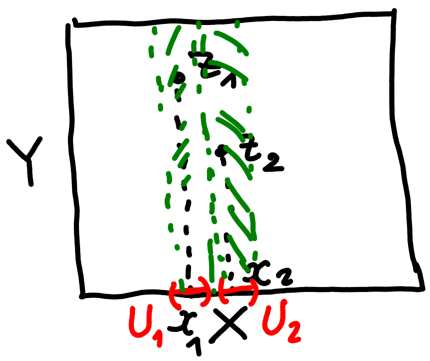
Home **Problem 1.49.** Give an example of non-Hausdorff topological space.

Class **Problem 1.50.** Prove that the Cartesian product of Hausdorff spaces is a Hausdorff space.

Home **Problem 1.51.** Prove that in any Hausdorff space each point is a closed set.



$z_1 = (x_1, y_1)$
$\neq$
$z_2 = (x_2, y_2)$

$\Rightarrow$ one of "coordinate" should be distinct. assume the first one:

$x_1 \neq x_2$

X is Haus. $\Longrightarrow$ $U_1 \ni x_1,\ U_2 \ni x_2,\ U_1 \cap U_2 = \emptyset$

$\underset{z_1}{\underset{\Psi}{U_1 \times Y}} \cap \underset{z_2}{\underset{\Psi}{U_2 \times Y}} = \emptyset$

Class **Problem 1.53.** Verify that any metric space is normal.

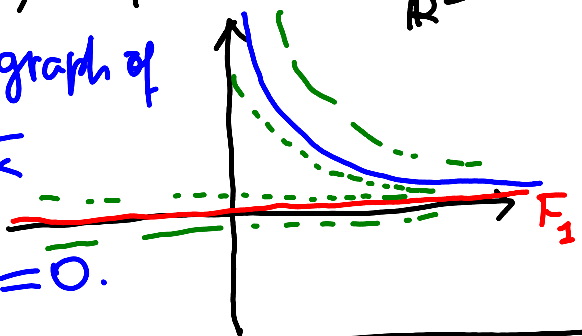
$F_1 \cap F_2 = \emptyset$, F_1, F_2 - closed

~~Take~~ suff ~~small~~ ε-~~neighborhoods~~ of F_1, F_2 ~~st they~~ do ~~not intersect.~~

impossible

Ex: $X = \mathbb{R}^2$, $F_1 = Ox$

F_2 = graph of $\frac{1}{x}$



$\operatorname{dist}(F_1, F_2) = 0$.

Consider $x \in F_1$. since F_2 is closed x is NOT an adherance point of F_2 $\Leftrightarrow$ $\exists\, \varepsilon_x > 0$ s. that $B_{\varepsilon_x}(x) \cap F_2 = \emptyset$.

Let $U_1 = \bigcup_{x \in F_1} B_{\varepsilon_x/3}(x)$

Let $U_2 = \bigcup_{y \in F_2} B_{\varepsilon_y/3}(y)$, if $B_{\varepsilon_y}(y) \cap F_1 = \emptyset$

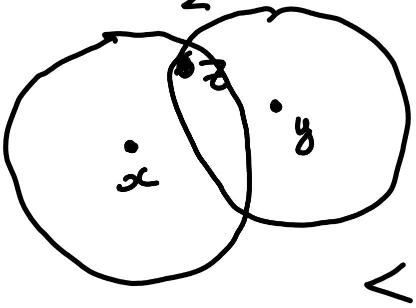
Evidently U_1, U_2 are open and contain F_1 and F_2, resp.

We claim that $U_1 \cap U_2 = \emptyset$.

Suppose the opposite: $\exists z \in U_1 \cap U_2$

$z \in U_1 \Rightarrow z \in B_{\varepsilon_x/3}(x)$ for some x.

$z \in U_2 \Rightarrow z \in B_{\varepsilon_y/3}(y)$ for some y.



Triangle inequality

$\rho(x,y) \le \rho(x,z) + \rho(z,y) <$

$< \varepsilon_x/3 + \varepsilon_y/3$

Suppose $\varepsilon_x \geq \varepsilon_y$ (otherwise similarly).

$\rho(x,y) \leq \frac{2}{3}\varepsilon_x$, so $y \in B_{\varepsilon_x}(x)$.

This contradicts the def. of ε_x.

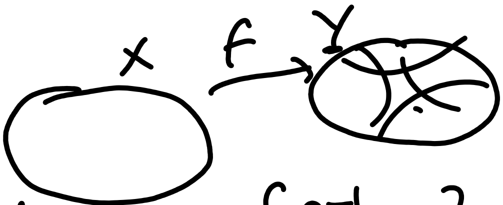
And of course, any metric space is Hausd. since for $x \neq y$, $\rho(x,y) =: d > 0$

$B_{d/3}(x) \cap B_{d/3}(y) = \emptyset$.

Home **Problem 1.70.** Prove that a continuous image of a compact is compact.

Class **Problem 1.71.** Let $f : X \to \mathbb{R}^1$ be a continuous function on a compact space X. Then f is bounded and reaches its maximal and minimal value.

1.70



$f(X) = Y$.

$\{U_\alpha\}$ open cover of Y

f cont. $\Rightarrow$ $\{f^{-1}U_\alpha\}$ form a cover of X

$\rightarrow$ $f^{-1}U_{\alpha_1}, \ldots, f^{-1}U_{\alpha_N}$ – a finite subcover, $U_{\alpha_1}, \ldots, U_{\alpha_N}$ form an open subcover of $\{U_\alpha\}$.

1.71. X is comp. $\overset{1.70}{\Longrightarrow}$ $f(X)$ is comp.

(to be proved) $\Longrightarrow$ $f(X)$ is a bounded closed set (a charac. of compact sets in $\mathbb{R}^n$).

then it has maximal and minimal points.

Class **Problem 1.73.** A cartesian product of compact spaces is compact.

For 2.) Observation. Suppose we have a cover $\{U_\alpha\}$ and its refinement $\{V_\beta\}$. And we can choose a finite subcover in V_β: $V_{\beta_1}, \dots, V_{\beta_N}$. Then we can find a finite subcov. in the initial cover $\{U_\alpha\}$.

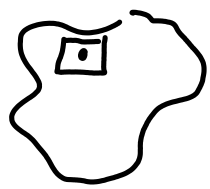
Indeed. $\alpha = \alpha(\beta)$, $U_\beta \subseteq U_{\alpha(\beta)}$

Consider $U_{\alpha(\beta_1)}, \dots, U_{\alpha(\beta_N)}$

$$\begin{array}{ccc} U_{\alpha(\beta_1)}, & \dots, & U_{\alpha(\beta_N)} \\ \cup\!\!\mid & & \cup\!\!\mid \\ V_{\beta_1}, & \dots, & V_{\beta_N} \end{array}$$

$$\bigcup_{i=1}^{N} U_{\alpha(\beta_i)} \supset \bigcup_{i=1}^{N} V_{\beta_i} = X.$$

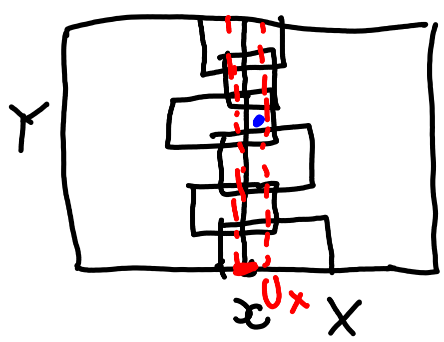
$X \times Y$, $\{\tilde{U}_\alpha\}$ open cover



using a base

a refinement $\{U_\beta \times V_\beta\}$, $U_\beta \in \tau_X$, $V_\beta \in \tau_Y$ of $\tilde{\mathcal{U}}$

By the "Observation" it is sufficient to find a finite subcover for $\{U_\beta \times V_\beta\}$



$Y \times \{x\}$ homeom to Y, hence compact.

$\Longrightarrow U_{\beta_{1(x)}} \times V_{\beta_{1(x)}}, \dots, U_{\beta_{N(x)}} \times V_{\beta_{N(x)}}$ a finite subcover for $Y \times \{x\}$

$U_x := \bigcap_{j=1}^{N} U_{\beta_{j(x)}}$ s.t. $U_x \times Y \subset \bigcup_{j=1}^{N} U_{\beta_{j(x)}} \times V_{\beta_{j(x)}}$.

These U_x form an open cover of X

$\rightsquigarrow U_{x_1}, \dots, U_{x_S}$ a finite subcover

We claim that the finite set

$$U_{\beta_j}(x_i)\times V_{\beta_j}(x_i), \quad i=1,\dots,s, \quad j=1,\dots,N(x_s)$$

is a cover. Indeed, take $\forall z=(x,y)$

$x\in U_{x_i}$ for some i

Then $(x,y)\in U_{\beta_j}(x_i)\times V_{\beta_j}(x_i)$ for some j.

Class **Problem 2.4.** Find an example of a manifold and two non-compatible smooth structures on it, i.e., two smooth atlases (U_i,φ_i) and (V_j,ψ_j) such that $\{(U_i,\varphi_i),(V_j,\psi_j)\}$ is not a smooth atlas.

Class **Problem 2.5.** Prove that the sphere S^n and the projective space $\mathbb{R}P^n$ are smooth manifolds.

$\mathbb{R}^1$ as a top. space

$(U,\varphi_1) \downarrow \mathbb{R}^1$ $\quad \varphi=id$, $U=\mathbb{R}^1$, $V=\mathbb{R}^1$

$\mathbb{R}^1$

$(\hat U,\hat\varphi)\downarrow \mathbb{R}^1$ $\quad U=\mathbb{R}^1$, $V=\mathbb{R}^1$, $\hat\varphi: t\to t^3$

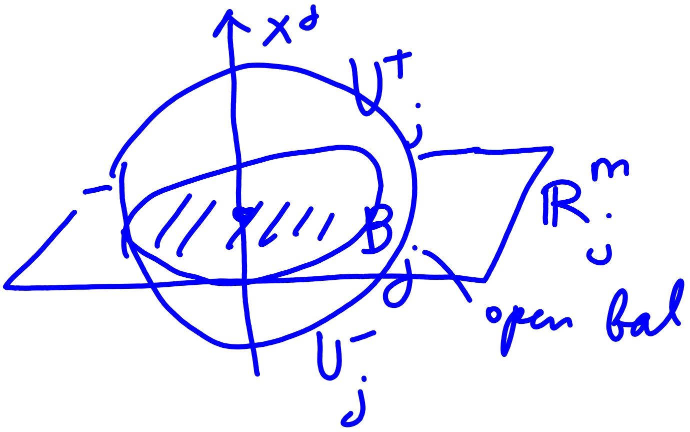
a homeomorphism from $\mathbb{R}^1$ to $\mathbb{R}^1$

We claim that $\{(U,\varphi),(\hat U,\hat\varphi)\}$ is not smooth.

$\hat\varphi\circ\varphi^{-1}: x\mapsto x^3$ $\quad$ smooth

$\varphi\circ\hat\varphi^{-1}: y\mapsto\sqrt[3]{y}$ $\quad$ not smooth in 0.

S^m, 1). $\quad S^m\subset\mathbb{R}^{m+1}$



U_j^+: $\vec x\in\mathbb{R}^{m+1}\cap S^m$, $x_j>0$

U_j^-: $\dots$ $x_j<0$

$\vec x\in S^m$ not all coord $=0$. $\Rightarrow\{U_j^\pm\}$ a cover

open since $x_j>0$ is open

$\varphi_j^\pm : U_j^\pm \to B_j$ $\quad (x_1,\dots,x_{m+1}) \mapsto (x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+1})$

$B_j =$ "V" open

cont. bij

$(\varphi_j^\pm)^{-1}(y_1,\dots,y_m) =$

$= (y_1,\dots,y_{j-1},\pm\sqrt{1-y_1^2-\dots-y_m^2},\ y_j,\dots,y_m)$

cont

So φ are homeom.

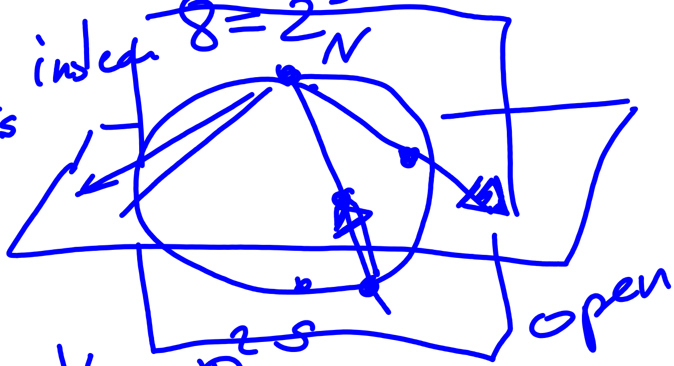
$(\varphi_i^-)(\varphi_j^+)^{-1}(y_1,\dots,y_m) =$ $\quad i<j$ other cases sim

$= (\varphi_i^-)(y_1,\dots,y_{j-1},\sqrt{1-y_1^2-\dots-y_m^2},\ y_j,\dots,y_m) =$

$= (y_1,\dots,y_{i-1},y_{i+1},\dots,y_{j-1},\sqrt{\dots},\ y_j,\dots,y_m)$ Smooth.

S^2 2 charts

instead ~~$8=2^3$~~



$U_N = S^2 \setminus \{S\}$

$U_S = S^2 \setminus \{N\}$

$V_N = V_S = \mathbb{R}^2$

$U_N \cap U_S \to \mathbb{R}^2 \setminus 0$

$\qquad\qquad\quad \mathbb{C}^2/ \quad z$

Polar coor

$\rho \mapsto \frac{1}{\rho}$

$\varphi \mapsto \varphi$

$\varphi_S \circ \varphi_N^{-1} : z \longmapsto \frac{1}{\bar z}$!

not analitic

$\varphi_S \rightsquigarrow J \circ \varphi_S = \hat{\varphi}_S$

compl..

$\hat{\varphi}_S \circ \varphi_N^{-1} : z \longmapsto \frac{1}{z}$

analm

S^2 is a compl. anl. mfd.

$\mathbb{R}P^n$ for homew