Class **Problem 2.15.** Find an example of smooth homeomorphism, which is not a diffeomorphism.

$\mathbb{R}^1$

$f:\mathbb{R}^1 \to \mathbb{R}^1 : t \mapsto t^3$

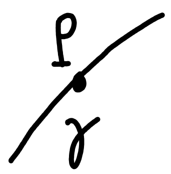
a smooth homeomorphism

$f^{-1}: s \mapsto \sqrt[3]{s}$ – not smooth at $\{0\}$.

Class **Problem 2.26.** (a justification of the definition) Suppose that $\gamma : (-1;1) \to M$ is a smooth mapping and $\gamma(0) = P$. Then the correspondence

$$\xi_\gamma : (x^1, \ldots, x^n) \rightsquigarrow \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right)\Big|_{t=0}$$

is a vector at P, where, for a local coordinate system $(x^1, \ldots, x^n)$, the mapping γ is defined as $(x^1(t), \ldots, x^n(t))$.



$\gamma:(-1,1) \to M$

$\gamma(0) = P$

$$\xi_\gamma : (x^1, \ldots, x^n) \rightsquigarrow \left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right)\Big|_{t=0}$$

U_α, U_β
U, U', U''
$x^i, x^{j'}, x^{k''}$

$$\boxed{x^{j'} = (x^{1'})}$$

$$\xi_\gamma : (x^{1'}, \ldots) \rightsquigarrow \left(\frac{dx^{1'}}{dt}, \ldots\right)\Big|_{t=0}$$

$$\xi_\gamma^{i'} = \frac{dx^{i'}}{dt} = \frac{\partial x^{i'}}{\partial x^i}\frac{dx^i}{dt} = \boxed{\frac{\partial x^{i'}}{\partial x^i}\xi_\gamma^i}$$

tensor law $\Rightarrow$ ξ_γ is a vector in 1 sense.

Home **Problem 2.33.** Prove the second equivalence in the above theorem ([Mishchenko, Fomenko], pp 125–127).

ξ is a tangent vector in the 1 sense (tensor)

$\xi \longrightarrow D_\xi$, defined by

$$D_\xi(f) := \sum_{j=1}^{m} \frac{\partial f}{\partial x^j}\xi^j$$

Right-hand side does not depend on the

choice of coordinate system. For (U',φ'):

$$\frac{\partial f}{\partial x^{j'}}\xi^{j'} = \underbrace{\frac{\partial f}{\partial x^{j}}\frac{\partial x^{j}}{\partial x^{j'}}}_{\text{compos.}}\underbrace{\frac{\partial x^{j'}}{\partial x^{\cancel{i}k}}\xi^{\cancel{i}k}}_{\text{tensor law}} = \frac{\partial f}{\partial x^{j}}\delta^{j}_{k}\xi^{k} = *$$

product of matrices inverse to each other

$$\delta^{j}_{k} = \begin{cases} 1, & j=k \\ 0, & j\neq k \end{cases}$$

Kronecker's symbol

$$* = \frac{\partial f}{\partial x^{j}}\xi^{j}$$

we are done.

So, D_ξ is a well-defined linear functional $D_\xi : C^\infty(U) \to \mathbb{R}$. Evidently D_ξ is defined on germs of functions.

$$D_\xi(f\cdot g) = \frac{\partial(f\cdot g)}{\partial x^i}\xi^i = \left(\frac{\partial f}{\partial x^i}\cdot g\Big|_p + f\Big|_p\frac{\partial g}{\partial x^i}\right)\xi^i =$$

Leibniz Rule for partial differentiation

$$= D_\xi(f)\cdot g\big|_p + f\big|_p\cdot D_\xi(g),$$

so D_ξ is a differentiation operator.

Evidently, $\xi \mapsto D_\xi$ is a linear map.

We wish to show that it is injective.

Suppose, $\xi \neq 0$, we wish to show that $D_\xi(f)\neq 0$ for some $f\in C^\infty(U)$.

$\xi\neq 0$ means that $\xi^i \neq 0$ for some i (after fixing a coord. syst. (U,φ))

$x^i : U \to \mathbb{R}^1$ (taking of i'th coordinate)

$\in C^\infty(U)$

Calculate: $D_\xi(x^i) = \frac{\partial x^i}{\partial x^j}\xi^j = \delta^i_j\xi^j = \xi^i \neq 0$

as we wish. So $\xi \mapsto D_\xi$ is injective.

Let's prove its surjectivity:

Lemma. Let $f : \underset{\mathbb{R}^n}{\underset{\cap}{U}} \to \mathbb{R}$ be a smooth function.

$P_0 = (x_0^1, \dots, x_0^n) \in U$. Then there exist smooth functions $h_{ij}(\vec{x})$, $i, j = 1, \dots, n$, such that

$$f(\vec{x}) = f(\vec{x_0}) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_{P_0} \cdot (x^i - x_0^i) +$$

$$+ \underbrace{\sum_{i,j=1}^{n} h_{ij}(\vec{x})\,(x^i - x_0^i)(x^j - x_0^j).}$$

Proof:

$$f(\vec{x}) = f(\vec{x_0}) + \underbrace{\int_0^1 \frac{d}{dt} f\left(x_0^1 + t(x^1 - x_0^1), \dots, x_0^n + t(x^n - x_0^n)\right) dt}_{N-L}$$

$$f\left(x_0^1 + t(x^1 - x_0^1), \dots\right)\Big|_0^1$$

$$\shortparallel$$

$$f(\vec{x}) - f(\vec{x_0})$$

$$= f(\vec{x_0}) + \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\left(x_0^1 + t(x^1 - x_0^1), \dots\right) \cdot (x^i - x_0^i)\, dt =$$

$$= f(\vec{x_0}) + \sum_{i=1}^{n} \underbrace{\left[\int_0^1 \frac{\partial f}{\partial x^i}\left(x_0^1 + t(x^1 - x_0^1), \dots\right) dt\right]}_{h_i(\vec{x})} (x^i - x_0^i)$$

Apply this to each h_i. substitute

$$h_i(\vec{x}) = h_i(\vec{x_0}) + \sum_{j=1}^{n} h_{ij}(\vec{x})\,(x^j - x_0^j).$$

Also, $h_i(\vec{x_0}) = \int_0^1 \frac{\partial f}{\partial x^i}\left(x_0^1 + t(\cancel{x_0^1 - x_0^1}), \dots\right) dt =$ 0

$$= \left.\frac{\partial f}{\partial x^i}\right|_{P_0}$$

Substitution:

$$f(\vec{x}) = f(\vec{x_0}) + \sum_{i=1}^{n} \left(h_i(\vec{x_0}) + \sum_{j=1}^{n} h_{ij}(\vec{x})(x^j - x_0^j)\right) \times$$

$$\times\,(x^i - x^i_0) = f(\vec{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{P_0} (x^i - x^i_0) +$$

$$+ \sum_{i,j=1}^{n} h_{ij}(\vec{x})(x^j - x^j_0)(x^i - x^i_0). \quad \square$$

We consider an arbitrary D (diff. operator in P_0). Fix (U, φ) in a neighbor. of P_0.

$$\underset{\substack{\uparrow \\ \text{const. } f \equiv 1}}{D(1)} = D(1 \cdot 1) \overset{\text{Leib.}}{=} D(1) \cdot 1\big|_{P_0} + 1\big|_{P_0} D(1) =$$

$$= 2D(1)$$

$\Rightarrow D(1) = 0$. $\quad D$ is linear $\Rightarrow$

$D(a) = 0$ for any constant function $\equiv a$

Take any $f \in C^\infty(U)$.

$$D(f) \underset{\text{Lemma}}{=} \underbrace{D(f(P_0))}_{=0} + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{P_0} D(x^i - x^i_0) +$$

$$+ \sum_{i,j=1}^{n} \Big\{ D(h_{ij}) \cdot \underbrace{(x^i - x^i_0)\big|_{P_0}}_{=0} \cdot \underbrace{(x^j - x^j_0)\big|_{P_0}}_{=0} +$$

$$+ h_{ij}\big|_{P_0} D(x^i - x^i_0) \cdot \underbrace{(x^j - x^j_0)\big|_{P_0}}_{=0} +$$

$$+ h_{ij}\big|_{P_0} \underbrace{(x^i - x^i_0)\big|_{P_0}}_{=0} \cdot D(x^j - x^j_0) \Big\} =$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{P_0} \Big(D(x^i) - \underbrace{D(x^i_0)}_{=0}\Big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{P_0} D(x^i). \quad (*)$$

Consider ξ_D: $(\xi_D)^i = D(x^i)$. Then

$$D_{\xi_D}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} (\xi_D)^i = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} D(x^i) =$$

$$= D(f) \quad \text{by } (*).$$

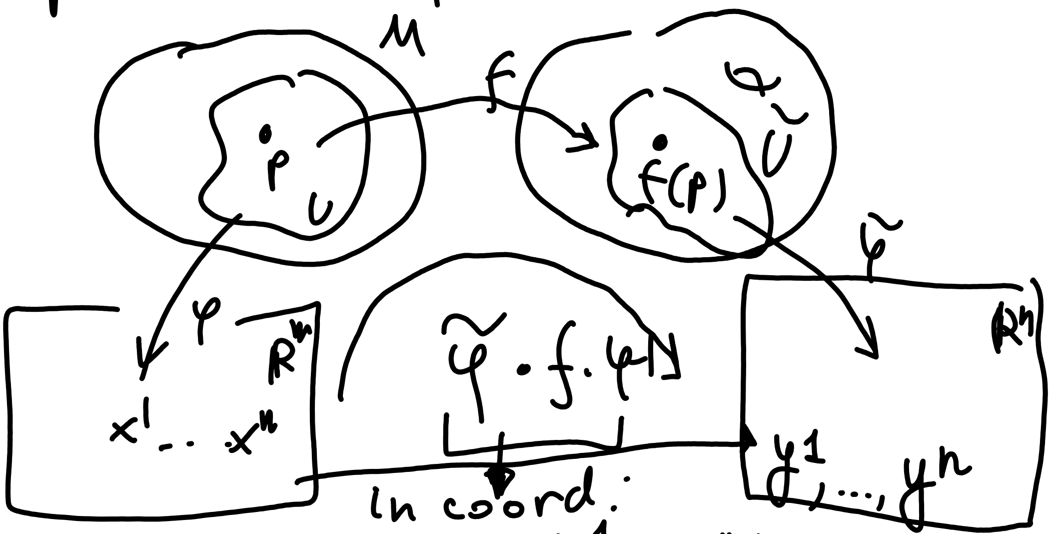
So, $\xi \mapsto D_\xi$ is epimorphism (surjection).

Theory Def The tangent map for a smooth map $f: M \to N$,

$$d_pf = T_pf : T_pM \to T_{f(p)}N ;$$

3 equivalent ways:

① $d_pf(\xi)=\eta$ such that for a local representation of f:



in coord.:

$$y^1=y^1(x^1,\dots,x^m)$$
$$\vdots$$
$$y^n=y^n(x^1\dots x^m)$$

$$\boxed{\eta^j = \frac{\partial y^j}{\partial x^i}\,\xi^i}$$

coord. of η in $(\tilde U, \tilde\varphi)$

Need to verify (now prob):

1) Well defined (does not dep on U,φ)

2) The image is a vector (tensor law) $\sim(\tilde U,\tilde\varphi)$

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② $d_pf[\gamma] = [f\circ\gamma]$.

③ $(d_pf)(D)(g) = D(g\circ f)$, $g \in C^\infty(N)$