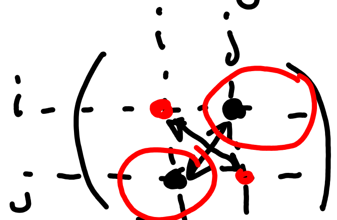
8.6. a^i_j inv. tensor field of type $(1,1)$ under ortogonal transf.

$e_i \longleftrightarrow e_j$



since the tensor is invariant $\Rightarrow$ the matrix is symmetric (in each ortonorm. base)

l. op. $C^{-1}AC$ b. form C^TBC, orthog. $\underline{C^{-1}=C^T}$

So there is an orthogone change s.t. it becomes diagonal. $\Rightarrow$ it is diagonal in any orth. base.

Then we apply $e_i \leftrightarrow e_j$ once again.

$\Rightarrow$ all diag. el. are the same $\Rightarrow \lambda\delta^i_j$

$\boxed{8.8}$ A $(1,1)$ $\quad L_A(\underset{\text{cov.}}{a}, \underset{\text{vec.}}{v})$, $\boxed{a(\mathring{A}(v))=L_A(a,v)}$

$\underline{e^i(\mathring{A}(e_j)) = A^i_j}$

taking of cov. $\mathcal{N}^o\, i$

$$\mathring{A}(\vec{e}_j) = A^i_j\, \vec{e}_i.$$

$$\underline{A^{i'}_{j'}} = A^i_j \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}$$

$$\underline{A' = C^{-1}AC}.$$

$\boxed{8.10}$ $\operatorname{grad} f \in (0,1)$,

$$(\operatorname{grad} f)_{i'} = \frac{\partial f}{\partial x^{i'}} = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial x^{i'}} =$$

$$= (\operatorname{grad} f)_i \frac{\partial x^i}{\partial x^{i'}} \quad \text{law}(0,1).$$

$\boxed{8.14}$ 1) How to substitute local fields?

globally defined: $\underbrace{h \cdot \frac{\partial}{\partial x^i}}$, $\underbrace{h \cdot dx^i}$

$$L\left(\frac{\partial}{\partial x^1}, \dots, dx^s\right)\Big|_p := L\left(h\frac{\partial}{\partial x^i}, \dots, h\, dx^s\right)$$

Why does not depend? $h, h', \dots \to hh'$

$$L\left(h\frac{\partial}{\partial x^i},\dots,h\,dx^s\right)\Big|_p \overset{?}{=} L\left(hh'\frac{\partial}{\partial x^i},\dots,hh'dx^s\right)\Big|_p$$

$$\overset{\text{mult.}}{=} \underbrace{\left(h'\right)^{p+q}\Big|_p}_{=1} L\left(h\frac{\partial}{\partial x^i},\dots,h\,dx^s\right).$$

2). Why L_T does not depend. on c.s.

$$L_T(a^1,\dots,a^p,v_1\dots v_q) = T^{i_1\dots i_p}_{j_1\dots j_q}(a^1)_{i_1}\cdots(a^p)_{i_p}\times$$

$$\times(v_1)^{j_1}\cdots(v_q)^{j_q} = T^{i_1'\dots i_p'}_{j_1'\dots j_q'}\frac{\partial x^{j_1'}}{\partial x^{j_1}}\cdots\frac{\partial x^{j_q'}}{\partial x^{j_q}}\times$$

$$\times\frac{\partial x^{i_1}}{\partial x^{i_1'}}\cdots\frac{\partial x^{i_p}}{\partial x^{i_p'}}(a^1)_{m_1'}\frac{\partial x^{m_1'}}{\partial x^{i_1}}\cdots(a^p)_{m_p'}\frac{\partial x^{m_p'}}{\partial x^{i_p}}$$

$$\times(v_1)^{s_1'}\frac{\partial x^{j_1}}{\partial x^{s_1'}}\cdots(v_q)^{s_q'}\frac{\partial x^{j_q}}{\partial x^{s_q'}} =$$

$$= T^{i_1'\dots i_p'}_{j_1'\dots j_q'}(a^1)_{m_1'}\cdots(a^p)_{m_p'}(v_1)^{s_1'}\cdots(v_q)^{s_q'}\cdot$$

$$\cdot\,\delta^{j_1'}_{s_1'}\cdots\delta^{j_q'}_{s_q'}\,\delta^{m_1'}_{i_1'}\cdots\delta^{m_p'}_{i_p'} = T^{i_1'\dots i_p'}_{j_1'\dots j_q'}(a^1)_{i_1'}\cdots(a^p)_{i_p'}\times$$

$$\times(v_1)^{j_1'}\cdots(v_q)^{j_q'}$$ as desired.

3). $$T_L{}^{i_1\dots i_p}_{j_1\dots j_q} = L\left(dx^{i_1},\dots,dx^{i_p},\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_q}}\right)$$

$$(T_L)^{i_1'\dots i_p'}_{j_1'\dots j_q'} = L\left(dx^{i_1'},\dots,dx^{i_p'},\frac{\partial}{\partial x^{j_1'}},\dots,\frac{\partial}{\partial x^{j_q'}}\right) =$$

$$= L\left(dx^{i_1}\frac{\partial x^{i_1'}}{\partial x^{i_1}},\dots,dx^{i_p}\frac{\partial x^{i_p'}}{\partial x^{i_p}},\frac{\partial}{\partial x^{j_1}}\frac{\partial x^{j_1}}{\partial x^{j_1'}},\dots\right)$$

$$= (T_L)^{i_1\dots i_p}_{j_1\dots j_q}\frac{\partial x^{i_1'}}{\partial x^{i_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j_q'}}$$

Why $\underline{dx^{i'} = dx^i\frac{\partial x^{i'}}{\partial x^i}}$ (this is not for comp., but for the entire tensor)

but this is the law of taking differential. $x^{i'}(x^1,\dots,x^n)$

$$d(\sin x) = dx \cdot \left(\frac{d \sin x}{dx}\right)$$

Similarly $\dfrac{\partial}{\partial x^{i'}} = \dfrac{\partial}{\partial x^{i}} \cdot \dfrac{\partial x^{i}}{\partial x^{i'}}$.

4). $L_{(T_L)}(a^1, \dots, a^p, v_1, \dots, v_q) =$

$$= (T_L)^{i_1 \dots i_p}_{j_1 \dots j_q} (a^1)_{i_1} \dots (a^p)_{i_p} (v_1)^{j_1} \dots (v_q)^{j_q} =$$

$$= L\left(dx^{i_1}, \dots, dx^{i_p}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_q}}\right) \cdot$$

$$\cdot (a^1)_{i_1} \dots (v_q)^{j_q} \underset{\text{mult}}{=} L(\underbrace{(a^1)_{i_1} dx^{i_1}}_{a^1}, \dots)$$

$\Rightarrow L_{(T_L)} = L$. Similarly $T_{(L_T)} = T$.

$\boxed{8.21}$ Locally $T \overset{?}{=} \underbrace{T^{i_1 \dots i_p}_{j_1 \dots j_q} dx^{j_1} \otimes \dots \otimes dx^{j_q} \otimes \frac{\partial}{\partial x^{i_1}} \dots \otimes \frac{\partial}{\partial x^{i_p}}}_{S}$

$$S^{m_1 \dots m_p}_{r_1 \dots r_q} =$$

$$= T^{i_1 \dots i_p}_{j_1 \dots j_q} \underbrace{(dx^{j_1})_{r_1}}_{\delta^{j_1}_{r_1}} \dots \underbrace{(dx^{j_q})_{r_q}}_{\delta^{j_q}_{r_q}} \underbrace{\left(\frac{\partial}{\partial x^{i_1}}\right)^{m_1}}_{\delta^{m_1}_{i_1}} \dots \underbrace{\left(\frac{\partial}{\partial x^{i_p}}\right)^{m_p}}_{\delta^{m_p}_{i_p}} =$$

$= T^{m_1 \dots m_p}_{r_1 \dots r_q}$. Why unique?

If $0 = \underbrace{T^{i_1 \dots i_p}_{j_1 \dots j_q}} dx^{j_1} \otimes \dots \otimes dx^{j_q} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}}$

then all $T^{i_1 \dots i_p}_{j_1 \dots j_q} = 0$. Suppose $T^{i_1^0 \dots i_p^0}_{j_1^0 \dots j_q^0} \neq 0$

$\underline{0 =} L_0\left(\frac{\partial}{\partial x^{j_1^0}}, \dots, \frac{\partial}{\partial x^{j_q^0}}, dx^{i_1^0}, \dots, dx^{i_p^0}\right) =$

$= T^{i_1 \dots i_p}_{j_1 \dots j_q} dx^{j_1}\left(\frac{\partial}{\partial x^{j_1^0}}\right) \dots dx^{j_q}\left(\frac{\partial}{\partial x^{j_q^0}}\right) \frac{\partial}{\partial x^{i_1}}(dx^{i_1^0}) \dots \frac{\partial}{\partial x^{i_p}}(dx^{i_p^0})$

$= T^{i_1 \dots i_p}_{j_1 \dots j_q} \delta^{j_1}_{j_1^0} \dots \delta^{j_q}_{j_q^0} \delta^{i_1^0}_{i_1} \dots \delta^{i_p^0}_{i_p} = T^{i_1^0 \dots i_p^0}_{j_1^0 \dots j_q^0}$.

A contradiction.

$\boxed{8.32}$ $dx^{i_1} \wedge \dots \wedge dx^{i_p}$ $(**)$ they form a basis for Λ^p, dim=?
$(i_1 < \dots < i_p)$

$* = dx^{i_1} \otimes \dots \otimes dx^{i_p}$ form a base for $(0,p)$

$\mathrm{Alt}(dx^{i_1} \otimes \dots \otimes dx^{i_p})$ form a generating set for Λ^p, because $\mathrm{Alt}: (0,p) \xrightarrow[\text{onto}]{} \Lambda^p$ is a projection. If there are repeated indexes, then $\mathrm{Alt}(*) = 0$, Otherwise $\pm \text{const}\ dx^{i_1} \wedge \dots dx^{i_p}$ $i_1 < \dots < i_p$

So $(**)$ generate Λ^p. Why they are ind.

Suppose $\sum\limits_{(i_1 < \dots < i_p)} T_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} = 0$

but some $T_{i_1^0 \dots i_p^0} \neq 0$

$$0 = L_0\left(\frac{\partial}{\partial x^{i_1^0}}, \dots, \frac{\partial}{\partial x^{i_p^0}}\right) = \sum_{\overline{i_1 < \dots < i_p}} T_{i_1 \dots i_p} \sum_{\sigma \in S^p} (-1)^\sigma$$

$$\underbrace{dx^{i_1} \otimes \dots \otimes dx^{i_p}\left(\frac{\partial}{\partial x^{i_1^0}}, \dots, \frac{\partial}{\partial x^{i_p^0}}\right)}_{\text{all these values are 0 except of } i_1 = i_1^0, \dots, i_p = i_p^0}$$

$= \pm T_{i_1^0 \dots i_p^0}$. A contradiction.

dim? $C_n^p = \underbrace{\frac{n!}{p!(n-p)!}}$ $= \binom{p}{n}$

$\boxed{8.35}$ The expression $\sqrt{\det \| g_{ij} \|} \cdot dx^1 \wedge \dots \wedge dx^n$ (form) defines a tensor (volume) for changes of coord. with positive Jacobian

$$|g_{i'j'}|^{1/2} dx^{1'}\wedge\ldots\wedge dx^{n'} = \left|g_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\right|^{1/2}\cdot$$

$$\cdot\left|\frac{\partial x^{i'}}{\partial x^i}\right| dx^1\wedge\ldots\wedge dx^n =$$

$$=|g_{ij}|^{1/2}\underbrace{\text{abs}\left|\frac{\partial x^i}{\partial x^{i'}}\right|\cdot\left|\frac{\partial x^{i'}}{\partial x^i}\right|}_{1} dx^1\wedge\ldots\wedge dx^n.$$

if this is positive.

$\boxed{8.38}$ $\langle\bar a,\bar b,\bar c\rangle = \langle[\bar a,\bar b],\bar c\rangle$ mixed or triple

vector prod.

$\cdot\; c_{ijk} = \langle\bar e_i,\bar e_j,\bar e_k\rangle$ 3-linear functo.

tensor of ty $(0,3)$

$[\bar a,\bar b]=\bar c$ $\quad V(\bar a,\bar b,\varphi) =$

$=\varphi([\bar a,\bar b])$, $(1,2)$:

$v^i_{jk} = [e_j,e_k]^i = e^i([e_j,e_k])$

$\underline{c_{ijk}} = g_{mp}[e_i,e_j]^m(e_k)^p =$

$= g_{mp}v^m_{ij}\delta^p_k = \underline{v^m_{ij}g_{mk}}$

lowering of index.

and vice versa

$$\underbrace{c_{ijk}g^{kp}} = v^m_{ij}\underbrace{g_{mk}g^{kp}}_{\delta^p_m} = \underbrace{v^p_{ij}}$$

$\boxed{8.20}$ W $(0,1)$ $\quad X$ $(1,0)$

$W\otimes X$ $(1,1)$ is a linear oper

$[(W\otimes X)(u)]^i = (W\otimes X)^i_j u^j = W_j X^i u^j =$

$= (W_j U^j) X^i = W(U) \cdot X^i$

$(W \otimes X) U = W(U) \cdot X$ The range is generated by X.

fixed.

its dim $= 0$ or 1.

dim $= 0$ if $X = 0$ or $W = 0$

otherwise dim $= 1$.