1.5. In a metric space (X, ρ), Y is open $\Leftrightarrow$ $X \setminus Y$ is closed.

$Y = \operatorname{Int} Y$ $\qquad$ $\overline{X \setminus Y} = X \setminus Y$

$\Updownarrow$

$\forall y \in Y \ \exists \varepsilon > 0$ s.t. $y \in B_\varepsilon(y) \subseteq Y$

$\boxed{\Rightarrow}$ $x \in \overline{X \setminus Y}$ we need to prove: $x \in X \setminus Y$

Suppose the opp.: $\begin{cases} x \in \overline{X \setminus Y} \\ x \notin X \setminus Y \end{cases}$ so $x \in Y$,

then $\exists_\varepsilon B_\varepsilon(x) \subseteq Y$. for $\forall x' \in X \setminus Y$

we have $x' \notin Y \Rightarrow x' \notin B_\varepsilon(x) \Rightarrow$

$\rho(x, x') \geq \varepsilon \Rightarrow \rho(x, X \setminus Y) \geq \varepsilon > 0$

$\Rightarrow x \notin \overline{X \setminus Y}$, a contrad.

$\boxed{\Leftarrow}$ $X \setminus Y = \overline{X \setminus Y}$ (is closed). We need to verify

$\forall y \in Y$, $\exists_\varepsilon B_\varepsilon(y) \subseteq Y$. Suppose the opp.:

for some y: and $\forall \varepsilon > 0$ $B_\varepsilon(y) \cap X \setminus Y \neq \emptyset$
$\varepsilon = \frac{1}{n}$ $\qquad \Downarrow x_n$

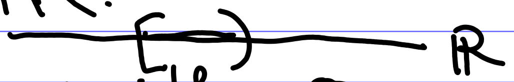
$\rho(y, x_n) < \frac{1}{n} \Rightarrow \rho(y, X \setminus Y) = 0$
$x_n \in X \setminus Y$

Contradiction $\qquad \Rightarrow y \in \overline{X \setminus Y} = X \setminus Y$.

Moreover: $\operatorname{Int} Y = X \setminus \overline{(X \setminus Y)}$
(for home work).

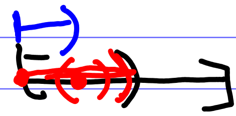
1.7. $X = [0, 1]$, $U_\alpha = [0, \frac{1}{\alpha})$

in $\mathbb{R}$. not $\mathbb{R}$



neither open nor closed

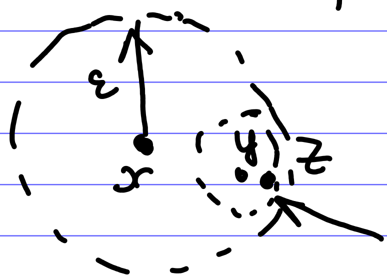
in X : Open.



$B_{1/4}(0)$

$$\bigcap_\alpha U_\alpha = \{0\}$$, which is closed because $\forall \varepsilon$ $\{0\} \neq B_\varepsilon(0)$

1.8. $B_\varepsilon(x)$ is open.



$$y \in B_\varepsilon(x) \Rightarrow \underbrace{\rho(x,y)}_{=\,\tau} < \varepsilon$$

$$B_{\varepsilon-\tau}(y) \subseteq B_\varepsilon(x)$$

Indeed, $\forall z \in B_{\varepsilon-\tau}(y)$

$$\rho(x,z) \overset{\text{TRIANGLE INEQ.}}{\leq} \rho(x,y)+\rho(y,z) < \tau + (\varepsilon-\tau) = \varepsilon.$$

1.9, 1.10 : $\operatorname{Int} Y = \operatorname{Int}(\operatorname{Int} Y) \Leftrightarrow \operatorname{Int} Y$ is open

$\overline{Y} = \overline{\overline{Y}}$. $\Leftrightarrow$ $\overline{Y}$ is closed.

Evid. $\overline{Y} \subset \overline{\overline{Y}}$. Consider $\forall \varepsilon > 0$ and $x \in \overline{\overline{Y}} \Rightarrow \exists y \in \overline{Y}$ s.t. $\rho(x,y) < \frac{\varepsilon}{2}$ and, for y, $\exists y_0 \in Y$ s.t. $\rho(y_0, y) < \frac{\varepsilon}{2}$

Then $\rho(\underset{\in Y}{y_0}, x) \underset{\text{the triangle ineq.}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow x \in \overline{Y}$

$\overline{Y} = \overline{\overline{Y}}$.

The another for homework.

1.12 was at the lecture

1.14. $X = \underset{a}{\bullet}$

open sets $:= \tau$

$\emptyset$ $\{a\} = X$ both should be open.

$X = \underset{a}{\bullet} \quad \underset{b}{\bullet}$

On X $\exists!$ (exists and only one) topology.

№ 1

$\emptyset$, a, b, $\{a, b\} = X$

(all sets are open)

2 discrete points

ALL THESE ARE TOPOLOGIES. (i.e. 1)–4) of the Def. are fulfilled).

$\emptyset$, a, $\{a, b\}$ +

$\emptyset$, b, $\{a, b\}$))

+

$\emptyset$, $\{a, b\}$ +

(semi-glued)

2 glued points, ~~to each other~~.

If we have a metric space on this set. We need to define only $\rho(a,b) \neq 0$

Take. $\rho(a,b) = 1$

Then

$a \xleftrightarrow{\;1\;} b$

$a = B_{1/2}(a)$ — is open

$b = B_{1/2}(b)$

even 1

So, the topology for metric:

$\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$.

So this top. is № 1.

We claim that the other 3 top-s are not

metrizable. Indeed, for a metriz., we have this list of open sets $= \tau$, and for 2–4 we have another lists.

Prob. 1.31.

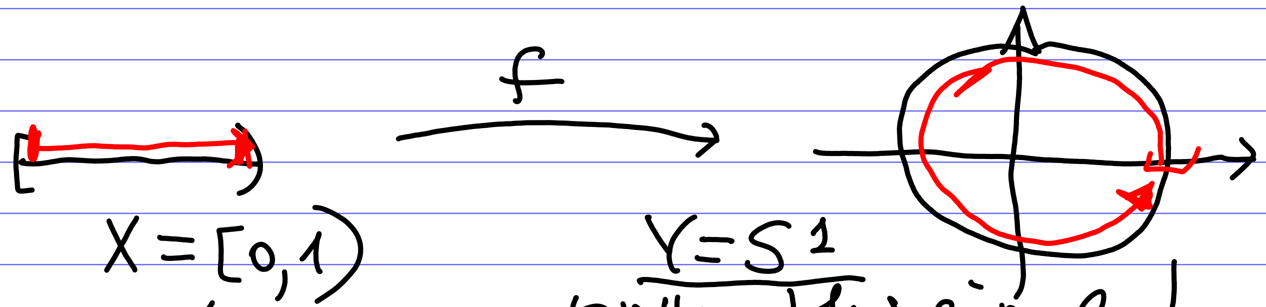
$f = Id$: $X = \{a, b\}$, τ: 1, discrete top. open: $\emptyset, \{a\}, \{b\}, \{a,b\}$

$\to X = \{a, b\}$, τ: 2. open: $\emptyset, \{a\}, \{a, b\}$.

Then, of course $f^{-1}(V)$ is open
V = any open set

So: f is a cont. bijection.
but f^{-1} is not: because $\{b\}$ is open in (X, τ_1)
but $(f^{-1})^{-1}(\{b\}) = \{b\}$ is not an open set in (X, τ_2).
f^{-1}: inverse; $(\,)^{-1}$: full pre-image.

A more geometric (for homework):



$X = [0,1)$ $Y = S^1$

ONLY the circle! induced from $\mathbb{R}^2$

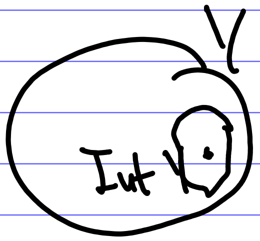
$f(x) = e^{2i\pi x}$

Prove!

The main: why f^{-1} is discont.

Prbs. 1.16–1.19.

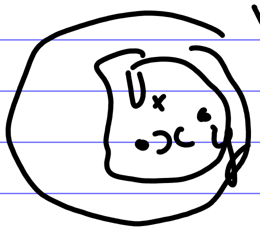
1.19. Int Y is open.



$y \in \operatorname{Int} Y \iff \exists\, U_x$ – (open) neighb. of x

$x \in U_x \subseteq Y$.

I claim: $U_x \subseteq \operatorname{Int} Y$.
all this set



$y \in U_x \Rightarrow y \in U_x \subset Y$

$\Rightarrow \operatorname{Int} Y = \bigcup\limits_{x \in \operatorname{Int} Y} \underbrace{U_x}_{\text{open}} \Rightarrow$ is open.

1.18. Y is open $\iff Y = \operatorname{Int} Y$.

$\Leftarrow$

$\Rightarrow$ Y is open, $\forall y \in Y$

$y \in \underbrace{Y}_{\text{open}} \subset Y \Rightarrow y \in \operatorname{Int} Y$

so $Y = \operatorname{Int} Y$.

1.17. $\bar{Y}$ is closed. $\iff X \setminus \bar{Y}$ is open.

$\bar{Y}$ = all adherence points. of Y.

$x \in \bar{Y}$, iff, any open neighborhood of x intersects Y.

Suppose, $X \setminus \bar{Y}$ is not open. i.e. $\exists\, x \in X \setminus \bar{Y}$

s.t. any open n. $x \in U_x \cap \bar{Y} \neq \emptyset$

$x_0 \in$

$\Rightarrow$ a poin $x_0 \in \bar{Y}$ and U_x is an open neigb. of x_0

$\Rightarrow$ by def. of $\bar{Y}$: $U_x \cap Y \neq \emptyset$

So any open neib. of x intersects Y

$\xRightarrow{\text{by def}}$ $x \in \overline{Y}$, a contrad.

Till 1.47 — for homework.

hopefully at 10:45 in a week!