9.37. $S^{2n-1} \subsetneq \mathbb{C}^n$

$$S^{2n-1} = \{ z = (z^1, \dots, z^n) \in \mathbb{C}^n \mid \|z\| = \left(\sum |z^i|^2\right)^{1/2} = \left(\sum \bar{z}^i z^i\right)^{1/2} = 1 \}$$

$S^1 = U(1) = \{ a \in \mathbb{C}^1 \mid |a| = 1 \}$. acts on S^{2n-1}

$$(a, z) \longmapsto az = (az^1, \dots, az^n) \in S^{2n-1}$$

$$\mathrm{Orb}(z) = \{ az \mid a \in U(1) \}.$$

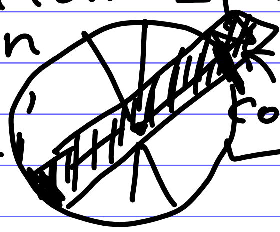
$z \sim w$ if z, w are in the same orbit:

$$\exists a \in U(1) \text{ s.t. } w = az.$$

The space of orbits = quotient space w.r.t. $\sim$

This is $\mathbb{C}P^{n-1}$.

$\mathbb{C}^n$



S^{2n-1} compl. dim $= 1$

A 1-dim. subspace $\underbrace{L_0 = \lambda z_0}_{\lambda \in \mathbb{C}}$

$$L_0 \cap S^{2n-1} = \{ \lambda z_0 \mid \text{s.t. } \|\lambda z_0\| = 1 \} \neq \emptyset$$

$\Rightarrow$ We can take $z_0 \in S^{2n-1}$ from the very beginning. Take $\lambda = \dfrac{1}{\|z_0\|}$

$$L_0 \cap S^{2n-1} = \{ \lambda z_0 \mid |\lambda| = 1 \}, \quad \|z_0\| = 1. \quad \Leftrightarrow \text{ any fiber of } \pi \text{ is } U(1)$$

So each orbit corresponds to L_0

The projection $\pi = \pi_n : S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$.

Hopf's map. We wish to explain that it is a principal $U(1)$-bundle.

The charts of $\mathbb{C}P^{n-1}$: $U_j = \{ [z^1 : \dots : z^n] \mid z^j \neq 0 \}$, $|z^1|^2 + \dots + |z^n|^2 = 1$

$$U_j \to \mathbb{C}^{n-1}, \quad \left(\frac{z^1}{z^j}, \dots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \dots, \frac{z^n}{z^j} \right)$$

$$\pi^{-1}(U_j) = \{ z = (z^1, \dots, z^n) \in \mathbb{C}^n \mid |z^1|^2 + \dots + |z^n|^2 = 1,\ z^j \neq 0 \}$$

Define

$$\Phi_j : \pi^{-1}(U_j) \to \underset{\cap\ \mathbb{C}P^{n-1}}{U_j} \times S^1, \quad \varphi_j(z) = \Big([z^1 : \dots : z^n],\ \underbrace{\frac{z^j}{|z^j|}}_{\in\ U(1) = S^1}\Big)$$

EVIDENTLY THIS IS A SMOOTH MAP.
IT HAS THE FOLLOWING SMOOTH INV. MAP:

$$\varphi_j^{-1}\Big(\underset{\substack{z^j \neq 0 \\ |z^1|^2 + \dots + |z^n|^2 = 1}}{[z^1 : \dots : z^n]},\ \underset{\in\ U(1)}{a}\Big) = \frac{a\,|z^j|}{z^j}(z^1, \dots, z^n). \quad (*)$$

φ_j^{-1} is well defined: if $[z^1 : \dots : z^n] = [bz^1 : \dots : bz^n]$ $\Rightarrow |b| = 1$

$$\frac{a\,|bz^j|}{bz^j}(bz^1, \dots, bz^n) = \frac{a\,|z^j|}{\cancel{b}z^j}(\cancel{b}z^1, \dots, \cancel{b}z^n)$$

THE SAME AS $(*)$.

AND CLEARLY SMOOTH.

$$\Phi_j \circ \varphi_j^{-1}\Big(\underset{z^j \neq 0,\ |z^1|^2 + \dots + |z^n|^2 = 1}{[z^1 : \dots : z^n],\ a}\Big) =$$

$$= \varphi_j\Big(\frac{a|z^j|}{z^j}(z^1, \dots, z^n)\Big) = \left(\Big[\frac{a|z^j|}{z^j} z^1 : \dots : \frac{a|z^j|}{z^j} z^n\Big],\ \frac{\frac{a|z^j|z^j}{z^j}}{\left|\frac{a|z^j|z^j}{z^j}\right|}\right) =$$

$$= \Big([z^1 : \dots : z^n],\ \frac{a\,\cancel{|z^j|}}{\cancel{|a||z^j|}}\Big), \text{ so } \varphi_j \circ \Phi_j^{-1} = Id$$

$$\varphi_j^{-1} \circ \varphi_j\Big((z^1, \dots, z^n),\ \Sigma|z^j|^2 = 1,\ z^j \neq 0\Big) =$$

$$= \Phi_j^{-1}\Big([z^1 : \dots : z^n],\ \frac{z^j}{|z^j|}\Big) = \frac{z^j}{|z^j|}\frac{|z^j|}{z^j}(z^1, \dots, z^n) =$$

$= (z^1, \dots, z^n)$, so $\varphi_j^{-1} \circ \varphi_j = Id$

What is the structure group and λ?
(wish $\overset{\shortparallel}{U(1)}$ left multipl.)

$$\Phi_k \varphi_j^{-1}\big(\underbrace{[z^1:\dots:z^n]}_{U_k\cap U_j},\ \underset{U(1)}{\underset{\Uparrow}{a}}\big) = \varphi_k\left(\frac{a|z^j|}{z^j}(z^1,\dots,z^n)\right) =$$

$z^k, z^j \neq 0$

$|z^1|^2+\dots+|z^n|^2 = 1$

$$= \left(\left[\frac{a|z^j|}{z^j}z^1 : \dots : \frac{a|z^j|}{z^j}z^n\right],\ \frac{\frac{a|z^j|}{z^j}z^k}{\left|\frac{a|z^j|}{z^j}z^k\right|}\right) =$$

$$= \left([z^1:\dots:z^n],\ a\cdot\frac{|z^j|\,z^k}{z^j\,|z^k|}\right)$$

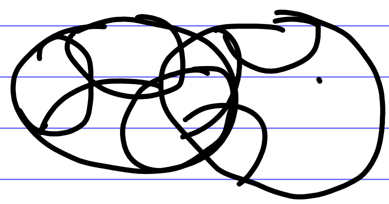
$$\Phi_{kj}([z^1:\dots:z^n])(a) = \underbrace{\frac{|z^j|\,z^k}{z^j\,|z^k|}}_{\in U(1)}\cdot a$$

So $G = U(1)$, and λ is the left multipl. ∎

9.46 | Let's use cocycles (9.45)

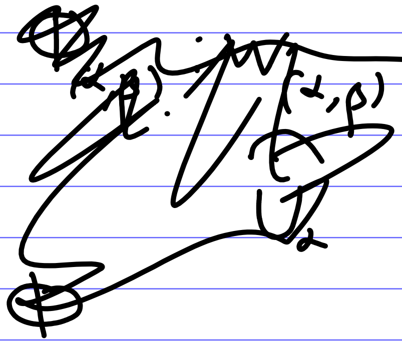
Two cocycles: $(\Phi_{\alpha\beta}, U_\alpha)$ (Ψ_{rs}, V_r) over distinct covers. How to compare the bundles?

Consider a refinement: $U_\alpha \cap V_r = W_{(\alpha,r)}$



$(\Phi_{(\alpha,r)(\beta,s)}, W_{(\alpha,r)})$

~~$\Phi_{(\alpha,\beta)}: W_{(\alpha,\beta)} \to G$, $\Phi_{(\alpha,\beta)} = \Phi_\alpha|_{W(\alpha,\beta)}$~~



$$\Phi_{(\alpha,r)(\beta,s)} : W_{(\alpha,r)(\beta,s)} \to G$$

$$W_{(\alpha,r)(\beta,s)} = (U_\alpha\cap V_r)\cap(U_\beta\cap V_s) \subset U_\alpha\cap U_\beta$$

$$\Phi_{(\alpha,r)(\beta,s)} = \Phi_{\alpha\beta}\Big|_{W_{(\alpha,r)(\beta,s)}}$$

Verify the prop. of Cocycle:

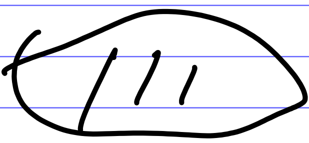
$$\Phi_{(\alpha,r)(\alpha,r)} = \Phi_{\alpha\alpha}\Big|_{W_{\dots}} = Id.$$

and so on.

π_1 and π_2

from the same $\{U_\alpha\}$

$$f_\alpha \otimes f_\beta \quad \otimes \quad \hat{f}_\alpha \otimes \hat{f}_\beta^{-1}$$

$(f_\alpha \otimes \hat{f}_\alpha)$  $(f_\beta^{-1} \otimes \hat{f}_\beta^{-1})$ —