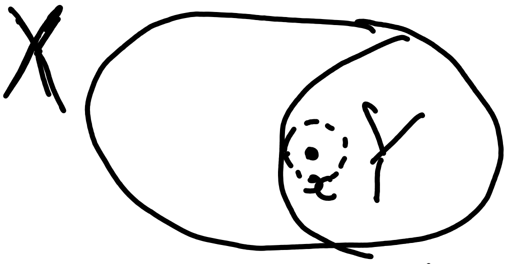
Literature: Lee's Book

Mishchenko-Fomenko

I'll send you some biblio refs.

Problem 1.5. Suppose, X is a metric space. Then $Y \subseteq X$ is open iff (if and only if) Class $X \setminus Y$ is closed. In fact, $\boxed{\operatorname{Int} Y = X \setminus \overline{X \setminus Y}}$. * Suppose, we have proved



Y open $\Leftrightarrow Y = \operatorname{Int} Y = X \setminus \overline{X \setminus Y}$

$\Leftrightarrow X \setminus Y = \overline{X \setminus Y}$

Why *? Suppose $x \in \operatorname{Int} Y$, so $\exists\, B_\varepsilon(x) \subseteq Y$

x is not an adherance point of $X \setminus Y$

$x \notin \overline{X \setminus Y}$, i.e. $x \in X \setminus (\overline{X \setminus Y})$

Convers. $x \notin \overline{X \setminus Y}$, $\Leftrightarrow \exists B_\varepsilon(x)$

$B_\varepsilon(x) \cap X \setminus Y = \emptyset$, $x \in \operatorname{Int} Y$.

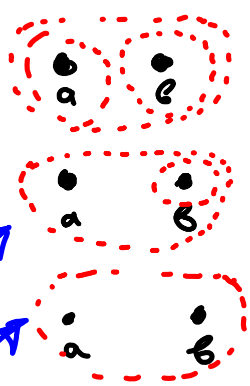
Class **Problem 1.14.** Find an example of a topological space (X, τ), which is not related to any metric (this is called: topology is not metrizable).

• All sets $\{\emptyset, X\}$

$\rho(a,b) = c > 0$

$\{a\} = B_{c/2}(a)$

$\{b\} = B_{c/2}(b)$ metrizable



$\tau = \{\emptyset, X, \{a\}, \{b\}\}$

$\tau = \{\emptyset, X, \{b\}\}$

$\tau = \{\emptyset, X\}$

not metriz topologies!

Non-Hausdorff

Class **Problem 1.28.** Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, which is uniformly convergent on X to some function f. Then f is continuous.

X, $f_n : X \to R$ $\qquad f_n \underset{X}{\rightrightarrows} f$

$$\left[\begin{array}{l} \forall \varepsilon > 0 \ \exists n_0, \text{ s.t. for any } n \geqslant n_0 \\ \sup\limits_{x \in X} \| f_n(x) - f(x) \| < \varepsilon. \end{array}\right] (U)$$

$\forall x_0 \in X \ \underline{\forall \varepsilon > 0} \ \exists U_0 \ni x_0$

$f(U_0) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

Use U with $\varepsilon/3 \longrightarrow$

- $\sup\limits_{x \in X} \| \underbrace{f_n(x)}_{0} - f(x) \| < \varepsilon/3 \quad (3)$

$\exists U_0(x_0) = U_0$ s.t. $\qquad (4)$

$f_{n_0}(\underline{U_0}) \subset (f_{n_0}(x_0) - \varepsilon/3, f_{n_0}(x_0) + \varepsilon/3)$

$\forall x \in \underline{U_0}$. $|f(x) - f(x_0)| \underset{\text{(wish)}}{<} \varepsilon$

we have: $|f(x) - f(x_0)| \leq$

$= | f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) +$

$+ f_{n_0}(x_0) - f(x_0) | \leq$

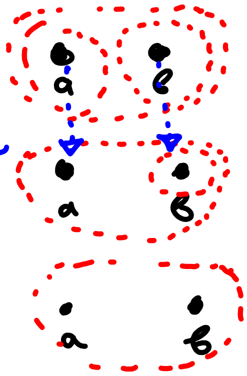
$\leq |\underline{f(x) - f_{n_0}(x)}| + |f_{n_0}(x) - f_{n_0}(x_0)| +$

$+ |\underline{f_{n_0}(x_0) - f(x_0)}| \overset{(3)+(4)}{<} \varepsilon/3 + \varepsilon/3 + \varepsilon/3$

$= \varepsilon.$

Class **Problem 1.31.** Give an example of a continuous bijection, which is not a homeomorphism.

J=Id



$\tau = \{\emptyset, X, \{a\}, \{b\}\}$

$\tau = \{\emptyset, X, \{b\}\}$

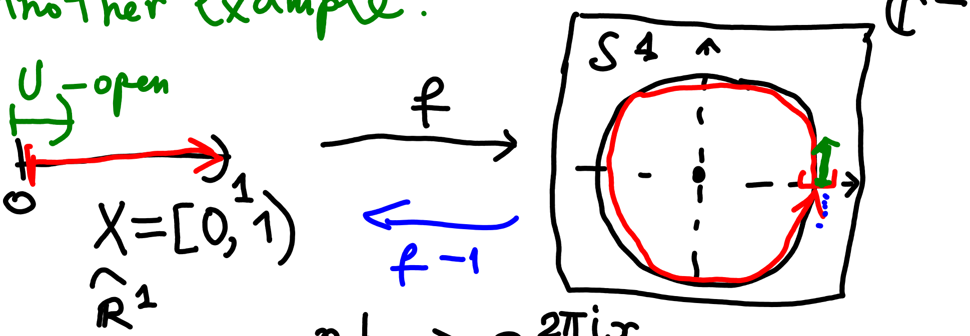
$\tau = \{\emptyset, X\}$

J^{-1}(any open set) is an open set $\Rightarrow$ J is a cont. bijection

$(J^{-1})^{-1}(\{a\}) = J(\{a\}) = \{a\}$ not open in the second space

so J^{-1} is not cont.
and J is not a homeomorphism.

Another example.



$x \longmapsto e^{2\pi i x}$

f is a continuous bijection.

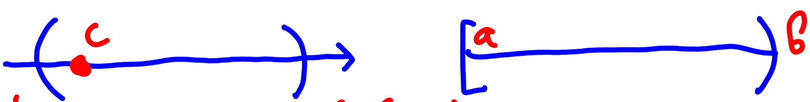
$(f^{-1})^{-1} = f$, so the inverse is continuous means f(open set) is open in S^1

$f(U)$ is not open in S^1

For Further: suppose that f is a homeomorphism. Then both spaces (X and S^1) should be both compact (or not). But S^1 is comp, and X is not.

Problem 1.39. Prove that (a,b), $[a,b)$ and $[a,b]$ (subsets of real line) are pair-wise non-homeomorphic. Class

non-comp $\leftrightarrow$ compact



take any point of (a,b). and consider

$\forall c: (a,b)\setminus\{c\}$ is not connected

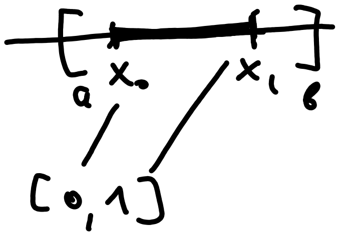
(a,c) (c,b)

open, not-empty, not-intersect.

in the same time $[a,b)\setminus\{a\} = (a,b)$

which is connected.

Problem 1.42. Any interval $[a,b] \subset \mathbb{R}$ ($=I$) is connected and arc connected. ✓



Suppose the opposite: $I = A \cup B$

not empty, open closed

$A \cap B = \emptyset$

$a \in A$.

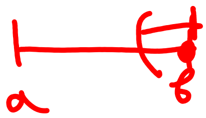
Consider all $a' \in I$ s.t. $[a,a'] \subset A$ $\quad \bar{a} = \sup a'$

$\forall c < a'$, $c \in A$ and A is closed

so $\bar{a} \in A$ and $[a,\bar{a}] \subseteq A$

But A is open, so there is an open interval $(\bar{a}-\varepsilon, \bar{a}+\varepsilon) \subset A$

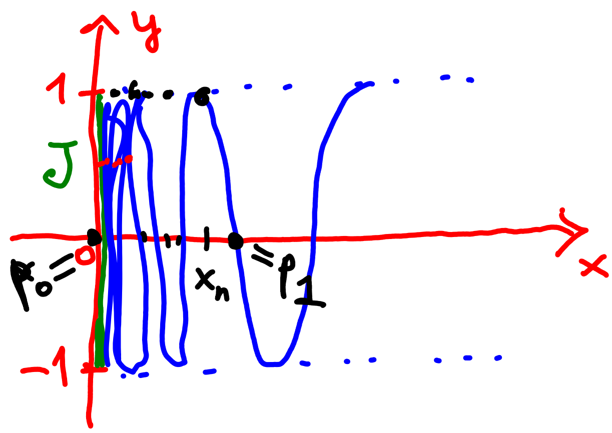
so $\bar{a}+\varepsilon \le \sup a' = \bar{a}$ except the case $\boxed{\bar{a}=b}$



$I = A$, $B = \emptyset$

Problem 1.47. Find an example of connected space, which is not arc-connected.

$\mathbb{R}^2$



$J = [-1,1] \subset Oy$.

$\Gamma = \Gamma(\sin\frac{1}{x})$

graph for $x > 0$

$\boxed{X = J \cup \Gamma}$

Let's prove that X is connected.

Evidently J is arc-connected

Γ is arc-connect.

$(f(x) = (x, \sin\frac{1}{x}))$ arc-cont.

from the def of

$\Rightarrow$ ~~The~~ A unique possibility of not-conn.:

$A = J$, $B = \Gamma$

J is not open. $\Rightarrow$ X is connected.

Now prove that X is not arc-connected

Suppose the opp.: $\exists\, f : [0,1] \to X = J \cup \Gamma$ continuous, such that $f(0) = p_0 = (0,0)$

$f(1) = p_1 = (x_1, 0)$.

$\pi : \mathbb{R}^2 \to \mathbb{R}^1$ – projection onto Ox continuous evidently

$\pi_X : X \to \mathbb{R}^1$, $\pi_X = \pi|_X$.

$\pi_X \circ f : [0,1] \to \underbrace{[0,\infty)}_{\text{may be not } [0,x_1]}$

$\pi_X \circ f(0) = 0$, $\pi_X(f)(1) = x_1$

Calculus: A continuous function takes all intermediate values; i.e.

as $x \mapsto 0$ (argument of $\pi_X \circ f$)

we have $\pi_X \circ f \longrightarrow 0$ taking all interm. values, in particular, those, where

$\sin\frac{1}{x} = 1$, ~~i.e.~~ $\frac{1}{x} = 2\pi n + \frac{\pi}{2}$

denote them $x_n = x = \dfrac{1}{2\pi n + \frac{\pi}{2}}$

$x_n \to 0$, $f(x_n) = (x_n, 1) \not\to (0,0)$

So f can not be continuous.

And this space X is not arc-connected

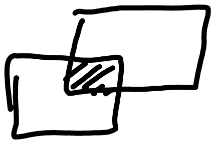
$\mathcal{B} = \{U\}$

1) $\emptyset \in \mathcal{B}$

2) $\bigcup_{U \in \mathcal{B}} U = X$

$$\boxed{3)\ \underset{\mathcal{B}}{\underset{\cup}{U_1}} \cap \underset{\mathcal{B}}{\underset{\cup}{U_2}} \in \mathcal{B}}$$

$\Rightarrow$ $\mathcal{B}$ is a base i.e. all unions form a topology



Pre-base instead of 3)

$$\forall x \in \underset{\mathcal{B}}{\underset{\cup}{U_1}} \cap \underset{\mathcal{B}}{\underset{\cup}{U_2}} \quad \exists \underset{\mathcal{B}}{\underset{\cup}{U}}$$

$$x \in U \subset U_1 \cap U_2$$

$\longrightarrow$ not only unions but also finite inter $\longrightarrow \tau$