assuming that they always give rise to an infinite number of twisted invariant functionals. But this is also wrong:

Example 6.2. (an idea of Fel'shtyn realized in [19]) Let $G = (Z \oplus Z) \rtimes_\theta Z$ be the semi-direct product by a hyperbolic action $\theta(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let ϕ be an automorphism of G

whose restriction to Z is $-id$ and restriction to $Z \oplus Z$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $R(\phi) = 4$, while the space \widehat{G}_p consists of a single II_1 -point once again (cf. [3, p. 94]).

These examples show that powerful methods of the decomposition theory do not work for more general classes of groups.

On the other hand Example 6.2 disproves the old conjecture of Fel'shtyn and Hill [12] who supposed that the Reidemeister numbers of an injective endomorphism for groups of exponential growth are always infinite. More precisely, this group is amenable and of exponential growth. Also, one has the following example

Example 6.3. In [32] D. Osin has constructed an infinite finitely generated group, which is not amenable, contains the free group in two generators, and has two (ordinary) conjugacy classes.

The role of this example will be clarified in Section 11.

In this relation the following example (to be discussed in Section 12) seems to be interesting.

Example 6.4. [11] For amenable and non-amenable Baumslag-Solitar groups Reidemeister numbers are always infinite.

For Example 6.2 recently we have found 4 fixed points of $\widehat{\phi}$ being finite dimensional irreducible representations. They give rise to 4 linear independent twisted invariant functionals. These functionals can also be obtained from the regular factorial representation. There also exist fixed points (at least one) that are infinite dimensional irreducible representations. The corresponding functionals are evidently linear dependent with the first 4. This example will be presented in detail in Section 7.

7. AN EXAMPLE FOR TYPE II_1 GROUPS

In the present section based on [17] it is shown that the twisted Burnside theorem (or Fel'shtyn-Hill conjecture) in the original formulation with \widehat{G} as a dual object is not true for non-type I groups. More precisely, an example of a group and its automorphism is constructed such that the number of fixed irreducible representations is greater than the Reidemeister number. But the number of fixed finite-dimensional representations (i.e. the number of invariant finite-dimensional characters) in this example coincides with the Reidemeister number. The directions for search of an appropriate formulation are indicated (another definition of the dual object). Some advances in this direction will be made in the next sections.

Let G be a semidirect product of \mathbb{Z}^2 and \mathbb{Z} by Anosov automorphism α with the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It consists by the definition of triples $((m, k), n)$ of integers with the following multiplication law:

$$((m, k), n) * ((m', k'), n') = ((m, k) + \alpha^n(m', k'), n + n').$$

In particular,

$$((m, k), 0) * ((0, 0), n) = ((m, k), n).$$

The inverse of $((m, k), n)$ is $(-\alpha^{-n}(m, k), -n)$. Indeed,

$$((m, k), n) * (-\alpha^{-n}(m, k), -n) = ((m, k) - \alpha^n \alpha^{-n}(m, k), n - n) = ((0, 0), 0).$$

The group G is a solvable (hence, amenable) group which is not of type I. Its regular representation is factorial. The irreducible representations can be obtained from ergodic orbits of the action of α on the torus \mathbb{T}^2 which is dual for the normal subgroup \mathbb{Z}^2 using appropriate cocycles [3, Sec. II.4], [2, Ch. 17, § 1].

Let us define an automorphism $\phi : G \rightarrow G$ by

$$\phi((m, k), n) = ((k, -m), -n),$$

i.e. the action on \mathbb{Z}^2 is defined by automorphism μ with the matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and on \mathbb{Z} by $n \mapsto -n$. The map ϕ is clearly a bijection,

$$\begin{aligned} \phi(((m, k), n) * ((m', k'), n')) &= \phi((m, k) + \alpha^n(m', k'), n + n') \\ &= \phi((k, -m) + \mu \alpha^n(m', k'), -n - n'), \end{aligned}$$

$$\begin{aligned} \phi((m, k), n) * \phi(((m', k'), n')) &= ((k, -m), -n) * ((k', -m'), -n') \\ &= ((k, -m) + \alpha^{-n}(k', -m'), -n - n'). \end{aligned}$$

Hence, to prove that ϕ is an automorphism, we need $\mu \alpha^n = \alpha^{-n} \mu$. This follows from $\mu \alpha = \alpha^{-1} \mu$. The further results in this direction can be found in [19].

Let us find the Reidemeister classes of ϕ , i.e. the classes of the equivalence relation $h \sim gh\phi(g^{-1})$. For $h = ((m, k), n)$ and $g = ((x, y), z)$ the right hand side of the relation takes the following form:

$$\begin{aligned} (12) \quad & ((x, y) + \alpha^z(m, k), z + n) * (-\mu \alpha^{-z}(x, y), z) \\ &= ((x, y) + \alpha^z(m, k) - \alpha^{z+n} \mu \alpha^{-z}(x, y), 2z + n) \\ &= (\alpha^z \{(m, k) + (\text{Id} - \alpha^n \mu) \alpha^{-z}(x, y)\}, 2z + n). \end{aligned}$$

Let us call *level* n (of G) the coset L_n of $\mathbb{Z}^2 \subset G$ of all elements of the form $((m, k), n)$. Let us first take an element $((m, k), 0)$ from the level 0 and describe elements from the same level, being equivalent to it. By (12) in this case $z = n = 0$ and they have the form

$$((m, k) + (\text{Id} - \mu)(x, y), 0) = ((m + (x - y), k + (x + y)), 0),$$

where $\text{Id} - \mu$ has the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Hence, the level 0 has intersections with 2 Reidemeister classes, say, B_1 and B_2 . The first intersection $B_1 \cap L_0$ is formed by elements $((u, v), 0)$ with even $u + v$, and $B_2 \cap L_0$ — with odd $u + v$. The elements from the other levels, which are equivalent to $((m, k), 0)$, have the form

$$(13) \quad (\alpha^z \{(m, k) + (\text{Id} - \mu) \alpha^{-z}(x, y)\}, 2z).$$

This means that B_1 and B_2 enter only even levels. Also, since α is an automorphism, we can rewrite (13) as

$$(14) \quad (\alpha^z \{(m, k) + (\text{Id} - \mu)(u, v)\}, 2z).$$

with arbitrary integers u and v . This means, that the intersections $B_i \cap L_{2z}$ have the form $\alpha^z(B_i)$, $i = 1, 2$. In particular, the other Reidemeister classes do not enter even levels.

In a similar way, the elements of L_1 equivalent to $((m, k), 1)$ have the form

$$((m, k) + (\text{Id} - \alpha\mu)(x, y), 1) = ((m + (2x - 2y), k + x), 1).$$

This means, that L_1 enters 2 classes: the intersection with B_3 is formed by elements with even first coordinate, and with B_4 — with the odd one. The elements from the other levels, which are equivalent to $((m, k), 1)$, have the form

$$(15) \quad (\alpha^z\{(m, k) + (\text{Id} - \alpha\mu)\alpha^{-z}(x, y)\}, 2z + 1).$$

Since α is an automorphism, we can rewrite (15) as

$$(16) \quad (\alpha^z\{(m, k) + (\text{Id} - \alpha\mu)(u, v)\}, 2z + 1).$$

with arbitrary integers u and v . This means, that the intersections $B_i \cap L_{2z+1}$ have the form $\alpha^z(B_i)$, $i = 3, 4$. In particular, these four classes cover G and $R(\phi) = 4$.

To obtain a complete description of B_i let us remark that directly from the definition of α

$$\alpha(x, y) = (2x + y, x + y)$$

one has the following properties:

- α maps the set of elements with an even (resp., odd) sum of coordinates onto the set of elements with an even (resp., odd) second coordinate,
- α maps the set of elements with an even (resp., odd) second coordinate onto the set of elements with an even (resp., odd) first coordinate,
- α maps the set of elements with an even (resp., odd) first coordinate onto the set of elements with an even (resp., odd) sum of coordinates.

Hence, the elements $((m, k), n)$ in intersections $B_i \cap L_j$ are of the form

i	1	2	3	4
$j \equiv 0 \pmod 6$	$m + k$ is even	$m + k$ is odd	\emptyset	\emptyset
$j \equiv 1 \pmod 6$	\emptyset	\emptyset	m is even	m is odd
$j \equiv 2 \pmod 6$	k is even	k is odd	\emptyset	\emptyset
$j \equiv 3 \pmod 6$	\emptyset	\emptyset	$m + k$ is even	$m + k$ is odd
$j \equiv 4 \pmod 6$	m is even	m is odd	\emptyset	\emptyset
$j \equiv 5 \pmod 6$	\emptyset	\emptyset	k is even	k is odd

Now we want to study the fixed points of the homeomorphism $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$, $[\rho] \mapsto [\rho\phi]$ of the unitary dual. Let us start from the finite-dimensional representations. As it was shown in [16] (see also Section 3 and after) in this case there exists exactly one twisted-invariant functional on $L^1(G)$, or ϕ -central L^∞ function, coming from a twisted-invariant functional on $\rho(L^1(G)) \cong M(\dim \rho, \mathbb{C})$ (up to scaling), namely

$$(17) \quad \varphi_\rho : g \mapsto \text{Tr}(S\rho(g)),$$

where S is the intertwining operator between ρ and $\rho\phi$.

First, we have to find μ -invariant finite α -orbits on \mathbb{T}^2 . One can notice that

$$\det(A^n - M) = \det A^n + 1 = 2$$

for any n . Hence, the mentioned orbits are formed by points with coordinates 0 and $1/2$. We have 2 orbits: one of them consists of 1 point $(0, 0)$ and gives rise to 1-dimensional trivial representation ρ_1 , and the other consists of $A_1 = (0, 1/2)$, $A_2 = (1/2, 0)$ and

$A_3 = (1/2, 1/2)$ and gives rise to a 3-dimensional (irreducible) representation ρ_2 . Also, one has the following 1-dimensional representation π :

$$\pi((m, k), 2n) = 1, \quad \pi((m, k), 2n + 1) = -1.$$

So, we have 4 representations

$$\rho_1, \quad \rho_2, \quad \pi, \quad \rho_2 \otimes \pi.$$

We claim that they give rise via (17) to 4 linear independent twisted-invariant functionals. In particular, there is no more finite-dimensional fixed points of $\widehat{\phi}$.

Clearly,

$$(18) \quad \varphi_{\rho_1} \equiv 1, \quad \varphi_{\pi} = \begin{cases} 1, & \text{on } \cup_n L_{2n} = B_1 \cup B_2, \\ -1, & \text{on } \cup_n L_{2n+1} = B_3 \cup B_4, \end{cases} \quad \varphi_{\rho_2 \otimes \pi} = \varphi_{\rho_2} \cdot \varphi_{\pi}.$$

Let us find φ_{ρ_2} . In the space $L^2(A_1, A_2, A_3)$ we take the base $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of characteristic functions in these points. One has

$$\begin{aligned} \alpha(\varepsilon_1) &= \varepsilon_3, & \alpha(\varepsilon_3) &= \varepsilon_2, & \alpha(\varepsilon_2) &= \varepsilon_1, \\ \mu(\varepsilon_1) &= \varepsilon_2, & \mu(\varepsilon_2) &= \varepsilon_1, & \mu(\varepsilon_3) &= \varepsilon_3. \end{aligned}$$

The representation (see [2, Ch. 17, § 1]) is defined by:

$$\begin{aligned} \rho_2(m, k, 0)(\varepsilon_i) &= \chi_{A_i}(m, k) \cdot \varepsilon_i, & \rho_2(0, 0, n)(\varepsilon_i) &= \alpha^{-n}(\varepsilon_i) = \varepsilon_{i+n \pmod 3}, \\ \rho_2(m, k, 0)(\varepsilon_1) &= e^{2\pi i(0 \cdot m + 1/2 \cdot k)} \cdot \varepsilon_1 = e^{\pi i k} \varepsilon_1, \\ \rho_2(m, k, 0)(\varepsilon_2) &= e^{2\pi i(1/2 \cdot m + 0 \cdot k)} \cdot \varepsilon_2 = e^{\pi i m} \varepsilon_2, \\ \rho_2(m, k, 0)(\varepsilon_3) &= e^{2\pi i(1/2 \cdot m + 1/2 \cdot k)} \cdot \varepsilon_3 = e^{\pi i(m+k)} \varepsilon_3. \end{aligned}$$

The representation $\widehat{\phi}\rho_2$ is defined by

$$\begin{aligned} \widehat{\phi}\rho_2(0, 0, n)(\varepsilon_i) &= \rho_2(0, 0, -n)(\varepsilon_i) = \varepsilon_{i-n \pmod 3}, \\ \widehat{\phi}\rho_2(m, k, 0)(\varepsilon_1) &= \rho_2(k, -m, 0)(\varepsilon_1) = e^{-\pi i m} \varepsilon_1 = e^{\pi i m} \varepsilon_1, \\ \widehat{\phi}\rho_2(m, k, 0)(\varepsilon_2) &= \rho_2(k, -m, 0)(\varepsilon_2) = e^{\pi i k} \varepsilon_2, \\ \widehat{\phi}\rho_2(m, k, 0)(\varepsilon_3) &= \rho_2(k, -m, 0)(\varepsilon_3) = e^{\pi i(k-m)} \varepsilon_3 = e^{\pi i(m+k)} \varepsilon_3. \end{aligned}$$

The intertwining operator is induced by ϕ and has the matrix $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence,

$$\begin{aligned} \varphi_{\rho_2}(m, k, n) &= \text{Tr}(S\rho_2(m, k, 0)\rho_2(0, 0, n)) \\ &= \text{Tr} \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\pi i k} & 0 & 0 \\ 0 & e^{\pi i m} & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \right] \\ &= \text{Tr} \left[\begin{pmatrix} 0 & e^{\pi i k} & 0 \\ e^{\pi i k} & 0 & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \right]. \end{aligned}$$

For $n \equiv 0 \pmod{3}$

$$\varphi_{\rho_2}(m, k, n) = \text{Tr} \left[\begin{pmatrix} 0 & e^{\pi im} & 0 \\ e^{\pi ik} & 0 & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix} \right] = e^{\pi i(k+m)} = \begin{cases} 1, & \text{if } m+k \text{ is even} \\ -1, & \text{if } m+k \text{ is odd} \end{cases},$$

for $n \equiv 1 \pmod{3}$

$$\begin{aligned} \varphi_{\rho_2}(m, k, n) &= \text{Tr} \left[\begin{pmatrix} 0 & e^{\pi im} & 0 \\ e^{\pi ik} & 0 & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] \\ &= \text{Tr} \begin{pmatrix} e^{\pi im} & 0 & 0 \\ 0 & 0 & e^{\pi ik} \\ 0 & e^{\pi i(k+m)} & 0 \end{pmatrix} = e^{\pi im} = \begin{cases} 1, & \text{if } m \text{ is even} \\ -1, & \text{if } m \text{ is odd} \end{cases}, \end{aligned}$$

for $n \equiv 2 \pmod{3}$

$$\begin{aligned} \varphi_{\rho_2}(m, k, n) &= \text{Tr} \left[\begin{pmatrix} 0 & e^{\pi im} & 0 \\ e^{\pi ik} & 0 & 0 \\ 0 & 0 & e^{\pi i(k+m)} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right] = \\ &= \text{Tr} \begin{pmatrix} 0 & 0 & e^{\pi im} \\ 0 & e^{\pi ik} & 0 \\ e^{\pi i(k+m)} & 0 & 0 \end{pmatrix} = e^{\pi ik} = \begin{cases} 1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

φ_{ρ_2} is 3-periodical in n , while the characteristic functions of B_i are 6-periodical. For $j = 0, \dots, 5$ one has

$$\begin{aligned} \varphi_{\rho_2}|_{B_1 \cap L_0} &\equiv 1, & \varphi_{\rho_2}|_{B_2 \cap L_0} &\equiv -1, & \varphi_{\rho_2}|_{B_3 \cap L_1} &\equiv 1, & \varphi_{\rho_2}|_{B_4 \cap L_1} &\equiv -1, \\ \varphi_{\rho_2}|_{B_1 \cap L_2} &\equiv 1, & \varphi_{\rho_2}|_{B_2 \cap L_2} &\equiv -1, & \varphi_{\rho_2}|_{B_3 \cap L_3} &\equiv 1, & \varphi_{\rho_2}|_{B_4 \cap L_3} &\equiv -1, \\ \varphi_{\rho_2}|_{B_1 \cap L_4} &\equiv 1, & \varphi_{\rho_2}|_{B_2 \cap L_4} &\equiv -1, & \varphi_{\rho_2}|_{B_3 \cap L_5} &\equiv 1, & \varphi_{\rho_2}|_{B_4 \cap L_5} &\equiv -1, \end{aligned}$$

so $\varphi_{\rho_2}|_{B_1 \cup B_3} \equiv 1$, $\varphi_{\rho_2}|_{B_2 \cup B_4} \equiv -1$. The determinant of the values of the functions φ_{ρ_1} , φ_{π} , φ_{ρ_2} , $\varphi_{\rho_2 \otimes \pi}$ on B_1, B_2, B_3, B_4 is

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = -8 \neq 0.$$

Hence, they are linearly independent.

Nevertheless, there are infinite-dimensional irreducible $\widehat{\phi}$ -invariant representations. E.g. we have a representation ρ of G on $L^2(\mathbb{T}^2)$ with the respect to the Lebesgue measure, with $\rho(m, k, 0)$ be the multiplier by characters in the appropriate points and $\rho(0, 0, 1)$ is α , in the same manner as for ρ_2 .

This disproves the conjecture of Fel'shtyn and Hill [12], who supposed that the Reide-meister number equals to the number of fixed points of $\widehat{\phi}$ on \widehat{G} .

This representation is not traceable, but one can nevertheless try to calculate (17). Let us choose an orthonormal base of $L^2(\mathbb{T}^2)$ formed by $\varepsilon_{st}(x, y) = e^{2\pi i(sx+ty)}$, $x, y \in [0, 1]$. The intertwining operator S is generated by μ . Then

$$\text{Tr}(S\rho(m, k, n)) = \sum_{s,t} \int_0^1 \int_0^1 (\rho(m, k, 0)\rho(0, 0, n)\varepsilon_{s,t})(\mu(x, y))\overline{\varepsilon_{s,t}}(x, y) dx dy$$

$$\begin{aligned}
 &= \sum_{s,t} \int_0^1 \int_0^1 e^{2\pi i \langle (m,k), \mu(x,y) \rangle} (\rho(0,0,n) \varepsilon_{s,t})(\mu(x,y)) \varepsilon_{-s,-t}(x,y) dx dy \\
 &= \sum_{s,t} \int_0^1 \int_0^1 e^{2\pi i \langle (m,k), \mu(x,y) \rangle} \varepsilon_{s,t}(\alpha^n \mu(x,y)) \varepsilon_{-s,-t}(x,y) dx dy \\
 &= \sum_{s,t} \int_0^1 \int_0^1 e^{2\pi i \langle (m,k) + (\alpha^n - \mu)(s,t), \mu(x,y) \rangle} dx dy = \sum_{(m,k) = (\mu - \alpha^n)(s,t)} 1.
 \end{aligned}$$

For $n = 0$ this equals 1 for $m + k$ even and 0 for $m + k$ odd. Hence, $\varphi_\rho|_{B_1 \cap L_0} = 1$, $\varphi_\rho|_{B_2 \cap L_0} = 0$. For $n = 1$ the equality takes the form $(m, k) = (-2s - 2t, -t)$. Hence, $\varphi_\rho|_{B_3 \cap L_1} = 1$, $\varphi_\rho|_{B_4 \cap L_1} = 0$. If $n = 2r$, the equality takes form

$$\begin{aligned}
 (m, k) &= (\mu - \alpha^{2r})(s, t), \quad (m, k) = (\alpha^r \mu \alpha^r - \alpha^{2r})(s, t), \\
 \alpha^{-r}(m, k) &= (\mu - \alpha^0) \alpha^r(s, t).
 \end{aligned}$$

Since α is an automorphism of $\mathbb{Z} \oplus \mathbb{Z}$ and by the description of B_i via the action of α , we obtain that $\varphi_\rho|_{B_1} = 1$, $\varphi_\rho|_{B_2} = 0$. Similarly, for n odd. So, φ_ρ is well defined and

$$\varphi_\rho = \chi_{B_1} + \chi_{B_3} = \frac{1}{2}(\varphi_{\rho_1} + \varphi_{\rho_2}).$$

Of course, there are also $\widehat{\phi}$ -invariant traceable factor representations of this group G , e.g. the regular representation. Since its kernel is trivial, evidently all twisted-invariant functionals can be pushed back from it.

This example shows that the conjecture of [12] for general groups can survive only after eliminating badly separated points in \widehat{G} .

8. ALGEBRAS OF OPERATOR FIELDS

Let us first remind the theory of operator fields following [9]. Let T be a topological space and for each point $t \in T$ a C^* -algebra (or more general — involutive Banach algebra) A_t is fixed.

Definition 8.1. A continuity structure for T and the $\{A_t\}$ is a linear space F of operator fields on T , with values in the $\{A_t\}$, (i.e. maps sending $t \in T$ to an element of A_t), satisfying

- (1) if $x \in F$, the real-valued function $t \mapsto \|x(t)\|$ is continuous on T ;
- (2) for each $t \in T$, $\{x(t) \mid x \in F\}$ is dense in A_t ;
- (3) F is closed under pointwise multiplication and involution.

Definition 8.2. An operator field x is *continuous* with respect to F at t_0 , if for each $\varepsilon > 0$, there is an element $y \in F$ and a neighborhood U of t_0 such that $\|x(t) - y(t)\| < \varepsilon$ for all $t \in U$. The field x is *continuous on T* if it is continuous at all points of T .

Definition 8.3. A *full algebra of operator fields* is a family A of operator fields on T satisfying:

- (1) A is a $*$ -algebra, i.e., it is closed under all the pointwise algebraic operations;
- (2) for each $x \in A$, the function $t \mapsto \|x(t)\|$ is continuous on T and vanishes at infinity;
- (3) for each t , $\{x(t) \mid x \in A\}$ is dense in A_t ;

(4) A is complete in the norm $\|x\| = \sup_t \|x(t)\|$.

A full algebra of operator fields is evidently a continuity structure. If F is any continuity structure, let us define $C_0(F)$ to be the family of all operator fields x which are continuous on T with respect to F , and for which $t \mapsto \|x\|$ vanishes at infinity. One can prove that $C_0(F)$ is a full algebra of operator fields — indeed, a maximal one.

Lemma 8.4. *For any full algebra A of operator fields on T , the following three conditions are equivalent:*

- (1) A is a maximal full algebra of operator fields;
- (2) $A = C_0(F)$ for some continuity structure F ;
- (3) $A = C_0(A)$.

Such a maximal full algebra A of operator fields may sometimes be called a *continuous direct sum* of the $\{A_t\}$. It is clearly *separating*, in the sense that, if $s, t \in T$, $s \neq t$, $\alpha \in A_s$, $\beta \in A_t$, there is an $x \in A$ such that $x(s) = \alpha$, $x(t) = \beta$. We will denote by \widehat{a} the section corresponding to an element $a \in A$. We will study the unital case and T will be compact without the property of vanishing at infinity. The corresponding algebra will be denoted by $\Gamma(\mathcal{A}) \cong A$. Moreover, we suppose that T is Hausdorff, hence, normal.

9. FUNCTIONALS AND MEASURES

Definition 9.1. A (*bounded additive*) *algebra of operator field measure* (BA AOFM) related to a maximal full algebra of operator fields $A = \Gamma(\mathcal{A})$ is a set function $\mu : S \rightarrow \Gamma(\mathcal{A})^* = A^*$, where $S \in \Sigma$, some algebra of sets,

- being additive: $\mu(\sqcup S_i)(a) = \sum_i \mu(S_i)(a)$
- $\mu(S)(a) = 0$ if $\text{supp } \widehat{a} \cap S = \emptyset$.
- bounded: sup over partitions $\{S_i\}$ of T of $\sum_i \|\mu(S_i)\|$ is finite and denoted by $\|\mu\|$

Definition 9.2. It is **-weak regular* (RBA AOFM) if for each $E \in \Sigma$, $a \in A$, and $\varepsilon > 0$ there is a set $F \in \Sigma$ whose closure is contained in E and a set $G \in \Sigma$ whose interior contains E such that $\|\mu(C)a\| < \varepsilon$ for every $C \in \Sigma$ with $C \subset G \setminus F$.

We will use as Σ all sets and the algebra generated by closed sets in T .

Definition 9.3. Let AOFM λ be defined on a field Σ of sets in T and $\lambda(\emptyset) = 0$. A set $E \in \Sigma$ is called λ -set if for any $M \in \Sigma$

$$\lambda(M) = \lambda(M \cap E) + \lambda(M \cap (T \setminus E)).$$

Lemma 9.4. *Let λ be an AOFM defined on a field Σ of sets in T with $\lambda(\emptyset) = 0$. The family of λ -sets is a subfield of Σ on which λ is additive. Furthermore, if E is the union of a finite sequence $\{E_n\}$ of disjoint λ -sets and $M \in \Sigma$, then $\lambda(M \cap E) = \sum_n \lambda(M \cap E_n)$.*

Proof. It is clear, that the void set, the whole space, and the complement of any λ -set are λ -sets. Now let X and Y be λ -sets, and $M \in \Sigma$. Then, since X is a λ -set,

$$(19) \quad \lambda(M \cap Y) = \lambda(M \cap Y \cap X) + \lambda(M \cap Y \cap (T \setminus X)),$$

and since Y is a λ -set,

$$(20) \quad \lambda(M) = \lambda(M \cap Y) + \lambda(M \cap (T \setminus Y)),$$

$\lambda(M \cap (T \setminus (X \cap Y))) = \lambda(M \cap (T \setminus (X \cap Y)) \cap Y) + \lambda(M \cap (T \setminus (X \cap Y)) \cap (T \setminus Y))$,
hence,

$$(21) \quad \lambda(M \cap (T \setminus (X \cap Y))) = \lambda(M \cap (T \setminus X) \cap Y) + \lambda(M \cap (T \setminus Y)).$$

From (19) and (20) it follows that

$$\lambda(M) = \lambda(M \cap Y \cap X) + \lambda(M \cap Y \cap (T \setminus X)) + \lambda(M \cap (T \setminus Y)),$$

and from (21) that

$$\lambda(M) = \lambda(M \cap Y \cap X) + \lambda(M \cap (T \setminus (X \cap Y))).$$

Thus $X \cap Y$ is a λ -set. Since $\cup X_n = T \setminus \cap (T \setminus X_n)$, we conclude that the λ -sets form a field. Now if E_1 , and E_2 are disjoint λ -sets, it follows, by replacing M by $M \cap (E_1 \cup E_2)$ in Definition 9.3, that

$$\lambda(M \cap (E_1 \cup E_2)) = \lambda(M \cap E_1) + \lambda(M \cap E_2).$$

The final conclusion of the lemma follows from this by induction. \square

As it is well known, any functional τ on a C^* -algebra B can be represented as a linear combination of 4 positive functionals in the following canonical way. First let us represent τ under the form $\tau = \tau_1 + i\tau_2$, where self-adjoint functionals τ_1 and τ_2 are defined by the formulas

$$(22) \quad \tau_1(a) = \frac{\tau(a) + \overline{\tau(a^*)}}{2}, \quad \tau_2(a) = \frac{\tau(a) - \overline{\tau(a^*)}}{2i}.$$

By the lemma about Jordan decomposition, any self-adjoint functional α can be represented in a unique way as a difference of two positive functionals $\alpha = \alpha_+ - \alpha_-$ under requirement

$$(23) \quad \|\alpha\| = \|\alpha_+\| + \|\alpha_-\|$$

(see [29, §3.3], [33, Theorem 3.2.5]). Let us decompose an AOFM in the related way. Since the decomposition is unique, the second property of AOFM will held. If we start from BA AOFM, then the additivity of summands will follow from the uniqueness of the decomposition, and the boundedness (with double constant) will follow from (22) and property (23). The same argument shows that the summands are $*$ -weak regular, if the initial AOFM was $*$ -weak regular. So, the AOFM's in the decomposition are *positive* AOFM, i.e., such that

$$\mu(E)[a^*a] \geq 0$$

for any $E \in \Sigma$. Such a set function is non-decreasing.

Lemma 9.5. *The sets F and G in the Definition 9.2 can be chosen in such a way that $\|\mu(C)(fa)\| < \varepsilon$ for any continuous function $f : T \rightarrow [0, 1]$.*

Proof. Let us take the decompositions $\mu = \sum_{i=1}^4 x_i \mu_i$, $a = \sum_{j=1}^4 y_j a_j$, where μ_i and a_j are positive, x_i, y_j are complex numbers of norm ≤ 1 . Let us choose the sets F and G , as in Definition 9.2, for $\varepsilon/16$ and for all pairs μ_i, a_j simultaneously. Then

$$0 \leq \mu_i(C)(f \cdot a_j) = \mu_i(C)((a_j)^{1/2} f (a_j)^{1/2}) \leq \mu_i(C)(a_j) \leq \frac{\varepsilon}{16},$$

and

$$\|\mu_i(C)(f \cdot a)\| \leq \sum_{i,j=1}^4 |x_i y_j| \cdot |\mu_i(C)(a_j)| \leq 16 \cdot \frac{\varepsilon}{16} = \varepsilon.$$

□

Theorem 9.6. *Let a unital separable C^* -algebra A be isomorphic to a full algebra of operator fields $\Gamma(\mathcal{A})$ over a Hausdorff space X . Then functionals on $A \cong \Gamma(\mathcal{A})$ can be identified with RBA AOFM of \mathcal{A} .*

Proof. Let us remark that these suppositions imply the following: T is a separable Hausdorff compact and the unit ball of the dual space of A is a metrizable compact in $*$ -weak topology.

Obviously RBA AOFM form a linear normed space with respect to $\|\cdot\|$.

First, we want to prove that the natural linear map $\mu \mapsto \mu(T)$ is an isometry of RBA AOFM into A^* . Since $\|\mu(T)\| \leq \|\mu\|$, it is of norm ≤ 1 . Let now take an arbitrary small $\varepsilon > 0$. Let E_1, \dots, E_n be a partition of T such that

$$\sum_{i=1}^n \|\mu(E_i)\| \geq \|\mu\| - \varepsilon.$$

Let $a_i \in A$ be such elements of norm 1, that $\mu(E_i)(a_i) \geq \|\mu(E_i)\| - \varepsilon/n$.

By $*$ -weak regularity of μ and normality of T one can take closed sets C_i , disjoint open sets G_i , and continuous functions $f_i : T \rightarrow [0, 1]$ such that $C_i \subset E_i$, $\|\mu(E_i \setminus C_i)(a_j)\| \leq \varepsilon/n^2$, $C_i \subset G_i$, $\|\mu(G_i \setminus C_i)(a_j)\| \leq \varepsilon/n^2$, (and estimations hold for multiplication by positive functions as well, as in Lemma 9.5) $f_i(s) = 0$ if $s \notin G_i$, $f_i(s) = 1$ if $s \in C_i$, $i, j = 1, \dots, n$.

Consider the element $a := \sum_i f_i a_i \in \Gamma(\mathcal{A}) = A$. Then $\|a\| \leq 1$ and

$$\begin{aligned} |\mu(S)(a) - \|\mu\|| &\leq \sum_{i=1}^n |\mu(E_i)(a) - \mu(E_i)| + \varepsilon \\ &\leq \sum_{i=1}^n |\mu(E_i \setminus C_i)(a) + \mu(C_i)(a) - \mu(E_i)(a_i)| + 2\varepsilon \\ &= \sum_{i=1}^n \left| \sum_{j=1}^n \mu(E_i \setminus C_i)(f_j a_j) + \mu(C_i)(a_i) - \mu(E_i)(a_i) \right| + 2\varepsilon \\ &\leq \sum_{i,j=1}^n |\mu(E_i \setminus C_i)(f_j a_j)| + \sum_{i=1}^n |\mu(E_i \setminus C_i)(a_i)| + 2\varepsilon \leq n^2 \frac{\varepsilon}{n^2} + n \frac{\varepsilon}{n^2} + 2\varepsilon \leq 4\varepsilon. \end{aligned}$$

Since ε is arbitrary small, $\|\mu\| = \|\mu(S)\|$.

It remains to represent the general functional φ by RBA AOFM. This functional on $\Gamma(\mathcal{A})$ can be extended by Hahn-Banach theorem to a continuous functional ψ on $B(\mathcal{A}) = \prod_{t \in T} A_t$ (the C^* -algebra of possibly discontinuous cross-sections of \mathcal{A} with sup-norm). This functional can be decomposed $\psi = \sum_{i=1}^4 \alpha_i \psi_i$, where ψ_i are positive functionals, $\alpha_i \in \mathbb{C}$, $|\alpha_i| = 1$, $\|\psi_i\| \leq \|\psi\|$. Let us define

$$\lambda(E)(a) := \psi(\chi_E a), \quad \lambda_i(a) := \psi_i(\chi_E a), \quad i = 1, \dots, 4.$$

where $a \in \Gamma(\mathcal{A})$, and χ_E is the characteristic function of E . Evidently, $\lambda(T)(a) = \psi(a)$ and λ is a BA AOFM. Indeed, the first two properties of Definition 9.1 are evident. The third one can be verified for each λ_i , $i = 1, \dots, 4$:

$$\sum_{j=1}^N |\lambda_i(E_j)| = \sum_{j=1}^N \lambda_i(E_j)(\mathbf{1}) = \sum_{j=1}^N \psi_i(\chi_{E_j} \mathbf{1}) = \psi_i(\mathbf{1}) \leq \|\psi_i\|,$$

hence,

$$\sum_{j=1}^N |\lambda(E_j)| \leq \sum_{i=1}^4 \sum_{j=1}^N |\lambda_i(E_j)| \leq \sum_{i=1}^4 \|\psi_i\| \leq 4 \cdot \|\psi\|.$$

Now we want to find an RBA AOFM μ such that $\mu(T)(a) = \lambda(T)(a)$. Without loss of generality it is sufficient to do this for a positive $\lambda = \lambda_i$.

Let F represent the general closed subset, G the general open subset, E the general subset of S . Define μ_1 and μ_2 by putting

$$\mu_1(F)(a^*a) = \inf_{G \supset F} \lambda(G)(a^*a), \quad \mu_2(E)(a^*a) = \sup_{F \subset E} \mu_1(F)(a^*a),$$

and then by taking linear extension. More precisely, due to separability one can take a cofinal sequence $\{G_i\}$. The unit ball in the dual space is weakly compact and one can take a weakly convergent sequence $\lambda(G_{i_k})$. Its limit ψ is a positive functional on A enjoying inf-property on positive elements. In particular, it is independent of the choice of $\{G_i\}$ and $\{G_{i_k}\}$. In a similar way for sup.

These set functions are non-negative and non-decreasing. Let G_1 be open and F_1 be closed. Then if $G \supset F_1 \setminus G_1$, it follows that $G_1 \cap G \supset F_1$ and $\lambda(G_1) \leq \lambda(G_1) + \lambda(G)$ so that $\mu_1(F_1) \leq \lambda(G_1) + \lambda(G)$. Since G is an arbitrary open set containing $F_1 \setminus G_1$ we have

$$\mu_1(F_1) \leq \lambda(G_1) + \mu_1(F_1 \setminus G_1).$$

If F is a closed set it follows from this inequality, by allowing G_1 to range over all open sets containing $F \cap F_1$, that

$$\mu_1(F_1) \leq \mu_1(F \cap F_1) + \mu_2(F_1 \setminus F).$$

If E is an arbitrary subset of T and F_1 ranges over the closed subsets of E , then it follows from the preceding inequality that

$$(24) \quad \mu_2(E) \leq \mu_2(E \cap F) + \mu_2(E \setminus F).$$

It will next be shown that for an arbitrary set E in T and arbitrary closed set F in T we have

$$(25) \quad \mu_2(E) \geq \mu_2(E \cap F) + \mu_2(E \setminus F).$$

To see this let F_1 and F_2 be disjoint closed sets. Since T is normal there are disjoint neighborhoods G_1 and G_2 of F_1 and F_2 respectively. If G is an arbitrary neighborhood of $F_1 \cup F_2$ then $\lambda(G) \geq \lambda(G \cap G_1) + \lambda(G \cap G_2)$ so that

$$\mu_1(F_1 \cap F_2) \geq \mu_1(F_1) + \mu_2(F_2).$$

We now let E and F be arbitrary sets in T with F closed and let F_1 range over closed subsets of $E \cap F$ while F_2 ranges over the closed subsets of $E \setminus F$. The preceding inequality

then proves (25). From (24) and (25) we have

$$(26) \quad \mu_2(E) = \mu_2(E \cap F) + \mu_2(E \cap (T \setminus F))$$

for any E in T and F closed. The function μ_2 is defined on the field of all subsets of T and it follows from (26) that every closed set F is a μ_2 -set. If μ is the restriction of μ_2 on the field determined by the closed sets, it follows from Lemma 9.4 that μ is additive on this field. It is clear from the definition of μ_1 and μ_2 that $\mu_1(F) = \mu_2(F) = \mu(F)$ if F is closed and thus $\mu(E) = \sup_{F \subset E} \mu(F)$. This shows that μ is $*$ -weak regular and since $\|\mu(T)\| < \infty$, we have μ is RBA AOFM.

Finally, by the definition, $\mu(S)(a) = \lambda(S)(a) = \psi(a) = \varphi(a)$ for $a \in \Gamma(\mathcal{A})$. \square

10. TWISTED-INVARIANT AOFM

The most part of the argument is valid for various representations of algebras by operator fields, but now we will pass to the case of group C^* -algebra of a discrete group and concentrate ourselves on the following important case due to Dauns and Hofmann [4, Corollary 8.13] (see also [4, Corollary 8.14]):

Theorem 10.1. *Consider a C^* -algebra A (with or without an identity) and the set B of all its primitive ideals in the hull-kernel topology q .*

(1) *The complete regularization $\varphi : (B, q) \rightarrow (M, t)$ can be taken to consist of closed ideals m of A satisfying $m = \bigcap \varphi^{-1}(m)$. The topology t of M contains the weak-star topology t^* induced on M by A .*

Denote by \mathcal{K} the family of all t -compact subsets of M of the form $\{m \in M \mid \|a+m\| \geq \varepsilon\}$, $a \in A$, $0 < \varepsilon$. Let $\pi'' : E' \rightarrow (M, t)$ be the uniform field of C^ -algebras obtained by first forming the canonical field from A , M and then enlarging the weak-star topology t^* of M up to t . For each $a \in A$, \hat{a} is the map $\hat{a} : M \rightarrow E'$, $\hat{a}(m) = a+m \in A/m$. The C^* -algebra of all sections vanishing at infinity with respect to the class \mathcal{K} is denoted by $\Gamma_0(\pi'')$.*

(2) *Then the map $A \rightarrow \Gamma_0(\pi'')$, $a \mapsto \hat{a}$ is an isometric star-isomorphism. In particular, if M is t -compact, then $M \in \mathcal{K}$ and $A \cong \Gamma(\pi'')$.*

Definition 10.2. Ideals m from the previous theorem, i.e., points of M , are called *Glimm ideals*.

We will use the notation T instead of M for Glimm spectrum to agree with the previous section.

Now we consider a countable discrete group G and its automorphism ϕ . Also, $A = C^*(G)$. One has the twisted action of G on A :

$$g[a] = L_g a L_{\phi(g^{-1})},$$

where L_g is the left shift, and respectively on functionals.

Definition 10.3. The dimension $R_*(\phi)$ of the space of twisted invariant functionals on $C^*(G)$ is called *generalized Reidemeister number* of ϕ .

Definition 10.4. A (Glimm) ideal I is a *generalized fixed point* of $\hat{\phi}$ if the linear span of elements $b - g[b]$ is not dense in $A_I = A/I$, where $g[\cdot]$ is the twisted action, i.e. its closure K_I does not coincide with A_I .

If we have only a finite number of such fixed points, then the twisted invariant RBA AOFM's are concentrated in these points. Indeed, let us describe the action of G on RBA AOFM's in more detail. The action of G on measures is defined by the identification of the measures and functionals on A .

Lemma 10.5. *If μ corresponds to a twisted invariant functional, then for any Borel $E \subset T$ the functional $\mu(E)$ is twisted invariant.*

Proof. This is an immediate consequence of *-weak regularity. Let $a \in A$, $g \in G$ and $\varepsilon > 0$ be an arbitrary small number, and U and F as in Lemma 9.5, for a and $g[a]$ simultaneously. Let us choose a continuous function $f : T \rightarrow [0, 1]$, with $f|_F = 1$ and $f|_{T \setminus U} = 0$. Then

$$\begin{aligned} |\mu(E)(a - g[a])| &= |\mu(E \setminus F)(a) + \mu(F)(a) - \mu(E \setminus F)(g[a]) - \mu(F)(g[a])| \\ &\leq |\mu(F)(a) - \mu(F)(g[a])| + 2\varepsilon = |\mu(F)(fa) - \mu(F)(g[fa])| + 2\varepsilon \\ &= |\mu(U)(fa) - \mu(U \setminus F)(fa) - \mu(U)(g[fa]) + \mu(U \setminus F)(g[fa])| + 2\varepsilon \\ &\leq |\mu(U)(fa) - \mu(U)(g[fa])| + 4\varepsilon = |\mu(T)(fa) - \mu(T)(g[fa])| + 4\varepsilon = 4\varepsilon. \end{aligned}$$

□

Lemma 10.6. *For any twisted-invariant functional φ on $C^*(G)$ the corresponding measure μ is concentrated in the set GFP of generalized fixed points.*

Proof. Let $\|\mu\| = 1$. Suppose opposite: there exists an element $a \in A$, $\|a\| = 1$, vanishing on generalized fixed points, such that $\varphi(a) \neq 0$. Let $\varepsilon := |\varphi(a)| > 0$. In each point $t \notin GFP$ we can find elements $b_t^i \in A$, $g_t^i \in G$, $i = 1, \dots, k(t)$, such that

$$\|a(t) - \sum_{i=1}^{k(t)} (g_t^i[b_t^i](t) - b_t^i(t))\| < \varepsilon/4.$$

Then there exists a neighborhood U_t such that for $s \in U_t$ one has

$$\|a(s) - \sum_{i=1}^{k(t)} (g_t^i[b_t^i](s) - b_t^i(s))\| < \varepsilon/2.$$

Let us choose a finite subcovering $\{U_{t_j}\}$, $j = 1, \dots, n$, of $\{U_t\}$ and a Borel partition E_1, \dots, E_n subordinated to this subcovering. Then

$$\varphi(a) = \sum_{j=1}^n \mu(E_j)(a) = \sum_{j=1}^n \mu(E_j) \left(a - \sum_{i=1}^{k(t_j)} (g_{t_j}^i[b_{t_j}^i] - b_{t_j}^i) \right) + \sum_{j=1}^n \sum_{i=1}^{k(t_j)} \mu(E_j)(g_{t_j}^i[b_{t_j}^i] - b_{t_j}^i).$$

By Lemma 10.5 each summand in the second term is zero. The absolute value of the first term is less than $\sum_j \|\mu(E_j)\| \varepsilon/2 \leq \|\mu\| \cdot \varepsilon/2 = \varepsilon/2$. A contradiction with $|\varphi(a)| = \varepsilon$. □

Since a functional φ on A_I is twisted-invariant if and only if $\text{Ker } \varphi \supset K_I$, the dimension of the space of these functionals equals the dimension of the space of functionals on A_I/K_I and is finite if and only if the space A_I/K_I is finite dimensional. In this case the dimension of the space of twisted-invariant functionals on A_I equals $\dim(A_I/K_I)$.

Definition 10.7. *Generalized number $S_*(\phi)$ of fixed points of $\widehat{\phi}$ on Glimm spectrum is*

$$S_*(\phi) := \sum_{I \in GFP} \dim(A_I/K_I).$$

Since functionals associated with measures, which are concentrated in different points, are linearly independent (the space is Hausdorff), the argument above gives the following statement.

Theorem 10.8 (Weak generalized Burnside).

$$R_*(\phi) = S_*(\phi)$$

if one of these numbers is finite.

In [16] (see also [10, 17] and the first part of the present survey paper) it was proved for groups of type I that the Reidemeister number $R(\phi)$, i.e., the number of twisted (or ϕ -)conjugacy classes, coincides with the number of fixed points of the corresponding action $\widehat{\phi}$ on the dual space \widehat{G} . This was a generalization of Burnside theorem. The proof used identification of $R(\phi)$ and the dimension of the space of (L^∞ -) twisted invariant functions on G , i.e. twisted invariant functionals on $L^1(G)$. Since only part of L^∞ -functions defines functionals on $C^*(G)$ (namely, Fourier-Stieltjes functions), so, *a priori* one has $R_*(\phi) \leq R(\phi)$. Nevertheless, the functions with some symmetry conditions very often are in Fourier-Stieltjes algebra, so one can conjecture that $R(\phi) = R_*(\phi)$ if $R(\phi) < \infty$. This is the case for all known examples.

11. FINITE-DIMENSIONAL REPRESENTATIONS AND ALMOST POLYCYCLIC GROUPS

Let us start from several useful facts.

Theorem 11.1 ([34, Theorem 1.41]). *If G is a finitely generated group and H is a subgroup with finite index in G , then H is finitely generated.*

Lemma 11.2. *Let G be finitely generated, and $H' \subset G$ its subgroup of finite index. Then there is a characteristic subgroup $H \subset G$ of finite index, $H \subset H'$.*

Proof. Since G is finitely generated, there is only finitely many subgroups of the same index as H' (see [23], [27, § 38]). Let H be their intersection. Then H is characteristic, in particular normal, and of finite index. \square

Theorem 11.3 (see [25]). *Let A be a finitely generated Abelian group, $\psi : A \rightarrow A$ its automorphism. Then $R(\psi) = \# \text{Coker}(\psi - \text{Id})$, i.e. to the index of subgroup generated by elements of the form $x^{-1}\psi(x)$.*

Proof. By Lemma 2.1, $R(\psi)$ is equal to the index of the subgroup $H = \{e\}_\psi$. This group consists by definition of elements of the form $x^{-1}\psi(x)$. \square

Let us remind the following definitions of a class of groups.

Definition 11.4. A group with finite conjugacy classes is called *FC-group*.

In a FC-group the elements of finite order form a characteristic subgroup with locally infinite abelian factor group; a finitely generated FC-group contains in its center a free abelian group of finite index in the whole group [30].

Lemma 11.5. *An automorphism ϕ of a finitely generated FC-group G with $R(\phi) < \infty$ has a finite number of fixed points.*

The same is true for $\tau_x \circ \phi$. Hence, the number of $g \in G$ such that for some $x \in G$

$$gx\phi(g^{-1}) = x,$$

is still finite.

Proof. Let A be the center of G . As it was indicated, A has a finite index in G and hence, by Theorem 11.1, is f.g. Since A is characteristic, one has an extension $A \rightarrow G \rightarrow G/A$ respecting ϕ . One has $R(\phi') \leq R(\phi) \cdot |G/A|$ by Lemma 4.3 and $\#\text{Fix}(\phi') \leq R(\phi) \cdot |G/A|$ by Theorem 4.5. Then $\#\text{Fix}(\phi) \leq R(\phi) \cdot |G/A|^2$ by Lemma 4.4. \square

Definition 11.6. We say that a group G has the property RP if for any automorphism ϕ with $R(\phi) < \infty$ the characteristic functions f of REIDEMEISTER classes (hence all ϕ -central functions) are PERIODIC in the following sense.

There exists a finite group W , its automorphism ϕ_W , and epimorphism $F : G \rightarrow W$ such that

(1) The diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ F \downarrow & & \downarrow F \\ W & \xrightarrow{\phi_W} & W \end{array}$$

commutes.

(2) $f = F^* f_W$, where f_W is a characteristic function of a subset of W .

If this property holds for a concrete automorphism ϕ , we will denote this by $\text{RP}(\phi)$.

Remark 11.7. By (2) there is only one class $\{g\}_\phi$ which maps onto $\{F(g)\}_{\phi_W}$. Hence, F induces a bijection of Reidemeister classes.

Lemma 11.8. *Suppose, G is f.g. and $R(\phi) < \infty$. Then characteristic functions of ϕ -conjugacy classes are periodic (i.e. G has $\text{RP}(\phi)$) if and only if their left shifts generate a finite dimensional space.*

Proof. From the supposition of finite dimension it follows that the stabilizer of each ϕ -conjugacy class has finite index. Hence, the common stabilizer of all ϕ -conjugacy classes under left shifts is an intersection of finitely many subgroups, each of finite index. Hence, its index is finite. By Lemma 11.2 there is some smaller subgroup G_S of finite index which is normal and ϕ -invariant. Then one can take $W = G/G_S$. Indeed, it is sufficient to verify that the projection F is one to one on classes. In other words, that each coset of G_S enters only one ϕ -conjugacy class, or any two elements of coset are ϕ -conjugated. Consider g and hg , $g \in G$, $h \in G_S$. Since h by definition preserves classes, $hg = xg\phi(x^{-1})$ for some $x \in G$, as desired.

Conversely, if G has $\text{RP}(\phi)$, the class $\{g\}_\phi$ is a full pre-image $F^{-1}(S)$ of some class $S \subset W$. Then its left shift can be described as

$$\begin{aligned} g'\{g\}_\phi &= g'F^{-1}(S) = \{g'g_1 | g_1 \in F^{-1}(S)\} = \{g|(g')^{-1}g \in F^{-1}(S)\} \\ &= \{g|F((g')^{-1}g) \in S\} = \{g|F(g) \in F(g')(S)\} = F^{-1}(F(g')(S)). \end{aligned}$$

Since W is finite, the number of these sets is finite. \square

Remark 11.9. 1) In this situation in accordance with Lemma 11.2 the subgroup G_S is characteristic, i.e. invariant under any automorphism.

2) Also, the group G/G_S will serve as W (i.e. give rise a bijection on sets of Reidemeister classes) not only for ϕ but for $\tau_g \circ \phi$ for any $g \in G$ because they have the same collection of left shifts of Reidemeister classes by Lemma 4.1.

Theorem 11.10. *Suppose, the extension (1) satisfies the following conditions:*

- (1) H has RP;
- (2) G/H is FC f.g.

Then G has $\text{RP}(\phi)$.

Proof. We have $R(\bar{\phi}) < \infty$, hence $\# \text{Fix}(\bar{\phi}) < \infty$ by Lemma 11.5 as well as $\# \text{Fix}(\tau_z \bar{\phi}) < \infty$ for any $z \in G/H$. Then by Lemma 4.3 $R(\tau_g \phi') < \infty$ for any $g \in G$. Let g_1, \dots, g_s , $s = R(\bar{\phi})$, be elements of G which are mapped by p to different Reidemeister classes of $\bar{\phi}$. Now we can apply the supposition that H has RP and find a characteristic subgroup $H_W := \text{Ker } F \subset H$ of finite index such that $F : H \rightarrow W$ gives rise a bijection for Reidemeister classes of each $\tau_{g_i} \circ \phi'$, $i = 1, \dots, s$, moreover, it is contained in the stabilizer of each twisted conjugacy class of each of these automorphisms. We choose it in a way to satisfy Remark 11.9. In particular, it is normal in G . Hence, we can take a quotient by H_W of the extension (1):

$$\begin{array}{ccccc}
 H \hookrightarrow & G & & & \\
 \downarrow F & \downarrow F_1 & \searrow p & & \\
 H/H_W \hookrightarrow & G/H_W & \xrightarrow{p} & G/H & \\
 \parallel & \parallel & & \parallel & \\
 W \hookrightarrow & G_1 & \xrightarrow{p} & \Gamma &
 \end{array}$$

The quotient map $F_1 : G \rightarrow G/H_W$ takes $\{g\}_\phi$ to $\{g\}_\phi$ and it is a unique class with this property (we conserve the notations e, g, ϕ for the quotient objects). Indeed, suppose two classes are mapped onto one. Hence, $g_i h$ and $g_i h \hat{h}_W$ belong to the same (different) classes as g and gh_W . Moreover, they can not be ϕ -conjugate by elements of H . Hence (cf. (4)), the elements h and $h \hat{h}_W$ are not $(\tau_{g_i} \circ \phi')$ -conjugate in H . But this contradicts the choice of $H_W \ni \hat{h}_W$.

By Lemma 11.8 (applied to G_1 and concrete automorphism ϕ) for the purpose to find a map $F_2 : G_1 \rightarrow W_1$ with properties (1) and (2) of the Definition 11.6 it is sufficient to verify that shifts of the characteristic function of $\{h\}_\phi \subset G_1$ form a finite dimensional space, i.e. the shifts of $\{h\}_\phi \subset G_1$ form a finite collection of subsets of G_1 . After that one can take the composition

$$G \xrightarrow{F_1} G_1 \xrightarrow{F_2} W_1$$

to complete the proof of theorem.

Let us observe, that we can apply Lemma 11.8 because the group G_1 is finitely generated: we can take as generators all elements of W and some pre-images $s(z_i) \in G_1$ under p of a finite system of generators z_i for Γ . Indeed, for any $x \in G_1$ one can find some product of z_i to be equal to $p(x)$. Then the same product of $s(z_i)$ differs from x by an element of W .

Let us prove that the mentioned space of shifts is finite-dimensional. By Lemma 4.1 these shifts of $\{h\}_\phi \subset G_1$ form a subcollection of

$$\{x\}_{\tau_y \circ \phi}, \quad x, y \in G_1.$$

Hence, by Corollary 4.2 it is sufficient to verify that the number of different automorphisms $\tau_y : G_1 \rightarrow G_1$ is finite.

Let x_1, \dots, x_n be some generators of G_1 . Then the number of different τ_y does not exceed

$$\prod_{j=1}^n \#\{\tau_y(x_j) \mid y \in G_1\} \leq \prod_{j=1}^n |W| \cdot \#\{\tau_z(p(x_j)) \mid z \in \Gamma\},$$

where the last numbers are finite by the definition of FC for Γ . \square

Now we describe using Theorem 11.10 some classes of RP groups. Of course these classes are only a small part of possible corollaries of this theorem.

Let $G' = [G, G]$ be the *commutator subgroup* or *derived group* of G , i.e. the subgroup generated by commutators. G' is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of G with an abelian factor group. Denoting $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(n)} := (G^{(n-1)})'$, $n \geq 2$, one obtains *derived series* of G :

$$(27) \quad G = G^{(0)} \supset G' \supset G^{(2)} \supset \dots \supset G^{(n)} \supset \dots$$

If $G^{(n)} = e$ for some n , i.e. the series (27) stabilizes by trivial group, the group G is *solvable*;

Definition 11.11. A solvable group with derived series with cyclic factors is called *polycyclic group*.

Theorem 11.12. *Any polycyclic group is RP.*

Proof. By Lemma 2.1 any commutative group is RP. Any extension with H being the commutator subgroup G' of G respects any automorphism ϕ of G , because G' is evidently characteristic. The factor group is abelian, in particular FC.

Since any polycyclic group is a result of finitely many such extensions with finitely generated (cyclic) factor groups, starting from Abelian group, applying inductively Theorem 11.10 we obtain the result. \square

Theorem 11.13. *Any finitely generated nilpotent group is RP.*

Proof. These groups are supersolvable, hence, polycyclic [35, 5.4.6, 5.4.12]. \square

Theorem 11.14. *Any finitely generated group of polynomial growth is RP.*

Proof. By [21] a finitely generated group of polynomial growth is just as finite extension of a f.g. nilpotent group H . The subgroup H can be supposed to be characteristic, i.e. $\phi(H) = H$ for any automorphism $\phi : G \rightarrow G$. Indeed, let $H' \subset G$ be a nilpotent subgroup of index j . Let H be the subgroup from Lemma 11.2. By Theorem 11.1 it is finitely generated. Also, it is nilpotent as a subgroup of nilpotent group (see [27, § 26]).

Since a finite group is a particular case of FC group and f.g. nilpotent group has RP by Theorem 11.13, we can apply Theorem 11.10 to complete the proof. \square

The proof of the following twisted Burnside theorem can be extracted from Theorem 11.18, but we formulate it separately due to importance of the classes under consideration.

Theorem 11.15. *The Reidemeister number of any automorphism ϕ of a f.g. group of polynomial growth or polycyclic group is equal to the number of finite-dimensional fixed points of $\widehat{\phi}$ on the unitary dual of this group if $R(\phi)$ is finite.*

Theorem 11.16. *Any almost polycyclic group, i.e. an extension of a polycyclic group with a finite factor group, is RP.*

Proof. The proof repeats almost literally the proof of Theorem 11.14. One has to use the fact that a subgroup of a polycyclic group is polycyclic [35, p. 147]. \square

Definition 11.17. Denote by \widehat{G}_f the subset of the unitary dual \widehat{G} related to finite-dimensional representations.

Theorem 11.18 (Twisted Burnside Theorem). *Let G be an RP group and ϕ its automorphism. Denote by $S_f(\phi)$ the number of fixed points of $\widehat{\phi}_f$ on \widehat{G}_f . Then*

$$R(\phi) = S_f(\phi),$$

if the Reidemeister number $R(\phi)$ is finite.

Proof. Let us start from the following observation. Let Σ be the universal compact group associated with G and $\alpha : G \rightarrow \Sigma$ the canonical morphism (see, e.g. [6, Sect. 16.1]). Then $\widehat{G}_f = \widehat{\Sigma}$ [6, 16.1.3]. The coefficients of (finite-dimensional) non-equivalent irreducible representations of Σ are linear independent by Peter-Weyl theorem as functions on Σ . Hence the coefficients of finite-dimensional non-equivalent irreducible representations of G as functions on G are linearly independent as well.

It is sufficient to verify the following three statements:

1) If $R(\phi) < \infty$, then each ϕ -class function is a finite linear combination of twisted-invariant functionals being coefficients of points of $\text{Fix } \widehat{\phi}_f$.

2) If $\rho \in \text{Fix } \widehat{\phi}_f$, there exists one and only one (up to scaling) twisted invariant functional on $\rho(C^*(G))$ (this is a finite full matrix algebra).

3) For different ρ the corresponding ϕ -class functions are linearly independent. This follows from the remark at the beginning of the proof.

Let us remark that the property RP implies in particular that ϕ -central functions (for ϕ with $R(\phi) < \infty$) are functionals on $C^*(G)$, not only $L^1(G)$, i.e. are in the Fourier-Stieltjes algebra $B(G)$.

The statement 1) follows from the RP property. Indeed, this ϕ -class function f is a linear combination of functionals coming from some finite collection $\{\rho_i\}$ of elements of \widehat{G}_f (these representations ρ_1, \dots, ρ_s are in fact representations of the form $\pi_i \circ F$, where π_i are irreducible representations of the finite group W and $F : G \rightarrow W$, as in the definition of RP). So,

$$f = \sum_{i=1}^s f_i \circ \rho_i, \quad \rho_i : G \rightarrow \text{End}(V_i), \quad f_i : \text{End}(V_i) \rightarrow \mathbb{C}, \quad \rho_i \neq \rho_j, \quad (i \neq j).$$

For any $g, \tilde{g} \in G$ one has

$$\sum_{i=1}^s f_i(\rho_i(\tilde{g})) = f(\tilde{g}) = f(g\tilde{g}\phi(g^{-1})) = \sum_{i=1}^s f_i(\rho_i(g\tilde{g}\phi(g^{-1}))).$$

By the observation at the beginning of the proof concerning linear independence,

$$f_i(\rho_i(\tilde{g})) = f_i(\rho_i(g\tilde{g}\phi(g^{-1}))), \quad i = 1, \dots, s,$$

i.e. f_i are twisted-invariant. For any $\rho \in \widehat{G}_f$, $\rho : G \rightarrow \text{End}(V)$, any functional $\omega : \text{End}(V) \rightarrow \mathbb{C}$ has the form $a \mapsto \text{Tr}(ba)$ for some fixed $b \in \text{End}(V)$. Twisted invariance implies twisted invariance of b (evident details can be found in [16, Sect. 3]). Hence, b is intertwining between ρ and $\rho \circ \phi$ and $\rho \in \text{Fix}(\widehat{\phi}_f)$. The uniqueness of intertwining operator (up to scaling) implies 2). \square

Remark 11.19. We were able to prove only one statement of the theorem in the terms of Σ because of difficulties with an extension of ϕ to Σ .

Now let us present some counterexamples to this statement for pathological (monster) discrete groups. Suppose, an infinite discrete group G has a finite number of conjugacy classes. Such examples can be found in [36] (HNN-group), [31, p. 471] (Ivanov group), and [32] (Osin group). Then evidently, the characteristic function of unity element is not almost-periodic and the argument above is not valid. Moreover, let us show, that these groups give rise counterexamples to above theorem.

Example 11.20. For the Osin group the Reidemeister number $R(\text{Id}) = 2$, while there is only trivial (1-dimensional) finite-dimensional representation. Indeed, Osin group is an infinite finitely generated group G with exactly two conjugacy classes. All nontrivial elements of this group G are conjugate. So, the group G is simple, i.e. G has no nontrivial normal subgroup. This implies that group G is not residually finite (by definition of residually finite group). Hence, it is not linear (by Mal'cev theorem [28], [35, 15.1.6]) and has no finite-dimensional irreducible unitary representations with trivial kernel. Hence, by simplicity of G , it has no finite-dimensional irreducible unitary representation with nontrivial kernel, except of the trivial one.

Let us remark that Osin group is non-amenable, contains the free group in two generators F_2 , and has exponential growth.

Example 11.21. For large enough prime numbers p , the first examples of finitely generated infinite periodic groups with exactly p conjugacy classes were constructed by Ivanov as limits of hyperbolic groups (although hyperbolicity was not used explicitly) (see [31, Theorem 41.2]). Ivanov group G is infinite periodic 2-generator group, in contrast to the Osin group, which is torsion free. The Ivanov group G is also a simple group. The proof (kindly explained to us by M. Sapir) is the following. Denote by a and b the generators of G described in [31, Theorem 41.2]. In the proof of Theorem 41.2 on [31] it was shown that each of elements of G is conjugate in G to a power of generator a of order s . Let us consider any normal subgroup N of G . Suppose $\gamma \in N$. Then $\gamma = ga^s g^{-1}$ for some $g \in G$ and some s . Hence, $a^s = g^{-1}\gamma g \in N$ and from periodicity of a , it follows that also $a \in N$ as well as $a^k \in N$ for any k , because p is prime. Then any element h of G also belongs to N being of the form $h = \tilde{h}a^k(\tilde{h})^{-1}$, for some k , i.e., $N = G$. Thus, the group G is simple. The discussion can be completed in the same way as in the case of Osin group.

Example 11.22. In paper [24], Theorem III and its corollary, G. Higman, B. H. Neumann, and H. Neumann proved that any locally infinite countable group G can be embedded into a countable group G^* in which all elements except the unit element are conjugate

to each other (see also [36]). The discussion above related Osin group remains valid for G^* groups.

Let us remark that almost polycyclic group are residually finite (see e.g. [35, 5.4.17]) while the groups from these counterexamples are not residually finite, as it is clear by definition. That is why we would like to complete this section with the following question.

Question. Suppose G is a residually finite group and ϕ is its endomorphism with finite $R(\phi)$. Does $R(\phi)$ equal $S_f(\phi)$?

The results of the present section are generalized to the case of coincidences in [14].

12. BAUMSLAG-SOLITAR GROUPS $B(1, n)$

In this section based on (a part of) [11] it is proved that any injective endomorphism of a Baumslag-Solitar group $B(1, n)$ has infinite Reidemeister number. In [11] the similar result for automorphisms of $B(m, n)$ is obtained as well.

Let $B(1, n) = \langle a, b : a^{-1}ba = b^n, n > 1 \rangle$ be the Baumslag-Solitar groups. These groups (see [8]) are finitely-generated solvable groups (in particular amenable) which are not virtually nilpotent. These groups have exponential growth [5] and they are not Gromov hyperbolic. Furthermore, these groups are torsion free and metabelian (an extension of an Abelian group by an Abelian). More precisely one has

Proposition 12.1. $B(1, n) \cong \mathbb{Z}[1/n] \rtimes_{\theta} \mathbb{Z}$, where the action of \mathbb{Z} on $\mathbb{Z}[1/n]$ is given by $\theta(1)(x) = x/n$.

Proof. The map defined by $\iota(a) = (0, 1)$ and $\iota(b) = (1, 0)$ extends to a unique homomorphism $\iota : B(1, n) \rightarrow \mathbb{Z}[1/n] \rtimes \mathbb{Z}$, because

$$\iota(a^{-1}) * \iota(b) * \iota(a) = (0, -1) * (1, 0) * (0, 1) = (0, -1) * (1, 1) = (n, 0) = \iota(b^n).$$

One has

$$(28) \quad \iota(a^r b^s a^{-r}) = (0, r) * (s, -r) = \left(\frac{s}{n^r}, 0 \right).$$

The map ι is clearly surjective. Let us show that this homomorphism is injective. Let us remark that the group relation implies $a^{-1}b^{-1}a = b^{-n}$. Hence for any s one has $a^{-1}b^s a = b^{ns}$. Thus we can move all a^{-1} to the right (until they annihilate with some a or take the extreme right place) and all a to the left, with an appropriate changing of powers of b . Hence any word in $B(1, n)$ is equivalent to a word of the form $w = a^{r_1} b^s a^{r_2}$, where $r_1 \geq 0, r_2 \leq 0$. Then $\iota(w) = (m, r_1 + r_2)$ for some $m \in \mathbb{Z}[1/n]$. Hence, if $r_1 + r_2 \neq 0$, then $\iota(w) \neq e$. Let now $r_1 + r_2 = 0$, then by (28) if $\iota(w) = e$, then $s = 0$ and $w = e$. \square

Consider the homomorphism $|\cdot|_a : B(1, n) \rightarrow \mathbb{Z}$ which associates to each word $w \in B(1, n)$ the sum of the exponents of a in this word. Since this sum for the relation is zero, this is a well defined map, which is evidently surjective.

Proposition 12.2. *We have a short exact sequence*

$$0 \longrightarrow K \longrightarrow B(1, n) \xrightarrow{|\cdot|_a} \mathbb{Z} \longrightarrow 1,$$

where K is the kernel of $|\cdot|_a$. Moreover, $B(1, n)$ equals a semidirect product $K \rtimes \mathbb{Z}$.

Proof. The first statement follows from surjectivity of $|\cdot|_a$. Since \mathbb{Z} is free, this sequence splits. \square

Proposition 12.3. *The kernel K coincide with the normalizer $N\langle b \rangle$ of the subgroup $\langle b \rangle$ generated by b in $B(1, n)$:*

$$(29) \quad 0 \longrightarrow N\langle b \rangle \longrightarrow B(1, n) \xrightarrow{| \cdot |_a} \mathbb{Z} \longrightarrow 1.$$

Proof. We have $N\langle b \rangle \subset K$. The quotient $B(1, n)/N\langle b \rangle$ has the following presentation: $\bar{a}^{-1}\bar{b}\bar{a} = \bar{b}^n$, $\bar{b} = 1$. Therefore this group is isomorphic to \mathbb{Z} under the identification $[a] \leftrightarrow 1_{\mathbb{Z}}$. Hence the natural projection coincides with the map $| \cdot |_a$ and we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N\langle b \rangle & \longrightarrow & B(1, n) & \longrightarrow & B(1, n)/(N\langle b \rangle) \longrightarrow 1 \\ & & \downarrow \cap & & \downarrow = & & \downarrow \cong \\ 0 & \longrightarrow & K & \longrightarrow & B(1, n) & \longrightarrow & \mathbb{Z} \longrightarrow 1. \end{array}$$

The five-lemma completes the proof. \square

Proposition 12.4. *Any homomorphism $\phi : B(1, n) \rightarrow B(1, n)$ is a homomorphism of the short exact sequence (29).*

Proof. Let $\bar{\phi}$ be the homomorphism induced by ϕ on the abelianization $B(1, n)_{ab}$ of $B(1, n)$. The group $B(1, n)_{ab}$ is isomorphic to $Z_{n-1} + Z$. The torsion elements of $B(1, n)_{ab}$ form a subgroup isomorphic to Z_{n-1} which is invariant under any homomorphism. The preimage of this subgroup under the projection $B(1, n) \rightarrow B(1, n)_{ab}$ is exactly the subgroup $N\langle b \rangle$, i.e. the elements represented by words where the sum of the powers of a is zero. So it follows that $N\langle b \rangle$ is mapped into $N\langle b \rangle$ and the result follows. \square

Theorem 12.5. *For any injective homomorphism ϕ of $B(1, n)$ the Reidemeister number is infinite.*

Proof. By Proposition 12.4 it is a homomorphism of short exact sequence. The induced map $\bar{\phi}$ on the quotient is an injective endomorphism of \mathbb{Z} . If $\bar{\phi} = \text{Id}_{\mathbb{Z}}$, then by (2) the number of Reidemeister classes is infinite. Hence, $\bar{\phi}$ is multiplication by $k \neq 0, 1$. But this is impossible. Indeed, when we apply ϕ to the relation $a^{-1}ba = b^n$, under the identification of Proposition 12.1 $\iota : B(1, n) \cong \mathbb{Z}[1/n] \rtimes \mathbb{Z}$ we have, because $\phi(b) \in N\langle b \rangle$ and hence $\iota(\phi(b)) = (d, 0)$ for some $d \in \mathbb{Z}[1/n]$,

$$(nd, 0) = \iota(\phi(b^n)) = \iota(\phi(a^{-1}ba)) = \iota(a^{-k}\phi(b)a^k) = (d \cdot n^k, 0).$$

This implies that either $n^{1-k} = 1$ or $\phi(b) = 0$. \square

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